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Integral Equations and Operator Theory

Difference of Weighted Composition Operators II

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Abstract. In the setting of the unit disk we have recently obtained characterizations in terms of Carleson measures for bounded/compact differences of weighted composition operators acting from a standard weighted Bergman space into the corresponding weighted Lebesgue space. In this paper we extend those results to the case when the exponents of the domain space and the target space are different.

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1. Introduction

Let $\mathbf{D}$ be the unit disk in the complex plane $\mathbf{C}$. Denote by $\mathcal{S}(\mathbf{D})$ the set of all holomorphic self-maps of $\mathbf{D}$. Given $\varphi \in \mathcal{S}(\mathbf{D})$ and a Borel function u on $\mathbf{D}$, the weighted composition operator $C_{\varphi,u}$ with symbol φ and weight u is defined by

$$C_{\varphi,u} f := u(f \circ \varphi)$$

for functions f holomorphic on $\mathbf{D}$. So, the classical composition operators correspond to the special case when u is the constant function 1.

It has been of growing interest to study on differences, or more generally linear combinations, of composition operators for the last three decades. For the background and historical remarks on such study, we refer to [1] and references therein. Quite recently, pursuing the same line of research, Acharyya and Wu [2] first considered differences of weighted composition operators with holomorphic weights satisfying certain growth rate and obtained characterizations for compactness of such operators acting from a standard

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weighted Bergman space into another. In particular, in the most basic case when the domain space and the target space are the same, their weights were restricted to bounded holomorphic functions. The current authors [1] then extended such a special case to the case of general weights which are possibly non-holomorphic and unbounded. In this paper we extend the results of [1] to the case when the exponents of the domain space and the target space are different.

To begin with, we recall the standard weighted Bergman spaces. Given $\alpha > -1$, we denote by A_α the normalized weighted measure defined by

$$dA_\alpha(z) := (\alpha+1)(1-|z|^2)^\alpha \, dA(z), \qquad z \in \mathbf{D}$$

where A denotes the area measure on $\mathbf{D}$ normalized to have the total mass 1.

For $0 < p < \infty$, the α-weighted Bergman space $A^p_\alpha(\mathbf{D})$ is the space of all holomorphic functions f on $\mathbf{D}$ for which the "norm"

$$\|f\|_{A^p_\alpha} := \left\{ \int_{\mathbf{D}} |f|^p \, dA_\alpha \right\}^{1/p}$$

is finite. As is well-known, the space $A^p_\alpha(\mathbf{D})$ is a closed subspace of $L^p_\alpha(\mathbf{D}) := L^p(\mathbf{D}, A_\alpha)$, the standard Lebesgue space associated with the measure A_α. So, it is a Banach space for $1 \le p < \infty$ and a complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f, g) \mapsto \|f - g\|^p_{A^p_\alpha}$.

To state our results we introduce several notation. We reserve symbol functions $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ and weights u, v to be considered throughout the paper. We put

$$\rho(z) := d\big(\varphi(z), \psi(z)\big), \qquad z \in \mathbf{D}$$

where d denotes the pseudohyperbolic distance on $\mathbf{D}$; see Sect. 2.2. Given a positive Borel measure μ on $\mathbf{D}$ and $\varphi \in \mathcal{S}(\mathbf{D})$, we denote by $\mu \circ \varphi^{-1}$ the pullback measure on $\mathbf{D}$ defined by $(\mu \circ \varphi^{-1})(E) = \mu[\varphi^{-1}(E)]$ for Borel sets $E \subset \mathbf{D}$. With these notation we now introduce below several pullback measures on $\mathbf{D}$ associated with φ, ψ, u and v.

First, for $\alpha > -1$ and $0 < q < \infty$, we define a pullback measure $\omega = \omega^{\alpha,q}_{\varphi,u;\psi,v}$ by

$$\omega := (|\rho u|^q \, dA_\alpha) \circ \varphi^{-1} + (|\rho v|^q \, dA_\alpha) \circ \psi^{-1}.$$

Also, for $\beta > 0$, we define a pullback measure $\sigma^\beta = \sigma^{\alpha,q,\beta}_{\varphi,u;\psi,v}$ by

$$\sigma^\beta := [(1-\rho)^\beta |u-v|^q \, dA_\alpha] \circ \varphi^{-1} + [(1-\rho)^\beta |u-v|^q \, dA_\alpha] \circ \psi^{-1}.$$

Finally, for $0 < r < 1$, we put

$$G_r := \{ z \in \mathbf{D} : \rho(z) < r \} \tag{1.1}$$

and define a pullback measure $\sigma_r = \sigma^{\alpha,q}_{\varphi,u;\psi,v;r}$ by

$$\sigma_r := (\chi_{G_r} |u-v|^q \, dA_\alpha) \circ \varphi^{-1} + (\chi_{G_r} |u-v|^q \, dA_\alpha) \circ \psi^{-1}$$

where χ_{G_r} denotes the characteristic function of the set G_r. Note that ω, σ^β and σ_r are finite measures if $u, v \in L^q_\alpha(\mathbf{D})$.

In this paper we obtain characterizations in terms of Carleson properties of measures involving those introduced above for bounded/compact differences of weighted composition operators acting from $A^p_\alpha(\mathbf{D})$ into $L^q_\alpha(\mathbf{D})$ for an arbitrary pair of p and q. The case $p = q$ was earlier studied by the current authors. As is well known, characterizations of Carleson measures in this context are split into two cases, namely, $p \le q$ and $q < p$; see Sect. 2.4. So, our characterizations also split into the corresponding two cases.

In case $p \le q$, we have the following characterization, which contains the main result of [1] as a special case. For the notions of (λ, α)-Carleson measures, see Sect. 2.4.

Theorem 1.1. *Let $\alpha > -1$, $0 < p \le q < \infty$, $\frac{\beta}{q} > \frac{\alpha+2}{p}$ and $0 < r < 1$. Put $\lambda = \frac{q}{p}$. Let $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ and $u, v \in L^q_\alpha(\mathbf{D})$. Then the following statements are equivalent:*

- (a) *$C_{\varphi,u} - C_{\psi,v} : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ is bounded(compact, resp.);*
- (b) *$\omega + \sigma^\beta$ is a (compact, resp.) (λ, α)-Carleson measure;*
- (c) *$\omega + \sigma_r$ is a (compact, resp.) (λ, α)-Carleson measure.*

In case $q < p$, boundedness and compactness turn out to be the same and characterizations are as follows.

Theorem 1.2. *Let $\alpha > -1$, $0 < q < p < \infty$, $\frac{\beta}{q} > \max\{1, \frac{1}{p}\} + \frac{\alpha+1}{p}$ and $0 < r < 1$. Put $\lambda = \frac{q}{p}$. Let $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ and $u, v \in L^q_\alpha(\mathbf{D})$. Then the following statements are equivalent:*

- (a) *$C_{\varphi,u} - C_{\psi,v} : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ is bounded;*
- (b) *$C_{\varphi,u} - C_{\psi,v} : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ is compact;*
- (c) *$\omega + \sigma^\beta$ is a (λ, α)-Carleson measure;*
- (d) *$\omega + \sigma_r$ is a (λ, α)-Carleson measure.*

In Sect. 2 we recall and collect some basic facts to be used in later sections. In Sect. 3 we prove a more detailed version of Theorem 1.1; see Theorem 3.1. As one may expect, the main ideas of proofs are essentially the same as those in [1]. In Sect. 4 we prove a more detailed version of Theorem 1.2; see Theorem 4.1. The proofs are quite different from those in [1], but still utilize some technical estimates from [1].

Constants Throughout the paper we use the same letter C to denote positive constants which may vary at each occurrence but do not depend on the essential parameters. Variables indicating the dependency of constants C will be often specified inside parentheses. For nonnegative quantities X and Y the notation $X \lesssim Y$ or $Y \gtrsim X$ means $X \le CY$ for some inessential constant C. Similarly, we write $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Preliminaries

In this section we collect well-known basic facts to be used in later sections. One may find details in standard references such as [3,4], unless otherwise specified.

2.1. Compact Operator

We clarify the notion of compact operators, since the spaces under consideration are not Banach spaces when $0 < p < 1$. Let X and Y be topological vector spaces whose topologies are induced by complete metrics. A continuous linear operator $L : X \to Y$ is said to be compact if the image of every bounded sequence in X has a convergent subsequence in Y.

For a linear combination of weighted composition operators with L^q_α-weights acting on the weighted Bergman spaces, we have the following convenient compactness criterion taken from [1, Lemma 2.1].

Lemma 2.1. *Let $\alpha > -1$ and $0 < p, q < \infty$. Let T be a linear combination of weighted composition operators with weights in $L^q_\alpha(\mathbf{D})$ and assume that $T : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ is bounded. Then $T : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ is compact if and only if $Tf_n \to 0$ in $L^q_\alpha(\mathbf{D})$ for any bounded sequence $\{f_n\}$ in $A^p_\alpha(\mathbf{D})$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$.*

2.2. Pseudohyperbolic Distance

The well-known pseudohyperbolic distance between $z, w \in \mathbf{D}$ is given by

$$d(z, w) := |\eta_w(z)|$$

where $\eta_w(z) := \frac{w-z}{1-z\overline{w}}$ is the involutive automorphism of $\mathbf{D}$ that exchanges 0 and w. The explicit expression of $d(z, w)$ is given by the identity

$$1 - d^2(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\overline{w}|^2}. \tag{2.1}$$

This yields an inequality

$$\frac{1 - d(z, w)}{1 + d(z, w)} \le \frac{1 - |z|^2}{1 - |w|^2} \le \frac{1 + d(z, w)}{1 - d(z, w)} \tag{2.2}$$

which is useful for our purpose. In fact, with $a := \eta_w(z)$, we have by (2.1)

$$\frac{1 - |z|^2}{1 - |w|^2} = \frac{1 - |\eta_w(a)|^2}{1 - |w|^2} = \frac{1 - |a|^2}{|1 - a\overline{w}|^2} \le \frac{1 + |a|}{1 - |a|} = \frac{1 + d(z, w)}{1 - d(z, w)}$$

so that (2.2) holds by symmetry.

We denote by $E_s(z)$ the pseudohyperbolic disk with center $z \in \mathbf{D}$ and radius $s \in (0, 1)$. By an elementary calculation one can see that $E_s(z)$ is a Euclidean disk with

$$(\text{center}) = \frac{1 - s^2}{1 - s^2|z|^2} z \quad \text{and} \quad (\text{radius}) = \frac{1 - |z|^2}{1 - s^2|z|^2} s. \tag{2.3}$$

Given $s \in (0, 1)$, we will frequently use the estimate

$$1 - |z|^2 \approx |1 - \overline{z}w| \approx 1 - |w|^2 \tag{2.4}$$

and

$$|1 - \overline{\xi}z| \approx |1 - \overline{\xi}w| \tag{2.5}$$

for all $\xi \in \mathbf{D}$ and $z, w \in \mathbf{D}$ with $d(z,w) < s$; constants suppressed in these estimates depend only on s. Given $\alpha > -1$ and $s \in (0,1)$, one may use the above estimate to verify

$$A_\alpha\left[E_s(z)\right] \approx (1-|z|^2)^{\alpha+2} \tag{2.6}$$

for $z \in \mathbf{D}$; constants suppressed in this estimate depend only on s and α. We refer to [5, Chapter 4] for details of these estimates.

2.3. Test Functions

Given $\alpha > -1$ and $s \in (0,1)$, we recall the submean value type inequality

$$|f(a)|^p \le \frac{C}{(1-|a|^2)^{\alpha+2}} \int_{E_s(a)} |f|^p \, dA_\alpha, \qquad a \in \mathbf{D} \tag{2.7}$$

valid for functions f holomorphic on $\mathbf{D}$ and $0 < p < \infty$ where $C > 0$ is a constant depending only on α and s. This is easily verified via (2.3), (2.4) and the subharmonicity of $|f|^p$.

Note from (2.7) with $p = 2$ that each point evaluation is a continuous linear functional on the Hilbert space $A^2_\alpha(\mathbf{D})$. Thus, to each $a \in \mathbf{D}$ corresponds a unique reproducing kernel whose explicit formula is known as $z \mapsto \tau_a^{\alpha+2}(z)$ where

$$\tau_a(z) := \frac{1}{1-\overline{a}z}. \tag{2.8}$$

Powers of these functions will be the source of test functions in conjunction with Lemma 2.1. The norms of such kernel-type functions are well known. Namely, when $tp > \alpha + 2$, we have

$$\|\tau_a^t\|_{A^p_\alpha} \approx (1-|a|^2)^{-t+\frac{\alpha+2}{p}}, \qquad a \in \mathbf{D}; \tag{2.9}$$

constants suppressed in this estimate are independent of a; see, for example, [5, Lemma 3.10]. We thus see that

$$\frac{\tau_a^t}{\|\tau_a^t\|_{A^p_\alpha}} \to 0 \ \text{ uniformly on compact subsets of } \ \mathbf{D} \tag{2.10}$$

as $|a| \to 1$.

2.4. Carleson Measure

Let μ be a positive Borel measure on $\mathbf{D}$. Let $\alpha > -1$ and $0 < p, q < \infty$. We say that μ is a (p,q,α)-*Carleson measure* if the embedding $A^p_\alpha(\mathbf{D}) \subset L^q(d\mu)$ is bounded, i.e., if there is a constant $C > 0$ such that

$$\left\{\int_{\mathbf{D}} |f|^q \, d\mu\right\}^{1/q} \le C\|f\|_{A^p_\alpha}$$

for all $f \in A^p_\alpha(\mathbf{D})$. If, in addition, such embedding is compact, then μ is called a *compact* (p,q,α) *-Carleson measure*. Note that (p,q,α)-Carleson measures are finite measures.

For $0 < s < 1$ and $\lambda > 0$ we put

$$\widehat{\mu}_{\alpha,s,\lambda}(z) := \frac{\mu[E_s(z)]}{\left(A_\alpha[E_s(z)]\right)^\lambda}, \qquad z \in \mathbf{D}$$

for the weighted averaging function of μ with respect to the measure A_α and the pseudohyperbolic s-disks. In case $\lambda = 1$, we put

$$\widehat{\mu}_{\alpha,s} := \widehat{\mu}_{\alpha,s,1}$$

for simplicity. We recall below the well-known characterizations for (p, q, α)-Carleson measures by means of these weighted averaging functions.

In case $0 < p \le q < \infty$, characterizations for (p, q, α)-Carleson measures are as follows:

$$\mu : (p, q, \alpha)\text{-Carleson measure} \iff \sup_{\mathbf{D}} \widehat{\mu}_{\alpha,s,\frac{q}{p}} < \infty \tag{2.11}$$

and

$$\mu : \text{compact } (p, q, \alpha)\text{-Carleson measure} \iff \lim_{|z|\to 1} \widehat{\mu}_{\alpha,s,\frac{q}{p}}(z) = 0. \tag{2.12}$$

In case $0 < q < p < \infty$, characterizations turn out to be quite different. In fact (p, q, α)-Carleson measures are the same as compact ones and their characterizations are as follows:

$$\mu : (\text{compact}) \ (p, q, \alpha)\text{-Carleson measure} \iff \widehat{\mu}_{\alpha,s} \in L_\alpha^{p/(p-q)}(\mathbf{D}) \tag{2.13}$$

see [6, Theorem 54] or [7, Theorem B]. Note that the notions of (compact) (p, q, α)-Carleson measures are independent of the parameter s and, when $\alpha > -1$ is fixed, depend only on the ratio $\frac{q}{p}$. So, setting $\lambda := \frac{q}{p}$, we simply say *(compact, resp.)* (λ, α)*-Carleson measure* instead of (compact, resp.) (p, q, α)-Carleson measure.

3. The Case $p \le q$

In this section we prove a more detailed version of Theorem 1.1. Before proceeding, we decompose the measures ω, σ^β and σ_r (associated with $\varphi, \psi, u, v, \alpha, q$) defined in the Introduction into two parts as follows:

$$\begin{aligned} \omega &= \omega_{\varphi,u} + \omega_{\psi,v}, \\ \sigma^\beta &= \sigma_\varphi^\beta + \sigma_\psi^\beta, \\ \sigma_r &= \sigma_{\varphi,r} + \sigma_{\psi,r} \end{aligned}$$

where measures $\omega_{\varphi,u}$, σ_φ^β, $\sigma_{\varphi,r}$ are defined by

$$\begin{aligned} \omega_{\varphi,u} &:= (|\rho u|^q \, dA_\alpha) \circ \varphi^{-1}, \\ \sigma_\varphi^\beta &:= [(1-\rho)^\beta |u - v|^q \, dA_\alpha] \circ \varphi^{-1}, \\ \sigma_{\varphi,r} &:= (\chi_{G_r} |u - v|^q \, dA_\alpha) \circ \varphi^{-1}; \end{aligned}$$

measures $\omega_{\psi,v}$, σ_ψ^β, $\sigma_{\psi,r}$ are defined similarly. Parameters omitted in these notation should be clear from the context.

The next theorem is a more detailed version of Theorem 1.1.

Theorem 3.1. *Let* $\alpha > -1$, $0 < p \le q < \infty$, $\frac{\beta}{q} > \frac{\alpha+2}{p}$ *and* $0 < r < 1$. *Put* $\lambda := \frac{q}{p}$. *Let* $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ *and* $u, v \in L_\alpha^q(\mathbf{D})$. *Then the following statements are equivalent:*

(a) $C_{\varphi,u} - C_{\psi,v} : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ *is bounded(compact, resp.);*
(b) $\omega + \sigma^\beta_\varphi$ *and* $\omega + \sigma^\beta_\psi$ *are (compact, resp.)* (λ, α)*-Carleson measures;*
(c) $\omega + \sigma^\beta_\varphi$ *or* $\omega + \sigma^\beta_\psi$ *is a (compact, resp.)* (λ, α)*-Carleson measure;*
(d) $\omega + \sigma_{\varphi,r}$ *and* $\omega + \sigma_{\psi,r}$ *are (compact, resp.)* (λ, α)*-Carleson measures;*
(e) $\omega + \sigma_{\varphi,r}$ *or* $\omega + \sigma_{\psi,r}$ *is a (compact, resp.)* (λ, α)*-Carleson measure.*

We will complete the Proof of Theorem 3.1 by proving the sequences of implications

$$\text{(b)} \implies \text{(c)} \implies \text{(e)} \implies \text{(a)} \implies \text{(b)}$$

and

$$\text{(b)} \implies \text{(d)} \implies \text{(e)}.$$

Note that the implications (b) $\implies$ (c) and (d) $\implies$ (e) are trivial. Also, since

$$1 \le \frac{1-\rho}{1-r} \quad \text{on} \quad G_r$$

for each $r \in (0, 1)$, the implications (b) $\Longrightarrow$ (d) and (c) $\Longrightarrow$ (e) are clear for any $\beta > 0$. Thus it remains to prove the implications

$$\text{(e)} \implies \text{(a)} \implies \text{(b)}. \tag{3.1}$$

General scheme of the proofs will be the same as those [1]. For that purpose we need to extend a couple of inequalities, which were used in [1] for the case $p = q$, to the current setting. First, the following lemma, taken from [8, Lemma 3.1], extends [1, Lemma 4.2].

Lemma 3.2. *Let* $\alpha > -1, 0 < p \le q < \infty$ *and* $0 < s_1 < s_2 < 1$*. Then there is a constant* $C = C(\alpha, p, q, s_1, s_2) > 0$ *such that*

$$|f(z) - f(w)|^q \le C\|f\|^{q-p}_{A^p_\alpha} \frac{d^q(z, w)}{(1 - |z|^2)^{(\alpha+2)\frac{q}{p}}} \int_{E_{s_2}(z)} |f|^p \, dA_\alpha$$

for functions $f \in A^p_\alpha(\mathbf{D})$ *and* $z, w \in \mathbf{D}$ *with* $d(z, w) < s_1$*.*

Next, we also need the following lemma which extends [1, Eq. (2.17)].

Lemma 3.3. *Let* $\alpha > -1, 0 < p \le q < \infty$ *and* $0 < s < 1$*. Then there is a constant* $C = C(\alpha, p, q, s) > 0$ *such that*

$$\int_{\mathbf{D}} |f|^q \, d\mu \le C\|f\|^{q-p}_{A^p_\alpha} \int_{\mathbf{D}} |f|^p \widehat{\mu}_{\alpha,s,\frac{q}{p}} \, dA_\alpha$$

for positive Borel measures μ *on* $\mathbf{D}$ *and functions* $f \in A^p_\alpha(\mathbf{D})$*.*

Proof. Let $f \in A^p_\alpha(\mathbf{D})$. We have by (2.7)

$$\begin{aligned}
|f(z)|^q &= (|f(z)|^p)^{q/p} \\
&\lesssim \frac{1}{(1 - |z|^2)^{(\alpha+2)\frac{q}{p}}} \left\{ \int_{E_s(z)} |f(w)|^p \, dA_\alpha(w) \right\}^{q/p} \\
&\le \frac{1}{(1 - |z|^2)^{(\alpha+2)\frac{q}{p}}} \|f\|^{q-p}_{A^p_\alpha} \int_{E_s(z)} |f(w)|^p \, dA_\alpha(w)
\end{aligned}$$

for all $z \in \mathbf{D}$; the last inequality holds by $p \le q$. Thus, integrating against the measure $d\mu(z)$ and then interchanging the order of integrations, we obtain

$$\begin{aligned}\int_{\mathbf{D}} |f(z)|^q \, d\mu(z) &\lesssim \|f\|_{A^p_\alpha}^{q-p} \int_{\mathbf{D}} |f(w)|^p \left\{ \int_{E_s(w)} \frac{d\mu(z)}{(1-|z|^2)^{(\alpha+2)\frac{q}{p}}} \right\} dA_\alpha(w) \\ &\approx \|f\|_{A^p_\alpha}^{q-p} \int_{\mathbf{D}} |f(w)|^p \widehat{\mu}_{\alpha,s,\frac{q}{p}}(w) \, dA_\alpha(w);\end{aligned}$$

the last estimate holds by (2.4) and (2.6). Note that the constants suppressed in the above estimates depend only on α, p, q and s. □

Having Lemmas 3.2 and 3.3, we may now proceed to the proof of the implications (3.1). In the proofs below most of the technical estimates, which are already established in [1], are still available and thus omitted.

Auxiliary notation Before proceeding, we set some auxiliary notation to be used for the rest of the paper.

First, as for the notation associated with $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ and $u, v \in L^q_\alpha(\mathbf{D})$, we put

$$T := C_{\varphi,u} - C_{\psi,v}$$

for simplicity. In case $T : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ is bounded, its "norm" is denoted by

$$\|T\|_{A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})}.$$

We also put

$$Q_b := \frac{1-\bar{b}\varphi}{1-\bar{b}\psi}$$

and

$$R^{\alpha,q}_{s,t}(a,b) := \int_{\varphi^{-1}[E_s(a)]} \left| u - vQ^t_b \right|^q \, dA_\alpha$$

for $a, b \in \mathbf{D}$, $0 < s < 1$ and $t > 0$.

Next, we will also use the notation

$$\Gamma_N(a) := \left\{ a\zeta : |\zeta| = 1 \text{ and } \ |\mathrm{Arg}\,\zeta| \le |N|(1-|a|) \right\}$$

and

$$a_N := a e^{-iN(1-|a|)}$$

for $a \in \mathbf{D}$ and N real. Finally, we denote by $\|\cdot\|_{L^q_\alpha}$ the "norm" on $L^q_\alpha(\mathbf{D})$.

In the proofs below auxiliary notation specified above will be used often without further references. First, we prove the implication (e) $\Longrightarrow$ (a).

Proof of (e) $\Longrightarrow$ (a) We first consider boundedness. Assume (e). Fix $s \in (r, 1)$. By symmetry we may assume that

$$\mu := \omega + \sigma_{\varphi,r}$$

is a (λ, α)-Carleson measure so that

$$\sup_{\mathbf{D}} \widehat{\mu}_{\alpha,s,\lambda} < \infty \tag{3.2}$$

by (2.11).

Fix an arbitrary $f \in A^p_\alpha(\mathbf{D})$ with $\|f\|_{A^p_\alpha} \le 1$. By the proof of [1, Theorem 4.1] we have

$$\|Tf\|^q_{L^q_\alpha} \lesssim \tfrac{1}{r^q} \int_{\mathbf{D}} |f|^q \, d\mu + \int_{G_r} |v(f\circ\varphi - f\circ\psi)|^q \, dA_\alpha; \tag{3.3}$$

recall that G_r is the set specified in (1.1).

For the first integral in (3.3), we note from Lemma 3.3

$$\int_{\mathbf{D}} |f|^q \, d\mu \lesssim \int_{\mathbf{D}} |f|^p \widehat{\mu}_{\alpha,s,\lambda} \, dA_\alpha. \tag{3.4}$$

For the second integral in (3.3), using Lemma 3.2 and proceeding as in the proof of [1, Theorem 4.1], we obtain

$$\begin{aligned}\int_{G_r} |v(f\circ\varphi - f\circ\psi)|^q \, dA_\alpha &\lesssim \int_{\mathbf{D}} |f|^p \widehat{(\omega_{\psi,v})}_{\alpha,s,\lambda} \, dA_\alpha \\ &\le \int_{\mathbf{D}} |f|^p \widehat{\mu}_{\alpha,s,\lambda} \, dA_\alpha.\end{aligned} \tag{3.5}$$

We now see from (3.3), (3.4) and (3.5) that

$$\|Tf\|^q_{L^q_\alpha} \lesssim \int_{\mathbf{D}} |f|^p \widehat{\mu}_{\alpha,s,\lambda} \, dA_\alpha \le \sup_{\mathbf{D}} \widehat{\mu}_{\alpha,s,\lambda}; \tag{3.6}$$

one may keep track of the constant suppressed in this estimate to find it depending only on α, p, q and r. Consequently, we conclude

$$\|T\|^q_{A^p_\alpha(\mathbf{D})\to L^q_\alpha(\mathbf{D})} \le C \left(\sup_{\mathbf{D}} \widehat{\mu}_{\alpha,s,\lambda} \right)$$

for some constant $C = C(\alpha, p, q, r) > 0$. This, together with (3.2), completes the proof for boundedness.

For the compactness part, one may use Lemma 2.1 and repeat the Proof of [1, Theorem 4.1]. □

Next, we prove the implication (a) $\Longrightarrow$ (b).

Proof of (a) $\Longrightarrow$ (b) Fix $s \in (0, 1)$. Also, as in the proof of [1, Theorem 4.1], fix $\gamma > 0$ such that

$$\frac{\alpha + 2}{p} < \gamma < \frac{\beta}{q}$$

and choose $N = N(s, \gamma) > 0$ so large that

$$\operatorname{Arg} \left[1 + \frac{8i}{N(1-s)} \right] < \min \left\{ \frac{\pi}{12}, \frac{\pi}{12\gamma} \right\}. \tag{3.7}$$

Put

$$\mu := \omega_{\varphi,u} + \sigma^\beta_\varphi$$

for simplicity. Since $u, v \in L^q_\alpha(\mathbf{D})$ by assumption, we see that μ is a finite measure.

Assume that $T : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ is bounded. In order to prove that μ is a (λ, α)-Carleson measure, we use (2.11). We will actually prove a bit more. Namely, we will establish

$$\sup_{\mathbf{D}} \widehat{\mu}_{\alpha,s,\lambda} \le C \|T\|^q_{A^p_\alpha(\mathbf{D})\to L^q_\alpha(\mathbf{D})} \tag{3.8}$$

for some constant $C = C(\alpha, \beta, p, q, s) > 0$.

To establish (3.8) we introduce our test functions. Let $a \in \mathbf{D}$. For $t > \frac{\alpha+2}{p}$ to be fixed later, we put

$$f_{b,t} := \frac{\tau_b^t}{\|\tau_b^t\|_{A_\alpha^p}}, \qquad b \in \Gamma_N(a)$$

where τ_b is the function specified in (2.8). Proceeding exactly as in the Proof of [1, Eq. (4.7)] and setting $R_t := R_{s,t}^{\alpha,q}$ for short, we have

$$\|Tf_{b,t}\|_{L_\alpha^q}^q \gtrsim \frac{R_t(a,b)}{(1-|a|^2)^{(\alpha+2)\lambda}} \tag{3.9}$$

for all a with $N(1-|a|) < \pi$. We now assume $N(1-|a|) < \pi$ for the rest of the proof. In conjunction with the above estimate, we can find from the Proof of [1, Theorem 4.1] a radius $\epsilon = \epsilon(N,s) = \epsilon(\alpha,\beta,p,q,s) \in (0,1)$ such that

$$\begin{aligned} \mu[E_s(a)] \lesssim R_\gamma(a,a_N) + R_{2\gamma}(a,a_N) + R_\gamma(a,\overline{a_N}) \\ + R_{2\gamma}(a,\overline{a_N}) + R_\gamma(a,a) + R_{\gamma+1}(a,a) \end{aligned} \tag{3.10}$$

for all a with $|a| \ge \epsilon$. This, together with (3.9), yields

$$\begin{aligned} \widehat{\mu}_{\alpha,s,\lambda}(a) \lesssim \|Tf_{a_N,\gamma}\|_{L_\alpha^q}^q + \|Tf_{a_N,2\gamma}\|_{L_\alpha^q}^q + \|Tf_{\overline{a_N},\gamma}\|_{L_\alpha^q}^q \\ + \|Tf_{\overline{a_N},2\gamma}\|_{L_\alpha^q}^q + \|Tf_{a,\gamma}\|_{L_\alpha^q}^q + \|Tf_{a,\gamma+1}\|_{L_\alpha^q}^q \end{aligned} \tag{3.11}$$

for all a with $|a| \ge \epsilon$. One may check that the constant suppressed in the above estimate depends only on α, β, p, q and s. Consequently,

$$\sup_{\mathbf{D}\setminus\epsilon\mathbf{D}} \widehat{\mu}_{\alpha,s,\lambda} \le C\|T\|_{A_\alpha^p(\mathbf{D})\to L_\alpha^q(\mathbf{D})}^q \tag{3.12}$$

for some constant $C = C(\alpha,\beta,p,q,s) > 0$.

Meanwhile, note from (2.3)

$$\bigcup_{|a|<\epsilon} E_s(a) = \delta\mathbf{D} \quad \text{where} \quad \delta := \frac{\epsilon+s}{1+\epsilon s}$$

and thus by (2.6)

$$\sup_{\epsilon\mathbf{D}} \widehat{\mu}_{\alpha,s,\lambda} \lesssim \frac{\mu(\delta\mathbf{D})}{(1-\epsilon^2)^{(\alpha+2)\lambda}}.$$

In addition, since

$$\rho|u| \le \frac{|\varphi-\psi||u|}{1-\delta} \le \frac{|u\varphi - v\psi| + |u-v|}{1-\delta} \quad \text{on} \quad \varphi^{-1}(\delta\mathbf{D}),$$

we have

$$\begin{aligned} \mu(\delta\mathbf{D}) &= \int_{\varphi^{-1}(\delta\mathbf{D})} |\rho u|^q \, dA_\alpha + \int_{\varphi^{-1}(\delta\mathbf{D})} (1-\rho)^\beta |u-v|^q \, dA_\alpha \\ &\lesssim \int_{\mathbf{D}} |u\varphi - v\psi|^q \, dA_\alpha + \int_{\mathbf{D}} |u-v|^q \, dA_\alpha \\ &= \|T(id)\|_{L_\alpha^q}^q + \|T(1)\|_{L_\alpha^q}^q \end{aligned}$$

where id denotes the identity map on $\mathbf{D}$. Combining these observations, we obtain

$$\sup_{\epsilon \mathbf{D}} \widehat{\mu}_{\alpha,s,\lambda} \leq C\|T\|^q_{A^p_\alpha(\mathbf{D})\to L^q_\alpha(\mathbf{D})}$$

for some constant $C = C(\alpha, \beta, p, q, s) > 0$. By this and (3.12) we conclude (3.8), as asserted.

Now, if, in addition, $T : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ is compact, then one may deduce from (3.11), (2.10), Lemma 2.1 and (2.12) that μ is a compact (λ, α)-Carleson measure.

Finally, by symmetry the same assertions also hold for the measure $\omega_{\psi,v} + \sigma^\beta_\psi$. □

4. The Case $q < p$

In this section we prove a more detailed version of Theorem 1.2. We continue using the notation introduced at the beginning of Sect. 3. The next theorem is a more detailed version of Theorem 1.2.

Theorem 4.1. *Let* $\alpha > -1$, $0 < q < p < \infty$, $\frac{\beta}{q} > \max\{1, \frac{1}{p}\} + \frac{\alpha+1}{p}$ *and* $0 < r < 1$. *Put* $\lambda := \frac{q}{p}$. *Let* $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ *and* $u, v \in L^q_\alpha(\mathbf{D})$. *Then the following statements are equivalent:*

(a) $C_{\varphi,u} - C_{\psi,v} : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ *is bounded;*
(b) $C_{\varphi,u} - C_{\psi,v} : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ *is compact;*
(c) $\omega + \sigma^\beta_\varphi$ *and* $\omega + \sigma^\beta_\psi$ *are* (λ, α)*-Carleson measures;*
(d) $\omega + \sigma^\beta_\varphi$ *or* $\omega + \sigma^\beta_\psi$ *is a* (λ, α)*-Carleson measure;*
(e) $\omega + \sigma_{\varphi,r}$ *and* $\omega + \sigma_{\psi,r}$ *are* (λ, α)*-Carleson measures;*
(f) $\omega + \sigma_{\varphi,r}$ *or* $\omega + \sigma_{\psi,r}$ *is a* (λ, α)*-Carleson measure.*

As in the Proof of Theorem 3.1, it is enough to show the implications

$$\text{(f)} \implies \text{(a)} + \text{(b)} \qquad \text{and} \qquad \text{(a)} \implies \text{(c)}.$$

Auxiliary notation introduced in the previous section will be used again without further references throughout the proofs.

Proof of (f) $\implies$ *(a) + (b)* Assume (f). Fix $s \in (r, 1)$. By symmetry we may assume that

$$\mu := \omega + \sigma_{\varphi,r}$$

is a (λ, α)-Carleson measure so that

$$\widehat{\mu}_{\alpha,s} \in L^{p/(p-q)}_\alpha(\mathbf{D}) \tag{4.1}$$

by (2.13).

We see from the first inequality (with $p = q$) in (3.6)

$$\|Tf\|^q_{L^q_\alpha} \lesssim \int_{\mathbf{D}} |f|^q \widehat{\mu}_{\alpha,s}\, dA_\alpha \tag{4.2}$$

for functions $f \in A^p_\alpha(\mathbf{D})$. The constant suppressed above depends only on α, p, q and r. Since $q < p$, we now apply Hölder's Inequality to conclude by (2.13) that $T : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ is bounded with norm estimate

$$\|T\|^q_{A^p_\alpha(\mathbf{D})\to L^q_\alpha(\mathbf{D})} \le C\|\widehat{\mu}_{\alpha,s}\|_{L^{p/(p-q)}_\alpha}$$

for some constant $C = C(\alpha, p, q, r) > 0$.

We now proceed to the proof of compactness. Our proof here relies on Lemma 2.1. So, consider an arbitrary sequence $\{f_n\}$ in $A^p_\alpha(\mathbf{D})$ such that $\sup_n \|f_n\|_{A^p_\alpha} \le 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. It is enough to show

$$Tf_n \to 0 \quad \text{in} \quad L^q_\alpha(\mathbf{D}) \tag{4.3}$$

by Lemma 2.1.

Let $t \in (0, 1)$. We have by (4.2)

$$\int_{\mathbf{D}} |Tf_n|^q \, dA_\alpha \lesssim \int_{t\mathbf{D}} + \int_{\mathbf{D}\setminus t\mathbf{D}} |f_n|^q \widehat{\mu}_{\alpha,s} \, dA_\alpha$$

for all n. Note that μ is a finite measure, being a (λ, α)-Carleson measure. We thus have

$$\sup_{t\mathbf{D}} \widehat{\mu}_{\alpha,s} < \infty.$$

Now, since $f_n \to 0$ uniformly on $t\mathbf{D}$, we obtain

$$\lim_{n\to\infty} \int_{t\mathbf{D}} |f_n|^q \widehat{\mu}_{\alpha,s} \, dA_\alpha = 0$$

for each t. For the integral over $\mathbf{D}\setminus t\mathbf{D}$, we apply Hölder's Inequality to obtain

$$\int_{\mathbf{D}\setminus t\mathbf{D}} |f_n|^q \widehat{\mu}_{\alpha,s} \, dA_\alpha \lesssim \|\widehat{\mu}_{\alpha,s}\chi_t\|_{L^{p/(p-q)}_\alpha}$$

where χ_t is the characteristic function of the annulus $\mathbf{D} \setminus t\mathbf{D}$. Accordingly, we obtain

$$\limsup_{n\to\infty} \int_{\mathbf{D}} |Tf_n|^q \; dA_\alpha \lesssim \|\widehat{\mu}_{\alpha,s}\chi_t\|_{L^{p/(p-q)}_\alpha};$$

the constant suppressed in this estimate is independent of t. Note from (4.1) that the right hand side of the above tends to 0 as $t \to 1$. Thus, taking the limit $t \to 1$, we conclude (4.3), as required. □

We now proceed to the proof of the implication (a) $\Longrightarrow$ (c), which is the hardest step. We need several preliminary lemmas.

To begin with, we recall the well-known notion of lattices. Let $0 < \delta < 1$ and $\{a_n\}$ be a sequence of distinct points in $\mathbf{D}$. We say that $\{a_n\}$ is δ-*separated* if the pseudohyperbolic disks $E_{\delta/2}(a_n)$ are pairwise disjoint. We say that $\{a_n\}$ is a δ-*lattice* (or simply a *lattice* if the size of δ does not matter) if it is δ-separated and

$$\mathbf{D} = \bigcup_n E_\delta(a_n).$$

We refer to [5, Lemma 4.8] for existence of δ-lattices for each $0 < \delta < 1$. In fact [5, Lemma 4.8] is stated in terms of the hyperbolic distance and one may easily modify its proof for the pseudohyperbolic distance.

Using the lattices, one may translate characterizations for Carleson measures described in Sect. 2.4 into the corresponding discrete versions. More precisely, given $\alpha > -1$, $0 < q < p < \infty$, $0 < s < 1$ and $0 < \delta < 1$, we have

$$\|\widehat{\mu}_{\alpha,s}\|_{L^{p/(p-q)}_\alpha} \approx \|\{\widehat{\mu}_{\alpha,s,\frac{q}{p}}(a_n)\}\|_{\ell^{p/(p-q)}} \tag{4.4}$$

for δ-lattices $\{a_n\}$; the constants suppressed in this estimate depend only on α, p, q, s and δ. See [7, Theorem B] and references therein. Here, and in what follows, ℓ^p stands for the standard sequence space with "norm" $\|\cdot\|_{\ell^p}$.

The following lemma can be found in [9, Lemma 2.15].

Lemma 4.2. *Given a δ-separated sequence $\mathfrak{a}$, denote by $M_{\mathfrak{a},\delta,s}(a)$ the number of points in $\mathfrak{a}$ that lie in the pseudohyperbolic disk $E_s(a)$. Then the inequlaity*

$$M_{\mathfrak{a},\delta,s}(a) \le \left(\frac{2}{\delta}+1\right)^2 \frac{1}{1-s^2}$$

holds for all $\delta, s \in (0,1)$ and $a \in \mathbf{D}$.

In what follows recall $a_N = ae^{-iN(1-|a|)}$.

Lemma 4.3. *Let $0 < s \le \frac{1}{3}$. Let $a, b \in \mathbf{D}$ and assume $E_s(a) \cap E_s(b) \ne \emptyset$. Then the inequality*

$$d(a_N, b_N) < 2(1 + 8|N|)s$$

holds for all N real.

Proof. Pick $\xi \in E_s(a) \cap E_s(b)$. Given N real, put

$$\xi_a := \xi e^{-iN(1-|a|)} \quad \text{and} \quad \xi_b := \xi e^{-iN(1-|b|)}$$

for short. Using the elementary inequality $|1 - e^{i\theta}| \le |\theta|$ for θ real, we note

$$\begin{aligned} d(\xi_a, \xi_b) &= \frac{|\xi|\,\left|1 - e^{-iN(|a|-|b|)}\right|}{\left|1 - |\xi|^2 e^{-iN(|a|-|b|)}\right|} \\ &\le \frac{|N|\big||a|-|b|\big|}{1-|\xi|^2} \le \frac{2|N|\big||a|-|b|\big|}{1-|a|^2}; \end{aligned}$$

the last inequality holds by (2.2), because $s \le \frac{1}{3}$. Meanwhile, since $d(a,b) < 2s$ and $s \le \frac{1}{3}$, we have

$$\big||a| - |b|\big| \le |a - b| \le \frac{4(1-|a|^2)}{1-4s^2|a|^2}s < 8s(1-|a|^2);$$

the second inequality holds by (2.3). Combining these observations, we obtain

$$d(\xi_a, \xi_b) \le 16|N|s.$$

Thus, we obtain

$$\begin{aligned} d(a_N, b_N) &\le d(a_N, \xi_a) + d(\xi_a, \xi_b) + d(\xi_b, b_N) \\ &= d(a, \xi) + d(\xi_a, \xi_b) + d(\xi, b) \\ &< 2s + 16|N|s \end{aligned}$$

as required. □

In what follows we put

$$a_{n,N} := (a_n)_N$$

for simplicity.

Lemma 4.4. *Given $0 < \delta < 1$ and N real, there is a constant $M = M(\delta, |N|) > 0$ with the following property: If $\{a_n\}$ is a δ-separated sequence in $\mathbf{D}$, then any collection of more than M of the pseudohyperbolic disks $E_s(a_{n,N})$ with $s = \frac{1}{3(1+8|N|)}$ contains no point in common.*

Proof. Let $\{a_n\}$ be an arbitrary δ-separated sequence in $\mathbf{D}$. If $E_s(a_{j,N}) \cap E_s(a_{k,N}) \neq \emptyset$, then we have by Lemma 4.3 (with $-N$ in place of N)

$$d(a_j, a_k) < \frac{2}{3}, \quad \text{i.e.,} \quad a_j \in E_{2/3}(a_k).$$

Thus the lemma holds by Lemma 4.2 (with $s = \frac{2}{3}$). □

The idea of the proof below of the next lemma comes from that of [5, Theorem 4.33].

Lemma 4.5. *Let $\alpha > -1$, $0 < p < \infty$ and*

$$t > \max\left\{1, \frac{1}{p}\right\} + \frac{\alpha+1}{p}. \tag{4.5}$$

For $0 < s < 1$ and a positive integer M, assume that $\{a_n\}$ is a sequence in $\mathbf{D}$ such that any collection of more than M of the pseudohyperbolic disks $E_s(a_n)$ contains no point in common. Let $\{c_n\} \in \ell^p$ and put

$$f(z) := \sum_{n=1}^{\infty} c_n \frac{(1-|a_n|^2)^{t-\frac{\alpha+2}{p}}}{(1-\overline{a_n}z)^t}.$$

Then $f \in A^p_\alpha(\mathbf{D})$ and

$$\|f\|^p_{A^p_\alpha} \leq C \sum_{n=1}^{\infty} |c_n|^p$$

for some constant $C = C(\alpha, p, s, M) > 0$.

Proof. Put

$$h_n(z) := \frac{(1-|a_n|^2)^{t-\frac{\alpha+2}{p}}}{(1-\overline{a_n}z)^t}$$

for positive integers n. Since $pt > \alpha + 2$ by (4.5), we have by (2.9)

$$\|h_n\|^p_{A^p_\alpha} \approx 1$$

for all n. Thus, in case $0 < p \leq 1$, the asserted inequality holds by the inequality

$$\|f\|^p_{A^p_\alpha} \leq \sum_{n=1}^{\infty} |c_n|^p \|h_n\|^p_{A^p_\alpha} \leq C \sum_{n=1}^{\infty} |c_n|^p$$

for some constant $C = C(\alpha, p) > 0$.

Now, assume $1 < p < \infty$ for the rest of the proof. Since $p > 1$, we have $p(t-1) > \alpha + 1$ by (4.5) and thus the integral operator Λ defined by

$$\Lambda h(z) := \int_{\mathbf{D}} \frac{(1-|w|^2)^{t-2}}{|1-\overline{w}z|^t} h(w)\, dA(w)$$

is bounded on $L^p_\alpha(\mathbf{D})$ by [5, Corollary 3.13]. Consider the function

$$g(z) := \sum_{n=1}^{\infty} \frac{|c_n| \chi_n(z)}{\{A_\alpha[E_s(a_n)]\}^{1/p}} \tag{4.6}$$

where χ_n denotes the characteristic function of the pseudohyperbolic disk $E_s(a_n)$. Applying Λ to g and integrating term by term, we have by (2.4), (2.5) and (2.6)

$$\begin{aligned}
\Lambda g(z) &= \sum_{n=1}^{\infty} \frac{|c_n|}{\{A_\alpha[E_s(a_n)]\}^{1/p}} \int_{E_s(a_n)} \frac{(1-|w|^2)^{t-2}}{|1-\overline{w}z|^t}\, dA(w) \\
&\approx \sum_{n=1}^{\infty} \frac{|c_n|}{\{A_\alpha[E_s(a_n)]\}^{1/p}} \cdot \frac{(1-|a_n|^2)^{t-2}}{|1-\overline{a_n}z|^t} \cdot A[E_s(a_n)] \\
&\approx \sum_{n=1}^{\infty} |c_n| \frac{(1-|a_n|^2)^{t-\frac{\alpha+2}{p}}}{|1-\overline{a_n}z|^t} \\
&\geq |f(z)|
\end{aligned}$$

for all $z \in \mathbf{D}$. So, since Λ is bounded on $L^p_\alpha(\mathbf{D})$, we have

$$\|f\|_{A^p_\alpha} \lesssim \|g\|_{L^p_\alpha}; \tag{4.7}$$

the constant suppressed in this estimate depends only on α, p and s.

Meanwhile, for each $z \in \mathbf{D}$, note that the series in (4.6) is actually a finite sum with at most M terms. Accordingly, we have by Jensen's Inequality

$$|g(z)|^p \leq M^{p-1} \sum_{n=1}^{\infty} \frac{|c_n|^p \chi_n(z)}{A_\alpha[E_s(a_n)]}$$

for all $z \in \mathbf{D}$. So, integrating term by term, we obtain

$$\int_{\mathbf{D}} |g|^p \, dA_\alpha \leq M^{p-1} \sum_{n=1}^{\infty} |c_n|^p.$$

This, together with (4.7), yields the asserted inequality. □

The sequence of Rademacher functions $\{r_n\}$ is defined by

$$r_n(x) := \operatorname{sgn}[\sin(2^n \pi x)], \qquad 0 \leq x \leq 1$$

for positive integers n. The well-known Khinchin's Inequality involving these Rademacher functions asserts the following; see, for example [10, Appendix C].

Lemma 4.6. *(Khinchin's Inequality) Given* $0 < p < \infty$*, there exists a constant* $C = C(p) > 0$ *such that*

$$C^{-1}\left(\sum_{n=1}^{\infty}|c_n|^2\right)^{p/2} \le \int_0^1\left|\sum_{n=1}^{\infty}c_n r_n(x)\right|^p dx \le C\left(\sum_{n=1}^{\infty}|c_n|^2\right)^{p/2}$$

for all $\{c_n\} \in \ell^2$*.*

The next lemma is contained in [1, Lemma 3.3]. In fact [1, Lemma 3.3] is proved under an additional restriction $a \neq 0$, but such a restriction can be easily removed.

Lemma 4.7. *Let* $s \in (0,1)$ *and* $N > 0$*. Then there is a constant* $C = C(s,N) > 0$ *such that*

$$1 \le \frac{|1-\bar{b}w|}{1-|a|^2} \le C$$

for $a \in \mathbf{D}$ *with* $N(1-|a|) < \pi$*,* $b \in \Gamma_N(a)$ *and* $w \in E_s(a)$*.*

Lemma 4.8. *Let* $\alpha > -1$ *and* $0 < q < p < \infty$*. For* $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ *and* $u, v \in L^q_\alpha(\mathbf{D})$*, assume that* $C_{\varphi,u} - C_{\psi,v} : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ *is bounded. For* $0 < \delta < 1$ *and* $N > 0$*, assume that* $\{a_n\}$ *is a* δ*-separated sequence in* $\mathbf{D}$ *such that* $N(1-|a_n|) < \pi$ *for all* n*. Assume*

$$t > \max\left\{1, \frac{1}{p}\right\} + \frac{\alpha+1}{p}$$

and let $\{b_n\}$ *be a sequence given by one of* $\{a_n\}$*,* $\{a_{n,N}\}$ *and* $\{\overline{a_{n,N}}\}$*. Put* $s := \frac{1}{3(1+8N)}$*. Then*

$$\left\|\left\{\frac{R^{\alpha,q}_{s,t}(a_n,b_n)}{(1-|a_n|^2)^{(\alpha+2)\frac{q}{p}}}\right\}\right\|_{\ell^{p/(p-q)}} \le C\|C_{\varphi,u} - C_{\psi,v}\|^q_{A^p_\alpha(\mathbf{D})\to L^q_\alpha(\mathbf{D})}$$

for some constant $C = C(\alpha, p, q, \delta, N) > 0$*.*

Proof. Noting that $\left(\ell^{p/q}\right)^* = \ell^{p/(p-q)}$, we will complete the proof by duality. More precisely, setting

$$\zeta_n := \frac{R^{\alpha,q}_{s,t}(a_n,b_n)}{(1-|a_n|^2)^{(\alpha+2)\frac{q}{p}}}$$

for each n, we will complete the proof by establishing

$$\sum_{n=1}^{\infty}\zeta_n|\eta_n| \le C\|T\|^q_{A^p_\alpha(\mathbf{D})\to L^q_\alpha(\mathbf{D})} \tag{4.8}$$

for all $\{\eta_n\} \in \ell^{p/q}$ with $\|\{\eta_n\}\|_{\ell^{p/q}} = 1$ and for some constant $C = C(\alpha, p, q, \delta, N) > 0$.

Fix an arbitrary $\{\eta_n\} \in \ell^{p/q}$ with $\|\{\eta_n\}\|_{\ell^{p/q}} = 1$. Put

$$f(z) := \sum_{n=1}^{\infty} c_n \frac{(1-|a_n|^2)^{t-\frac{\alpha+2}{p}}}{(1-\overline{b_n}z)^t} \quad \text{where} \quad c_n := |\eta_n|^{1/q}.$$

Since $\|\{c_n\}\|_{\ell^p} = 1$, we see by Lemmas 4.2, 4.4 and 4.5 that $f \in A^p_\alpha(\mathbf{D})$ with norm bounded by some constant depending only on α, p, δ, and N. Also, note

$$\|Tf\|^q_{L^q_\alpha} = \int_{\mathbf{D}} \left| \sum_{n=1}^{\infty} c_n \frac{(1-|a_n|^2)^{t-\frac{\alpha+2}{p}}}{(1-\overline{b_n}\varphi)^t} \left(u - vQ^t_{b_n}\right) \right|^q dA_\alpha.$$

Since $T : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ is bounded by assumption, it follows that

$$\int_{\mathbf{D}} \left| \sum_{n=1}^{\infty} c_n \frac{(1-|a_n|^2)^{t-\frac{\alpha+2}{p}}}{(1-\overline{b_n}\varphi)^t} \left(u - vQ^t_{b_n}\right) \right|^q dA_\alpha \lesssim \|T\|^q_{A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})}.$$

Replace c_n by $r_n(x)c_n$ in the left hand side of the above, integrate both sides of the above over $[0, 1]$ against the measure dx, interchange the order of integrations and apply Khinchin's Inequality. As a result, setting

$$\Phi := \sum_{n=1}^{\infty} \left| c_n \frac{(1-|a_n|^2)^{t-\frac{\alpha+2}{p}}}{(1-\overline{b_n}\varphi)^t} \left(u - vQ^t_{b_n}\right) \right|^2,$$

we obtain

$$\int_{\mathbf{D}} \Phi^{\frac{q}{2}} \, dA_\alpha \lesssim \|T\|^q_{A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})}; \tag{4.9}$$

the constant suppressed in this estimate depends only on α, p, q, δ and N.

We now turn to the lower estimate of the integral in (4.9). Denote by χ_n the characteristic function of the pseudohyperbolic disk $E_s(a_n)$. Note from Lemma 4.2

$$\sum_{n=1}^{\infty} \chi_n \le M \tag{4.10}$$

for some positive integer $M = M(\delta, N)$. We claim

$$\Phi^{\frac{q}{2}} \ge M^{\frac{q}{2}-1} \sum_{n=1}^{\infty} \left| c_n \frac{(1-|a_n|^2)^{t-\frac{\alpha+2}{p}}}{(1-\overline{b_n}\varphi)^t} \left(u - vQ^t_{b_n}\right) \right|^q (\chi_n \circ \varphi). \tag{4.11}$$

To see this, we consider two cases (i) $q \ge 2$ and (ii) $0 < q < 2$ separately. First, the case $q \ge 2$ is easily seen, because $\frac{q}{2} \ge 1$. In case $0 < q < 2$ so that $\frac{2}{q} > 1$, we note by (4.10)

$$\Phi^{\frac{q}{2}} \ge \left[\sum_{n=1}^{\infty} \left| c_n \frac{(1-|a_n|^2)^{t-\frac{\alpha+2}{p}}}{(1-\overline{b_n}\varphi)^t} \left(u - vQ^t_{b_n}\right) \right|^2 \right]^{\frac{q}{2}} \left[\frac{1}{M} \sum_{n=1}^{\infty} (\chi_n \circ \varphi) \right]^{1-\frac{q}{2}}$$

and thus obtain (4.11) by Hölder's Inequality.

Note $b_n \in \Gamma_N(a_n)$ for each n by assumption. So, having verified (4.11), we now see by Lemma 4.7 that the right hand side of (4.11) dominates some constant (depending only on α, p, q, δ and N) times

$$\sum_{n=1}^{\infty} |\eta_n| \frac{\left|u - vQ^t_{b_n}\right|^q}{(1-|a_n|^2)^{(\alpha+2)\frac{q}{p}}} (\chi_n \circ \varphi).$$

Thus, integrating over $\mathbf{D}$ against the measure dA_α, we see from (4.9) that (4.8) holds. □

We are now ready to prove the implication (a) $\Longrightarrow$ (c).

Proof of (a) $\Longrightarrow$ (c) Fix a lattice $\{a_n\}$. Fix $\gamma > 0$ such that

$$\max\left\{1, \frac{1}{p}\right\} + \frac{\alpha+1}{p} < \gamma < \frac{\beta}{q}$$

and pick $N = N(\gamma) > 0$ so such that

$$\operatorname{Arg}\left[1 + \frac{8i}{N} \cdot \frac{1}{1 - \frac{1}{3(1+8N)}}\right] < \frac{\pi}{12\gamma}.$$

Thus, setting

$$s := \frac{1}{3(1+8N)},$$

which is the number specified by Lemma 4.4, we also see that (3.7) holds. Put

$$\mu := \omega_{\varphi,u} + \sigma_\varphi^\beta$$

for simplicity. Since $u, v \in L^q_\alpha(\mathbf{D})$ by assumption, we see that μ is a finite measure. We note that the estimate (3.10) is still available with suitably adjusted $\epsilon = \epsilon(\alpha, \beta, p, q, N) \in (0, 1)$. The sequence $\{a_n\}$ being a lattice, we also note $|a_n| \to 1$. Pick a positive integer $K = K(\epsilon) = K(\alpha, \beta, p, q)$ such that $|a_n| \geq \epsilon$ for all $n \geq K$. So, writing $R_t := R^{\alpha,q}_{s,t}$ for short, we have

$$\begin{aligned}\mu[E_s(a_n)] \lesssim\, & R_\gamma(a_n, a_{n,N}) + R_{2\gamma}(a_n, a_{n,N}) + R_\gamma(a_n, \overline{a_{n,N}}) \\ & + R_{2\gamma}(a_n, \overline{a_{n,N}}) + R_\gamma(a_n, a_n) + R_{\gamma+1}(a_n, a_n)\end{aligned}$$

for all $n \geq K$. Moreover, we may assume $|a_n| < \epsilon$ for all $n < K$ after renumbering indices if necessary. Consequently, assuming the boundedness of $T : A^p_\alpha(\mathbf{D}) \to L^q_\alpha(\mathbf{D})$ and slightly modifying the proof of the implication (a) $\Longrightarrow$ (b) of Theorem 3.1, we obtain by Lemma 4.8 and (2.6)

$$\|\{\widehat{\mu}_{\alpha,s,\lambda}(a_n)\}\|_{\ell^{p/(p-q)}} \leq C\|T\|^q_{A^p_\alpha(\mathbf{D})\to L^q_\alpha(\mathbf{D})}$$

for some constant $C = C(\alpha, \beta, p, q) > 0$. We now conclude by (2.13) and (4.4) that μ is a (λ, α)-Carleson measure. By symmetry the same assertion also holds for the measure $\omega_{\psi,v} + \sigma^\beta_\psi$. □

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