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Integral Equations and Operator Theory

Essential Spectrum and Feller Type Properties

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Abstract. We give necessary and sufficient conditions for a regular semi-Dirichlet form to enjoy a new Feller type property, which we call *weak Feller property*. Our characterization involves potential theoretic as well as probabilistic aspects and seems to be new even in the symmetric case. As a consequence, in the symmetric case, we obtain a new variant of a decomposition principle of the essential spectrum for (the self-adjoint operators induced by) regular symmetric Dirichlet forms and a Persson type theorem, which applies e.g. to Cheeger forms on RCD^* spaces.

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1. Introduction

Let X be a locally compact separable metrizable space equipped with a positive Radon measure m with full support. Let $\mathscr{E}$ be a regular semi-Dirichlet form on $L^2(X)$ with H the associated sectorial operator and $T_t := e^{-tH}$ the associated semigroup. Given a subset $A \subset X$ we denote by $\operatorname{cap}(A)$ the induced capacity, and given an open subset $U \subset X$ we denote by H_U the restriction of H to U with a certain Dirichlet condition, which is a sectorial operator in $L^2(U) \subset L^2(X)$. We refer the reader to Sect. 2 for detailed definitions. The starting point for our investigations is the following result from [19].

Theorem 1.1. *Assume $\mathscr{E}$ is symmetric and spatially locally compact, in the sense that $\mathbf{1}_A T_t$ is compact for all $t > 0$ and all Borel sets $A \subset X$ with $m(A) < \infty$. Then for all open $U \subset X$ with $\operatorname{cap}(X \setminus U) < \infty$ one has $\sigma_{\mathrm{ess}}(H_U) = \sigma_{\mathrm{ess}}(H)$.*

Although the class of U's considered in this result is very large (in fact, somewhat optimal), the restriction on the underlying geometry through the spatial local compactness assumption is rather strong: for example, there exist RCD^* spaces of finite measure such that the (operator induced by the)

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Cheeger form (cf. Example 4.3 for the definitions) does not have a purely discrete spectrum and thus is not spatially locally compact. Nontrivial examples of such spaces are provided by the natural Dirichlet forms (cf. Example 4.2) of certain complete Riemannian manifolds with Ricci curvature bounded from below: indeed, noncompact hyperbolic manifolds with finite volume are never spatially locally compact (as these always have nonempty essential spectrum [23]). The main goal of this paper is to obtain a variant of Theorem 1.1, which allows to treat geometries such as arbitrary RCD^* spaces. To this end, we found a surprising connection between such results and probability theory. Namely, we say that $\mathscr{E}$ has the *weak Feller property*, if the following two properties are satisfied:

(α) with $L_0^\infty(X)$ the space of all $u \in L^\infty(X)$ such that for all $\epsilon > 0$ there exists $K \subset X$ compact such that $\|1_{X\setminus K} u\|_\infty < \epsilon$, one has

$$T_t(L_0^\infty(X)) \subset L_0^\infty(X) \quad \text{for all } t > 0.$$

(β) For any compact $K \subset X$ there exists a function $0 \le w \in L^2(X) \cap L_0^\infty(X)$ with $(H+1)^{-1} w \ge \mathbf{1}_K$.

Recall that $\mathscr{E}$ is said to *induce a Feller semigroup*, if

$$T_t(C_0(X)) \subset C_0(X) \qquad \text{for all } t > 0, \tag{1.1}$$

$$\|T_t\phi - \phi\|_\infty \to 0 \qquad \text{as } t \to 0^+ \text{ for all } \quad \phi \in C_0(X), \tag{1.2}$$

with $C_0(X)$ the space of continuous functions on X vanishing at ∞. In general, this property implies the weak Feller property and these notions are equivalent on Riemannian manifolds, but there exist $\mathscr{E}$'s which have the weak Feller property but not the Feller property (cf. Sects. 3 and 4). We provide a list of equivalent characterizations of the weak Feller property under the above condition (β), one of which is the traditional probabilistic condition that compact sets are hard to hit from close to infinity by the underlying diffusion: namely, whenever K is compact and σ_K denotes the induced first hitting time, then for all $t \ge 0$ one has

$$\mathbb{P}^\bullet\{\sigma_K \le t\} \in L_0^\infty(X), \tag{1.3}$$

keeping in mind that by the seminal work of Azencott [1] it is well-known that *in the Riemannian case*, (1.3) is *equivalent* to the Feller property.
With this definition, one of our main results is:

Theorem 1.2. *Let $\mathscr{E}$ be symmetric and satisfy the weak Feller property, and let $\mathscr{E}$ be* weakly *spatially local compact, in the sense that $\mathbf{1}_K T_t$ is compact for all compact $K \subset X$ and all $t > 0$. Then for every open $U \subset X$ with $X \setminus U$ compact one has $\sigma_{\mathrm{ess}}(H_U) = \sigma_{\mathrm{ess}}(H)$.*

The weak Feller property enters the proof of Theorem 1.2 precisely in the form (1.3), through a method from [19].

Concerning the assumptions of Theorem 1.2 we remark that the natural Dirichlet form on a connected Riemannian manifold is automatically weakly spatially locally compact and satisfies the condition (β), and has the (weak) Feller property, e.g., if the manifold is complete and its Ricci curvature does not decay to fast to $-\infty$; likewise, the Cheeger energy of an arbitrary RCD^*

space as well as many jump diffusion Dirichlet forms satisfy all assumptions of the above theorem (cf. Examples 4.2, 4.3, 4.4 and Sect. 5).

Finally, we obtain a variant of Persson's theorem, which states that under the same assumptions on $\mathscr{E}$ as in Theorem 1.2 one has

$$\inf \sigma_{\text{ess}}(H) = \lim_{K \to X} \inf \sigma(H_{X\setminus K}),$$

and we show that these assumptions on $\mathscr{E}$ are satisfied, if $\mathscr{E}$ has the doubly Feller property in the sense of [17], that is, if $\mathscr{E}$ has the Feller property in addition to $T_t(L^\infty(X)) \subset C_0(X)$ for all $t > 0$.

For completeness, we have also included an "Appendix", where we show that every semi-Dirichlet form whose semigroup satisfies (1.1) is automatically regular, a result that comes in handy e.g. for possibly nonsymmetric jump diffusions.

2. Preliminaries

For the standard terminology concerning Dirichlet forms we refer to [9] for the symmetric and to [25,26] for the non-symmetric case. In the sequel, $\|\cdot\|_p$ for $p \in [1,\infty]$ denotes the norm on $L^p(X)$. Following [26], we say that a pair $(\mathscr{E},\mathscr{F})$ (or shortly $\mathscr{E}$, if there is no danger of confusion) is a (non-negative definite) *semi-Dirichlet form on* $L^2(X)$, if the following assumptions are satisfied:

- $\mathscr{F}$ is a dense subspace of $L^2(X)$ and $\mathscr{E} : \mathscr{F} \times \mathscr{F} \to \mathbb{R}$ is bilinear and $\mathscr{E}[u] := \mathscr{E}(u,u) \geq 0$.
- Sector condition: there exists a constant $K \geq 1$ such that
$$|\mathscr{E}(u,v)| \leq K\mathscr{E}(u,u)^{\frac{1}{2}}\mathscr{E}(v,v)^{\frac{1}{2}}.$$
- $\mathscr{F}$ is complete w.r.t. $\|\cdot\|_{\mathscr{E}}$, where
$$\|u\|_{\mathscr{E}}^2 := \mathscr{E}[u] + \|u\|_2^2.$$
- Markovian property: for all $u \in \mathscr{F}$, $a \geq 0$, one has
$$u \wedge a \in \mathscr{F} \text{ and } \mathscr{E}(u \wedge a, u - u \wedge a) \geq 0. \tag{2.1}$$

In the above situation, we get a maximally sectorial operator H on $L^2(X)$ associated with $\mathscr{E}$ by Kato's first form representation theorem, [14, VI,Thm 2.1, p. 322] and for $t \geq 0$, $\alpha > 0$ we write

$$T_t := e^{-tH}, \quad G_\alpha := (H+\alpha)^{-1}$$

for the corresponding contraction semigroup and resolvent, respectively, see [26, Thm 1.1.2, p. 4]. As a consequence of (2.1), the semigroup $(T_t; t \geq 0)$ is a sub-Markov semigroup [26, Thm 1.1.5, p. 7] and so extends in a p-consistent way to a contraction semigroup $L^p(X)$ for all $p \in [1,\infty]$ which is strongly continuous for $p < \infty$ and weak-*-continuous for $p = \infty$. Likewise, G_α is bounded in $L^p(X)$ for all $p \in [1,\infty]$. To ease notation, we do not distinguish between these semigroups and the resolvents just write T_t resp. G_α for all of them.

The semi-Dirichlet form $(\mathscr{E},\mathscr{F})$ is called *regular*, if $\mathscr{F} \cap C_c(X)$ is dense in $\mathscr{F}$ with respect to $\|\cdot\|_{\mathscr{E}}$ and in $C_c(X)$ with respect to $\|\cdot\|_\infty$.

We fix once for all a regular semi-Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(X)$.

Then $\mathscr{E}$ is called *symmetric*, if one has $\mathscr{E}(u,v) = \mathscr{E}(v,u)$ for all $u, v \in \mathscr{F}$, noting that a regular symmetric semi-Dirichlet in our sense is automatically a regular symmetric Dirichlet form in the standard sense of Fukushima [10]. Moreover, $\mathscr{E}$ is called *strongly local*, if $\mathscr{E}(u,v) = 0$, whenever $u, v \in \mathscr{F}$ are such that u is constant on the support of v.

We set

$$\mathscr{E}_\alpha(u,v) := \mathscr{E}(u,v) + \alpha \int_X uv \, dm, \quad \mathscr{E}_\alpha(u) := \mathscr{E}_\alpha(u,u), \qquad \text{for all } \alpha > 0,$$

and remark that the induced Choquet capacity is defined on open subsets $U \subset X$ by

$$\operatorname{cap}(U) := \inf\{\mathscr{E}_1(v) \mid v \in \mathscr{F}, \mathbf{1}_U \le v\},$$

with the usual convention $\inf \emptyset = \infty$, and for arbitrary $A \subset X$ one then sets

$$\operatorname{cap}(A) := \inf\{\operatorname{cap}(U) \mid A \subset U, U \text{ open}\}.$$

With respect to this capacity, every $u \in \mathscr{F}$ has a (quasi-)unique quasi-continuous representative $\tilde{u}$, and keeping this in mind, for every set B one sets

$$\tilde{\mathscr{F}}_{B,1} := \{u \in \mathscr{F} \mid \tilde{u} \ge \mathbf{1}_B \text{ q.e.}\},$$

so that for open $U \subset X$ one has

$$\operatorname{cap}(U) = \inf\{\mathscr{E}_1(v) \mid v \in \tilde{\mathscr{F}}_{U,1}\}.$$

The regularity of $\mathscr{E}$ implies (through the existence of cut-off functions) that

$$\operatorname{cap}(K) < \infty \quad \text{for all compact } K \subset X.$$

We will also be concerned with the Hunt process

$$\mathscr{M} := (\Omega, (\mathbb{P}^x; x \in X), (X_t; t \ge 0))$$

associated with $\mathscr{E}$ (see [9,25], [26, Section 3.3] and the groundbreaking [10]), with lifetime $\zeta \in (0, \infty]$; it gives the following probabilistic representation for the semigroup: for all $t > 0$, $f \in L^2(X)$ and m-a.e. $x \in X$ one has

$$T_t f(x) = \mathbb{E}^x\{\mathbf{1}_{\{t<\zeta\}} f(X_t)\}.$$

The restriction of $\mathscr{E}$ to $\overline{\mathscr{F} \cap C_c(U)}^{\|\cdot\|_{\mathscr{E}}}$ is a regular semi-Dirichlet form on $L^2(U) \subset L^2(X)$, [26, Section 3.5, in particular Thm 3.5.7]. This form will be denoted by $\mathscr{E}_U$. The maximally sectorial operator associated to $\mathscr{E}_U$ will be denoted by H_U and its semigroup with $(T_t^U; t \ge 0)$.

The *first exit time* of $\mathscr{M}$ from a Borel set B is defined by

$$\tau_B := \inf\{t > 0 \mid X_t \notin B\}$$

and the *first hitting time* of $\mathscr{M}$ of B is defined by

$$\sigma_B := \inf\{t > 0 \mid X_t \in B\}.$$

The following form of the Feynman-Kac formula allows a probabilistic interpretation of the semigroup generated by H_U: for all $f \in L^2(U)$, $t > 0$, m-a.e. $x \in U$ one has

$$T_t^U f(x) = \mathbb{E}^x\left[\mathbf{1}_{\{t<\tau_U\}} f(X_t)\right] \tag{2.2}$$

(see [26, Theorem 3.5.7, p. 100 and its proof], see also [4], Section 3.3, in particular Theorem 3.3.8, p. 109f. for the symmetric case). We will view bounded operators in $L^p(U)$ to be acting on $L^p(X)$, by defining them to be 0 on $L^p(X\backslash U)$.

We finally introduce some potential theoretic notions that will be needed in the sequel: let $u \in L^p(X)$ for some $1 \leq p \leq \infty$. We say that u is 1-*excessive*, if $e^{-t}T_t u \leq u$ for all $t > 0$.

The 1-*equilibrium potential* e_B for a Borel set B is defined by

$$e_B(x) = \mathbb{E}^x(e^{-\sigma_B}). \tag{2.3}$$

Actually, this is a little shortcut that is convenient for what we have in mind. Typically, e_B is introduced by a variational principle, and (2.3) is then deduced as an important link between the stochastic and the analytic world. In this spirit, the right-hand-side of the definition of e_B is typically denoted as $p_B^1(x)$ or $H_B 1(x)$, see [4, Lemma 3.1.1] or [26, Lemma 3.4.3].

Remark 2.1. (1) In the symmetric case, for any Borel set B such that $\tilde{\mathscr{F}}_{B,1} \neq \emptyset$, the function e_B is the unique function in $\tilde{\mathscr{F}}_{B,1}$ which satisfies

$$\mathscr{E}_1(e_B, e_B) = \min\{\mathscr{E}_1(u) \mid u \in \tilde{\mathscr{F}}_{B,1}\},$$

and one then has

$$\text{cap}(B) = \mathscr{E}_1(e_B, e_B),$$

see [4, p. 78, p. 105].

(2) In the non-symmetric case, at least for open B, [26, Lemma 3.4.3] gives an analogous statement, however, the variational problem is somewhat different then.

(3) [22, Proposition 2.8 (iii)] gives, that for all open B and all 1-excessive $u \in \tilde{\mathscr{F}}_{B,1}$ one has $e_B \leq u$, another variational property of e_B that will be of prime importance later.

(4) If X is a connected Riemannian manifold with m its volume measure and $\mathscr{E}$ is given by

$$\mathscr{E}(u, v) = \int_X (\nabla u, \nabla v) dm \quad \text{with domain of definition } \mathscr{F} := W_0^{1,2}(X),$$

where $W_0^{1,2}(X)$ is the closure of $C_c^\infty(X)$ with respect to the norm $\|\cdot\|_{\mathscr{E}}$, then for any open relatively compact $U \subset X$ the function e_U is the minimal nonnegative solution of the exterior boundary value problem

$$\begin{aligned} \Delta u &= u \text{ in } X \setminus \overline{U} \\ u &> 0 \text{ on } U^c \\ u &= 1 \text{ on } \partial U. \end{aligned} \tag{2.4}$$

Here $\Delta = -H$ is the Laplace-Beltrami operator.

(5) Let $U \subset X$ be open and relatively compact. Then there is a unique positive σ-finite Borel measure μ_U on X, supported in $\overline{U}$, such that e_U is the 1-potential of μ_U, that is,

$$\mathscr{E}_1(e_U, u) = \int_X u \, d\mu_U \quad \text{for all } u \in \mathscr{F},$$

and one has $\mathrm{cap}(U) = \mu_U(\overline{U})$. If G_1 has an integral kernel $\kappa_1(x,y)$, then

$$e_U(x) = \int_X \kappa_1(x,y)\, d\mu_U(y). \tag{2.5}$$

Indeed, let $\kappa_1^*(x,y) := \kappa_1(y,x)$ and let G_1^* be the dual resolvent of G_1 (G_1^* is in fact the resolvent of the dual form $\mathscr{E}^*(u,v) = \mathscr{E}(v,u)$). Obviously κ_1^* is the kernel of G_1^*. Let $u \in L^2(X), u \geq 0$. Then

$$\begin{aligned}
\mathscr{E}_1(e_U, G_1^* u) &= \int_X G_1^* u(x)\, d\mu_U(x) \\
&= \mathscr{E}_1^*(G_1^* u, e_U) \\
&= \int_X u(y) e_U(y)\, dm(y) \\
&= \int_X \int_X \kappa_1^*(x,y) u(y)\, dm(y)\, d\mu_U(x) \\
&= \int_X u(y) \cdot \Big(\int_X \kappa_1^*(x,y)\, d\mu_U(x) \Big)\, dm(y),
\end{aligned}$$

and the claim follows, by extending the latter identity to every $u \in L^2(X)$ through $u = u^+ - u^-$.

3. The Weak Feller Property

Recall that by definition, if $\mathscr{E}$ induces a Feller semigroup, then one has (1.1). The generalization of this property that we seek for relies on the space

$$\begin{aligned}
L_0^\infty(X) := \big\{ u \in L^\infty(X) \;\big|\; &\text{for all } \epsilon > 0, \exists\, K \subset X \\
&\text{compact s.t. } \|u 1_{X\setminus K}\|_\infty < \epsilon \big\}
\end{aligned}$$

of bounded functions vanishing at ∞ in the measure theoretic sense.

Definition 3.1. (α) We say that $\mathscr{E}$ satisfies the L_0^∞-*diffusion property*, if

$$T_t(L_0^\infty(X)) \subset L_0^\infty(X) \text{ for all } t > 0. \tag{3.1}$$

(β) We say that $\mathscr{E}$ satisfies property **(P)**, if for any compact $K \subset X$ there is $w \in L^2(X) \cap L_0^\infty(X), w \geq 0$ such that $G_1 w \geq \mathbf{1}_K$.
(γ) We say that $\mathscr{E}$ satisfies the *weak Feller property*, if it has the L_0^∞-diffusion property and if **(P)** holds.

We discuss a number of sufficient conditions:

Remark 3.2. (1) The following property implies **(P)**:
(P"): for any compact $K \subset X$ there exists $w \in L^2(X) \cap L_0^\infty(X), w \geq 0$ and $t' > 0$ such that for all $0 < t < t'$ one has $T_t w \geq \mathbf{1}_K$, which follows from

$$G_1 f = \int_{[0,\infty)} e^{-t} T_t f dt, \quad f \in L^p(X),$$

where integral is in the $L^p(X)$-Bochner sense if $p \in [1,\infty)$, and in the weak-*-sense for $p = \infty$.

(2) If $\mathscr{E}$ induces a Feller semigroup, then it also has the weak Feller property: indeed, given a compact $K \subset X$ pick $0 \leq w \in C_c(X)$ with $w \geq c$ in K for some $c > 0$. Then, since $t \mapsto T_t$ is strongly continuous at $t = 0$ in $C_0(X)$, we can pick a $c' > 0$ and $t' > 0$ such that $T_t w \geq c'$ in K for all $0 < t < t'$, so that property **(P)** follows from the previous item. Furthermore by positivity of T_t we get

$$T_t(1_K) \leq \frac{1}{c'} T_t w \in C_0(X) \text{ by (1.1).}$$

Hence $T_t(1_K) \in L_0^\infty(X)$ and the L_0^∞-diffusion property follows from item (4).

(3) Property **(P)** is equivalent to:
(P'): there is $w \in L^2(X) \cap L_0^\infty(X), w \geq 0$ such that for any compact $K \subset X$ there is $c_K > 0$ such that

$$G_1 w \geq c_K \mathbf{1}_K. \tag{3.2}$$

(4) By local compactness and the semigroup property, the L_0^∞-diffusion property is equivalent to the following property:

$$T_t \mathbf{1}_K \in L_0^\infty(X) \quad \text{for all compact } K \subset X, 0 < t < 1.$$

The L_0^∞-diffusion property of the semigroup implies the analogous property of the resolvent. This fact is well known for $C_0(X)$ instead of L_0^∞, where it is an equivalence, see, e.g. [17], p. 638 and [15]. Since we have a more general situation at hand, we present a proof. We thank the referee for the suggestion to do so:

Proposition 3.3. *For the following assertions*

(1) *$\mathscr{E}$ satisfies the L_0^∞-diffusion property.*
(2) *There is some $\alpha > 0$ such that $G_\alpha(L_0^\infty(X)) \subset L_0^\infty(X)$.*
(3) *For all $\alpha > 0$ one has $G_\alpha(L_0^\infty(X)) \subset L_0^\infty(X)$.*

we have that (1)⇒(2)⇔(3).

The proofs we know for $C_0(X)$ in place of $L_0^\infty(X)$ make use of the denseness of the range of the resolvent in $C_0(X)$ with respect to the uniform norm, see [18] and the references in there. Such a denseness is not valid for $L_0^\infty(X)$ in many cases, e.g. when the semigroup has the strong Feller property. Therefore we are not completely convinced that in our setting (3) implies (1).

Proof. Pick an exhaustion $(U_n)_{n\in\mathbb{N}}$ of X by relatively compact open sets and set $\eta_n := \mathbf{1}_{X\setminus U_n}$ for $n \in \mathbb{N}$. Obviously, $f \in L_0^\infty(X)$ if and only $f \in L^\infty(X)$ and

$$\|\eta_n f\|_\infty \to 0 \text{ for } n \to \infty,$$

which in turn can be expressed through duality with $L^1(X)$. Semigroups and resolvents are related through:

$$G_\alpha f = \int_{[0,\infty)} \mathrm{e}^{-\alpha t} T_t f \mathrm{d}t \tag{3.3}$$

and

$$T_t f = \lim_{\beta\to\infty} \mathrm{e}^{-\beta t} \sum_{k=0}^{\infty} \frac{(\beta t)^k}{k!} (\beta G_\beta)^k f \tag{3.4}$$

for $f \in L^\infty(X)$. Note however, that both right hand sides are to be understood in the w-$*$- sense and this adds a little subtelty to the argument. If we had norm convergence in these equations, the equivalence of (1) and (3) would be not too hard since $L_0^\infty(X)$ is a closed subspace of $L^\infty(X)$.

(1)⇒(3): Fix α and $f \in L_0^\infty(X)$ and note that by w-$*$-continuity of the semigroup, and the resulting measurability,

$$\|\eta_n G_\alpha f\|_\infty \le \int_{[0,\infty)} \mathrm{e}^{-\alpha t} \|\eta_n T_t f\|_\infty \mathrm{d}t \to 0 \text{ as } n \to \infty$$

by dominated convergence, as the integrand is Lebesgue measurable and goes to 0 pointwise by the L_0^∞-diffusion property.

(3)⇒(2) is clear.

(3)⇐(2): Note that by positivity, it remains to check the convergence of $\eta_n G_\alpha f$ for given positive $f \in L_0^\infty(X)$; the $\eta_n G_\alpha f$ are nonincreasing in α, so whenever $G_\alpha(L_0^\infty(X)) \subset L_0^\infty(X)$, we get the same mapping property for G_β, for $\beta > \alpha$. Using that

$$G_\beta f = \lim_{T\to\infty} \int_{[0,T]} \mathrm{e}^{-\beta t} T_t f \mathrm{d}t$$

in $\|\cdot\|_\infty$, with the integral understood in the w-$*$- sense, and that the integral can be majorized by $\mathrm{e}^{(\alpha-\beta)T} G_\alpha f$ pointwise, we are done. □

We are now in position to give a characterization of the weak Feller property:

Theorem 3.4. *Under assumption **(P)**, the following assertions are equivalent:*

1. *$\mathscr{E}$ satisfies the L_0^∞-diffusion property.*
2. *For any compact $K \subset X$, $t \ge 0$ one has $\mathbb{P}^\bullet\{\sigma_K \le t\} \in L_0^\infty(X)$.*
3. *For any compact $K \subset X$ one has $e_K \in L_0^\infty(X)$.*
4. *For any compact $K \subset X$ there exists a 1-excessive function $\phi \in \mathscr{F}$ with $\mathbf{1}_K \le \phi \in L_0^\infty(X)$.*
5. *There is some $\alpha > 0$ such that $G_\alpha(L_0^\infty(X)) \subset L_0^\infty(X)$.*
6. *For all $\alpha > 0$ one has $G_\alpha(L_0^\infty(X)) \subset L_0^\infty(X)$.*

Proof. (4)⇒(3): Let K as in (3) and pick a relatively compact, open $B \supset K$. By (4) there exists a 1-excessive $\phi \in L_0^\infty(X) \cap \mathscr{F}$ such that $\mathbf{1}_B \le \phi$. From Remark 2.1(3) above, we get that $e_B \le \phi$ and so

$$e_K \le e_B \in L_0^\infty(X).$$

(3)⇒(2): follows from

$$\mathbb{P}^\bullet\{\sigma_K \le t\} \le e^t e_K.$$

(2)⇒(1): follows from Remark 3.2(4) and

$$T_t \mathbf{1}_K = \mathbb{E}^\bullet[\mathbf{1}_K \circ X_t] \le \mathbb{P}^\bullet\{\sigma_K \le t\}.$$

(1)⇒(6) and (5)⇔(6) are true without **(P)** as follows from the previous Proposition.

(6)⇒(4): Pick w as in **(P)** and set

$$\phi := G_1 w = \int_0^\infty e^{-t} T_t w dt. \tag{3.5}$$

Condition **(P)** ensures that $\phi \geq \mathbf{1}_K$ and evidently, ϕ is 1-excessive and belongs to $\mathscr{F}$. Finally, by (6) it follows that $\phi \in L_0^\infty(X)$ as asserted. □

Although in most concrete applications (see the next section) the weak Feller property is most conveniently checked in terms of the semigroup, the above result allows in principle to check this property directly through the equilibrium potential. Assume for example that X is a metric space with metric $d(x, y)$, that G_1 has an integral kernel $\kappa_1(x, y)$ and that for every open relatively compact $U \subset X$ the function $G_1 \mathbf{1}_U$ is bounded from below by a strictly positive lower semicontinuous function (so that one has property (**P**)). If then for any such U there exist constants $c, \delta, \gamma > 0$ such that

$$\kappa_1(x, y) \leq c d(x, y)^{-\gamma} \text{ for all } y \in U \text{ and all } x \in X \text{ with } d(x, y) > \delta, \tag{3.6}$$

then one has $e_U \in L_0^\infty(X)$, which clearly entails the weak Feller property. Indeed, let $n \in \mathbb{N}$ be large enough and let

$$K_n := \{x \in X : d(x, \overline{U}) \leq n\}.$$

From formula (2.5) we obtain that for $x \notin K_n$

$$\begin{aligned} e_U(x) &= \int_X \kappa_1(x, y)\, d\mu_U(y) = \int_{\overline{U}} \kappa_1(x, y)\, d\mu_U(y) \leq c \int_{\overline{U}} d(x, y)^{-\gamma}\, d\mu_U(y) \\ &\leq \frac{c}{n^\gamma} \mu_U(\overline{U}) = \frac{c}{n^\gamma} \mathrm{cap}(U), \end{aligned}$$

which, by letting $n \to \infty$ proves the claim. Situations as above can be inferred for example from results such as [29, Section 5.6], [11, Lemma 7.7] and [5, Lemma 2.4].

4. Examples of Weakly Feller $\mathscr{E}'s$

In this section we give some classes of examples to illustrate the weak Feller property.

Example 4.1. (Multiplication operators) Let $h : X \to [0, \infty)$ be measurable and define a regular symmetric Dirichlet form by

$$\mathscr{F} = \left\{ f \in L^2(X) \;\middle|\; h^{\frac{1}{2}} f \in L^2(X) \right\}$$

$$\mathscr{E}(f, g) = \int_X h \cdot f \cdot g \; dm.$$

Then H is the maximally defined multiplication operator induced by h and T_t is the bounded multiplication operator induced by e^{-th}.

(i) The L^∞_0-diffusion property always holds, in view of $|T_t f| \le |f|$ for all $t \ge 0$.
(ii) Property **(P)** is equivalent to $h \in L^\infty_{\mathrm{loc}}(X)$.
(iii) The Feller property is equivalent to $h \in C(X)$, keeping in my mind that the required strong continuity in (1.2) automatically follows from Dini's theorem.
(iv) One has $T_t(C_0(X)) \subset C_0(X)$ for all $t \ge 0$, if and only if $T_t(C_b(X)) \subset C_b(X)$ for all $t \ge 0$, which in turn is equivalent to $h \in C(X)$, as mentioned already.

Example 4.2. (Riemannian manifolds) Let X be a connected Riemannian manifold with m its volume measure, $d(x, y)$ the geodesic distance and $\mathscr{E}$ is the regular strongly local symmetric Dirichlet form in $L^2(X)$ given by

$$\mathscr{E}(u, v) = \int_X (\nabla u, \nabla v) dm \quad \text{with domain of definition } \mathscr{F} := W_0^{1,2}(X). \tag{4.1}$$

Then H is the Friedrichs realization of the Laplace-Beltrami operator $-\Delta$ and T_t has a jointly smooth integral kernel $p_t(x, y)$ for all $t > 0$, which is strictly positive by the parabolic maximum principle. Accordingly, $\mathscr{E}$ satisfies property **(P")**, as given $K \subset X$ compact one has $T_t w \ge \mathbf{1}_K$ for

$$w := \big(m(K) \inf_{x,y\in K} p_t(x, y)\big)^{-1} \mathbf{1}_K.$$

Moreover, $\mathscr{E}$ induces a Feller semigroup, if and only if one has $T_t(C_0(X)) \subset C_0(X)$: indeed, the required strong continuity follows from writing

$$P_t f - f = \int_0^t P_s \Delta f ds \quad \text{for all } t > 0, f \in C_c^\infty(X),$$

and using that P_s is a contraction in $L^\infty(X)$. We have borrowed this argument from the proof of Proposition 4.3 in [24]. Moreover, $\mathscr{E}$ induces a Feller semigroup, if and only if [27, Theorem 2.2] the unique minimal solution to (2.4) vanishes at ∞, for all open relatively compact $U \subset X$ (see also [33, Theorem 3.3] for analogous result on graphs). Since by Remark 2.1(4) this solution is precisely e_U, it follows from Theorem 3.4 that $\mathscr{E}$ is weakly Feller, if and only if it has the Feller property. This is the case, e.g., if X is geodesically complete and its Ricci curvature satisfies [12]

$$\mathrm{Ric} \ge -C_1 - C_2 d(x, y)^2 \quad \text{for some } C_1, C_2 \ge 0.$$

Example 4.3. (RCD^* **spaces)** Let X be a complete separable geodesic metric measure space with metric $d(x, y)$ and let m be an inner regular Borel measure on X with full support, which is finite on all open balls $B(x, r)$. The *Cheeger form*

$$\mathrm{Ch} : L^2(X) \longrightarrow [0, \infty]$$

on X is defined to be the L^2-lower semicontinuous relaxation of the functional

$$\widetilde{\mathrm{Ch}} : L^2(X) \cap \mathrm{Lip}(X) \longrightarrow [0, \infty]$$

given by

$$\widetilde{\mathrm{Ch}}(f) := \int_X \Big(\limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x,y)} \Big)^2 \, dm(x),$$

One gets a functional $\mathscr{E}$ on $L^2(X)$ with domain of definition $\mathscr{F}$ given by all f with $\mathrm{Ch}(f) < \infty$ by setting $\mathscr{E}(f) := \mathrm{Ch}(f)$ for such f's. The metric measure space X is called *infinitesimally Hilbertian*, if $\mathscr{E}$ is a quadratic form. For example, if X is a complete connected Riemannian manifold with its geodesic distance and Riemannian volume measure, then $\mathscr{E}$ is given by (4.1).

Next, let us give the following definition of the $\mathsf{CD}^*(K,N)$ condition from [3], which is a slight generalization of the original $\mathsf{CD}(K,N)$ condition from [21,30] having the advantage of admitting a natural local-to-global principle: let $\mathsf{P}(X)$ denote the set of all Borel probability measures on X and let $\mathsf{P}_2(X)$ denote the elements of $\mathsf{P}(X)$ that have finite second moments. Given $\mu_0, \mu_1 \in \mathsf{P}_2(X)$, the L^2-*Wasserstein distance* is defined by

$$W_2(\mu_0,\mu_1) := \inf \left\{ \int_{X \times X} d(x,y)^2 dq(x,y) : \ q \in \mathscr{C}(\mu_0,\mu_1) \right\},$$

with $\mathscr{C}(\mu_0,\mu_1)$ the set of all couplings between μ_0 and μ_1. Any minimizer of the above infimum is called an *optimal coupling* of μ_0 and μ_1. Then $\mathsf{P}_2(X)$ together with the L^2-Wasserstein distance is again a complete separable geodesic space (as X is so). Given $K \in \mathbb{R}$, $t \in [0,1]$, $N \in [1,\infty)$ define the function

$$\sigma^{(t)}_{K,N} : [0,\infty) \longrightarrow \mathbb{R} \cup \{\infty\}$$

by

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \geq N\pi^2 \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})}, & \text{if } 0 < K\theta^2 < N\pi^2 \\ t, & \text{if } K\theta^2 = 0 \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})}, & \text{if } K\theta^2 < 0. \end{cases}$$

Given $K \in \mathbb{R}$, $N \in [1,\infty)$, the metric measure space X is called a $\mathsf{CD}^*(K,N)$ *space*, if for all $\mu_0, \mu_1 \in \mathsf{P}(X)$ with bounded support and $\mu_0, \mu_1 \ll m$, there exists an optimal coupling q of them and a geodesic $(\mu_t)_{t \in [0,1]}$ in $\mathsf{P}_2(X)$ connecting them, such that

$$\mu_t \ll m \text{ and } \mu_t \text{ has a bounded support for all } t \in [0,1],$$

and such that for all $N' \in [N,\infty)$, $t \in [0,1]$ one has

$$\begin{aligned} &\int_X \rho_t(x)^{1-1/N'} m(dx) \\ &\quad \geq \int_{X \times X} \Big(\sigma^{(1-t)}_{K,N'}(d(x_0,x_1)) \rho_0(x_0)^{-1/N'} \\ &\qquad + \sigma^{(t)}_{K,N'}(d(x_0,x_1)) \rho_1(x_1)^{-1/N'} \Big) \, dq(x_0,x_1), \end{aligned}$$

where ρ_t denotes the Radon-Nikodym density of μ_t with respect to m for all $t \in [0,1]$. Finally, given $K \in \mathbb{R}$, $N \in [1,\infty)$, the metric measure space X is called an $\mathsf{RCD}^*(K,N)$ *space*, if it is an infinitesimally Hilbertian $\mathsf{CD}^*(K,N)$ space. This definition is shown to be equivalent to weak Bochner type inequality in [8]. Examples of such space include complete connected Riemannian manifolds of dimension $n \le N$ and Ricci curvature bounded from below by K with their geodesic distance and volume measure, but also possibly very singular spaces such as Alexandrov spaces of dimension $n \le N$ with curvature $\ge K/(n-1)$, (≥ 0 if $n=1$) and their n-dimensional Hausdorff measure.

We fix an $\mathsf{RCD}^*(K,N)$ space X in the sequel.

Then, by the validity of the Bishop-Gromov volume estimate [3], there exists $C>0$ such that for all $0<r\le R$ and all $x\in X$ one has the volume local doubling

$$\frac{m(x,R)}{m(x,r)} \le C\mathrm{e}^{CR}(R/r)^N, \tag{4.2}$$

which implies that X is locally compact. Here, we have set $m(y,s) := m(B(y,s))$. Moreover, by a standard argument one gets the following local volume comparison inequality from the latter estimate: there is a constant $C'>0$ such that for all $t>0$, $x_1,x_2\in M$ and $\varepsilon>0$,

$$\frac{m(x_2,\sqrt{t})}{m(x_1,\sqrt{t})} \le C'\mathrm{e}^{\frac{C't}{\epsilon}}\mathrm{e}^{\epsilon\frac{d(x_1,x_2)^2}{t}}. \tag{4.3}$$

As shown in [16], $\mathscr{E}$ becomes a regular strongly local symmetric Dirichlet form. The semigroup T_t has a jointly continuous integral kernel $p_t(x,y)$ for all $t>0$, which satisfies [13] for all $x,y\in X$, $0<t<1$ the following Gaussian bounds

$$C_1 m(x,\sqrt{t})^{-1}e^{-\frac{d(x,y)^2}{C_2 t}} \le p_t(x,y) \le C_3 m(x,\sqrt{t})^{-1}e^{-\frac{d(x,y)^2}{C_4 t}}.$$

Clearly, this implies $p_t(x,y)>0$, so that property $(\mathbf{P}'')$ is satisfied as in the Riemannian case. For the proof of the L^∞_0-diffusion property we can assume that X is noncompact. We are going to show that $T_t 1_K \in L^\infty_0(X)$ for all $0<t<1$. Indeed, the Gaussian upper bound implies

$$|T_t\mathbf{1}_K(x)| \le C_5 C_3 \int_K e^{\frac{-d(x,y)^2}{C_4 t}}\, dm(y),$$

which clearly proves the claim, noting that

$$C_5 := \sup_{y\in K} m(y,\sqrt{t})^{-1} < \infty,$$

as for every fixed $z\in X\setminus K$, by local volume comparison,

$$\inf_{y\in K} m(y,\sqrt{t}) \ge \inf_{y\in K} C' e^{-C't} e^{-d(z,y)^2/t} m(z,\sqrt{t}) > 0.$$

While the previous two examples were both strongly local and symmetric, we finally provide an example which is nonlocal and nonsymmetric:

Example 4.4. (Nonsymmetric jump diffusions) Assume $X=\mathbb{R}^m$ with Lebesgue measure dx, and fix $0<\alpha<2$ and a Borel function

$$\kappa:\mathbb{R}^m\times\mathbb{R}^m\longrightarrow\mathbb{R},\quad\text{with}$$
$$\kappa_0\le\kappa(x,z)\le\kappa_1,\quad \kappa(x,z)=\kappa(x,-z),$$

$$|\kappa(x,z)-\kappa(y,z)| \le \kappa_2|x-y|^{\beta}, \quad \text{for all } x,z,y\in\mathbb{R}^m,$$

for some constants $\kappa_0,\kappa_1,\kappa_2>0$, $0<\beta<1$. Define the operator $\tilde{H}$ on functions $f:\mathbb{R}^m\to\mathbb{R}$ by

$$\tilde{H}f(x) := \lim_{\epsilon\to 0}\int_{\{|z|>\epsilon\}}(f(x+z)+f(z))\frac{\kappa(x,z)}{|z|^{m+\alpha}}dz, \tag{4.4}$$

whenever the expression makes sense.

Then it follows from Theorem 1.1 in [6] (and a scaling argument) that there exists a unique jointly continuous function

$$p:(0,\infty)\times\mathbb{R}^m\times\mathbb{R}^m \longrightarrow [0,\infty), \quad (t,x,y)\longmapsto p_t(x,y)$$

which has the following three properties:

(i) one has

$$\partial_t p_t(x,y) = \tilde{H}p_t(\bullet,y)(x), \quad \text{for all } t>0, x\neq y,$$

(ii) for all $t_0>0$ there exists a constant $c_1>0$ such that for all $0<t<t_0$ and all x,y one has

$$p_t(x,y)\le c_1 t(t^{1/\alpha}+|x-y|)^{-m-\alpha},$$

(iii) for every $0<\gamma<\min(\alpha,1)$ and $t_0>0$ there exists a constant $c_2>0$ such that for all $x,x',y\in\mathbb{R}^m$, $0<t<t_0$ one has

$$\begin{aligned}&|p_t(x,y)-p_t(x',y)|\\ &\quad\le c_1|x-x'|^{\gamma}t^{1-\gamma/\alpha}\left(t^{1/\alpha}+\min(|x-y|,|x'-y|)\right)^{-m-\alpha},\end{aligned}$$

(iv) the map $t\mapsto \tilde{H}p_t(\cdot,y)(x)$ is continuous for all $x\neq y$, and for all $t_0>0$ there exists a constant $c_3>0$ such that for all $0<t<t_0$ and all $x\neq y$ one has

$$|\tilde{H}p_t(\bullet,y)(x)|\le c_3(t^{1/\alpha}+|x-y|)^{-m-\alpha},$$

(v) for all bounded uniformly continuous functions $f:\mathbb{R}^m\to\mathbb{R}$ one has

$$\lim_{t\to 0+}\left\|\int_{\mathbb{R}^m}p_t(\bullet,y)f(y)dy-f\right\|_{\infty}=0.$$

Moreover, $(t,x,y)\mapsto p_t(x,y)$ satisfies the Chapman-Kolmogorov identities, and with

$$T_tf(x) := \int_{\mathbb{R}^m}p_t(x,y)f(y)dy,$$

the semigroup $(T_t;t\ge 0)$ is analytic in $L^2(\mathbb{R}^m)$, and one has the conservativeness

$$\int_{\mathbb{R}^m}p_t(x,y)dy=1 \quad \text{for all } t>0, x,y\in\mathbb{R}^m.$$

Assuming that the function $\kappa^*(x,y):=\kappa(y,x)$ satisfies the the same assumptions as κ, then with H the generator of this semigroup, we can define [25] a semi-Dirichlet form $\mathscr{E}$ in $L^2(\mathbb{R}^m)$ by defining $\mathscr{F}$ to be the completion of $\mathrm{Dom}(H)$ with respect to the norm

$$u \longmapsto \sqrt{\langle Hu, u\rangle + \|u\|_2},$$

and defining $\mathscr{E}$ to be the unique bilinear extension of

$$(u, v) \longmapsto \langle Hu, v\rangle$$

to $\mathscr{F}$. It follows from the continuity of the heat kernel $p_t(x, y)$, (ii), (iii), (v) and Theorem 6.1 that $\mathscr{E}$ is regular and induces a Feller semigroup (and so a weak Feller semigroup). Moreover, $\mathscr{E}$ is symmetric, if and only if κ is so. This example generalizes the usual α-stable Dirichlet form in $\mathbb{R}^m$, where κ is a constant. Even more general nonlocal semi-Dirichlet forms have been treated along the same lines in [18], where the function $z \mapsto |z|^{-m-\alpha}$ in (4.4) has been replaced by a general class of functions subject to certain growth conditions.

5. The Decomposition Principle

Here, we state some spectral theoretic consequences of Feller type properties, mainly addressing the issue of stability of the essential spectrum. Since functional calculus will be involved, we fix a regular symmetric Dirichlet form $\mathscr{E}$ throughout this section, with H, T_t denoting, as usual, the associated self adjoint operator and semigroup and $\mathscr{E}_U$, H_U, T_t^U the corresponding restriction to an open $U \subset X$, as discussed in Sect. 2. We denote by $\|\cdot\|$ the operator norm in $L^2(X)$.

Our main aim is to show that

$$\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_{X\setminus K})$$

provided K is compact and H is suitable. The essential spectrum of a self-adjoint operator S can be characterized as

$$\sigma_{\mathrm{ess}}(S) = \{\lambda \in \mathbb{C} \mid S - \lambda \text{ is not a Fredholm operator}\},$$

see [28, Chapter XIII.14]. It was introduced by H. Weyl in a slightly different form in [32], who also proved its essential property, namely that it is invariant under compact perturbations, see [31].

Definition 5.1. We say that $\mathscr{E}$ satisfies the *weak spatial local compactness property*, if $\mathbf{1}_K T_t$ is a compact operator for every compact $K \subset X$ and every $t > 0$.

Clearly, this property is weaker than spatial local compactness, introduced in [19, Definition 2.1] which requires that $\mathbf{1}_A T_t$ is a compact operator for every Borel set $A \subset X$ with $m(A) < \infty$.

The main point is the following variant of the corresponding result in [19].

Lemma 5.2. *Let $\mathscr{E}$ satisfy the weak Feller property. Let $K \subset X$ be compact. Then there is a sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets such that for every $t > 0$*

$$\lim_{n\to\infty} \left\| \mathbf{1}_{K_n} \left(T_t - T_t^{X\setminus K} \right) - \left(T_t - T_t^{X\setminus K} \right) \right\| = 0. \tag{5.1}$$

This result is proved exactly as [19, Corollary 2.5.], where

$$K_n \supset \{e_K > 1/n\}$$

can be chosen compact in view of the L_0^∞-diffusion property of $\mathscr{E}$, for n large enough. With exactly the same argument as in [19], one can now prove:

Theorem 5.3. (Decomposition principle) *Let $\mathscr{E}$ satisfy the weak Feller property and H the weak spatial local compactness property. Let $K \subset X$ be compact. Then the operator $\varphi(H_{X\setminus K}) - \varphi(H)$ is compact for every $\varphi \in C_0(\mathbb{R})$. In particular,*

$$\sigma_{\mathrm{ess}}(H_{X\setminus K}) = \sigma_{\mathrm{ess}}(H).$$

Remark 5.4. The $\mathscr{E}$'s from Examples 4.2, 4.3 and 4.4 (assuming symmetry in the last case) are weakly locally compact: indeed, in each case T_t has a jointly continuous integral kernel, which shows that the Hilbert-Schmidt norm of the integral operator $\mathbf{1}_K T_t$ is finite for all compact $K \subset X$, $t > 0$. Thus, for RCD^*-spaces and symmetric jump diffusions Theorem 5.3 applies directly. In the Riemannian case, one has to guarantee the Feller property, which holds, as noted above, if the manifold is complete with a Ricci curvature that does not decay to fast to ∞. In fact, as shown in [7] with completely different methods, the decomposition principle holds in the Riemannian case without any assumptions on the geometry. From this point of view, the main strength of Theorem 5.3 stems from the fact that it can deal with many local and nonlocal situations *simultaneously.*

We get the following variant of the Persson type theorem from [19], where $\lim_{K\to X}$ stands for the limit along the net of compact subsets of X:

Theorem 5.5. (Persson's theorem) *Let $\mathscr{E}$ satisfy the weak Feller property and H the weak spatial local compactness property. Then,*

$$\inf \sigma_{\mathrm{ess}}(H) = \lim_{K\to X} \inf \sigma(H_{X\setminus K}).$$

Finally, we present further results that connect Feller type properties with the stability of the essential spectrum. The first one shows that the compactness of $\varphi(H_{X\setminus K}) - \varphi(H)$ deduced in Theorem 5.3 actually implies the corresponding spatial local compactness property:

Proposition 5.6. *Let $B \subset X$ be closed, $U := X \setminus B$. If the operator $\varphi(H_U) - \varphi(H)$ is compact for every $\varphi \in C_0(\mathbb{R})$ then $\mathbf{1}_B T_t$ is a compact operator as well.*

We have chosen the assumption and the conclusion so as to fit the above results, many different equivalent formulations are at hand:

Remark 5.7. Let $U \subset X$ be open and $B \subset X$ be closed.
(1) The following properties are equivalent:

- $\varphi(H_U) - \varphi(H)$ is compact for every $\varphi \in C_0(\mathbb{R})$,
- $T_t^U - T_t$ is compact for one (every) $t > 0$,
- $D_\lambda := G_\lambda - (H_U + \lambda)^{-1}$ is compact for one (every) $\lambda > 0$.

(2) The following properties are equivalent:

- $\mathbf{1}_B \varphi(H)$ is compact for every $\varphi \in C_0(\mathbb{R})$,
- $\mathbf{1}_B T_t$ is compact for one (every) $t > 0$,
- $\mathbf{1}_B G_\lambda$ is compact for one (every) $\lambda > 0$.

For the proof of the second statement, see [20, Theorem 1.3]; the first statement can be deduced with the help of the Stone-Weierstrass theorem, see proof of Theorem 2.3 in [19].

Proof of Prop. 5.6. By [2, Lemma 3]

$$D_1 := G_1 - (H_U + 1)^{-1} = (\check{H}^{1/2} J G_1)^* \check{H}^{1/2} J G_1,$$

where

$$J : (\mathscr{F}, \mathscr{E}_1) \to L^2(B, \mathbf{1}_B m),\ u \mapsto u|_B,$$

and $\check{H}$ is the selfadjoint operator of the trace of $\mathscr{E}_1$ w.r.t. the map J. Accordingly, D_1 is compact if and only if $\check{H}^{1/2} J G_1$ is compact as well. Since $\check{H} \geq 1$, we get that JG_1 is compact, whence the assertion in view of the preceding remark, part (2). □

Proposition 5.8. *Let $\mathscr{E}$ satisfy the Feller property. If T_1^V is compact for some relatively compact open V, then $\mathbf{1}_K T_1$ is compact for every compact subset $K \subset V$. Consequently, if T_1^V is compact for all relatively compact open V, then H satisfies the weak spatial local compactness property.*

Proof. Let K and V be as in the assertion. By [17, Lemma 2.2] we get:

$$\sup_{x \in K} \mathbb{P}^x \{\tau_V \leq s\} \to 0 \text{ as } s \searrow 0.$$

With an argument just as in [19, Proof of Proposition 2.4] this gives that

$$\|\mathbf{1}_K (T_s - T_s^V)\| \to 0 \text{ as } s \searrow 0. \tag{5.2}$$

Since T_1^V is compact, T_s^V is compact as well by Remark 5.7(2), applied to H^V and $B = V$, we finally conclude that

$$\begin{aligned} \mathbf{1}_K T_1 &= \mathbf{1}_K T_s T_{1-s} \text{ for } s < 1 \\ &= \lim_{s \searrow 0} \mathbf{1}_K T_s^V T_{1-s} \end{aligned}$$

is compact, as a norm limit (see (5.2)) of compact operators.

The second assertion is evident. □

Using a compactness result [17, Corollary 4.1] for doubly Feller forms, we get:

Corollary 5.9. *Let $\mathscr{E}$ be* doubly Feller*, i.e., $\mathscr{E}$ induces a semigroup and one has the smoothing property $T_t(L^\infty(X)) \subset C_b(X)$ for every $t > 0$. Then H satisfies the weak spatial local compactness property. In particular, the conclusions of Theorems 5.3 and 5.5 hold.*

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Declarations

Conflict of interest There is no conflict of interest.

6. Appendix

Let again X be a locally compact separable metrizable space which is equipped with a positive Radon measure m with full support. We fix a semi-Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(X)$, with H the associated sectorial operator, $T_t := e^{-tH}$ the semigroup, and G_α the resolvent, both considered to be acting as contractions in $L^p(X)$. Finally, we are going to need the norm

$$\|u\|_{\mathscr{E}} := \sqrt{\mathscr{E}(u,u) + \|u\|_2^2} \quad \text{on } \mathscr{F}$$

and the norm

$$\|u\|_H := \sqrt{\|Hu\|_2^2 + \|u\|_2^2} \quad \text{on } \operatorname{Dom}(H).$$

Our goal here is to prove the following result, which should be known to experts:

Theorem 6.1. *Assume one has*

$$T_t(C_0(X)) \subset C_0(X) \quad \textit{for all } t > 0, \tag{6.1}$$

$$\|T_t\phi - \phi\|_\infty \to 0 \quad \textit{as } t \to 0+ \textit{ for all } \phi \in C_0(X). \tag{6.2}$$

Then $\mathscr{E}$ is regular.

The proof relies on the following auxiliary results:

Proposition 6.2. *The following statements are equivalent:*

- $\mathscr{E}$ *is regular.*
- $\mathscr{F} \cap C_0(X)$ *is dense in* $(\mathscr{F}, \|\bullet\|_{\mathscr{E}})$ *and in* $(C_0(X), \|\bullet\|_\infty)$.

Proof. It suffices to show that every $\phi \in \mathscr{F} \cap C_0(X)$ can be approximated by a sequence in $\mathscr{F} \cap C_c(X)$ with respect to $\|\bullet\|_{\mathscr{E}}$. To this end, by the Markovian property of $\mathscr{E}$, we can assume $\phi \geq 0$. Then one has $\phi_n := (\phi - 1/n)_+ \in \mathscr{F} \cap C_c(X)$ by the Markovian property, $\phi_n \to \phi$ in $L^2(X)$, and moreover $\sup_n \mathscr{E}(\phi_n, \phi_n) < \infty$. Thus, a subsequence of ϕ_n does the job. □

Proposition 6.3. *Under* (6.1) *and* (6.2)*, the space* $D_0 := \mathrm{Dom}(H) \cap C_0(X)$ *is dense in* $(\mathrm{Dom}(H), \|\bullet\|_H)$ *and in* $(C_0(X), \|\bullet\|_\infty)$.

Proof. Note first that in this situation $(T_t; t \geq 0)$ becomes a strongly continuous contraction semigroup in $(C_0(X), \|\bullet\|_\infty)$. Thus, one has

$$G_\alpha \phi = \int_0^\infty e^{-\alpha t} T_t \phi \, dt \quad \text{for every } \alpha > 0, \phi \in C_0(X), \tag{6.3}$$

where the integral converges in the uniform norm.
(1) With

$$D_1 := \{G_\alpha \phi \mid \phi \in C_c(X), \alpha > 0\},$$

in view of (6.3), one has $D_1 \subset D_0$. Given $g \in \mathrm{Dom}(H)$ we have $f := G_1^{-1} g \in L^2(X)$, so there exists a sequence f_n in $C_c(X)$ with $f_n \to f$ in $L^2(X)$. It follows that with $g_n := G_1 f_n \in D_1$ we have

$$g_n \to g \quad \text{in } (\mathrm{Dom}(H), \|\bullet\|_H),$$

showing that D_1 is dense in $(\mathrm{Dom}(H), \|\bullet\|_H)$.
(2) Given $\phi \in C_c(X)$ we have $\alpha G_\alpha \phi \in D_1$ for all $\alpha > 0$ and

$$\alpha G_\alpha \phi \to \phi \quad \text{in } (C_0(X), \|\bullet\|_\infty), \text{ as } \alpha \to \infty,$$

in view of formula (6.3): indeed, given $\epsilon > 0$ we can pick $\delta > 0$ such that for all $t \in [0, \delta]$ one has $\|T_t \phi - \phi\|_\infty < \epsilon/2$. It then follows easily from decomposing the integral in (6.3) as $\int_0^\infty \cdots = \int_0^\delta \cdots + \int_\delta^\infty \cdots$ that for all $\alpha > 0$ one has

$$\|\alpha G_\alpha \phi - \phi\|_\infty \leq \epsilon/2 + 2 \|\phi\|_\infty \alpha e^{-\alpha\delta}.$$

□

Proof of Theorem 6.1. As the embedding

$$(\mathrm{Dom}(H), \|\bullet\|_H) \hookrightarrow (\mathscr{F}, \|\bullet\|_{\mathscr{E}})$$

has a dense image, the second proposition gives that $\mathscr{F} \cap C_0(X)$ is dense in both, $(\mathrm{Dom}(H), \|\bullet\|_H)$ and $(C_0(X), \|\bullet\|_\infty)$, so that the claim follows from the first proposition. □

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