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Integral Equations and Operator Theory

An Improved Discrete p-Hardy Inequality

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Abstract. We improve the classical discrete Hardy inequality for $1 < p < \infty$ for functions on the natural numbers. For integer values of p the Hardy weight is shown to have a series expansion with strictly positive coefficients. Notably, this weight is optimal, i.e. critical and null-critical.

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1. Introduction and Main Result

In 1918 Hardy was looking for a simple and elegant proof of Hilbert's theorem in the context of the convergence of double sums, [12]. Although it is not explicitly mentioned, the paper contains the essential argument for his then famous inequality. In a letter to Hardy in 1921, [20], Landau gave a proof with the sharp constant

$$\sum_{n=1}^{\infty} a_n^p \geq \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{\infty}\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^p$$

for $p > 1$ where (a_n) is an arbitrary sequence of non-negative real numbers. This inequality was first highlighted in [13] and is referred to as a *p-Hardy inequality*. Since then various proofs of this inequality were given, where short and elegant ones are due to Elliott [6] and Ingham, see [13, p. 243] and by Lefèvre [21]. See also [15] for a beautiful historical survey about the origins of Hardy's inequality.

It is not hard to see that the inequality above can be derived from the following inequality for compactly supported $\phi \in C_c(\mathbb{N})$ with $\phi(0) = 0$

$$\sum_{n=1}^{\infty} |\phi(n) - \phi(n-1)|^p \geq \sum_{n=1}^{\infty} w_p^H(n)|\phi(n)|^p,$$

where

$$w_p^H(n) = \left(\frac{p-1}{p}\right)^p \frac{1}{n^p}.$$

In this work, we show that the classical weight w_p^H can be replaced by a pointwise strictly larger weight w_p, and the aforementioned Hardy inequality holds with w_p^H being replaced by w_p. Recently, the concept of optimality was studied for general $1 < p < \infty$ on weighted graphs in [7]. Employing these results, we conclude that w_p is *optimal*, which means that

- for every function w with $w \geq w_p$ and $w \neq w_p$, the p-Hardy inequality does not hold (criticality), and in addition
- $u \notin \ell^p(\mathbb{N}, w_p)$, where $u(n) = n^{(p-1)/p}$, $n \in \mathbb{N}$ (null-criticality).

The first optimality criterion says that there cannot be a p-Hardy weight dominating w_p. The second optimality criterion says that the underlying ground state u is not an eigenfunction and so, the Hardy inequality does not admit a minimizer.

Consequently, by [7, Theorem 2.6], the weight w_p is also *optimal at infinity*, which means that for every $\lambda > 0$ and each finite set $K \subset \mathbb{N}$, the weight $(1+\lambda)w_p$ does not yield a p-Hardy inequality for functions φ supported outside of K. This criterion in particular shows that the constant $((p-1)/p)^p$ is optimal, a fact already known for a century, cf. [15].

We formulate the main theorem of this paper.

Theorem 1. *Let $p > 1$. Then, for all $\phi \in C_c(\mathbb{N})$ with $\phi(0) = 0$,*

$$\sum_{n=1}^{\infty} |\phi(n) - \phi(n-1)|^p \geq \sum_{n=1}^{\infty} w_p(n)\, |\phi(n)|^p ,$$

where w_p is a strictly positive function given by

$$w_p(n) = \left(1 - \left(1 - \frac{1}{n}\right)^{\frac{p-1}{p}}\right)^{p-1} - \left(\left(1 + \frac{1}{n}\right)^{\frac{p-1}{p}} - 1\right)^{p-1} .$$

Furthermore, w_p is optimal, and we have for all $n \in \mathbb{N}$

$$w_p(n) > w_p^H(n).$$

Moreover, for integer $p \geq 2$, we have $w_p(n) = \sum_{k \in 2\mathbb{N}_0} c_k n^{-k-p}$ with $c_k > 0$.

Example 2. The case $p = 2$ was already covered in [16] (with optimality proven in [17]), and via a different method in [11,19]. In this case one gets $w_2(1) = 2 - \sqrt{2}$ and for $n \geq 2$

$$w_2(n) = -\sum_{k \in 2\mathbb{N}} \binom{1/2}{k} \frac{2}{n^k} = \frac{1}{4}\frac{1}{n^2} + \frac{5}{64}\frac{1}{n^4} + \frac{21}{512}\frac{1}{n^6} + \frac{429}{16384}\frac{1}{n^8} + \ldots$$

In the case $p = 3$, one obtains $w_3(1) = 1 - (2^{2/3} - 1)^2$ and for $n \geq 2$

$$w_3(n) = \sum_{k \in 2\mathbb{N}+1} \left(2\binom{2/3}{k} - \binom{4/3}{k}\right) \frac{2}{n^k} = \frac{8}{27}\frac{1}{n^3} + \frac{8}{81}\frac{1}{n^5} + \frac{112}{2187}\frac{1}{n^7} + \ldots$$

In the case $p = 4$, one gets $w_4(1) = 1 - (2^{3/4} - 1)^3$ and for $n \geq 2$

$$\begin{aligned} w_4(n) &= \sum_{k \in 2\mathbb{N}+2} \left(3\binom{3/2}{k} - 3\binom{3/4}{k} - \binom{9/4}{k}\right) \frac{2}{n^k} \\ &= \frac{81}{256}\frac{1}{n^4} + \frac{891}{8192}\frac{1}{n^6} + \frac{58653}{1048576}\frac{1}{n^8} + \ldots \end{aligned}$$

For general $p > 1$ one obtains the asymptotics

$$w_p(n) = \left(\frac{p-1}{pn}\right)^p \left(1 + \epsilon_p(n)\right),$$

where

$$\epsilon_p(n) = \left(\frac{3}{8} - \frac{1}{8p}\right)\frac{1}{n^2} + \left(\frac{215p^3 - 38p^2 - 31p + 6}{1152p^3}\right)\frac{1}{n^4} + O\left(\frac{1}{n^6}\right).$$

From this formula it is clear that $w_p(n)$ is strictly larger than the classical Hardy weight for large n. Note however that the theorem above states that $\epsilon_p(n) > 0$ at all places $n \in \mathbb{N}$. It is not hard to check that $\epsilon_p(n)$ can be expanded into a power series with respect to $1/n$ where all odd coefficients vanish. Theorem 1 states that for integer $p \geq 2$ these coefficients are positive. We conjecture that all these coefficients are strictly positive for all $p > 1$.

Remark 3. It is easy to see that for $1 < p < \infty$, our p-Hardy inequality can be stated as follows: for every real-valued sequence $a = (a_n)$, one has

$$\sum_{n=1}^{\infty} |a_n|^p \geq \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{\infty} \left(1 + \epsilon_p(n)\right) \left|\frac{1}{n}\sum_{j=1}^{n} a_j\right|^p,$$

where the function ϵ_p is as in the previous example. Denoting by C the *Cesàro mean operator* on $\ell^p(\mathbb{N})$, defined as $C(a) = \frac{1}{n}\sum_{j=1}^{n} a_j$, the above inequality says that $C : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}, \rho)$ is bounded, where $\rho = 1 + \epsilon_p$ is understood as a measure on $\mathbb{N}$. Equivalently, one obtains ℓ^p-boundedness of a weighted version of the Cesàro mean operator. For certain weights, such boundedness phenomena were studied recently in [22].

2. Proof of the Hardy Inequality

The *combinatorial p-Laplacian* Δ_p for real valued functions on $\mathbb{N}_0$ is given by

$$\Delta_p f(n) = \sum_{m=n\pm 1} \operatorname{sgn}\left(f(n) - f(m)\right) |f(n) - f(m)|^{p-1}$$

for all functions f and $n \geq 1$, where sgn is the function which takes the value -1 on $(-\infty, 0)$, the value 1 on $(0, \infty)$ and 0 at 0.

The following proposition shows that the existence of a suitable positive supersolution of $\Delta_p u \geq 0$ implies the non-negativity of the corresponding energy functional. This is one of the implications of the so-called Allegretto-Piepenbrink-type theorem (see [1,2,23] for linear versions in the continuum, [4,18] for a linear version in the discrete setting, [24] for a non-linear version in the continuum and [8] for a recent version in the quasi-linear discrete setting). This statement is used to show that the weight w_p is in fact a p-Hardy weight.

Proposition 4. *Let $p > 1$ and let $u\colon \mathbb{N}_0 \to [0, \infty)$ be strictly positive on $\mathbb{N}$ and such that $u(0) = 0$. Suppose that $w\colon \mathbb{N} \to \mathbb{R}$ satisfies $\Delta_p u = wu^{p-1}$ on $\mathbb{N}$.*

Then for all $\phi \in C_c(\mathbb{N})$ *with* $\phi(0) = 0$ *we have*

$$\sum_{n\in\mathbb{N}} |\phi(n) - \phi(n-1)|^p \geq \sum_{n\in\mathbb{N}} w(n)\,|\phi(n)|^p \,.$$

The proof follows along the lines of the proof of Proposition 2.2 in [10].

Proof. Let $p > 1$. From Lemma 2.6 in [10], we obtain for all $0 \leq t \leq 1$ and $a \in \mathbb{C}$

$$|a - t|^p \geq (1-t)^{p-1}(|a|^p - t).$$

Let w be such that $\Delta_p u = wu^{p-1}$. For given $\varphi \in C_c(\mathbb{N})$, we can consider $\psi = \varphi/u \in C_c(\mathbb{N})$, by strict positivity of u on $\mathbb{N}$. We assume for a moment that $m, n \in \mathbb{N}$ are such that $u(n) \geq u(m)$ and $\psi(m) \neq 0$. We apply the above inequality with the choice $t = u(m)/u(n)$ and $a = \psi(n)/\psi(m)$ in order to obtain

$$\big|(u\psi)(n) - (u\psi)(m)\big|^p \geq \big|u(n) - u(m)\big|^{p-1}\big(|\psi(n)|^p u(n) - |\psi(m)|^p u(m)\big).$$

Further, since $u^p(n) \geq |u(n) - u(m)|^{p-1}u(n)$, the above inequality remains true even if $\psi(m) = 0$. Summing over $\mathbb{N}$, we obtain

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} |(u\psi)(n) - (u\psi)(n-1)|^p \\
&\geq \sum_{n\in\mathbb{N}} \operatorname{sgn}(u(n) - u(n-1))\,|u(n) - u(n-1)|^{p-1} \cdot \\
&\qquad \big(|\psi(n)|^p\, u(n) - |\psi(n-1)|^p\, u(n-1)\big) \\
&= \sum_{n\in\mathbb{N}} \Delta_p u(n)\,|\psi(n)|^p\, u(n) \;=\; \sum_{n\in\mathbb{N}} w(n) u^p(n)\,|\psi(n)|^p \,.
\end{aligned}$$

Note that the latter two equalities follow from rearranging the involved sums while recalling that $u(0) = 0$, and using the assumption $\Delta_p u = wu^{p-1}$. Recalling that $\phi = u\psi$, we infer the statement. □

Next we show that for the weight w_p on $\mathbb{N}$ taken from Theorem 1

$$w_p(n) = \Big(1 - (1 - 1/n)^{(p-1)/p}\Big)^{p-1} - \Big((1 + 1/n)^{(p-1)/p} - 1\Big)^{p-1},$$

there is a suitable positive function u such that $\Delta_p u = w_p u^{p-1}$.

Proposition 5. *Let* $p > 1$*. Then, the function* $u\colon \mathbb{N}_0 \to [0, \infty)$*,* $u(n) = n^{(p-1)/p}$ *satisfies*

$$\Delta_p u = w_p u^{p-1} \qquad \text{on } \mathbb{N}.$$

Proof. One directly checks that for all $n \in \mathbb{N}$

$$\frac{\Delta_p u(n)}{u^{p-1}(n)} = \frac{\Delta_p n^{(p-1)/p}}{n^{(p-1)^2/p}} = w_p(n)$$

which immediately yields the statement. □

The choice of the function u in the previous proposition is motivated by the so-called supersolution construction which yields optimal p-Hardy weights both in the continuum case for $p > 1$, cf. [3,5], as well as for graphs with $p = 2$, cf. [17]. Moreover, for $p = 2$, the function $u(n) = n^{1/2}$ arises naturally in the method applied in [11,19] for a proof of the optimal Hardy inequality on the line graph.

Combining the two propositions above already yields the p-Hardy inequality with the weight w_p. Next we show that w_p is strictly larger than the classical Hardy weight $w_p^H(n) = \big((p-1)/p\big)^p n^{-p}$ for all $n \in \mathbb{N}$.

3. Proof of $w_p > w_p^H$

In this section we show that the weight

$$w_p(n) = \left(1 - \left(1 - \frac{1}{n}\right)^{\frac{p-1}{p}}\right)^{p-1} - \left(\left(1 + \frac{1}{n}\right)^{\frac{p-1}{p}} - 1\right)^{p-1}$$

from the main theorem, Theorem 1, is strictly larger than the classical p-Hardy weight

$$w_p^H(n) = \left(\frac{p-1}{p}\right)^p \frac{1}{n^p}.$$

In fact, for fixed $p \in (1, \infty)$, we analyze the function $w\colon [0,1] \to [0,\infty)$

$$w(x) = \Big(1 - (1-x)^{1/q}\Big)^{p-1} - \Big((1+x)^{1/q} - 1\Big)^{p-1}$$

for $x \in [0, 1/2]$ and $x = 1$, where $q \in (1, \infty)$ is such that $1/p + 1/q = 1$. Specifically, we show

$$w(x) > \left(\frac{x}{q}\right)^p.$$

The case $x = 1$ is simple and is treated at the end of the section. The proof for $x \leq 1/2$ is also elementary but more involved. We proceed by bringing w_p into form for which we then analyze its parts. This will be eventually done by a case distinction depending on p.

Recall the binomial theorem for $r \in [0, \infty)$ and $0 \leq x \leq 1$

$$(1 \pm x)^r = \sum_{k=0}^{\infty} \binom{r}{k} (\pm 1)^k x^k$$

where $\binom{r}{0} = 1$, $\binom{r}{1} = r$ and $\binom{r}{k} = r(r-1)\cdots(r-k+1)/k!$ for $k \geq 2$ which is derived from the Taylor expansion of the function $x \mapsto (1 \pm x)^r$. Applying this formula to the function w from above we obtain

$$
\begin{aligned}
w(x) &= \left(-\sum_{k=1}^{\infty}\binom{1/q}{k}(-x)^k\right)^{p-1} - \left(\sum_{k=1}^{\infty}\binom{1/q}{k}x^k\right)^{p-1}\\
&= \left(\frac{x}{q}\right)^{p-1}\left(\left(q\sum_{k=0}^{\infty}\binom{1/q}{k+1}(-x)^k\right)^{p-1} - \left(q\sum_{k=0}^{\infty}\binom{1/q}{k+1}x^k\right)^{p-1}\right)
\end{aligned}
$$

To streamline notation we set

$$
g(x) = q\sum_{k=1}^{\infty}\binom{1/q}{k+1}x^k.
$$

Note that since $q\binom{1/q}{1} = 1$ and $q\left|\binom{1/q}{k}\right| < 1$ for $k \geq 2$, we have $0 < |g(\pm x)| < 1$ for $0 < x \leq 1/2$. Thus, we can apply the binomial theorem to $\big(1+g(\pm x)\big)^{p-1}$ in order to get

$$
\begin{aligned}
w(x) &= \left(\frac{x}{q}\right)^{p-1}\left(\big(1+g(-x)\big)^{p-1} - \big(1+g(x)\big)^{p-1}\right)\\
&= \left(\frac{x}{q}\right)^{p-1}\left(\sum_{n=0}^{\infty}\binom{p-1}{n}\big(g^n(-x) - g^n(x)\big)\right)\\
&= \left(\frac{x}{q}\right)^{p-1}\left(\binom{p-1}{1}\big(g(-x) - g(x)\big) + \sum_{n=2}^{\infty}\binom{p-1}{n}\big(g^n(-x) - g^n(x)\big)\right)
\end{aligned}
$$

Thus, we have to show that the second factor on the right hand side is strictly larger than x/q. Using $q = p/(p-1)$ we compute the first term in the parenthesis on the left hand side

$$
\begin{aligned}
\binom{p-1}{1}\big(g(-x) - g(x)\big) &= q(p-1)\sum_{k=1}^{\infty}\binom{1/q}{k+1}\left((-x)^k - x^k\right)\\
&= \frac{q(p-1)(1/q)(1/q-1)}{2}(-2x) + q(p-1)\sum_{k=2}^{\infty}\binom{1/q}{k+1}\left((-x)^k - x^k\right)\\
&= \frac{x}{q} - 2p\sum_{k\in 2\mathbb{N}+1}\binom{1/q}{k+1}x^k\\
&= \frac{x}{q} + E_p(x),
\end{aligned}
$$

with

$$
E_p(x) = -2p\sum_{k\in 2\mathbb{N}+1}\binom{1/q}{k+1}x^k > 0
$$

since $-2p\binom{1/q}{k+1} > 0$ for odd k and $x > 0$. So, it remains to show that for the term

$$
F_p(x) = \sum_{n=2}^{\infty}\binom{p-1}{n}\big(g^n(-x) - g^n(x)\big)
$$

we have for $0 < x \leq 1/2$

$$E_p(x) + F_p(x) > 0.$$

Specifically, we then get with the substitution $x = 1/n$

$$w_p(n) = w(1/n) = \left(\frac{1}{nq}\right)^{p-1}\left(\frac{1}{nq} + E_p(1/n) + F_p(1/n)\right) > \frac{1}{(nq)^p} = w_p^H(n)$$

for $n \geq 2$.

Remark 6. It is not hard to see that $F_p \geq 0$ whenever $p \in \mathbb{N}$ is integer valued. Indeed, $g(-x) \geq g(x)$ as all terms in the sum $g(-x)$ are positive since $-\binom{1/q}{k+1} \geq 0$ for odd k, while the terms in $g(x)$ alternate, (they are positive for even k and negative for odd k). Moreover for positive integers p the binomial coefficients $\binom{p-1}{n}$ are positive. Thus, the Hardy weight we computed is larger than the classical one for integer p.

Let us now turn to the proof of

$$E_p(x) + F_p(x) > 0$$

for $p \in (1, \infty)$ and $0 < x \leq 1/2$.

We collect the following basic properties of the function g which were partially already discussed above and will be used subsequently.

Lemma 7. *For $p \in (1, \infty)$ and $0 < x \leq 1/2$, we have*

$$-1 < g(x) < 0 < -g(x) < g(-x) < 1.$$

Proof. The function g is given by $g(x) = q\sum_{k=1}^{\infty}\binom{1/q}{k+1}x^k$. Since $q > 1$, the coefficients $b_k = q\binom{1/q}{k+1}$ are negative for odd k and positive for even k. Furthermore, the sequence $(|b_k|)$ takes values strictly less than 1 and decays monotonically. Thus, the asserted inequalities follow easily. □

We distinguish the following three cases depending on p for which the arguments are quite different:

- p lies between an odd and an even number with the subcases:
    - $p \in [3, \infty)$
    - $p \in (1, 2]$
- p lies between an even and an odd number.

Here, for $a, b \in \mathbb{N}$, we say that p is between a and b if $a \leq p \leq b$.

We start with investigating the case of p lying between an odd and an even number. To this end we consider two subsequent summands as they appear in the sum given by F_p and show that they are positive. (Indeed the sum in F_p starts at $n = 2$ but we also consider the corresponding term for $n = 1$.)

Lemma 8. *Let p be between an odd and an even integer. Then, for all $0 < x \leq 1/2$ and odd $n \in 2\mathbb{N} - 1$*

$$\binom{p-1}{n}\left(g^n(-x) - g^n(x)\right) + \binom{p-1}{n+1}\left(g^{n+1}(-x) - g^{n+1}(x)\right) \geq 0.$$

Proof. Let p be between an odd and an even integer. We first consider $n < p-1$. In the case $k \leq p-1$, we have $\binom{p-1}{k} \geq 0$. So, the statement for $n < p-1$ follows directly from Lemma 7 as $|g(x)| < 1$ for $0 < x \leq 1/2$. (Observe that $n+1 \leq p-1$ for $n < p-1$ and $n \in 2\mathbb{N}-1$ as p is between an odd and an even integer.)

On the other hand, for odd $n \in 2\mathbb{N}-1$ with $n \geq p-1$,

$$\binom{p-1}{n} \geq -\binom{p-1}{n+1} \geq 0.$$

From Lemma 7 we know that $g^{n+1}(x) \geq 0 \geq g^n(x)$ for odd $n \in 2\mathbb{N}-1$ and $0 \leq x \leq 1/2$.

We obtain

$$\begin{aligned}
&\binom{p-1}{n}\big(g^n(-x) - g^n(x)\big) + \binom{p-1}{n+1}\big(g^{n+1}(-x) - g^{n+1}(x)\big) \\
&\quad = \binom{p-1}{n}\big(g^n(-x) - g^n(x)\big) - \left|\binom{p-1}{n+1}\right|\big(g^{n+1}(-x) - g^{n+1}(x)\big) \\
&\quad \geq \binom{p-1}{n} g^n(-x) - \left|\binom{p-1}{n+1}\right| g^{n+1}(-x) \\
&\quad \geq \left|\binom{p-1}{n+1}\right|\big(g^n(-x) - g^{n+1}(-x)\big) \\
&\quad \geq 0,
\end{aligned}$$

where the last inequality follows from $0 \leq g(-x) < 1$ for $0 \leq x \leq 1/2$ thanks to Lemma 7. □

With Lemma 8 we can treat the case of $p \geq 3$ lying between an odd and an even number. This is done in the next proposition.

Proposition 9. *Let $p \geq 3$ be between an odd and an even integer. Then, for all $0 < x \leq 1/2$ we have $F_p(x) \geq 0$ and*

$$E_p(x) + F_p(x) > 0.$$

In particular, $w_p(n) > w_p^H(n)$ for $n \geq 2$.

Proof. We can write $F_p(x) = \sum_{n=2}^{\infty} \binom{p-1}{n}\big(g^n(-x) - g^n(x)\big)$ as

$$\begin{aligned}
F_p(x) &= \binom{p-1}{2}\big(g^2(-x) - g^2(x)\big) \\
&+ \sum_{n\in 2\mathbb{N}+1}^{\infty} \left(\binom{p-1}{n}\big(g^n(-x) - g^n(x)\big) + \binom{p-1}{n+1}\big(g^{n+1}(-x) - g^{n+1}(x)\big)\right)
\end{aligned}$$

By Lemma 8 the terms in the sum on the right hand side are all positive. Furthermore, $\binom{p-1}{2} \geq 0$ for $p \geq 3$ and $g(-x) \geq |g(x)|$ by Lemma 7. Thus, also the first term on the right hand side is positive as well and $F_p \geq 0$ follows. From the discussion in the beginning in the section we take $E_p(x) > 0$ for $0 < x \leq 1/2$. The "in particular" follows from the discussion above Lemma 7. □

Note that we cannot treat the case $1 \leq p \leq 2$ in the same way since the sum in F_p starts at the index $n = 2$. Hence, there is still a negative term $\binom{p-1}{2}\left(g^2(-x) - g^2(x)\right)$. We deal with this case, $1 \leq p \leq 2$, next.

We denote the Taylor coefficients of $x \mapsto g(-x)$ by a_k, i.e.,

$$g(-x) = q \sum_{k=1}^{\infty} \binom{1/q}{k+1} (-x)^k = \sum_{k=1}^{\infty} a_k x^k,$$
$$g(x) = q \sum_{k=1}^{\infty} \binom{1/q}{k+1} x^k = \sum_{k=1}^{\infty} a_k (-1)^k x^k.$$

The function $E_p(x) = -2p \sum_{k \in 2\mathbb{N}+1} \binom{1/q}{k+1} x^k$ is odd and, therefore, we have

$$E_p(x) = 2(p-1) \sum_{n=1}^{\infty} a_{2n+1} x^{2n+1}.$$

Furthermore, recall that $E_p(x) > 0$ for $x > 0$, since $-2p\binom{1/q}{k+1} > 0$ for odd k.

Lemma 10. *Let $p \geq 1$ and $0 \leq x \leq 1/2$. Then,*

$$g(-x) + g(x) \leq \frac{4}{9} \cdot \frac{(p+1)}{p^2} x^2.$$

Proof. We calculate using $a_2 \geq a_n$ for $n \geq 2$, the geometric series, $x \leq 1/2$ and the specific value of the Taylor coefficient $a_2 = q\binom{1/q}{3} = \frac{(p+1)}{6p^2}$

$$g(-x) + g(x) = 2 \sum_{k=1}^{\infty} a_{2k} x^{2k} \leq 2a_2 \frac{x^2}{1-x^2} \leq \frac{8}{3} a_2 x^2 = \frac{4}{9} \cdot \frac{(p+1)}{p^2} x^2. \qquad \square$$

With the help of this lemma and Lemma 8 we can treat the case $p \in (1, 2]$.

Proposition 11. *Let $p \in (1, 2]$. Then, for all $0 < x \leq 1/2$, we have*

$$E_p(x) + F_p(x) > 0.$$

In particular, $w_p(n) > w_p^H(n)$ for $n \geq 2$.

Proof. We show $E_p + F_p > 0$ and deduce the "in particular" from the discussion above Lemma 7. By Lemma 8 we have for all $0 < x \leq 1/2$

$$\begin{aligned}
F_p(x) =& \binom{p-1}{2}\big(g^2(-x) - g^2(x)\big) + \sum_{n\in 2\mathbb{N}+1}^{\infty} \left(\binom{p-1}{n}\big(g^n(-x) - g^n(x)\big)\right. \\
&\left. + \binom{p-1}{n+1}\big(g^{n+1}(-x) - g^{n+1}(x)\big)\right) \\
\geq& \binom{p-1}{2}\big(g^2(-x) - g^2(x)\big) \\
=& \frac{p-2}{2}\big(g(-x) + g(x)\big)\left(E_p(x) + \frac{p-1}{p}x\right) \\
\geq& \frac{2}{9}\cdot\frac{(p-2)(p+1)}{p^2}\left(E_p(x) + \frac{p-1}{p}x\right)\cdot x^2 \\
\geq& -\frac{1}{9}E_p(x) + \frac{2}{9}\cdot\frac{(p-2)(p-1)(p+1)}{p^3}\cdot x^3
\end{aligned}$$

where we used the definition of E_p, i.e., $(p-1)\big(g(-x)-g(x)\big) = E_p(x) + \frac{p-1}{p}x$ and Lemma 10 which is justified since $E_p(x) > 0$ and $p - 2 < 0$. Moreover, in the last step we estimated the coefficient of the first term in its minimum in $p = 1$ and $x = 1/2$.

Now, we use the representation of E_p as a power series to estimate

$$E_p(x) = -2p \sum_{k\in 2\mathbb{N}+1} \binom{1/q}{k+1} x^k \geq -2p\binom{1/q}{4}x^3 = \frac{(p-1)(p+1)(2p+1)}{12p^3}x^3.$$

Putting this together with the estimate on F_p above, we arrive at

$$\begin{aligned}
E_p(x) + F_p(x) &\geq \frac{8}{9}E_p(x) + \frac{2}{9}\cdot\frac{(p-2)(p-1)(p+1)}{p^3}\cdot x^3 \\
&\geq \left(\frac{8}{9}\frac{(p-1)(p+1)(2p+1)}{12p^3} + \frac{2}{9}\cdot\frac{(p-2)(p-1)(p+1)}{p^3}\right)\cdot x^3 \\
&= \frac{10(p-1)^2(p+1)}{27p^3}\cdot x^3
\end{aligned}$$

Hence, it remains to consider the case of p between an even and an odd integer for which we need the following three lemmas.

Lemma 12. *Let $p, q \geq 1$ such that $1/p + 1/q = 1$ and $k \geq 2$. Then,*

$$a_k = q\left|\binom{1/q}{k+1}\right| \geq \frac{1}{pk(k+1)} = \frac{1}{q(p-1)k(k+1)}.$$

Proof. We calculate using $1/p + 1/q = 1$

$$\begin{aligned} q\left|\binom{1/q}{k+1}\right| &= \frac{(1-1/q)(2-1/q)(3-1/q)\cdots(k-1/q)}{(k+1)!} \\ &= \frac{1}{pk(k+1)}\frac{(1+1/p)(2+1/p)\cdots((k-1)+1/p)}{(k-1)!} \\ &= \frac{1}{pk(k+1)}\left(1+\frac{1}{p}\right)\left(1+\frac{1}{2p}\right)\cdots\left(1+\frac{1}{(k-1)p}\right) \\ &\geq \frac{1}{pk(k+1)}. \end{aligned}$$

Lemma 13. *Let $p, q \in (1,\infty)$ such that $1/p+1/q = 1$ and $k \in \mathbb{N}, k > p$. Then,*

$$\left|\binom{p-1}{k}\right| \leq \frac{1}{4(p-1)} = \frac{(q-1)}{4}.$$

Proof. Let $n \in \mathbb{N}$ be such that $n-1 \leq p \leq n$. Moreover, let $\gamma = p-(n-1)$, i.e., $1-\gamma = n-p$, so, $\gamma \in [0,1]$. Since $k > p$ and $n, k \in \mathbb{N}$, we have that $k \geq n$ and therefore,

$$\begin{aligned} \left|\binom{p-1}{k}\right| &= \left|\frac{(p-1)(p-2)\cdots(p-(n-1))(p-n)\cdots(p-k)}{k!}\right| \\ &= \left|\left(\frac{p-1}{n-1}\right)\left(\frac{p-2}{n-2}\right)\cdots\frac{(p-(n-1))}{1}\left(\frac{p-n}{n}\right)\cdots\left(\frac{p-k}{k}\right)\right| \\ &\leq \left|\frac{(p-(n-1))(p-n)}{n}\right| = \frac{\gamma(1-\gamma)}{n} \leq \frac{1}{4(p-1)} = \frac{(q-1)}{4}. \end{aligned}$$

Lemma 14. *For $0 < x \leq 1/2$ and $q > 1$, we get*

$$g(-x) \leq \frac{(q-1)(5q-1)}{6q^2}x.$$

Proof. We calculate using $a_2 \geq a_k$ for $k \geq 2$

$$\begin{aligned} g(-x) &= q\left(\left|\binom{1/q}{2}\right| + \sum_{k=1}^{\infty}\left|\binom{1/q}{k+2}\right|x^k\right)x \\ &\leq q\left(\left|\binom{1/q}{2}\right| + \sum_{k=1}^{\infty}\left|\binom{1/q}{k+2}\right|2^{-k}\right)x \\ &\leq q\left(\left|\binom{1/q}{2}\right| + \left|\binom{1/q}{3}\right|\right)x \\ &= \frac{(q-1)(5q-1)}{6q^2}x. \end{aligned}$$

With the help of these lemmas we can finally treat the case where p lies between an even and an odd number.

Proposition 15. *Let $p \in [2, \infty)$ be between an even and an odd integer. Then, for all $0 < x \leq 1/2$ we have*

$$E_p(x) + F_p(x) > 0.$$

In particular, $w_p(n) > w_p^H(n)$ for $n \geq 2$.

Proof. Clearly, we have $\binom{p-1}{n} \geq 0$ for $n \leq p$ and for $n \in 2\mathbb{N}$. Since we have $g(-x) \geq |g(x)|$ by Lemma 7, we obtain for the first $n \leq p$ terms and the terms for even n in $F_p(x)$ that

$$\binom{p-1}{n}\big(g^n(-x) - g^n(x)\big) \geq 0.$$

Note that $E_p(x) = 2(p-1)\sum_{n=1}^{\infty} a_{2n+1}x^{2n+1} > 2(p-1)\sum_{n=k}^{\infty} a_{2n+1}x^{2n+1}$ since the coefficients a_k are positive. With the observation made at the beginning of the proof, this leads to

$$E_p(x) + F_p(x) > \sum_{n \in 2\mathbb{N}+1, n \geq p} \left(2(p-1)a_n x^n + \binom{p-1}{n}\big(g^n(-x) - g^n(x)\big)\right).$$

We continue to show that all the terms in the sum are strictly positive which finishes the proof. To this end note that for $n \geq p$ with $n \in 2\mathbb{N}+1$, we use $g(-x) \geq |g(x)|$, see Lemma 7, as well as $\binom{p-1}{n} \leq 0$ in the first step and the estimate on $\binom{p-1}{n}$, see Lemma 13, and the estimate on $g(-x)$, see Lemma 14, in the second step in order to get

$$\begin{aligned}\binom{p-1}{n}\big(g^n(-x) - g^n(x)\big) &\geq 2\binom{p-1}{n} g^n(-x) \\ &\geq -\frac{(q-1)}{2}\left(\frac{(q-1)(5q-1)}{6q^2}\right)^n x^n\end{aligned}$$

We use the estimate on a_n, Lemma 12,

$$2(p-1)a_n x^n \geq 2\frac{1}{qn(n+1)}x^n.$$

Next, we put these two estimates together and find that the minimum in the coefficient is clearly assumed at $q = 2$ since $p \geq 2 \geq q$

$$\begin{aligned}&\left(2(p-1)a_n x^n + \binom{p-1}{n}\big(g^n(-x) - g^n(x)\big)\right) \\ &\geq 2\left(\frac{1}{qn(n+1)} - \frac{(q-1)}{4}\left(\frac{(q-1)(5q-1)}{6q^2}\right)^n\right)x^n \\ &\geq \left(\frac{1}{n(n+1)} - \frac{1}{2}\left(\frac{3}{8}\right)^n\right)x^n \geq \left(\frac{1}{n(n+1)} - \frac{1}{2^{n+1}}\right)x^n > 0,\end{aligned}$$

where the positivity follows by a simple induction argument. This concludes the proof by noticing that the "in particular" part follows from the discussion above Lemma 7. □

In summary, the above considerations yield

$$E_p(x) + F_p(x) > 0$$

for $p \in (1,\infty)$ and $0 < x \leq 1/2$. By the discussion at the beginning of the section this yields $w_p(n) > w_p^H(n)$ for $n \geq 2$.

We finish the section by treating the case $n = 1$ which corresponds to $x = 1$. With this we finally conclude that $w_p(n) > w_p^H(n)$ for all $n \geq 1$ in the next section.

Proposition 16. *Let $p \in (1,\infty)$. Then, $w_p(1) > w_p^H(1)$.*

Proof. Recall that $w_p(1) = 1 - (2^{1-1/p} - 1)^{p-1}$ and $w_p^H = (1 - 1/p)^p$. By the mean value theorem applied to the function $[1, 2] \to [1, 2^{1-1/p}]$, $t \mapsto t^{1-1/p}$ we find

$$2^{1-1/p} - 1 < 1 - \frac{1}{p}.$$

Therefore,

$$w_p(1) - w_p^H(1) > 1 - \left(1 - \frac{1}{p}\right)^{p-1} - \left(1 - \frac{1}{p}\right)^p.$$

Now the function $\psi\colon (1,\infty) \to (0,\infty)$, $p \mapsto (1 - 1/p)^{p-1} + (1 - 1/p)^p$ is strictly monotonically decreasing because

$$\psi'(p) = \frac{1}{p-1}\left(\frac{p-1}{p}\right)^p \left((2p-1)\log\left(\frac{p-1}{p}\right) + 2\right) < 0,$$

since $\theta\colon p \mapsto (2p - 1)\log(p - 1)/p$ is strictly monotonically increasing and we have $\lim_{p\to\infty} \theta(p) = -2$. Hence, we conclude

$$w_p(1) - w_p^H(1) > 1 - \psi(p) > 1 - \lim_{t\to 1} \psi(t) = 0.$$

4. Proof of Theorem 1

Proof of Theorem 1. Combining Propositions 4 and 5 from Sect. 2 yields that w_p satisfies the Hardy inequality. In Sect. 3, one obtains from Proposition 9, Proposition 11, Proposition 15 and Proposition 16 that $w_p > w_p^H$ on $\mathbb{N}$.

The optimality of the p-Hardy weight, i.e., for every $w \gneq w_p$ the p-Hardy inequality fails (criticality), and in addition $u \notin \ell^p(\mathbb{N}, w_p)$, where $u(n) = n^{(p-1)/p}, n \in \mathbb{N}$ (null-criticality) can be deduced from [7, Theorem 2.3]. To this end one has to check that the function $v > 0$ which gives rise to the Hardy weight via the supersolution construction, i.e., $u = v^{(p-1)/p}$ and $w_p = \Delta_p u/u^{(p-1)}$ is proper, i.e., the preimage of every compact set in $(0,\infty)$ is compact, and of bounded oscillation, i.e., $C^{-1} \leq v(n)/v(n+1) \leq C$ for some $C > 0$ and all n. These two properties are clearly satisfied for the identity function $\mathrm{id}(n) = n$ which was used in Proposition 5.

Note that given criticality, null-criticality is also a simple consequence of $w_p > w_p^H$ on $\mathbb{N}$ and since $u(n) = n^{(p-1)/p}$ is not in $\ell^p(\mathbb{N}, w_p^H)$. Moreover, in the remark below, Remark 17, we give a sketch of the proof of criticality to give the reader an idea of the arguments involved.

To see the statement about the coefficients in the series expansion of w_p for integer $p \geq 2$, recall the function

$$w(x) = \big(1-(1-x)^{1/q}\big)^{p-1} - \big((1+x)^{1/q}-1\big)^{p-1}$$

with $1/p + 1/q = 1$ on $[0,1]$ from the previous section. It is easy to check that the function $w_+ \colon x \mapsto \big(1-(1-x)^{1/q}\big)^{p-1}$ is absolutely monotonic on $[0,1)$ for integer $p \geq 2$ with strictly positive derivatives. On the other hand, expanding the function $w_- \colon x \mapsto \big((1+x)^{1/q}-1\big)^{p-1}$ at $x = 0$, we observe that it has the same Taylor coefficients in absolute value as w_+. However, the signs alternate such that for the difference $w_+ - w_-$ of these two functions the even/odd coefficients cancel for odd/even p. Furthermore, in the Taylor expansion of w at $x = 0$ the first non-zero coefficient is the one for x^p (confer Sect. 3). □

Remark 17. (Sketch of the proof of criticality) Criticality of the weight w_p is equivalent to existence of a null sequence, i.e., existence of $0 \leq \varphi_N \in C_c(\mathbb{N})$ with $\varphi(0) = 0$, $\varphi(1) = 1$ and φ_N converging pointwise to $u(n) = n^{1/q}$ and

$$h(\varphi_N) := \sum_{n=0}^{\infty} |\varphi_N(n) - \varphi_N(n+1)|^p - \sum_{n=1}^{\infty} w_p^H(n)|\varphi_N(n)|^p \to 0, \quad N \to \infty$$

cf. [8, Theorem 5.1]. We denote $a \vee b = \max\{a,b\}$ for $a, b \in \mathbb{R}$ and choose

$$\varphi_N = u\psi_N \qquad \text{with} \qquad \psi_N(n) = 0 \vee \left(1 - \frac{\log n}{\log N}\right)$$

for $n, N \in \mathbb{N}$ and $u(n) = n^{1/q}$. We now use the simplified energy from [9, Theorem 3.1] which serves as a substitute of a ground state transform for $p \neq 2$

$$\begin{aligned} h(u\psi) \asymp h_u(\psi) := \sum_{n=0}^{\infty} & u(n)u(n+1)|\psi(n) - \psi(n+1)|^2 \cdot \\ & \times \Bigg((u(n)u(n+1))^{\frac{1}{2}} |\psi(n) - \psi(n+1)| \\ & \qquad + \frac{|\psi(n)| + |\psi(n+1)|}{2} |u(n) - u(n+1)| \Bigg)^{p-2}, \end{aligned}$$

where $\asymp$ means that there are two sided estimates with positive constants independent of ψ. We employ $u(n) = n^{1/q}$ and the definition of φ_N, ψ_N to obtain

$$h(\varphi_N) \asymp h_u(\psi_N) \leq \frac{C}{\log^p N} \left(\sum_{n=1}^{N} n^{p-1} \log^p \left(1 + \frac{1}{n}\right) + \sum_{n=1}^{N} n^{-1} \log^{p-2} \left(\frac{N}{n}\right) \right),$$

where we used $a(b+c)^r \leq C(ab^r + ac^r)$ for all $a, b, c \geq 0$, $r \in \mathbb{R}$ and some $C = C(r)$ to split the sum into two sums, $|(0 \vee a) - (0 \vee b)| \leq |a - b|$ for all

$a, b \in \mathbb{R}$ and $|n^r - (n+1)^r| \asymp n^{r-1}$ for $r \in (0, \infty)$ cf. e.g. [14, Lemma 2.28]. Now, using $\log(1 + 1/n) \leq 1/n$ and again $a(b+c)^r \leq C(ab^r + ac^r)$, we infer

$$h_u(\psi_N) \leq \frac{C}{\log^p N}\left(\log N + \log^{p-1} N\right) \to 0, \qquad N \to \infty$$

which finishes the proof.

While we know that all Taylor coefficients of the Hardy weight w_p are strictly positive for integer $p \geq 2$, we only know $w_p > w_p^H$ for non-integer $p > 1$. This leads us to the following conjecture.

Conjecture. We conjecture that $x \mapsto w(x)$ as defined in Sect. 3 is absolutely monotonic.

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