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Integral Equations and Operator Theory

L^p and H^1 Boundedness of Oscillatory Singular Integral Operators with Hölder Class Kernels

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Abstract. For oscillatory singular integrals with polynomial phases and Hölder class kernels, we establish their uniform boundedness on L^p spaces as well as a sharp logarithmic bound on the Hardy space H^1. These results improve the ones in (Pan in Forum Math 31: 535–542, 2019) by removing the restriction that the phase polynomials be quadratic.

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1. Introduction

Let $n, d \in \mathbb{N}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let

$$P(x) = \sum_{|\alpha| \le d} a_\alpha x^\alpha \tag{1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $a_\alpha \in \mathbb{R}$. For each nonconstant polynomial $P(x)$ we let

$$\|P\|_o = \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2 \le |\alpha| \le d} |a_\alpha|^{1/|\alpha|}}. \tag{2}$$

When $\deg(P) = 1$, the value of $\|P\|_o$ shall be interpreted as ∞.

For a Calderón–Zygmund type singular kernel $K(x)$, let $T_{P,K}$ be the oscillatory singular integral operator defined by

$$T_{P,K} f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x-y)} K(x-y) f(y) dy. \tag{3}$$

In [7] the author proved the following:

Theorem 1.1. *Suppose that* $\deg(P) = 2$ *and there exist* $q > 2$ *and* $\delta > 0$ *such that*

$$\boldsymbol{CZ}(q,\delta) \quad \begin{cases} \text{(a)} \left(\int_{s\le|x|\le 2s} |K(x)|^q dx \right)^{1/q} \le As^{-n/q'} \quad \textit{for} \quad s > 0; \\ \text{(b)} \ |K(x-y) - K(x)| \le \dfrac{A|y|^\delta}{|x|^{n+\delta}} \quad \textit{for} \quad |x| \ge 2|y|; \\ \text{(c)} \left| \int_{s_1\le|x|\le s_2} K(x)dx \right| \le A \quad \textit{for} \quad 0 < s_1 < s_2. \end{cases}$$

Then,
(i) *For* $1 < p < \infty$, *there exists a positive constant* C_p *such that*

$$\|T_{P,K}f\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)} \tag{4}$$

for all $f \in L^p(\mathbb{R}^n)$. *The constant* C_p *may depend on* n, p, δ, q *and* A, *but is otherwise independent of* K *and the coefficients of* P;
(ii) *There exists a positive constant* C *such that*

$$\|T_{P,K}f\|_{H^1(\mathbb{R}^n)} \le C(1 + \log^+ \|P\|_o)\|f\|_{H^1(\mathbb{R}^n)} \tag{5}$$

for all $f \in H^1(\mathbb{R}^n)$. *The constant* C *may depend on* n, δ, q *and* A, *but is otherwise independent of* K *and the coefficients of* P. *The bound given in* (5) *is the best possible in the sense that the logarithmic function cannot be replaced by any function with a slower rate of growth.*

The above conditions (a)–(c) are commonly referred to as the size, smoothness and cancellation conditions for singular kernels, respectively. In classical Calderón–Zygmund theory of singular integrals, one assumes that condition (a) holds for $q = \infty$ together with the C^1 condition $|\nabla K(x)| \le C|x|^{-n-1}$ instead of the weaker Hölder continuity condition (b), as well as (c).

The restriction that $P(x)$ be quadratic (i.e. $\deg(P) = 2$) in the above theorem is clearly a severe one. For the operator $T_{P,K}$ with a C^1 kernel K, both the L^p bound in (1) and the H^1 bound in (2) have been known to be true when the phase polynomial P is of arbitrary degree (for L^p see [8]; for H^1 see [1]). The main purpose of this paper is to show that, for $T_{P,K}$ with a kernel K in the Hölder class, the same L^p and H^1 bounds are true when the degree of the phase polynomial P is arbitrary. We have the following:

Theorem 1.2. *Let* $P(x)$ *be a real-valued polynomial of any positive degree. Suppose that* $K(x)$ *satisfies* $\boldsymbol{CZ}(q,\delta)$(a)–(c) *for some* $q > 1$ *and* $\delta > 0$. *Then,*
(i) *For* $1 < p < \infty$, *there exists a positive constant* C_p *such that*

$$\|T_{P,K}f\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)} \tag{6}$$

for all $f \in L^p(\mathbb{R}^n)$. *The constant* C_p *may depend on* n, p, δ, q, $\deg(P)$ *and* A, *but is otherwise independent of* K *and the coefficients of* P;
(ii) *There exists a positive constant* C *such that*

$$\|T_{P,K}f\|_{H^1(\mathbb{R}^n)} \le C(1 + \log^+ \|P\|_o)\|f\|_{H^1(\mathbb{R}^n)} \tag{7}$$

for all $f \in H^1(\mathbb{R}^n)$. *The constant* C *may depend on* n, δ, q, $\deg(P)$ *and* A, *but is otherwise independent of* K *and the coefficients of* P. *The bound given* *in* (7) *is the best possible in the sense that the logarithmic function cannot be replaced by any function with a slower rate of growth.*

We point out that, aside from lifting the restriction that $\deg(P) = 2$ from Theorem 1.1, Theorem 1.2 also improves the range of q from $q > 2$ to the more natural range $q > 1$.

In the rest of the paper we shall use $A \lesssim B$ to mean that $A \leq cB$ for a certain constant c which depends on some essential parameters only. A subscript may be added to the symbol $\lesssim$ to indicate a particular dependence as appropriate.

2. L^p Boundedness

In this section we will establish part (i) of Theorem 1.2. For $u \in \mathbb{R}^n$ and $r > 0$ we let $B(u, r)$ denote the ball $\{x \in \mathbb{R}^n : |x - u| \leq r\}$. An important tool will be the following lemma from [6].

Lemma 2.1. *Let $P(x)$ be given as in* (1)*. Let $R > 0$ and let $\psi : \mathbb{R}^n \to \mathbb{C}$ be an integrable function supported in $B(0, R/2)$. Then*

$$\left| \int_{\mathbb{R}^n} e^{iP(x)} \psi(x) dx \right| \lesssim_{d,n} \sup_{v \in B(0, R\Lambda^{-1/d})} \int_{\mathbb{R}^n} |\psi(x) - \psi(x - v)| dx, \tag{8}$$

where $\Lambda := \sum_{1 \leq |\alpha| \leq d} |a_\alpha| R^{|\alpha|}$.

For $h > 0$, we let $T_{P,K,h}$ denote the truncation of $T_{P,K}$ given by

$$T_{P,K,h} f(x) = \text{p.v.} \int_{|x-y| \leq h} e^{iP(x-y)} K(x - y) f(y) dy. \tag{9}$$

We will establish the following uniform L^p boundedness theorem:

Theorem 2.1. *Let $P(x)$ be a real-valued polynomial of any degree and $h > 0$. Suppose that $K(x)$ satisfies **CZ**(q, δ)(a)–(c) for some $q > 1$ and $\delta > 0$. Then, for $1 < p < \infty$, there exists a positive constant C_p such that*

$$\|T_{P,K,h} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{10}$$

for all $f \in L^p(\mathbb{R}^n)$. The constant C_p may depend on n, p, δ, q, $\deg(P)$ and A, but is otherwise independent of h, K and the coefficients of P.

Proof. Without loss of generality we may assume that $P(x)$ is nonconstant and $P(0) = 0$. In order to prove (10) we shall use induction on $\deg(P)$. When $\deg(P) = 1$, by $P(x-y) = P(x) - P(y)$, (10) follows from the L^p boundedness of singular integrals ([4], page 300). Suppose that for a $d \geq 2$, (10) holds for all P with $\deg(P) \leq d - 1$.

We now assume that $\deg(P) = d$, i.e.

$$P(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$$

with $\sum_{|\alpha|=d} |a_\alpha| \neq 0$.

It is easy to see that for $t > 0$, $t^n K(tx)$ satisfies conditions $\mathbf{CZ}(q,\delta)$(a)–(c) with the same q, δ and A. Thus, by rescaling if necessary we may assume that $\sum_{|\alpha|=d} |a_\alpha| = 1$.

Let

$$R(x) = P(x) - \sum_{|\alpha|=d} a_\alpha x^\alpha. \tag{11}$$

Then $\deg(R) \leq d-1$ and for $0 < h \leq 8$

$$|T_{P,K,h}f(x) - T_{R,K,h}f(x)| \lesssim \int_{|x-y|\leq 8} |x-y|^d |K(x-y)||f(y)|dy. \tag{12}$$

By (12), Hölder's inequality and $\mathbf{CZ}(q,\delta)$(a),

$$\begin{aligned}
\|T_{P,K,h}f - T_{R,K,h}f\|_{L^p(\mathbb{R}^n)} &\lesssim \left(\int_{|x|\leq 8} |x|^d |K(x)|dx\right) \|f\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \left(\|f\|_{L^p(\mathbb{R}^n)}\right) \sum_{j=-\infty}^{2} \left(\int_{2^j\leq|x|\leq 2^{j+1}} |x|^{dq'} dx\right)^{1/q'} \\
&\quad \times \left(\int_{2^j\leq|x|\leq 2^{j+1}} |K(x)|^q dx\right)^{1/q} \\
&\lesssim \left(\|f\|_{L^p(\mathbb{R}^n)}\right) \sum_{j=-\infty}^{2} \left(2^{jdq'} 2^{jn}\right)^{1/q'} 2^{-jn/q'} \\
&\lesssim \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Thus, for $0 < h \leq 8$

$$\|T_{P,K,h}f\|_{L^p(\mathbb{R}^n)} \lesssim \|T_{R,K,h}f\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{13}$$

For the rest of this proof we assume that $h > 8$. For $j \leq 2$ let $I_j = [2^j, 2^{j+1}]$,

$$\psi_j(x) = \chi_{I_j}(|x|)K(x)$$

and let S_j denote the following operator

$$S_j f(x) = \int_{\mathbb{R}^n} e^{iP(x-y)} \psi_j(x-y) f(y)dy. \tag{14}$$

Then

$$\widehat{S_j f}(\xi) = m_j(\xi)\hat{f}(\xi) \tag{15}$$

where

$$m_j(\xi) = \int_{\mathbb{R}^n} e^{i(P(x)-2\pi\xi\cdot x)} \psi_j(x)dx. \tag{16}$$

For each j, we will now apply Lemma 2.1 to the estimate of $\|m_j\|_\infty$ with $R = 2^{j+2}$. By

$$P(x) - 2\pi\xi\cdot x = \sum_{k=1}^{n} \left(\frac{\partial P}{\partial x_k}(0) - 2\pi\xi_k\right) x_k + \sum_{2\leq|\alpha|\leq d} a_\alpha x^\alpha,$$

and

$$\Lambda := \sum_{k=1}^{n}\left|\frac{\partial P}{\partial x_k}(0) - 2\pi\xi_k\right|R + \sum_{2\le|\alpha|\le d}|a_\alpha|R^{|\alpha|} \ge \left(\sum_{|\alpha|=d}|a_\alpha|\right)R^d = R^d,$$

we have $R\Lambda^{-1/d} \le 1$. Thus, for any $\xi \in \mathbb{R}^n$,

$$|m_j(\xi)| \lesssim \sup_{v\in B(0,1)}\int_{\mathbb{R}^n}|\psi_j(x) - \psi_j(x-v)|dx. \tag{17}$$

For any $v \in B(0,1)$, $j \ge 2$ and $x \in B(0,2^{j+1})\backslash B(0,2^j)$ we have $|x| \ge 2|v|$ and

$$|B(0,2^j)\Delta B(v,2^j)| + |B(0,2^{j+1})\Delta B(v,2^{j+1})| \lesssim 2^{j(n-1)}. \tag{18}$$

Thus, by $\mathbf{CZ}(q,\delta)$(b),

$$\begin{aligned}|\psi_j(x) - \psi_j(x-v)| &\le \chi_{I_j}(|x|)|K(x) - K(x-v)| \\ &\quad + |\chi_{I_j}(|x|) - \chi_{I_j}(|x-v|)||K(x-v)| \\ &\lesssim \chi_{I_j}(|x|)\left(\frac{|v|^\delta}{|x|^{n+\delta}}\right) \\ &\quad + (\chi_{B(0,2^j)\Delta B(v,2^j)}(x) + \chi_{B(0,2^{j+1})\Delta B(v,2^{j+1})}(x))|K(x-v)|.\end{aligned} \tag{19}$$

Since $(B(0,2^j)\Delta B(v,2^j))\cup(B(0,2^{j+1})\Delta B(v,2^{j+1})) \subseteq B(v,2^{j+2})\backslash B(v,2^{j-1})$, by (17)–(19) and $\mathbf{CZ}(q,\delta)$(a) we have

$$\begin{aligned}|m_j(\xi)| &\lesssim \int_{2^j\le|x|\le 2^{j+1}}\frac{dx}{|x|^{n+\delta}} \\ &\quad + \int_{\mathbb{R}^n}(\chi_{B(0,2^j)\Delta B(v,2^j)}(x) + \chi_{B(0,2^{j+1})\Delta B(v,2^{j+1})}(x))|K(x-v)|dx \\ &\lesssim 2^{-j(n+\delta)}2^{jn} + (|B(0,2^j)\Delta B(v,2^j)| + |B(0,2^{j+1})\Delta B(v,2^{j+1})|)^{1/q'} \\ &\quad \times\left(\int_{B(v,2^{j+2})\backslash B(v,2^{j-1})}|K(x-v)|^q dx\right)^{1/q} \\ &\lesssim 2^{-j\delta} + 2^{j(n-1)/q'}2^{-jn/q'} \\ &\lesssim 2^{-j\mu}\end{aligned}$$

where $\mu = \min\{\delta, 1/q'\}$. It follows from Plancherel's theorem that

$$\|S_j\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim 2^{-j\mu}. \tag{20}$$

By $\mathbf{CZ}(q,\delta)$(a), for any t_1, t_2 satisfying $0 < t_1 < t_2$ and $t_2/t_1 \lesssim 1$,

$$\|\chi_{B(0,t_2)\backslash B(0,t_1)}|K|\|_{L^1(\mathbb{R}^n)} \lesssim 1. \tag{21}$$

Thus,

$$\begin{aligned}&\|S_j\|_{L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} + \|S_j\|_{L^\infty(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \\ &\lesssim \|\chi_{B(0,2^{j+1})\backslash B(0,2^j)}K\|_{L^1(\mathbb{R}^n)} \lesssim 1.\end{aligned} \tag{22}$$

By the Riesz–Thorin interpolation theorem, for $1 < p < \infty$,

$$\|S_j\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim 2^{-j\mu_p}, \tag{23}$$

where

$$\mu_p = \mu(1 - |1 - 2/p|) > 0.$$

Let $m = [\log_2 h]$. Then

$$|T_{P,K,h}f| \le |T_{P,K,4}f| + \sum_{j=2}^{m-1} |S_j f| + |(\chi_{B(0,h)\setminus B(0,2^m)}|K|) * |f|. \tag{24}$$

It follows from (13), (21) and (23) that

$$\|T_{P,K,h}f\|_{L^p(\mathbb{R}^n)} \lesssim \left(1 + \sum_{j=2}^{m-1} 2^{-j\mu_p} + \|\chi_{B(0,h)\setminus B(0,2^m)}|K|\|_{L^1(\mathbb{R}^n)}\right)\|f\|_{L^p(\mathbb{R}^n)}$$
$$\lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

The proof of Theorem 2.1 is now complete. □

By using

$$T_{P,K}f = \lim_{h\to\infty} T_{P,K,h}f$$

interpreted in the distributional sense, we obtain (6) for all test functions f. Part (i) of Theorem 1.2 then follows by standard arguments.

3. $H^1 \to H^1$ Estimates

As for the L^p boundeness, the H^1 arguments in [7] relied both on the phase being quadratic as well as the condition $\mathbf{CZ}(q,\delta)$ with a $q > 2$. To prove part (ii) of Theorem 1.2, we will let the degree of the phase polynomial be any positive integer while assuming that K satisfies $\mathbf{CZ}(q,\delta)$ with a $q > 1$.

Lemma 3.1. *Let $d \ge 2$, $P(x) = \sum_{|\alpha|\le d} a_\alpha x^\alpha$ and $K(x)$ satisfy*

$$\sup_{s>0} s^{n/q'} \|\chi_{B(0,2s)\setminus B(0,s)}K\|_{L^q(\mathbb{R}^n)} \lesssim 1$$

for some $q > 1$. Then,
(i) *for any $0 < a < b$ and $\nu \ge 1$,*

$$\int_{a\le|x|\le b} |x|^\nu |K(x)|dx \lesssim (b-a)^{1/q'} b^{\nu-1/q'}; \tag{25}$$

$$\int_{a\le|x|\le b} |K(x)|dx \lesssim 1 + \ln(b/a); \tag{26}$$

(ii) *for any $\lambda \ge 1$,*

$$\int_{|x|\ge\lambda} \left|K(x)\int_{B(0,1)} e^{iP(x-y)} f(y)dy\right| dx$$
$$\lesssim \left[\lambda\left(\sum_{2\le|\alpha|\le d} |a_\alpha|^{1/|\alpha|}\right)^2\right]^{-1/(2\gamma' d)} \|f\|_{L^{\gamma'}(B(0,1))} \tag{27}$$

where $\gamma = \min\{q, 2\}$.

Proof. (i) Let $N = [\log_2(b/a)]$. Then

$$\begin{aligned}\int_{a\le|x|\le b}|x|^{\nu}|K(x)|dx &\lesssim \left(\int_{a\le|x|\le b}|x|^{1-n}dx\right)^{1/q'}\\ &\quad\times\left(\int_{a\le|x|\le 2^{N+1}a}|x|^{(\nu+(n-1)/q')q}|K(x)|^q dx\right)^{1/q}\\ &\lesssim (b-a)^{1/q'}\left(\sum_{j=0}^{N}(2^j a)^{(\nu+(n-1)/q')q}\|\chi_{B(0,2^{j+1}a)\setminus B(0,2^j a)}K\|^q_{L^q(\mathbb{R}^n)}\right)^{1/q}\\ &\lesssim (b-a)^{1/q'}\left(\sum_{j=0}^{N}(2^j a)^{(\nu-1/q')q+nq/q'}((2^j a)^{-n/q'})^q\right)^{1/q}\\ &\lesssim (b-a)^{1/q'}(2^N a)^{\nu-1/q'}\lesssim (b-a)^{1/q'}b^{\nu-1/q'},\end{aligned}$$

which proves (25). The proof of (26) is simpler and will be omitted.

(ii) Since $1<\gamma\le q$, we have for any $s>0$,

$$\begin{aligned}\|\chi_{B(0,2s)\setminus B(0,s)}K\|_{L^\gamma(\mathbb{R}^n)} &\lesssim \|\chi_{B(0,2s)\setminus B(0,s)}K\|_{L^q(\mathbb{R}^n)}|B(0,2s)|^{1/\gamma-1/q}\\ &\lesssim s^{-n/q'+n(1/\gamma-1/q)} = s^{-n/\gamma'}.\end{aligned}$$

For any $\lambda\ge 1$, by Hölder's inequality and applying Lemma 2.3 in [1] (taking p to be $\gamma'\ge 2$),

$$\begin{aligned}&\int_{|x|\ge\lambda}\left|K(x)\int_{B(0,1)}e^{iP(x-y)}f(y)dy\right|dx\\ &\lesssim \sum_{j=0}^{\infty}\|\chi_{B(0,2^{j+1}\lambda)\setminus B(0,2^j\lambda)}K\|_{L^\gamma(\mathbb{R}^n)}\\ &\qquad\times\left(\int_{B(0,2^{j+1}\lambda)}\left|\int_{B(0,1)}e^{iP(x-y)}f(y)dy\right|^{\gamma'}dx\right)^{1/\gamma'}\\ &\lesssim \left(\sum_{j=0}^{\infty}(2^j\lambda)^{-n/\gamma'}(2^j\lambda)^{(2nd-1)/(2\gamma' d)}\right)\\ &\qquad\times\left(\sum_{2\le|\alpha|\le d}|a_\alpha|^{1/|\alpha|}\right)^{-1/(\gamma' d)}\|f\|_{L^{\gamma'}(B(0,1))}\\ &\lesssim \left(\sum_{j=0}^{\infty}2^{-j/(2\gamma' d)}\right)\left[\lambda\left(\sum_{2\le|\alpha|\le d}|a_\alpha|^{1/|\alpha|}\right)^2\right]^{-1/(2\gamma' d)}\|f\|_{L^{\gamma'}(B(0,1))}\\ &\lesssim \left[\lambda\left(\sum_{2\le|\alpha|\le d}|a_\alpha|^{1/|\alpha|}\right)^2\right]^{-1/(2\gamma' d)}\|f\|_{L^{\gamma'}(B(0,1))}.\end{aligned}$$

□

Lemma 3.2. *Let $K(x)$ be given as in Lemma 3.1 and $Q(x)$ be a polynomial satisfying $\nabla Q(0)=0$. Let f be a Lebesgue measurable function satisfying*

$$supp(f)\subseteq B(0,1); \tag{28}$$

$$\|f\|_\infty \le 1; \tag{29}$$

$$\int_{B(0,1)} f(y)dy = 0. \tag{30}$$

Then, there exists a $C > 0$ such that

$$\int_{|x|\ge 2} \left| K(x) \int_{B(0,1)} e^{iQ(x-y)} f(y)dy \right| dx \le C. \tag{31}$$

The constant C may depend on $\deg(Q)$ *but is otherwise independent of the coefficients of $Q(x)$.*

Proof. When $\deg(Q) \le 1$, by $\nabla Q(0) = 0$, (31) follows from (30) trivially.

Suppose that $d \ge 2$ and (31) holds for all $Q(x)$ satisfying $\deg(Q) \le d-1$ and $\nabla Q(0) = 0$.

Assume that $\deg(Q) = d$ and $\nabla Q(0) = 0$. Then

$$Q(x) = \sum_{|\alpha|=d} q_\alpha x^\alpha + R(x)$$

where $\deg(R) \le d - 1$ and $\nabla R(0) = 0$. Thus,

$$\int_{|x|\ge 2} \left| K(x) \int_{B(0,1)} e^{iR(x-y)} f(y)dy \right| dx \lesssim 1.$$

Let

$$\beta = \max \left\{ 2, \left(\sum_{|\alpha|=d} |q_\alpha| \right)^{-1/(d-1)} \right\}.$$

Then

$$\begin{aligned}
&\int_{2\le|x|\le\beta} \left| K(x) \int_{B(0,1)} e^{iQ(x-y)} f(y)dy \right| dx \\
&\quad \lesssim \int_{2\le|x|\le\beta} \left| K(x) \int_{B(0,1)} \left(e^{iQ(x-y)} - e^{i(\sum_{|\alpha|=d} q_\alpha x^\alpha + R(x-y))} \right) f(y)dy \right| \\
&dx + 1 \\
&\quad \lesssim \left(\sum_{|\alpha|=d} |q_\alpha| \right) \int_{2\le|x|\le\beta} |x|^{d-1} |K(x)| dx + 1 \\
&\quad \lesssim \left(\sum_{|\alpha|=d} |q_\alpha| \right) (\beta - 2)^{1/q'} \beta^{d-1-1/q'} + 1 \lesssim 1.
\end{aligned} \tag{32}$$

Let $\gamma = \min\{q, 2\}$. By Hölder's inequality, (29) and Lemma 4.3 in [5] (after interpolating between the $L^2 \to L^2$ bound there and a trivial $L^1 \to L^\infty$ bound), we have

$$\begin{aligned}
\int_{|x|\ge\beta} \left| K(x) \int_{B(0,1)} e^{iQ(x-y)} f(y)dy \right| dx &\lesssim \sum_{j=0}^{\infty} \left(\int_{2^j\beta \le |x| \le 2^{j+1}\beta} |K(x)|^\gamma dx \right)^{1/\gamma} \\
&\times \left(\int_{|x|\le 2^{j+1}\beta} \left| \int_{B(0,1)} e^{iQ(x-y)} f(y)dy \right|^{\gamma'} dx \right)^{1/\gamma'}
\end{aligned}$$

$$\lesssim \sum_{j=0}^{\infty}(2^j\beta)^{-n/\gamma'}\left[(2^j\beta)^{n/2}\left((2^j\beta)^{d-1}\sum_{|\alpha|=d}|q_\alpha|\right)^{-1/(4(d-1))}\right]^{2/\gamma'} \tag{33}$$

$$\times\|f\|_{L^\gamma(B(0,1))}$$

$$\lesssim \left[\beta\left(\sum_{|\alpha|=d}|q_\alpha|\right)^{1/(d-1)}\right]^{-1/2\gamma'} \lesssim 1.$$

Now (31) follows from (32) and (33). □

We will now prove part (ii) of Theorem 1.2.

Proof. Let $P(x)$ be a real-valued polynomial of any positive degree. Suppose that $K(x)$ satisfies $\boldsymbol{CZ}(q,\delta)$ *(a)–(c)* for some $q>1$ and $\delta>0$. Let $\gamma=\min\{q,2\}$. For P with $\deg(P)=1$, we have $\|P\|_o=\infty$ in which case (7) holds trivially. Thus we may assume that $d=\deg(P)\geq 2$.

Since $T_{P,K}$ is translation invariant, by the standard atomic theory of Hardy spaces, it suffices to prove that

$$\|T_{P,K}f\|_{L^1(\mathbb{R}^n)} \lesssim 1+\log^+\|P\|_o \tag{34}$$

holds for every $H^1(\mathbb{R}^n)$ atom $f(\cdot)$ which is supported in a ball centered at the origin (see [2,3,9,10]). Additionally, due to the invariance of the $\mathbf{CZ}(q,\delta)$ conditions under $K(x)\to t^nK(tx)$ and the invariance of $\|\ \ \|_o$ under $P(x)\to P(tx)$, we may assume that f satisfies (28)–(30).

First we will prove that

$$\int_{|x|\geq 2}\left|K(x)\int_{B(0,1)}e^{iP(x-y)}f(y)dy\right|dx \lesssim 1+\log^+\|P\|_o. \tag{35}$$

Let

$$a=\max\left\{2,\left(\sum_{|\alpha|=1}|a_\alpha|\right)^{-1}\right\},\qquad b=\max\left\{a,\left(\sum_{2\leq|\alpha|\leq d}|a_\alpha|^{1/|\alpha|}\right)^{-2}\right\}.$$

By Lemma 3.1(ii) and (29),

$$\begin{aligned}&\int_{|x|\geq b}\left|K(x)\int_{B(0,1)}e^{iP(x-y)}f(y)dy\right|dx\\ &\lesssim \left[b\left(\sum_{2\leq|\alpha|\leq d}|a_\alpha|^{1/|\alpha|}\right)^2\right]^{-1/(2\gamma'd)}\|f\|_{L^{\gamma'}(B(0,1))}\lesssim 1.\end{aligned} \tag{36}$$

Let

$$Q(x)=P(0)+\sum_{2\leq|\alpha|\leq d}a_\alpha x^\alpha.$$

By Lemmas 3.2 and 3.1(i),

$$\int_{2\leq|x|\leq a}\left|K(x)\int_{B(0,1)}e^{iP(x-y)}f(y)dy\right|dx$$

$$\lesssim \int_{2\leq|x|\leq a}\left|K(x)\int_{B(0,1)}\left(e^{iP(x-y)}-e^{iQ(x-y)}\right)f(y)dy\right|dx+1$$

$$\lesssim \Big(\sum_{|\alpha|=1} |a_\alpha|\Big) \int_{2\le|x|\le a} |x||K(x)|dx$$
$$\lesssim \Big(\sum_{|\alpha|=1} |a_\alpha|\Big)(a-2)^{1/q'} a^{1-1/q'} \lesssim 1. \tag{37}$$

Thus, by (36) and (37), (35) would follow if we can prove that

$$\int_{a\le|x|\le b} \Big| K(x) \int_{B(0,1)} e^{iP(x-y)} f(y)dy \Big| dx \lesssim 1 + \log^+ \|P\|_o. \tag{38}$$

Since (38) holds trivially when $a = b$, we may assume that $a < b$. Then

$$\Big(\sum_{|\alpha|=1} |a_\alpha|\Big)^{-1} \le a < b = \Big(\sum_{2\le|\alpha|\le d} |a_\alpha|^{1/|\alpha|}\Big)^{-2}. \tag{39}$$

For $y \in B(0,1)$,

$$\Big| e^{iP(x-y)} - e^{i(\sum_{|\alpha|=1} a_\alpha x^\alpha + Q(x-y))} \Big| \le \min\Big\{2, \sum_{|\alpha|=1} |a_\alpha|\Big\} \lesssim a^{-1}. \tag{40}$$

By (40), (29) ,(26) and (39),

$$\int_{a\le|x|\le b} \Big| K(x) \int_{B(0,1)} e^{iP(x-y)} f(y)dy \Big| dx$$
$$\lesssim \int_{a\le|x|\le b} \Big| K(x) \int_{B(0,1)}$$
$$\times\Big(e^{iP(x-y)} - e^{i(\sum_{|\alpha|=1} a_\alpha x^\alpha + Q(x-y))} \Big) f(y)dy \Big| dx + 1$$
$$\lesssim a^{-1} \int_{a\le|x|\le b} |K(x)|dx + 1 \lesssim a^{-1} \ln(b/a) + 1$$
$$\lesssim a^{-1} \ln(b/a^2) + \sup_{t\ge 2}(t^{-1} \ln t) \lesssim 1 + \log^+ \|P\|_o.$$

This proves (38) and, in turn, (35).

By part (i) of Theorem 1.2, $\mathbf{CZ}(q,\delta)$(b), (35) and (29),

$$\|T_{P,K} f\|_{L^1(\mathbb{R}^n)} \lesssim \int_{|x|\le 2} |T_{P,K} f(x)|dx$$
$$+ \int_{|x|\ge 2} \Big| \int_{B(0,1)} e^{iP(x-y)}(K(x-y) - K(x)) f(y)dy \Big| dx$$
$$+ \int_{|x|\ge 2} \Big| K(x) \int_{B(0,1)} e^{iP(x-y)} f(y)dy \Big| dx$$
$$\lesssim \|T_{P,K} f\|_{L^2(\mathbb{R}^n)} + \int_{|x|\ge 2} \int_{B(0,1)} \frac{|y|^\delta |f(y)|}{|x|^{n+\delta}} dydx + (1 + \log^+ \|P\|_o)$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^1(\mathbb{R}^n)} + (1 + \log^+ \|P\|_o) \lesssim +(1 + \log^+ \|P\|_o).$$

The proof of part (ii) of Theorem 1.2 is now complete. □

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