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Algebras of Convolution Type Operators with Continuous Data do Not Always Contain All Rank One Operators

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Dedicated to Bernd Silbermann on the occasion of his 80th birthday.

Abstract. Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. The algebra $C_X(\dot{\mathbb{R}})$ of continuous Fourier multipliers on $X(\mathbb{R})$ is defined as the closure of the set of continuous functions of bounded variation on $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with respect to the multiplier norm. It was proved by C. Fernandes, Yu. Karlovich and the first author [11] that if the space $X(\mathbb{R})$ is reflexive, then the ideal of compact operators is contained in the Banach algebra $\mathcal{A}_{X(\mathbb{R})}$ generated by all multiplication operators aI by continuous functions $a \in C(\dot{\mathbb{R}})$ and by all Fourier convolution operators $W^0(b)$ with symbols $b \in C_X(\dot{\mathbb{R}})$. We show that there are separable and non-reflexive Banach function spaces $X(\mathbb{R})$ such that the algebra $\mathcal{A}_{X(\mathbb{R})}$ does not contain all rank one operators. In particular, this happens in the case of the Lorentz spaces $L^{p,1}(\mathbb{R})$ with $1 < p < \infty$.

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1. Introduction

We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz class of all infinitely differentiable and rapidly decaying functions (see, e.g., [14, Section 2.2.1]). Let F denote the Fourier transform, defined on $\mathcal{S}(\mathbb{R})$ by

$$(Ff)(x) := \widehat{f}(x) := \int_{\mathbb{R}} f(t)e^{itx}\, dt, \quad x \in \mathbb{R},$$

and let F^{-1} be the inverse of F defined on $\mathcal{S}(\mathbb{R})$ by

$$(F^{-1}g)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} g(x)e^{-itx}\,dx, \quad t \in \mathbb{R}.$$

It is well known that these operators extend uniquely to the space $L^2(\mathbb{R})$. As usual, we will use the symbols F and F^{-1} for the direct and inverse Fourier transform on $L^2(\mathbb{R})$. The Fourier convolution operator

$$W^0(a) := F^{-1}aF$$

is bounded on the space $L^2(\mathbb{R})$ for every $a \in L^\infty(\mathbb{R})$.

In this paper, we study algebras of operators generated by operators of multiplication aI and Fourier convolution operators $W^0(b)$ on so-called Banach function spaces in the case when both a and b are continuous. We postpone a formal definition of a Banach function space $X(\mathbb{R})$ and its associate space $X'(\mathbb{R})$ until Sect. 2.1. The Lebesgue spaces $L^p(\mathbb{R})$ with $1 \le p \le \infty$ constitute the most important example of Banach function spaces. The class of Banach function spaces includes classical Orlicz spaces $L^\Phi(\mathbb{R})$, Lorentz spaces $L^{p,q}(\mathbb{R})$, all other rearrangement-invariant spaces, as well as (non-rearrangement-invariant) weighted Lebesgue spaces $L^p(\mathbb{R}, w)$ and variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R})$.

Let $X(\mathbb{R})$ be a separable Banach function space. Then $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$ (see Lemma 2.2 below). A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $X(\mathbb{R})$ if the convolution operator $W^0(a) := F^{-1}aF$ maps $L^2(\mathbb{R}) \cap X(\mathbb{R})$ into $X(\mathbb{R})$ and extends to a bounded linear operator on $X(\mathbb{R})$. The function a is called the symbol of the Fourier convolution operator $W^0(a)$. The set $\mathcal{M}_{X(\mathbb{R})}$ of all Fourier multipliers on $X(\mathbb{R})$ is a unital normed algebra under pointwise operations and the norm

$$\|a\|_{\mathcal{M}_{X(\mathbb{R})}} := \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))},$$

where $\mathcal{B}(X(\mathbb{R}))$ denotes the Banach algebra of all bounded linear operators on the space $X(\mathbb{R})$. Let $\mathcal{K}(X(\mathbb{R}))$ denote the ideal of all compact operators in the Banach algebra $\mathcal{B}(X(\mathbb{R}))$.

Recall that the (non-centered) Hardy-Littlewood maximal function Mf of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is defined by

$$(Mf)(x) := \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the supremum is taken over all intervals $Q \subset \mathbb{R}$ of finite length containing x. The Hardy-Littlewood maximal operator M defined by the rule $f \mapsto Mf$ is a sublinear operator.

Suppose that $a : \mathbb{R} \to \mathbb{C}$ is a function of bounded variation $V(a)$ given by

$$V(a) := \sup \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})|,$$

where the supremum is taken over all partitions of $\mathbb{R}$ of the form

$$-\infty < x_0 < x_1 < \cdots < x_n < +\infty$$

with $n \in \mathbb{N}$. The set $V(\mathbb{R})$ of all functions of bounded variation on $\mathbb{R}$ with the norm

$$\|a\|_{V(\mathbb{R})} := \|a\|_{L^\infty(\mathbb{R})} + V(a)$$

is a unital non-separable Banach algebra.

Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. It follows from [16, Theorem 4.3] that if a function $a : \mathbb{R} \to \mathbb{C}$ has a bounded variation $V(a)$, then the convolution operator $W^0(a)$ is bounded on the space $X(\mathbb{R})$ and

$$\|W^0(a)\|_{\mathcal{B}(X(\mathbb{R}))} \le c_X \|a\|_{V(\mathbb{R})}, \tag{1.1}$$

where c_X is a positive constant depending only on $X(\mathbb{R})$.

For Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, inequality (1.1) is usually called Stechkin's inequality. We refer to [8, Theorem 2.11] for the proof of (1.1) in the case of Lebesgue spaces $L^p(\mathbb{R})$ with $c_{L^p} = \|S\|_{\mathcal{B}(L^p(\mathbb{R}))}$, where S is the Cauchy singular integral operator.

Let $C(\dot{\mathbb{R}})$ denote the C^*-algebra of continuous functions on the one-point compactification $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of the real line. For a subset $\mathfrak{S}$ of a Banach space $\mathcal{E}$, we denote by $\operatorname{clos}_{\mathcal{E}}(\mathfrak{S})$ the closure of $\mathfrak{S}$ with respect to the norm of $\mathcal{E}$. Consider the following algebra of continuous Fourier multipliers:

$$C_X(\dot{\mathbb{R}}) := \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R})}} \big(C(\dot{\mathbb{R}}) \cap V(\mathbb{R})\big). \tag{1.2}$$

It follows from Theorem 2.3 below that $C_X(\dot{\mathbb{R}}) \subset C(\dot{\mathbb{R}})$. The aim of this paper is to continue the study of the smallest Banach subalgebra

$$\mathcal{A}_{X(\mathbb{R})} := \operatorname{alg}\{aI, W^0(b) \ : \ a \in C(\dot{\mathbb{R}}),\ b \in C_X(\dot{\mathbb{R}})\}$$

of the algebra $\mathcal{B}(X(\mathbb{R}))$ that contains all operators of multiplication aI by functions $a \in C(\dot{\mathbb{R}})$ and all Fourier convolution operators $W^0(b)$ with symbols $b \in C_X(\dot{\mathbb{R}})$ started in the setting of reflexive Banach function spaces in [11]. The main result of that paper says the following.

Theorem 1.1. *([11, Theorem 1.1]) Let $X(\mathbb{R})$ be a reflexive Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Then the ideal of compact operators $\mathcal{K}(X(\mathbb{R}))$ is contained in the Banach algebra $\mathcal{A}_{X(\mathbb{R})}$.*

Note that results of this kind are well known in the setting of (weighted) Lebesgue spaces (see, e.g., [24, Lemma 6.1], [29, Theorem 5.2.1 and Proposition 5.8.1] and also [4, Lemma 8.23], [29, Theorem 4.1.5]). They constitute the first step in the Fredholm study of more general algebras of convolution type operators with more general function algebras in place of $C(\dot{\mathbb{R}})$ and $C_X(\dot{\mathbb{R}})$, respectively (see, e.g., [23–25]), by means of local principles (see, e.g., [5, Sections 1.30–1.35]).

Let $\mathfrak{A}$ be a Banach algebra with unit e. The center $\operatorname{Cen}\mathfrak{A}$ of $\mathfrak{A}$ is the set of all elements $z \in \mathfrak{A}$ with the property that $za = az$ for all $a \in \mathfrak{A}$. One can successfully apply the Allan-Douglas local principle [5, Section 1.35] to the algebra $\mathfrak{A}$ if it possesses a (hopefully large) closed subalgebra $\mathfrak{C}$ lying in its center. Having applications of the Allan-Douglas local principle in mind, the authors of [11] asked whether the quotient algebra

$$\mathcal{A}^{\pi}_{X(\mathbb{R})} := \mathcal{A}_{X(\mathbb{R})}/\mathcal{K}(X(\mathbb{R}))$$

is commutative under the assumptions of Theorem 1.1. Our first result is the positive answer to [11, Question 1.2].

Theorem 1.2. *Let $X(\mathbb{R})$ be a reflexive Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Then the quotient algebra $\mathcal{A}^{\pi}_{X(\mathbb{R})}$ is commutative.*

It is well known that a Banach function space $X(\mathbb{R})$ is reflexive if and only if the space $X(\mathbb{R})$ and its associate space $X'(\mathbb{R})$ are separable (see [27, Chap. 1, §2, Theorem 4 and §3, Corollary 1 to Theorem 7] or [3, Chap. 1, Corollaries 4.4 and 5.6]). So, it is natural to ask whether the assumption of the reflexivity of the space $X(\mathbb{R})$ in Theorem 1.1 can be relaxed to the assumption of the separability of the space $X(\mathbb{R})$. Our main result says that this is impossible.

Theorem 1.3. *(Main result) There exists a separable non-reflexive Banach function space $X(\mathbb{R})$ such that*

(a) the Hardy-Littlewood maximal operator is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$;
(b) the algebra $\mathcal{A}_{X(\mathbb{R})}$ does not contain all rank one operators.

This theorem means that the usual methods of the Fredholm study of algebras of convolution type operators with discontinuous data on non-reflexive separable Banach function spaces will require a modification to overcome an obstacle that certain compact operators do not belong to the algebra $\mathcal{A}_{X(\mathbb{R})}$ and, therefore, the quotient algebra $\mathcal{A}_{X(\mathbb{R})}/\mathcal{K}(X(\mathbb{R}))$ cannot be defined.

In fact, Theorem 1.3 holds for a familiar example of separable and non-reflexive Banach function spaces, namely the classical Lorentz spaces $L^{p,1}(\mathbb{R})$ with $1 < p < \infty$. Let us recall their definition. The distribution function μ_f of a measurable function $f : \mathbb{R} \to \mathbb{C}$ is given by

$$\mu_f(\lambda) := |\{x \in \mathbb{R} : |f(x)| > \lambda\}|, \quad \lambda \geq 0.$$

The non-increasing rearrangement of f is the function f^* defined on $[0,\infty)$ by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \quad t \geq 0$$

(see, e.g., [3, Chap. 3, Definitions 1.1 and 1.5]).

For given $1 < p < \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L^{p,q}(\mathbb{R})$ consist of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that the norm

$$\|f\|_{(p,q)} := \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{1/p} f^{**}(t)\right)^q \frac{dt}{t}\right)^{1/q}, & q < \infty, \\ \displaystyle\sup_{0<t<\infty} \left(t^{1/p} f^{**}(t)\right), & q = \infty, \end{cases}$$

is finite, where

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(x)\,dx$$

(see [3, Chap. 4, Lemma 4.5]).

Theorem 1.4. *Let $1 < p < \infty$. The Lorentz space $L^{p,1}(\mathbb{R})$ is a separable and non-reflexive Banach function space satisfying assumption* (a) *of Theorem* 1.3 *and such that the algebra $\mathcal{A}_{L^{p,1}(\mathbb{R})}$ does not contain all rank one operators.*

The paper is organized as follows. In Sect. 2, we collect definitions of a Banach function space and its associate space $X'(\mathbb{R})$, recall that the set of Fourier multipliers $\mathcal{M}_{X(\mathbb{R})}$ on a separable Banach function space $X(\mathbb{R})$, such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$, is continuously embedded into $L^\infty(\mathbb{R})$. Consequently, $\mathcal{M}_{X(\mathbb{R})}$ is a unital Banach algebra. Further, we prove several lemmas on approximation of continuous functions (or Fourier multipliers) vanishing at infinity by compactly supported continuous functions (or Fourier multipliers, respectively).

In Sect. 3, we show that if $X(\mathbb{R})$ is a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$ and $a \in C(\dot{\mathbb{R}})$, $b \in C_X(\dot{\mathbb{R}})$, then the commutator $aW^0(b) - W^0(b)aI$ is compact on the space $X(\mathbb{R})$. Combining this result with Theorem 1.1, we arrive at Theorem 1.2.

Section 4 is devoted to the proof of Theorems 1.3 and 1.4. For $R > 0$, let $\chi_{\{R\}} := \chi_{\mathbb{R}\setminus[-R,R]}$. We show that if a is a compactly supported continuous function and b is a compactly supported function of bounded variation, then the norm of the operator $aW^0(b)\chi_{\{R\}}I$ goes to zero as $R \to \infty$. If a Banach function space $X(\mathbb{R})$ is separable and non-reflexive, its associate space $X'(\mathbb{R})$ may contain a function g such that $\|g\chi_{\{R\}}\|_{X'(\mathbb{R})}$ is bounded away from zero for all $R > 0$ (this cannot happen if $X(\mathbb{R})$ is reflexive). If, in addition, the Hardy-Littlewood operator is bounded on $X(\mathbb{R})$ and on its associate space $X(\mathbb{R})$, then we show that for every $h \in X(\mathbb{R}) \setminus \{0\}$ the rank one operator

$$(T_{g,h}f)(x) := h(x)\int_{\mathbb{R}} g(y)f(y)\,dy$$

does not belong to the algebra $\mathcal{A}_{X(\mathbb{R})}$, which implies Theorem 1.3 under the assumption that the function $g \in X'(\mathbb{R})$ mentioned above does indeed exist. Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Finally, we prove Theorem 1.4 first recalling that the classical Lorentz space $L^{p,1}(\mathbb{R})$ is a separable non-reflexive Banach function space with the associate space $L^{p',\infty}(\mathbb{R})$, that the Hardy-Littlewood maximal operator is bounded on both $L^{p,1}(\mathbb{R})$ and $L^{p',\infty}(\mathbb{R})$; and then showing that the function $g(x) = |x|^{-1/p'}$ belongs to $L^{p',\infty}(\mathbb{R})$ and $\|\chi_{\{R\}}g\|_{(p',\infty)}$ is bounded away from zero for all $R > 0$. This completes the proof of Theorem 1.4 and, thus, of Theorem 1.3.

In Sect. 5, we define the algebra of continuous Fourier multipliers $C_X^0(\dot{\mathbb{R}})$ as the closure of $\mathbb{C}\dot{+}C_c^\infty(\mathbb{R})$, where $C_c^\infty(\mathbb{R})$ is the set of smooth compactly supported functions and $\mathbb{C}$ denotes the set of constant functions. It is not difficult to see that $C_X^0(\dot{\mathbb{R}}) \subset C_X(\dot{\mathbb{R}})$. We do not know whether these algebras coincide, in general. We prove a possible refinement of Theorem 1.1 for the algebra $\mathcal{A}^0_{X(\mathbb{R})}$, where the latter algebra is defined in the same way as the algebra $\mathcal{A}_{X(\mathbb{R})}$ with $C_X(\dot{\mathbb{R}})$ replaced by $C_X^0(\dot{\mathbb{R}})$. Further, we recall the definition of the set of slowly oscillating functions $SO^\diamond$ and slowly oscillating Fourier multipliers $SO^\diamond_{X(\mathbb{R})}$ (see [12,18]). Since $C(\dot{\mathbb{R}}) \subset SO^\diamond$ and $C_X^0(\dot{\mathbb{R}}) \subset SO^\diamond_{X(\mathbb{R})}$, the ideal of compact operators $\mathcal{K}(X(\mathbb{R}))$ is contained in the algebra $\mathcal{D}_{X(\mathbb{R})}$ generated by the operators aI with $a \in SO^\diamond$ and $b \in SO^\diamond_{X(\mathbb{R})}$ under the assumptions that $X(\mathbb{R})$ is a reflexive Banach function space such that the Hardy-Littlewood maximal operator is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. We conclude the paper with an open question on whether or not the quotient algebra $\mathcal{D}^\pi_{X(\mathbb{R})} := \mathcal{D}_{X(\mathbb{R})}/\mathcal{K}(X(\mathbb{R}))$ is commutative in this case.

2. Auxiliary Results

2.1. Banach Function Spaces

The set of all Lebesgue measurable complex-valued functions on $\mathbb{R}$ is denoted by $\mathfrak{M}(\mathbb{R})$. Let $\mathfrak{M}^+(\mathbb{R})$ be the subset of functions in $\mathfrak{M}(\mathbb{R})$ whose values lie in $[0,\infty]$. For a measurable set $E \subset \mathbb{R}$, its Lebesgue measure and the characteristic function are denoted by $|E|$ and χ_E, respectively. Following [27, p. 3] (see also [3, Chap. 1, Definition 1.1] and [28, Definition 6.1.5]), a mapping $\rho : \mathfrak{M}^+(\mathbb{R}) \to [0,\infty]$ is called a Banach function norm if, for all functions f, g, f_n $(n \in \mathbb{N})$ in $\mathfrak{M}^+(\mathbb{R})$, for all constants $a \geq 0$, and for all measurable subsets E of $\mathbb{R}$, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f+g) \leq \rho(f) + \rho(g)$,

(A2) $0 \leq g \leq f$ a.e. $\Rightarrow$ $\rho(g) \leq \rho(f)$ (the lattice property),

(A3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow$ $\rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(A4) E is bounded $\Rightarrow \rho(\chi_E) < \infty$,

(A5) E is bounded $\Rightarrow \displaystyle\int_E f(x)\,dx \leq C_E \rho(f)$,

where $C_E \in (0,\infty)$ may depend on E and ρ but is independent of f. When functions differing only on a set of measure zero are identified, the set $X(\mathbb{R})$ of functions $f \in \mathfrak{M}(\mathbb{R})$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\mathbb{R})$, the norm of f is defined by

$$\|f\|_{X(\mathbb{R})} := \rho(|f|).$$

With this norm and under natural linear space operations, the set $X(\mathbb{R})$ becomes a Banach space (see [27, Chap. 1, §1, Theorem 1] or [3, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathfrak{M}^+(\mathbb{R})$ by

$$\rho'(g) := \sup\left\{ \int_{\mathbb{R}} f(x)g(x)\,dx \ : \ f \in \mathfrak{M}^+(\mathbb{R}),\ \rho(f) \leq 1 \right\}, \quad g \in \mathfrak{M}^+(\mathbb{R}).$$

Then ρ' is itself a Banach function norm (see [27, Chap. 1, §1] or [3, Chap. 1, Theorem 2.2]). The Banach function space $X'(\mathbb{R})$ determined by the Banach function norm ρ' is called the associate space (Köthe dual) of $X(\mathbb{R})$. The associate space $X'(\mathbb{R})$ is a subspace of the (Banach) dual space $[X(\mathbb{R})]^*$.

Remark 2.1. We note that our definition of a Banach function space is slightly different from that found in [3, Chap. 1, Definition 1.1] and [28, Definition 6.1.5]. In particular, in Axioms (A4) and (A5) we assume that the set E is a bounded set, whereas it is sometimes assumed that E merely satisfies $|E| < \infty$. We do this so that the weighted Lebesgue spaces with Muckenhoupt weights satisfy Axioms (A4) and (A5). Moreover, it is well known that all main elements of the general theory of Banach function spaces work with (A4) and (A5) stated for bounded sets [27] (see also the discussion at the beginning of Chapter 1 on page 2 of [3]). Unfortunately, we overlooked that the definition of a Banach function space in our previous works [9–12, 16–18, 20, 21] had to be changed by replacing in Axioms (A4) and (A5) the requirement of $|E| < \infty$ by the requirement that E is a bounded set to include weighted Lebesgue spaces in our considerations. However, the results proved in the above papers remain true.

2.2. Density of Nice Functions in Banach Function Spaces

Let $C_c(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ denote the sets of continuous compactly supported functions on $\mathbb{R}$ and of infinitely differentiable compactly supported functions on $\mathbb{R}$, respectively.

Lemma 2.2. *Let $X(\mathbb{R})$ be a separable Banach function space. Then the sets $C_c(\mathbb{R})$, $C_c^\infty(\mathbb{R})$ and $L^2(\mathbb{R}) \cap X(\mathbb{R})$ are dense in the space $X(\mathbb{R})$.*

The density of $C_c(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ in $X(\mathbb{R})$ is shown in [21, Lemma 2.12]. Since $C_c(\mathbb{R}) \subset L^2(\mathbb{R}) \cap X(\mathbb{R}) \subset X(\mathbb{R})$, we conclude that $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$.

2.3. Banach Algebra $\mathcal{M}_{X(\mathbb{R})}$ of Fourier Multipliers

The following result plays an important role in this paper.

Theorem 2.3. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ or on its associate space $X'(\mathbb{R})$. If $a \in \mathcal{M}_{X(\mathbb{R})}$, then*

$$\|a\|_{L^\infty(\mathbb{R})} \le \|a\|_{\mathcal{M}_{X(\mathbb{R})}}. \tag{2.1}$$

The constant 1 on the right-hand side of (2.1) is best possible.

Proof. If the Hardy-Littlewood maximal operator M is bounded on the space $X(\mathbb{R})$ or on its associate space $X'(\mathbb{R})$, then in view of [15, Lemma 3.2] we have

$$\sup_{-\infty<a<b<\infty} \frac{1}{b-a} \|\chi_{(a,b)}\|_{X(\mathbb{R})} \|\chi_{(a,b)}\|_{X'(\mathbb{R})} < \infty.$$

If this condition is fulfilled, then inequality (2.1) follows from [20, inequality (1.2) and Corollary 4.2]. □

Inequality (2.1) was established earlier in [17, Theorem 1] with some constant on the right-hand side that depends on the space $X(\mathbb{R})$ under the assumption that the operator M is bounded on both $X(\mathbb{R})$ and $X'(\mathbb{R})$ (see also [10, Theorem 2.4]).

Since (2.1) is available, an easy adaptation of the proof of [14, Proposition 2.5.13] leads to the following (we refer to the proof of [17, Corollary 1] for details).

Corollary 2.4. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ or on its associate space $X'(\mathbb{R})$. Then the set of Fourier multipliers $\mathcal{M}_{X(\mathbb{R})}$ is a Banach algebra under pointwise operations and the norm $\|\cdot\|_{\mathcal{M}_{X(\mathbb{R})}}$.*

2.4. Approximation of Continuous Functions Vanishing at Infinity

Let $C_0(\mathbb{R})$ denote the set of all continuous functions on $\mathbb{R}$ that vanish at $\pm\infty$.

Lemma 2.5. *For a function $v \in C_c^\infty(\mathbb{R})$ such that $0 \le v \le 1$ and $v(x) = 1$ when $|x| \le 1$, let*

$$v_n(x) := v(x/n), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

(a) If $a \in C_0(\mathbb{R})$, then

$$\lim_{n\to\infty} \|a - v_n a\|_{L^\infty(\mathbb{R})} = 0. \tag{2.2}$$

(b) If $a \in C_0(\mathbb{R}) \cap V(\mathbb{R})$, then

$$\lim_{n\to\infty} \|a - v_n a\|_{V(\mathbb{R})} = 0. \tag{2.3}$$

Proof. (a) If $a \in C_0(\mathbb{R})$, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}\setminus[-N,N]} |a(x)| < \frac{\varepsilon}{2}.$$

For all $n > N$ and $x \in [-N, N]$, we have $v_n(x) = 1$. Since $0 \le v_n \le 1$, for $n > N$, we get

$$\|a - v_n a\|_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}\setminus[-N,N]} |a(x) - v_n(x)a(x)| \le 2 \sup_{x\in\mathbb{R}\setminus[-N,N]} |a(x)| < \varepsilon,$$

which completes the proof of equality (2.2).

(b) Let $V(g;\Omega)$ denote the total variation of a function g over a union of intervals $\Omega \subset \mathbb{R}$. Then for all $n \in \mathbb{N}$,

$$\begin{aligned} V(a - v_n a) &= V(a(1 - v_n); \mathbb{R} \setminus [-n, n]) \\ &\le V(a; \mathbb{R} \setminus [-n, n])\|1 - v_n\|_{L^\infty(\mathbb{R}\setminus[-n,n])} \\ &\quad + \|a\|_{L^\infty(\mathbb{R}\setminus[-n,n])} V(1 - v_n; \mathbb{R} \setminus [-n, n]) \\ &\le V(a; \mathbb{R} \setminus [-n, n]) + \|a\|_{L^\infty(\mathbb{R}\setminus[-n,n])} V(v). \end{aligned} \tag{2.4}$$

Since $a \in C_0(\mathbb{R})$, we have

$$\lim_{n\to\infty} \|a\|_{L^\infty(\mathbb{R}\setminus[-n,n])} = 0 \tag{2.5}$$

(see the proof of part (a)). On the other hand,

$$\lim_{n\to\infty} V(a;\mathbb{R}\setminus[-n,n]) = \lim_{n\to\infty}\big(V(a)-V(a;[-n,n])\big) = V(a)-V(a)=0. \tag{2.6}$$

It follows from (2.4)– (2.6) that

$$\lim_{n\to\infty} V(a-v_n a)=0. \tag{2.7}$$

Combining equalities (2.2) and (2.7), we arrive at equality (2.3). □

2.5. Approximation of Continuous Fourier Multipliers Vanishing at Infinity

Lemma 2.6. *Let $X(\mathbb{R})$ be a Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$.*

(a) If $b\in C_0(\mathbb{R})\cap V(\mathbb{R})$ and $\{v_n\}_{n=1}^\infty$ is the sequence of functions in $C_c^\infty(\mathbb{R})$ defined in Lemma 2.5, then

$$\lim_{n\to\infty}\|b-v_n b\|_{\mathcal{M}_{X(\mathbb{R})}}=0.$$

(b) If $b\in C_X(\dot{\mathbb{R}})$ is such that $b(\infty)=0$, then there exists a sequence $\{b_n\}_{n=1}^\infty$ of functions in $C_0(\mathbb{R})\cap V(\mathbb{R})$ such that

$$\lim_{n\to\infty}\|b_n-b\|_{\mathcal{M}_{X(\mathbb{R})}}=0.$$

Proof. Part (a) follows from Lemma 2.5(b) and inequality (1.1).

(b) It follows from the definition of $C_X(\dot{\mathbb{R}})$ that there exists a sequence $\{d_n\}_{n=1}^\infty$ in $C(\dot{\mathbb{R}})\cap V(\mathbb{R})$ such that

$$\lim_{n\to\infty}\|d_n-b\|_{\mathcal{M}_{X(\mathbb{R})}}=0. \tag{2.8}$$

Take $b_n := d_n - d_n(\infty)$. Then $b_n\in C_0(\mathbb{R})\cap V(\mathbb{R})$. It follows (2.8) and Theorem 2.3 that $\{d_n\}_{n=1}^\infty$ converges uniformly to b on $\mathbb{R}$. In particular,

$$\lim_{n\to\infty} d_n(\infty)=b(\infty)=0. \tag{2.9}$$

Combining (2.8) and (2.9), we see that

$$\lim_{n\to\infty}\|b_n-b\|_{\mathcal{M}_{X(\mathbb{R})}} = \lim_{n\to\infty}\|d_n-d_n(\infty)-b\|_{\mathcal{M}_{X(\mathbb{R})}} \le \lim_{n\to\infty}\|d_n-b\|_{\mathcal{M}_{X(\mathbb{R})}} + \lim_{n\to\infty}|d_n(\infty)| = 0,$$

which completes the proof. □

3. Commutativity of the Algebra $\mathcal{A}^\pi_{X(\mathbb{R})}$

3.1. Compactness of Convolution Operators from a Subspace of Compactly Supported Functions of $L^1(\mathbb{R})$ to a Subspace of Compactly Supported Functions of $C(\mathbb{R})$

Let $C^k(\mathbb{R})$, $k=0,1,2,\dots$ be the space of functions with continuous bounded derivatives of all orders up to k, $C(\mathbb{R})=C^0(\mathbb{R})$. For any space of functions $Y(\mathbb{R})$ and any $R>0$, let $Y_{[R]}(\mathbb{R})$ denote the subspace of $Y(\mathbb{R})$ consisting of functions with supports in $[-R, R]$. As usual, the support of a function $f : \mathbb{R} \to \mathbb{C}$ will be denoted by $\operatorname{supp} f$.

Lemma 3.1. *Suppose that $R_1, R_2 > 0$. If $k \in C^1(\mathbb{R})$ is such that $\operatorname{supp} k \subset [-R_1, R_1]$, then the convolution operator with the kernel k defined by*

$$(Kf)(x) := (k * f)(x) = \int_{\mathbb{R}} k(x-y) f(y)\, dy, \quad x \in \mathbb{R}, \tag{3.1}$$

is compact from the space $L^1_{[R_2]}(\mathbb{R})$ to the space $C_{[R_1+R_2]}(\mathbb{R})$.

Proof. It follows from [6, Propositions 4.18 and 4.20] that the operator K is bounded from the space $L^1_{[R_2]}(\mathbb{R})$ to the space $C^1_{[R_1+R_2]}(\mathbb{R})$. Further, by the Arzelà-Ascoli theorem (see, e.g., [28, Theorems 2.2.12 and 2.5.10]), the space $C^1_{[R_1+R_2]}(\mathbb{R})$ is compactly embedded into the space $C_{[R_1+R_2]}(\mathbb{R})$, which completes the proof. □

3.2. Compactness of Products of Fourier Convolution Operators and Multiplication Operators

The main step in the proof of Theorem 1.2 consists of proving the following.

Theorem 3.2. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If $a \in C(\dot{\mathbb{R}})$ and $b \in C_X(\dot{\mathbb{R}})$ are such that $a(\infty) = b(\infty) = 0$, then*

$$aW^0(b), W^0(b)aI \in \mathcal{K}(X(\mathbb{R})).$$

Proof. A part of the proof is quite standard (see, e.g., [29, Theorem 5.3.1(i)]). It follows from Lemma 2.6(b) that there exists a sequence $\{b_n\}_{n=1}^\infty$ of functions in $C_0(\mathbb{R}) \cap V(\mathbb{R})$ such that $\|b_n - b\|_{\mathcal{M}_{X(\mathbb{R})}} \to 0$ as $n \to \infty$. Then

$$\|aW^0(b) - aW^0(b_n)\|_{\mathcal{B}(X(\mathbb{R}))} \to 0, \quad \|W^0(b)aI - W^0(b_n)aI\|_{\mathcal{B}(X(\mathbb{R}))} \to 0$$

as $n \to \infty$. So, we can assume without loss of generality that $b \in C_0(\mathbb{R}) \cap V(\mathbb{R})$. Let $\{v_n\}_{n=1}^\infty$ be the sequence of functions in $C_c^\infty(\mathbb{R})$ as in Lemma 2.5. Since

$$\begin{aligned}
&\|aW^0(b) - v_n aW^0(v_n b)\|_{\mathcal{B}(X(\mathbb{R}))} \\
&\quad \le \|(a - v_n a)W^0(b)\|_{\mathcal{B}(X(\mathbb{R}))} + \|v_n aW^0(b - v_n b)\|_{\mathcal{B}(X(\mathbb{R}))} \\
&\quad \le \|a - v_n a\|_{L^\infty(\mathbb{R})} \|b\|_{\mathcal{M}_{X(\mathbb{R})}} + \|a\|_{L^\infty(\mathbb{R})} \|b - v_n b\|_{\mathcal{M}_{X(\mathbb{R})}},
\end{aligned}$$

Lemmas 2.5(a) and 2.6(a) imply that

$$\lim_{n\to\infty} \|aW^0(b) - v_n aW^0(v_n b)\|_{\mathcal{B}(X(\mathbb{R}))} = 0.$$

Analogously we can show that

$$\lim_{n\to\infty} \|W^0(b)aI - W^0(v_n b) v_n aI\|_{\mathcal{B}(X(\mathbb{R}))} = 0.$$

Taking into account that $v_n \in C_c^\infty(\mathbb{R})$ and

$$v_n aW^0(v_n b) = a(v_n W^0(v_n))W^0(b), \; W^0(v_n b)v_n aI = W^0(b)(W^0(v_n)v_n)aI,$$

it is enough to prove that $a_0 W^0(b_0)$ and $W^0(b_0)a_0 I$ are compact operators for all $a_0, b_0 \in C_c^\infty(\mathbb{R})$.

Since $F^{-1}b_0 \in \mathcal{S}(\mathbb{R})$, it is easy to see that $\upsilon_n F^{-1}b_0 \to F^{-1}b_0$ in $\mathcal{S}(\mathbb{R})$ as $n \to \infty$. Then $b_n := F\left(\upsilon_n F^{-1}b_0\right) \to b_0$ in $\mathcal{S}(\mathbb{R})$ as $n \to \infty$. It is easy to see that the convergence in $\mathcal{S}(\mathbb{R})$ implies the convergence in $V(\mathbb{R})$. Therefore $\|b_n - b_0\|_{V(\mathbb{R})} \to 0$ as $n \to \infty$. It follows from the Stechkin type inequality (1.1) that

$$\begin{aligned}
&\lim_{n\to\infty} \|a_0 W^0(b_n) - a_0 W^0(b_0)\|_{\mathcal{B}(X(\mathbb{R}))} \\
&\quad \le c_X \|a_0\|_{L^\infty(\mathbb{R})} \lim_{n\to\infty} \|b_n - b_0\|_{V(\mathbb{R})} = 0, \\
&\lim_{n\to\infty} \|W^0(b_n) a_0 I - W^0(b_0) a_0 I\|_{\mathcal{B}(X(\mathbb{R}))} \\
&\quad \le c_X \|a_0\|_{L^\infty(\mathbb{R})} \lim_{n\to\infty} \|b_n - b_0\|_{V(\mathbb{R})} = 0.
\end{aligned}$$

Thus, it is sufficient to prove that $a_0 W^0(b_n), W^0(b_n) a_0 I \in \mathcal{K}(X(\mathbb{R}))$ for all $n \in \mathbb{N}$. Let $k_n := F^{-1}b_n = \upsilon_n F^{-1}b_0 \in C_c^\infty(\mathbb{R})$. It follows from the convolution theorem for the inverse Fourier transform (see, e.g., [14, Proposition 2.2.11, statement (12)]) that for all $n \in \mathbb{N}$ and $f \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
W^0(b_n) f &= F^{-1}(b_n F f) = (F^{-1}b_n) * F^{-1}(Ff) \\
&= (F^{-1}b_n) * f = k_n * f =: K_n f, \qquad (3.2)
\end{aligned}$$

where K_n is the convolution operator with the kernel k_n defined by (3.1). In view of Lemma 2.2, equality (3.2) remains valid for all $f \in X(\mathbb{R})$.

Take $R_1, R_2 > 0$ such that $\operatorname{supp} k_n \subset [-R_1, R_1]$ and $\operatorname{supp} a_0 \subset [-R_2, R_2]$. Equality (3.2) implies that

$$a_0 W^0(b_n) = a_0 K_n \chi_{[-R_1-R_2, R_1+R_2]} I.$$

It follows from Axiom (A5) that there exists $C_{[-R_1-R_2,R_1+R_2]} \in (0, \infty)$ such that for all $f \in X(\mathbb{R})$,

$$\int_{-R_1-R_2}^{R_1+R_2} |f(x)| dx \le C_{[-R_1-R_2,R_1+R_2]} \|f\|_{X(\mathbb{R})},$$

which means that the operator $\chi_{[-R_1-R_2,R_1+R_2]} I$ is bounded from the space $X(\mathbb{R})$ to the space $L^1_{[R_1+R_2]}(\mathbb{R})$. By Lemma 3.1, the operator K_n is compact from the space $L^1_{[R_1+R_2]}(\mathbb{R})$ to the space $C_{[2R_1+R_2]}(\mathbb{R})$. It follows from Axiom (A2) that the operator $a_0 I : C_{[2R_1+R_2]}(\mathbb{R}) \to X(\mathbb{R})$ is bounded. Thus, for every $n \in \mathbb{N}$, the operator $a_0 W^0(b_n) : X(\mathbb{R}) \to X(\mathbb{R})$ is compact as the composition of the bounded operator $\chi_{[-R_1-R_2,R_1+R_2]} I : X(\mathbb{R}) \to L^1_{[R_1+R_2]}(\mathbb{R})$, the compact operator $K_n : L^1_{[R_1+R_2]}(\mathbb{R}) \to C_{[2R_1+R_2]}(\mathbb{R})$, and the bounded operator $aI : C_{[2R_1+R_2]}(\mathbb{R}) \to X(\mathbb{R})$.

Similarly, for every $n \in \mathbb{N}$, the operator $W^0(b_n) a_0 I : X(\mathbb{R}) \to X(\mathbb{R})$ is compact as the composition of the bounded operator $a_0 I : X(\mathbb{R}) \to L^1_{[R_2]}(\mathbb{R})$, the compact operator $K_n : L^1_{[R_2]}(\mathbb{R}) \to C_{[R_1+R_2]}(\mathbb{R})$, and the bounded operator $I : C_{[R_1+R_2]}(\mathbb{R}) \to X(\mathbb{R})$. □

3.3. Compactness of Commutators of Fourier Convolution Operators and Multiplication Operators

The previous theorem implies the following.

Corollary 3.3. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If $a \in C(\dot{\mathbb{R}})$ and $b \in C_X(\dot{\mathbb{R}})$, then*

$$[aI, W^0(b)] := aW^0(b) - W^0(b)aI \in \mathcal{K}(X(\mathbb{R})).$$

Since $a = a(\infty) + \widetilde{a}$ and $b = b(\infty) + \widetilde{b}$, where $\widetilde{a} \in C(\dot{\mathbb{R}})$, $\widetilde{b} \in C_X(\dot{\mathbb{R}})$, and $\widetilde{a}(\infty) = 0 = \widetilde{b}(\infty)$, Theorem 3.2 implies that

$$\begin{aligned}[aI, W^0(b)] &= (a(\infty) + \widetilde{a})(b(\infty) + W^0(\widetilde{b})) - (b(\infty) + W^0(\widetilde{b}))(a(\infty) + \widetilde{a})I \\ &= \widetilde{a}W^0(\widetilde{b}) - W^0(\widetilde{b})\widetilde{a}I \in \mathcal{K}(X(\mathbb{R})).\end{aligned}$$

□

3.4. Proof of Theorem 1.2

Since a Banach function space $X(\mathbb{R})$ is reflexive if and only if the space $X(\mathbb{R})$ and its associate space $X'(\mathbb{R})$ are separable (see [27, Chap. 1, §2, Theorem 4 and §3, Corollary 1 to Theorem 7] or [3, Chap. 1, Corollaries 4.4 and 5.6]), Theorem 1.2 follows from Theorem 1.1 and Corollary 3.3. □

4. Proof of the Main Result

4.1. Estimate for the Norm of a Product of Multiplication Operators and a Fourier Convolution Operator

For $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let

$$\ell_n(x) := \frac{\log^n(1 + |x|)}{1 + |x|}, \quad x \in \mathbb{R}.$$

Lemma 4.1. *If $Y(\mathbb{R})$ is a Banach function space such that the Hardy-Littlewood maximal operator M is bounded on it, then $\ell_n \in Y(\mathbb{R})$ for all $n \in \mathbb{N}_0$.*

Proof. Since $\chi_{[-1,1]} \in Y(\mathbb{R})$ by Axiom (A4), $M\chi_{[-1,1]} \in Y(\mathbb{R})$. It is easy to see that $0 \le \ell_0 \le M\chi_{[-1,1]}$ (see [14, Example 2.1.4]). Hence $\ell_0 \in Y(\mathbb{R})$ in view of Axiom (A2).

Now let $k \in \mathbb{N}_0$. It follows from the definition of the Hardy-Littlewood maximal operator that for $x \ne 0$,

$$(M\ell_k)(x) \ge \begin{cases} \dfrac{1}{x+\varepsilon} \displaystyle\int_0^{x+\varepsilon} \dfrac{\log^k(1+|t|)}{1+|t|} dt & \text{if } x, \varepsilon > 0, \\ \dfrac{1}{-x-\varepsilon} \displaystyle\int_{x+\varepsilon}^0 \dfrac{\log^k(1+|t|)}{1+|t|} dt & \text{if } x, \varepsilon < 0. \end{cases}$$

Passing to the limit as $\varepsilon \to 0^{\pm}$, we obtain for $x \neq 0$,

$$
\begin{aligned}
(M\ell_k)(x) &\geq \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x \dfrac{\log^k(1+|t|)}{1+|t|}dt & \text{if } x > 0, \\ \dfrac{1}{-x}\displaystyle\int_x^0 \dfrac{\log^k(1+|t|)}{1+|t|}dt & \text{if } x < 0 \end{cases} \\
&= \frac{1}{|x|}\int_0^{|x|} \frac{\log^k(1+t)}{1+t}dt \\
&= \frac{1}{(k+1)|x|}\log^{k+1}(1+|x|) \geq \frac{1}{k+1}\ell_{k+1}(x).
\end{aligned}
$$

So

$$
0 \leq \ell_{k+1} \leq (k+1)M\ell_k, \quad k \in \mathbb{N}_0,
$$

and one gets by induction that $\ell_n \in Y(\mathbb{R})$ for all $n \in \mathbb{N}_0$. □

For $R > 0$, let $\chi_{\{R\}} := \chi_{\mathbb{R}\setminus[-R,R]}$.

Theorem 4.2. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Let $a \in C_c(\mathbb{R})$ and $b \in C_c(\mathbb{R}) \cap V(\mathbb{R})$. Then for every $n \in \mathbb{N}_0$, there exists a constant $c_n(a,b) \in (0,\infty)$ depending only on a, b and n, such that for all $R > 0$,*

$$
\|aW^0(b)\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} \leq \frac{c_n(a,b)}{\log^n(R+2)}. \tag{4.1}
$$

Proof. Since $b \in C_c(\mathbb{R}) \subset L^1(\mathbb{R})$, it follows from the convolution theorem for the inverse Fourier transform (see, e.g., [1, Theorem 11.66]) that for $f \in C_c^\infty(\mathbb{R})$,

$$
W^0(b)f = F^{-1}(b \cdot Ff) = (F^{-1}b) * F^{-1}(Ff) =: k * f, \tag{4.2}
$$

where $k := F^{-1}b$. In view of Lemma 2.2, formula (4.2) remains valid for all $f \in X(\mathbb{R})$. Since $b \in V(\mathbb{R})$, using integration by parts, similarly to the proof of [26, Chap. I, Theorem 4.5], we get for $x \in \mathbb{R}$,

$$
k(x) = (F^{-1}b)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}b(\xi)\,d\xi = \frac{1}{2\pi ix}\int_{\mathbb{R}} e^{-ix\xi}db(\xi),
$$

and hence

$$
|k(x)| \leq \frac{V(b)}{2\pi|x|}, \quad x \in \mathbb{R}. \tag{4.3}
$$

Take $R_1 > 0$ such that $\operatorname{supp} a \subset [-R_1, R_1]$. If $x \in [-R_1, R_1]$ and $|y| > R \geq \max\{2R_1, 1\}$, then

$$
|x-y| \geq |y| - |x| \geq |y| - R_1 \geq |y| - \frac{|y|}{2} = \frac{|y|}{2} \geq \frac{|y|+1}{4} \tag{4.4}
$$

and

$$
\log(R+1) \geq \frac{1}{2}\log(R+2). \tag{4.5}
$$

Combining (4.3)–(4.5) and taking into account the definition of ℓ_n, we get for every $x \in [-R_1, R_1]$, $R \geq \max\{2R_1, 1\}$, and $n \in \mathbb{N}_0$,

$$\begin{aligned} |k(x-y)|\chi_{\{R\}}(y) &\leq \frac{V(b)}{2\pi|x-y|}\chi_{\{R\}}(y) \leq \frac{2V(b)}{\pi(1+|y|)}\chi_{\{R\}}(y) \\ &\leq \frac{2V(b)}{\pi \log^n(R+1)}\ell_n(y) \leq \frac{2^{n+1}V(b)}{\pi\log^n(R+2)}\ell_n(y). \end{aligned} \tag{4.6}$$

It follows from (4.6), Lemma 4.1 and Hölder's inequality for Banach function spaces (see [27, Chap. 1, §1, Lemma 2] or [3, Chap. 1, Theorem 2.4]) that for $x \in [-R_1, R_1]$, $R \geq \max\{2R_1, 1\}$, $n \in \mathbb{N}_0$ and $f \in X(\mathbb{R})$,

$$\begin{aligned} |k * (\chi_{\{R\}}f)(x)| &= \left| \int_{\mathbb{R}} k(x-y)\chi_{\{R\}}(y)f(y)\,dy \right| \\ &\leq \|k(x-\cdot)\chi_{\{R\}}\|_{X'(\mathbb{R})}\|f\|_{X(\mathbb{R})} \\ &\leq \frac{2^{n+1}V(b)\|\ell_n\|_{X'(\mathbb{R})}\|f\|_{X(\mathbb{R})}}{\pi\log^n(R+2)}. \end{aligned} \tag{4.7}$$

It follows from Axiom (A4) that $\chi_{[-R_1,R_1]} \in X(\mathbb{R})$. Since $\operatorname{supp} a \subset [-R_1, R_1]$, in view of Axiom (A2), equality (4.2) and inequality (4.7), we obtain for $R \geq \max\{2R_1, 1\}$, $f \in X(\mathbb{R})$ and $n \in \mathbb{N}_0$,

$$\begin{aligned} \|aW^0(b)\chi_{\{R\}}f\|_{X(\mathbb{R})} &\leq \|a\|_{L^\infty(\mathbb{R})}\|\chi_{[-R_1,R_1]}\|_{X(\mathbb{R})} \operatorname*{ess\,sup}_{x\in[-R_1,R_1]} |k*(\chi_{\{R\}}f)(x)| \\ &\leq \frac{2^{n+1}\|a\|_{L^\infty(\mathbb{R})}V(b)\|\ell_n\|_{X'(\mathbb{R})}\|\chi_{[-R_1,R_1]}\|_{X(\mathbb{R})}}{\pi\log^n(R+2)}\|f\|_{X(\mathbb{R})}. \end{aligned} \tag{4.8}$$

If $R \in (0, \max\{2R_1, 1\})$, then $\log(R+2) \leq \log(2+\max\{2R_1, 1\})$ and

$$\begin{aligned} \|aW^0(b)\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} &\leq \|aW^0(b)\|_{\mathcal{B}(X(\mathbb{R}))} \\ &\leq \frac{\log^n(2+\max\{2R_1,1\})\|aW^0(b)\|_{\mathcal{B}(X(\mathbb{R}))}}{\log^n(R+2)}. \end{aligned} \tag{4.9}$$

It follows from (4.8) and (4.9) that (4.1) is fulfilled with

$$\begin{aligned} c_n(a,b) := \max\Bigg\{ &\frac{2^{n+1}}{\pi}\|a\|_{L^\infty(\mathbb{R})}V(b)\|\ell_n\|_{X'(\mathbb{R})}\|\chi_{[-R_1,R_1]}\|_{X(\mathbb{R})}, \\ &\log^n(2+\max\{2R_1,1\})\|aW^0(b)\|_{\mathcal{B}(X(\mathbb{R}))} \Bigg\}, \end{aligned}$$

which completes the proof. □

4.2. Sufficient Condition on the Space $X(\mathbb{R})$ Implying that the Algebra $\mathcal{A}_{X(\mathbb{R})}$ does Not Contain All Rank One Operators

Now we prove a conditional statement, which will lead to the proof of Theorem 1.3.

Theorem 4.3. *Let $X(\mathbb{R})$ be a separable non-reflexive Banach function space such that the Hardy-Littlewood maximal operator is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Suppose that there exist a function $g \in X'(\mathbb{R})$ and* *a constant* $\delta > 0$ *such that* $\|\chi_{\{R\}} g\|_{X'(\mathbb{R})} \geq \delta$ *for all* $R > 0$. *Then for any function* $h \in X(\mathbb{R}) \setminus \{0\}$, *the rank one operator* $T_{g,h} \in \mathcal{B}(X(\mathbb{R}))$, *defined by*

$$(T_{g,h} f)(x) := h(x) \int_{\mathbb{R}} g(y) f(y)\, dy,$$

does not belong to the algebra $\mathcal{A}_{X(\mathbb{R})}$.

Proof. Fix $h \in X(\mathbb{R}) \setminus \{0\}$. Suppose the contrary: $T_{g,h} \in \mathcal{A}_{X(\mathbb{R})}$. Fix $\varepsilon > 0$. By the definition of the algebra $\mathcal{A}_{X(\mathbb{R})}$ there exist numbers $N, M \in \mathbb{N}$ and operators

$$A_{ij} \in \{aI, W^0(b) \ : \ a \in C(\dot{\mathbb{R}}),\ b \in C_X(\dot{\mathbb{R}})\}$$

for $i \in \{1, \dots, N\}$, $j \in \{1, \dots, M\}$ such that

$$\left\| T_{g,h} - \sum_{i=1}^{N} A_{i1} \dots A_{iM} \right\|_{\mathcal{B}(X(\mathbb{R}))} < \frac{\varepsilon}{6}. \tag{4.10}$$

Put

$$L := 2 \max \left\{ \|A_{ij}\|_{\mathcal{B}(X(\mathbb{R}))} \ : \ i \in \{1, \dots, N\},\ j \in \{1, \dots, M\} \right\}. \tag{4.11}$$

Let $b_1, \dots, b_r \in C_X(\dot{\mathbb{R}})$ be such that for $k \in \{1, \dots, r\}$,

$$W^0(b_k) \in \left\{ A_{ij} \ : \ i \in \{1, \dots, N\},\ j \in \{1, \dots, M\} \right\} \setminus \left\{ aI \ : \ a \in C(\dot{\mathbb{R}}) \right\}$$

and $a_1, \dots, a_s \in C(\dot{\mathbb{R}})$ be such that for $l \in \{1, \dots, s\}$,

$$a_l I \in \left\{ A_{ij} : i \in \{1, \dots, N\},\ j \in \{1, \dots, M\} \right\} \setminus \left\{ W^0(b_k) : k \in \{1, \dots, r\} \right\}.$$

It follows from the definition of the algebra $C_X(\dot{\mathbb{R}})$ that for each $k \in \{1, \dots, r\}$ there exists a function $c_k \in C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$ such that

$$\begin{aligned} \|W^0(b_k) - W^0(c_k)\|_{\mathcal{B}(X(\mathbb{R}))} &= \|b_k - c_k\|_{\mathcal{M}_{X(\mathbb{R})}} \\ &< \min \left\{ \frac{\varepsilon}{6NML^{M-1}}, \frac{L}{4} \right\}. \end{aligned} \tag{4.12}$$

Further, in view of Lemma 2.6(a), there exists a function $\widetilde{b}_k \in C_c(\mathbb{R}) \cap V(\mathbb{R})$ such that

$$\begin{aligned} \|W^0(c_k) - c_k(\infty) I - W^0(\widetilde{b}_k)\|_{\mathcal{B}(X(\mathbb{R}))} &= \|c_k - c_k(\infty) - \widetilde{b}_k\|_{\mathcal{M}_{X(\mathbb{R})}} \\ &< \min \left\{ \frac{\varepsilon}{6NML^{M-1}}, \frac{L}{4} \right\}. \end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), we get

$$\|W^0(b_k) - c_k(\infty) I - W^0(\widetilde{b}_k)\|_{\mathcal{B}(X(\mathbb{R}))} < \min \left\{ \frac{\varepsilon}{3NML^{M-1}}, \frac{L}{2} \right\}.$$

Analogously, by Lemma 2.5(a), for every $l \in \{1, \dots, s\}$, there exists $\widetilde{a}_l \in C_c(\mathbb{R})$ such that

$$\begin{aligned} \|a_l I - a_l(\infty) I - \widetilde{a}_l I\|_{\mathcal{B}(X(\mathbb{R}))} &\leq \|a_l - a_l(\infty) - \widetilde{a}_l\|_{L^\infty(\mathbb{R})} \\ &< \min \left\{ \frac{\varepsilon}{3NML^{M-1}}, \frac{L}{2} \right\}. \end{aligned}$$

We have shown that for every $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$ there exists an operator

$$B_{ij} \in \left\{ cI + \widetilde{a}I, cI + W^0(\widetilde{b}) \ : \ c \in \mathbb{C}, \ \widetilde{a} \in C_c(\mathbb{R}), \ \widetilde{b} \in C_c(\mathbb{R}) \cap V(\mathbb{R}) \right\} \tag{4.14}$$

such that

$$\|A_{ij} - B_{ij}\|_{\mathcal{B}(X(\mathbb{R}))} < \min\left\{ \frac{\varepsilon}{3NML^{M-1}}, \frac{L}{2} \right\}.$$

Then, taking into account (4.11), we get

$$\begin{aligned}
&\left\| \sum_{i=1}^{N} A_{i1} \dots A_{iM} - \sum_{i=1}^{N} B_{i1} \dots B_{iM} \right\|_{\mathcal{B}(X(\mathbb{R}))} \\
&= \left\| \sum_{i=1}^{N} \sum_{j=1}^{M} A_{i1} \dots A_{i,j-1}(A_{ij} - B_{ij}) B_{i,j+1} \dots B_{iM} \right\|_{\mathcal{B}(X(\mathbb{R}))} \\
&\le \sum_{i=1}^{N} \sum_{j=1}^{M} \left(\prod_{k=1}^{j-1} \|A_{ik}\|_{\mathcal{B}(X(\mathbb{R}))} \right) \|A_{ij} - B_{ij}\|_{\mathcal{B}(X(\mathbb{R}))} \left(\prod_{l=j+1}^{M} \|B_{il}\|_{\mathcal{B}(X(\mathbb{R}))} \right) \\
&< \sum_{i=1}^{N} \sum_{j=1}^{M} \left(\frac{L}{2} \right)^{j-1} \frac{\varepsilon}{3NML^{M-1}} \left(\frac{L}{2} + \frac{L}{2} \right)^{M-j} < \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{\varepsilon}{3NM} = \frac{\varepsilon}{3}.
\end{aligned} \tag{4.15}$$

It follows from (4.10) and (4.15) that

$$\|T_{g,h} - T_\varepsilon\|_{\mathcal{B}(X(\mathbb{R}))} < \frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \frac{\varepsilon}{2}, \tag{4.16}$$

where

$$T_\varepsilon := \sum_{i=1}^{N} B_{i1} \dots B_{iM}.$$

Taking into account (4.14), we can rearrange terms and write the operator T_ε in the form

$$T_\varepsilon = cI + W^0(\widetilde{b}_0) + \sum_{i=1}^{p} D_{1,i} \widetilde{a}_{1,i} I + \sum_{j=1}^{t} D_{2,j} \widetilde{a}_{2,j} W^0(\widetilde{b}_j), \tag{4.17}$$

where $c \in \mathbb{C}$, $\widetilde{b}_j \in C_c(\mathbb{R}) \cap V(\mathbb{R})$ for $j \in \{0, \dots, t\}$, $\widetilde{a}_{1,i}, \widetilde{a}_{2,j} \in C_c(\mathbb{R})$ and $D_{1,i}, D_{2,j}$ are some operators in $\mathcal{A}_{X(\mathbb{R})} \setminus \{0\}$ for $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, t\}$.

Since the space $X(\mathbb{R})$ is separable, it follows from [27, Chap. 1, §2, Definition 1 and §3, Corollary 1 to Theorem 7] (or [3, Chap. 1, Definition 3.1 and Corollary 5.6]) that there exists $R_1 > 0$ such that $\|\chi_{\{R_1\}} h\|_{X(\mathbb{R})} \le \frac{1}{2} \|h\|_{X(\mathbb{R})}$. Then

$$\|\chi_{R_1} h\|_{X(\mathbb{R})} \ge \|h\|_{X(\mathbb{R})} - \|\chi_{\{R_1\}} h\|_{X(\mathbb{R})} \ge \frac{1}{2} \|h\|_{X(\mathbb{R})}, \tag{4.18}$$

where

$$\chi_{R_1} := 1 - \chi_{\{R_1\}} = \chi_{[-R_1, R_1]}.$$

Since $\widetilde{a}_{1,i} \in C_c(\mathbb{R})$ for $i = 1, \dots, p$, there exists $R_2 > R_1$ such that for $R \geq R_2$,

$$\chi_{R_1}(cI)\chi_{\{R\}}I + \chi_{R_1}\sum_{i=1}^{p} D_{1,i}\widetilde{a}_{1,i}\chi_{\{R\}}I = 0. \tag{4.19}$$

Let $\widetilde{a}_0 \in C_c(\mathbb{R})$ be such that $\widetilde{a}_0 = 1$ for $x \in [-R_1, R_1]$. Then

$$\chi_{R_1}W^0(\widetilde{b}_0) = \chi_{R_1}\widetilde{a}_0 W^0(\widetilde{b}_0).$$

It follows from Theorem 4.2 that there exists $R_0 > R_2$ such that for all $R \geq R_0$ and $j \in \{1, \dots, t\}$,

$$\|\chi_{R_1}\widetilde{a}_0 W^0(\widetilde{b}_0)\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} \leq \|\widetilde{a}_0 W^0(\widetilde{b}_0)\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} < \frac{\varepsilon}{2(t+1)}, \tag{4.20}$$

$$\|\widetilde{a}_{2,j}W^0(\widetilde{b}_j)\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} < \frac{\varepsilon}{2(t+1)\|D_{2,j}\|_{\mathcal{B}(X(\mathbb{R}))}}. \tag{4.21}$$

Combining (4.17) and (4.19)–(4.21), we see that for all $R \geq R_0$,

$$\|\chi_{R_1}T_\varepsilon\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} < \frac{\varepsilon}{2(t+1)} + \sum_{j=1}^{t}\frac{\varepsilon}{2(t+1)} = \frac{\varepsilon}{2}. \tag{4.22}$$

It follows from (4.16) and (4.22) that for all $R \geq R_0$,

$$\begin{aligned}\|\chi_{R_1}T_{g,h}\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} &\leq \|\chi_{R_1}(T_{g,h} - T_\varepsilon)\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} \\ &\quad + \|\chi_{R_1}T_\varepsilon\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} \\ &\leq \|T_{g,h} - T_\varepsilon\|_{\mathcal{B}(X(\mathbb{R}))} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned} \tag{4.23}$$

On the other hand, in view of [3, Chap. 1, Lemma 2.8] (see also [27, Chap. 1, §1, Remark (2) after Theorem 2]), we have

$$\begin{aligned}&\|\chi_{R_1}T_{g,h}\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} \\ &\quad = \sup\left\{\left\|\chi_{R_1}h\int_{\mathbb{R}} g(y)\chi_{\{R\}}(y)f(y)\,dy\right\|_{X(\mathbb{R})} : f \in X(\mathbb{R}),\ \|f\|_{X(\mathbb{R})} \leq 1\right\} \\ &\quad = \sup\left\{\left|\int_{\mathbb{R}} g(y)\chi_{\{R\}}(y)f(y)\,dy\right| \|\chi_{R_1}h\|_{X(\mathbb{R})} : f \in X(\mathbb{R}),\ \|f\|_{X(\mathbb{R})} \leq 1\right\} \\ &\quad = \|\chi_{R_1}h\|_{X(\mathbb{R})}\sup\left\{\left|\int_{\mathbb{R}} g(y)\chi_{\{R\}}(y)f(y)\,dy\right| : f \in X(\mathbb{R}),\ \|f\|_{X(\mathbb{R})} \leq 1\right\} \\ &\quad = \|\chi_{R_1}h\|_{X(\mathbb{R})}\|g\chi_{\{R\}}\|_{X'(\mathbb{R})}.\end{aligned}$$

This equality, inequality (4.18) and inequality $\|\chi_{\{R\}}g\|_{X'(\mathbb{R})} \geq \delta$ imply that

$$\|\chi_{R_1}T_{g,h}\chi_{\{R\}}I\|_{\mathcal{B}(X(\mathbb{R}))} \geq \frac{\delta}{2}\|h\|_{X(\mathbb{R})}. \tag{4.24}$$

Inequalities (4.23) and (4.24) yield a contradiction for $\varepsilon \leq \frac{\delta}{2}\|h\|_{X(\mathbb{R})}$. □

Remark 4.4. Note that a Banach function spaces $X(\mathbb{R})$ is reflexive if and only if $X(\mathbb{R})$ and its associate space $X'(\mathbb{R})$ are separable (see [27, Chap. 1, §2, Theorem 4 and §3, Corollary 1 to Theorem 7] or [3, Chap. 1, Corollaries 4.4 and 5.6]). In turn, if $X'(\mathbb{R})$ is separable, then for any $g \in X'(\mathbb{R})$ one has

$\|\chi_{\{R\}}g\|_{X'(\mathbb{R})} \to 0$ as $R \to \infty$ in view of [27, Chap. 1, §2, Definition 1 and §3, Corollary 1 to Theorem 7] (or [3, Chap. 1, Definition 3.1 and Corollary 5.6]).

To complete the proof of Theorem 1.3, we have to show that there exists a separable non-reflexive Banach function space satisfying the hypotheses of Theorem 4.3. In the next subsection, we will show that the classical Lorentz spaces $L^{p,1}(\mathbb{R})$, $1 < p < \infty$, perfectly fit our needs.

4.3. Proof of Theorem 1.4

The space $X(\mathbb{R}) = L^{p,1}(\mathbb{R})$ is separable and

$$\left[L^{p,1}(\mathbb{R})\right]^* = \left(L^{p,1}\right)'(\mathbb{R}) = L^{p',\infty}(\mathbb{R}),$$

where $1/p + 1/p' = 1$ (see [3, Chap. 1, Corollaries 4.3 and 5.6, Chap. 4, Corollary 4.8]). It is also known that

$$L^{p,1}(\mathbb{R}) \subsetneqq [L^{p',\infty}(\mathbb{R})]^* = [L^{p,1}(\mathbb{R})]^{**}$$

(see [7, p. 83]). Hence $L^{p,1}(\mathbb{R})$ is non-reflexive. The lower and upper Boyd indices of $L^{p,1}(\mathbb{R})$ (resp., of $L^{p',\infty}(\mathbb{R})$) are both equal to $1/p$ (resp., to $1/p'$); see [3, Chap. 4, Theorem 4.6]. Hence the Hardy-Littlewood maximal operator is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$ in view of the Lorentz-Shimogaki theorem (see [3, Chap. 3, Theorem 5.17]). Thus, the space $L^{p,1}(\mathbb{R})$ is a separable non-reflexive Banach function space satisfying condition (a) of Theorem 1.3.

Consider the function $g(x) = |x|^{-1/p'}$. Its distribution function is

$$\mu_g(\lambda) = |\{x \in \mathbb{R} : |x|^{-1/p'} > \lambda\}| = |\{x \in \mathbb{R} : |x| < \lambda^{-p'}\}| = 2\lambda^{-p'}, \quad \lambda \geq 0,$$

and its non-increasing rearrangement is

$$\begin{aligned} g^*(t) &= \inf\{\lambda \geq 0 : 2\lambda^{-p'} \leq t\} = \inf\{\lambda \geq 0 : 2^{1/p'}t^{-1/p'} \leq \lambda\} \\ &= 2^{1/p'}t^{-1/p'}, \quad t \geq 0. \end{aligned}$$

Then

$$g^{**}(t) = \frac{1}{t}\int_0^t 2^{1/p'}y^{-1/p'}\,dy = \frac{2^{1/p'}t^{-1/p'}}{1-1/p'} = 2^{1/p'}pt^{-1/p'}, \quad t \geq 0$$

and

$$\|g\|_{(p',\infty)} = 2^{1/p'}p < \infty.$$

The distribution function of $\chi_{\{R\}}g$ for every $R > 0$ is given by

$$\begin{aligned} \mu_{\chi_{\{R\}}g}(\lambda) &= |\{x \in \mathbb{R} : \chi_{\{R\}}(x)g(x) > \lambda\}| \\ &= \begin{cases} 2\lambda^{-p'} - 2R & \text{if } 0 \leq \lambda < R^{-1/p'}, \\ 0 & \text{if } \lambda \geq R^{-1/p'}. \end{cases} \end{aligned}$$

Then

$$\begin{aligned} (\chi_{\{R\}}g)^*(t) &= \inf\{\lambda \geq 0 : 2\lambda^{-p'} - 2R \leq t\} = \inf\left\{\lambda \geq 0 : \lambda^{-p'} \leq \frac{t}{2} + R\right\} \\ &= \inf\left\{\lambda \geq 0 : \frac{2}{t+2R} \leq \lambda^{p'}\right\} = 2^{1/p'}(t+2R)^{-1/p'}, \quad t \geq 0. \end{aligned}$$

Since $(\chi_{\{R\}}g)^*$ is non-increasing, we have $(\chi_{\{R\}}g)^{**} \geq (\chi_{\{R\}}g)^*$ and

$$\begin{aligned}\|\chi_{\{R\}}g\|_{(p',\infty)} &\geq \sup_{0<t<\infty}\Big(t^{1/p'}(\chi_{\{R\}}g)^*(t)\Big)\\ &= 2^{1/p'}\sup_{0<t<\infty}\left(\frac{t}{t+2R}\right)^{1/p'} = 2^{1/p'}.\end{aligned}$$

Thus, the conditions of Theorem 4.3 are satisfied for $X(\mathbb{R}) = L^{p,1}(\mathbb{R})$, $g(x) = |x|^{-1/p'}$ and $\delta = 2^{1/p'}$. The desired result now follows from that theorem. □

5. Final Remarks on Algebras of Convolution Type Operators with Continuous and Slowly Oscillating Data

5.1. Algebra $C_X^0(\dot{\mathbb{R}})$ of Continuous Fourier Multipliers

Let $\mathbb{C}$ stand for the constant complex-valued functions on $\mathbb{R}$. Notice that $C(\dot{\mathbb{R}})$ decomposes into the direct sum $C(\dot{\mathbb{R}}) = \mathbb{C}\dot{+}C_0(\mathbb{R})$. It follows from the mean value theorem that

$$\mathbb{C}\dot{+}C_c^\infty(\mathbb{R}) \subset C(\dot{\mathbb{R}}) \cap V(\mathbb{R}). \tag{5.1}$$

Suppose $X(\mathbb{R})$ is a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Along with the algebra $C_X(\dot{\mathbb{R}})$ of continuous Fourier multipliers defined by (1.2), consider the following algebra of continuous Fourier multipliers:

$$C_X^0(\dot{\mathbb{R}}) := \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R})}}\big(\mathbb{C}\dot{+}C_c^\infty(\mathbb{R})\big). \tag{5.2}$$

It follows from embeddings (5.1) and definitions (1.2) and (5.2) that

$$C_X^0(\dot{\mathbb{R}}) \subset C_X(\dot{\mathbb{R}}). \tag{5.3}$$

For large classes of Banach function spaces, including separable rearrangement-invariant Banach function with nontrivial Boyd indices, weighted Lebesgue spaces with Muckenhoupt weights, reflexive variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R})$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R})$, the above embedding becomes equality (see [9, Theorem 3.3] and [19, Theorem 1.1]). Proofs of [9, Theorem 3.3] and [19, Theorem 1.1] are based on an interpolation argument. Unfortunately, interpolation tools are not available in the general setting of Banach function spaces. So, we arrive at the following.

Question 5.1. *Is it true that $C_X^0(\dot{\mathbb{R}}) = C_X(\dot{\mathbb{R}})$ for an arbitrary separable Banach function space $X(\mathbb{R})$ such that the Hardy-Littlewood maximal operator is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$?*

5.2. The Ideal of Compact Operators is Contained in the Algebra of Convolution Type Operators with Continuous Data

Since we do not know the answer on Question 5.1, along with the Banach algebra $\mathcal{A}_{X(\mathbb{R})}$, we will also consider the smallest Banach subalgebra

$$\mathcal{A}^0_{X(\mathbb{R})} := \operatorname{alg}\{aI, W^0(b) \;:\; a \in C(\dot{\mathbb{R}}),\ b \in C^0_X(\dot{\mathbb{R}})\}$$

of the algebra $\mathcal{B}(X(\mathbb{R}))$ that contains all operators of multiplication aI by functions $a \in C(\dot{\mathbb{R}})$ and all Fourier convolution operators $W^0(b)$ with symbols $b \in C^0_X(\dot{\mathbb{R}})$.

If the answer to Question 5.1 is negative, then the following result provides a refinement of Theorem 1.1.

Theorem 5.2. *Let $X(\mathbb{R})$ be a reflexive Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Then the ideal of compact operators $\mathcal{K}(X(\mathbb{R}))$ is contained in the Banach algebra $\mathcal{A}^0_{X(\mathbb{R})}$.*

The proof of Theorem 5.2 repeats word-by-word the proof of Theorem 1.1 with [11, Lemma 4.2] replaced by the following.

Lemma 5.3. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Suppose $a, b \in C_c(\mathbb{R})$ and a one-dimensional operator T_1 is defined on the space $X(\mathbb{R})$ by*

$$(T_1 f)(x) = a(x) \int_{\mathbb{R}} b(y) f(y)\, dy. \tag{5.4}$$

Then there exists a function $c \in C^0_X(\dot{\mathbb{R}})$ such that $T_1 = aW^0(c)bI$.

Proof. The idea of the proof is borrowed from [24, Lemma 6.1] (see also [29, Proposition 5.8.1]). Since $a, b \in C_c(\mathbb{R})$, there exists a number $M > 0$ such that the set $\{x - y : x \in \operatorname{supp} a, y \in \operatorname{supp} b\}$ is contained in the segment $[-M, M]$. By the smooth version of Urysohn's lemma (see, e.g., [13, Proposition 6.5]), there exists $k \in C_c^\infty(\mathbb{R})$ such that $0 \le k(x) \le 1$ for $x \in \mathbb{R}$, $k(x) = 1$ for $x \in [-M, M]$ and $k(x) = 0$ for $x \in \mathbb{R} \setminus (-2M, 2M)$. Then (5.4) can be rewritten in the form

$$(T_1 f)(x) = a(x) \int_{\mathbb{R}} k(x - y) b(y) f(y)\, dy = \big(aW^0(\widehat{k}) bf\big)(x), \quad x \in \mathbb{R}.$$

By [14, Example 2.2.2 and Proposition 2.2.11], $C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ and $\widehat{k} \in \mathcal{S}(\mathbb{R})$. Since $\mathcal{S}(\mathbb{R}) \subset C_0(\mathbb{R}) \cap V(\mathbb{R})$, it follows from Lemma 2.6(a) that

$$\mathcal{S}(\mathbb{R}) \subset \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R})}} \big(C_c^\infty(\mathbb{R})\big) \subset C^0_X(\dot{\mathbb{R}}).$$

Hence $c := \widehat{k} \in C^0_X(\dot{\mathbb{R}})$. $\square$

5.3. Slowly Oscillating Fourier Multipliers

For a set $E \subset \dot{\mathbb{R}}$ and a function $f : \dot{\mathbb{R}} \to \mathbb{C}$ in $L^\infty(\mathbb{R})$, let the oscillation of f over E be defined by

$$\operatorname{osc}(f, E) := \operatorname*{ess\,sup}_{s,t \in E} |f(s) - f(t)|.$$

Following [2, Section 4] and [23, Section 2.1], [24, Section 2.1], we say that a function $f \in L^\infty(\mathbb{R})$ is slowly oscillating at a point $\lambda \in \dot{\mathbb{R}}$ if for every $r \in (0, 1)$ or, equivalently, for some $r \in (0, 1)$, one has

$$\begin{aligned} &\lim_{x \to 0+} \operatorname{osc}\big(f, \lambda + ([-x, -rx] \cup [rx, x])\big) = 0 \text{ if } \lambda \in \mathbb{R}, \\ &\lim_{x \to +\infty} \operatorname{osc}\big(f, [-x, -rx] \cup [rx, x]\big) = 0 \qquad \text{if } \lambda = \infty. \end{aligned}$$

For every $\lambda \in \dot{\mathbb{R}}$, let SO_λ denote the C^*-subalgebra of $L^\infty(\mathbb{R})$ defined by

$$SO_\lambda := \Big\{ f \in C_b(\dot{\mathbb{R}} \setminus \{\lambda\}) \ : \ f \text{ slowly oscillates at } \lambda \Big\},$$

where $C_b(\dot{\mathbb{R}} \setminus \{\lambda\}) := C(\dot{\mathbb{R}} \setminus \{\lambda\}) \cap L^\infty(\mathbb{R})$.

Let $SO^\diamond$ be the smallest C^*-subalgebra of $L^\infty(\mathbb{R})$ that contains all the C^*-algebras SO_λ with $\lambda \in \dot{\mathbb{R}}$. The functions in $SO^\diamond$ are called slowly oscillating functions.

For a point $\lambda \in \dot{\mathbb{R}}$, let $C^3(\mathbb{R} \setminus \{\lambda\})$ be the set of all three times continuously differentiable functions $a : \mathbb{R} \setminus \{\lambda\} \to \mathbb{C}$. Following [23, Section 2.4] and [24, Section 2.3], consider the commutative Banach algebras

$$SO_\lambda^3 := \left\{ a \in SO_\lambda \cap C^3(\mathbb{R} \setminus \{\lambda\}) \ : \ \lim_{x \to \lambda} (D_\lambda^k a)(x) = 0, \ k = 1, 2, 3 \right\}$$

equipped with the norm

$$\|a\|_{SO_\lambda^3} := \sum_{k=0}^{3} \frac{1}{k!} \left\| D_\lambda^k a \right\|_{L^\infty(\mathbb{R})},$$

where $(D_\lambda a)(x) = (x - \lambda) a'(x)$ for $\lambda \in \mathbb{R}$ and $(D_\lambda a)(x) = x a'(x)$ for $\lambda = \infty$.

The following result leads us to the definition of slowly oscillating Fourier multipliers.

Theorem 5.4. *([18, Theorem 2.5]) Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If $\lambda \in \dot{\mathbb{R}}$ and $a \in SO_\lambda^3$, then the convolution operator $W^0(a)$ is bounded on the space $X(\mathbb{R})$ and*

$$\|W^0(a)\|_{\mathcal{B}(X(\mathbb{R}))} \le d_X \|a\|_{SO_\lambda^3},$$

where d_X is a positive constant depending only on $X(\mathbb{R})$.

Let $SO_{\lambda, X(\mathbb{R})}$ denote the closure of SO_λ^3 in the norm of $\mathcal{M}_{X(\mathbb{R})}$. Further, let $SO_{X(\mathbb{R})}^\diamond$ be the smallest Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$ that contains all the Banach algebras $SO_{\lambda, X(\mathbb{R})}$ for $\lambda \in \dot{\mathbb{R}}$. The functions in $SO_{X(\mathbb{R})}^\diamond$ will be called slowly oscillating Fourier multipliers.

5.4. The Ideal of Compact Operators is Contained in the Algebra of Convolution Type Operators with Slowly Oscillating Data

Consider the smallest Banach subalgebra

$$\mathcal{D}_{X(\mathbb{R})} := \operatorname{alg}\{aI, W^0(b) \ : \ a \in SO^\diamond, \ b \in SO^\diamond_{X(\mathbb{R})}\}$$

of the algebra $\mathcal{B}(X(\mathbb{R}))$ that contains all operators of multiplication aI by slowly oscillating functions $a \in SO^\diamond$ and all Fourier convolution operators $W^0(b)$ with slowly oscillating symbols $b \in SO^\diamond_{X(\mathbb{R})}$.

Now we are in a position to formulate the main result of this section.

Theorem 5.5. *Let $X(\mathbb{R})$ be a reflexive Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Then the ideal of compact operators $\mathcal{K}(X(\mathbb{R}))$ is contained in the Banach algebra $\mathcal{D}_{X(\mathbb{R})}$.*

This result follows from Theorem 5.2.

Under the assumptions of Theorem 5.5, we can define the quotient algebra

$$\mathcal{D}^\pi_{X(\mathbb{R})} := \mathcal{D}_{X(\mathbb{R})}/\mathcal{K}(X(\mathbb{R})).$$

We conclude this section with the following.

Question 5.6. *Let $X(\mathbb{R})$ be a reflexive Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Is it true that the quotient algebra $\mathcal{D}^\pi_{X(\mathbb{R})}$ is commutative?*

We know that the answer is positive for some particular cases of Banach function spaces. For Lebesgue spaces $L^p(\mathbb{R}, w)$, $1 < p < \infty$, with Muckenhoupt weights w, the positive answer to the above question follows from [24, Theorem 4.6], whose proof relies on a version of the Krasnosel'skii interpolation theorem for compact operators (see, e.g., [22, Corollary 5.3]). The answer is also positive for reflexive variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R})$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R})$. It is based on a similar interpolation argument (see [16, Lemma 6.4]). However, as far as we know, for arbitrary Banach function spaces, interpolation tools are not available.

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