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Path Components of the Space of (Weighted) Composition Operators on Bergman Spaces

Alexander V. Abanin, Le Hai Khoi and Pham Trong Tien

Abstract. The topological structure of the set of (weighted) composition operators has been studied on various function spaces on the unit disc such as Hardy spaces, the space of bounded holomorphic functions, weighted Banach spaces of holomorphic functions with sup-norm, Hilbert Bergman spaces. In this paper we consider this problem for all Bergman spaces A^p_α with $p \in (0, \infty)$ and $\alpha \in (-1, \infty)$. In this setting we establish a criterion for two composition operators to be linearly connected in the space of composition operators; furthermore, for the space of weighted composition operators, we prove that the set of compact weighted composition operators is path connected, but it is not a component.

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Keywords. Bergman spaces, Composition operators, Weighted composition operators, Topological structure, Carleson measure.

1. Introduction

Let $H(\mathbb{D})$ be the space of all holomorphic functions on the unit disc $\mathbb{D}$. Given $p \in (0, \infty)$ and $\alpha \in (-1, \infty)$, the Bergman space A^p_α is defined as follows

$$A^p_\alpha := \left\{ f \in H(\mathbb{D}) : \|f\|_{p,\alpha} := \left(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\}$$

with

$$dA_\alpha(z) := \frac{1+\alpha}{\pi} \left(1 - |z|^2\right)^\alpha dA(z),$$

where $dA(z)$ is the Lebesgue measure on $\mathbb{D}$. It is well known that A^p_α with $1 \le p < \infty$ is a Banach space, while for $0 < p < 1$, A^p_α is a complete metric space with the distance $d(f, g) := \|f - g\|^p_{p,\alpha}$.

Let $\mathcal{S}(\mathbb{D})$ be the set of all holomorphic self-maps of $\mathbb{D}$. For two functions $\varphi \in \mathcal{S}(\mathbb{D})$ and $\psi \in H(\mathbb{D})$, a *weighted composition operator* $W_{\psi,\varphi}$ is defined by $W_{\psi,\varphi} f := \psi \cdot (f \circ \varphi),\ f \in H(\mathbb{D})$. In particular, when ψ is identically 1, $W_{\psi,\varphi}$ reduces to a *composition operator* C_φ. According to Littlewood's Subordinate Theorem, each composition operator C_φ is bounded on Bergman spaces A^p_α, while compactness of C_φ on A^p_α was firstly characterized by MacCluer and Shapiro [16] in terms of angular derivative of φ. In details, the operator C_φ is compact on A^p_α if and only if φ has no finite angular derivative at any point ζ of $\partial\mathbb{D}$, which is equivalent to

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0. \tag{1.1}$$

Boundedness and compactness of weighted composition operators $W_{\psi,\varphi}$ on Bergman spaces A^p_α were investigated by Čučković and Zhao [7] in terms of Berezin type transforms, which are rather difficult to use.

When the basic questions on topological properties of (weighted) composition operators were completely solved, many researchers have paid more attention to the study of the topological structure of the space of such operators endowed with the operator norm topology. This problem was initiated by Berkson [2] with his isolation result on composition operators on the Hardy space H^2, and then developed by MacCluer [15], Shapiro and Sundberg [19]. Thereafter, the topological structure problem has been intensively investigated on various function spaces during the past few decades (see, for instance, [9,12] on Hardy spaces, [8,17] on Bergman spaces, [13,14] on the space H^∞ of all bounded holomorphic functions on $\mathbb{D}$, [11] on Bloch spaces, [1] on weighted Banach spaces with sup-norm, and [20] on Fock spaces).

In this paper we are interested in the topological structure problem on Bergman spaces A^p_α. Recall that the authors in [8,15,17] studied this problem only for composition operators on Hilbert Bergman spaces A^2_α and obtained some partial results. Firstly, MacCluer [15] gave a sufficient condition for isolated composition operators and a necessary condition for a composition operator to be in the path component of another one. Later, Moorhouse [17] established a sufficient condition under which two composition operators belong to the same path component. Recently, Dai [8] stated a criterion for two composition operators to be linearly connected. It is worth mentioning that in the Bergman space setting, till now there is no complete description of (path) components in the space of composition operators; moreover, the space of weighted composition operators has not yet been studied.

The aim of this paper is, firstly, to continue studying the topological structure problem for composition operators on Bergman spaces A^p_α with $p \in (0,\infty)$; secondly, to initiate this problem for weighted composition operators on these spaces A^p_α. Note that the technique of adjoint operators on Hilbert Bergman spaces A^2_α, played an essential role in [8,15,17], does not work for general spaces A^p_α. So we develop a new approach based on Carleson measure.

The paper is organized as follows. In Sect.2 we recall some preliminary results on pseudo-hyperbolic distance and Carleson measure for Bergman spaces A^p_α.

Section 3 is devoted to the topological structure of the space $\mathcal{C}(A^p_\alpha)$ of composition operators on A^p_α endowed with the operator norm topology. In Theorem 3.3, we prove that two operators C_φ and C_ϕ are *linearly connected* in the space $\mathcal{C}(A^p_\alpha)$, i.e. the path C_{φ_t} with $\varphi_t(z) := (1-t)\varphi(z)+t\phi(z),\ t \in [0,1]$, is continuous in $\mathcal{C}(A^p_\alpha)$, if and only if

$$\limsup_{\rho(\varphi(z),\phi(z))\to 1} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\phi(z)|^2} \right) = 0,$$

where $\rho(\varphi(z), \phi(z))$ is the pseudo-hyperbolic distance between $\varphi(z)$ and $\phi(z)$. This result implies that the set $[C_\varphi]$ of all composition operators that differ from the given one C_φ by a compact operator is path connected in $\mathcal{C}(A^p_\alpha)$ (Theorem 3.6). Moreover, the set $\mathcal{C}_0(A^p_\alpha)$ of all compact composition operators on A^p_α forms a path component in $\mathcal{C}(A^p_\alpha)$ (Corollary 3.8). However, as in the setting of other function spaces, such as the Hardy space H^2, the space H^∞, and weighted Banach space with sup-norm, the set of such a type, in general, is not always a component of $\mathcal{C}(A^p_\alpha)$ (Example 3.10). Finally, we also give sufficient conditions for isolated and non-isolated points in $\mathcal{C}(A^p_\alpha)$ (Proposition 3.11).

The space $\mathcal{C}_w(A^p_\alpha)$ of all nonzero weighted composition operators on A^p_α under the operator norm topology is studied in Sect. 4. We show that the set $\mathcal{C}_{w,0}(A^p_\alpha)$ of all nonzero compact weighted composition operators on A^p_α is path connected in $\mathcal{C}_w(A^p_\alpha)$; nevertheless, it is not a component in this space (Theorem 4.2). Moreover, we provide two path connected sets of the same type in the space $\mathcal{C}_w(A^p_\alpha)$, one of which is a path component, while another one is not (Examples 4.5 and 4.6).

Notation. Throughout this paper we always assume that $p \in (0,\infty)$ and $\alpha \in (-1,\infty)$ unless otherwise is stated. We denote constants by $c, c_0, c_1, \ldots$ to distinguish from composition operators C_{z_0} induced by $\varphi(z) \equiv z_0$. We also use the notation $A \lesssim B$ (and $A \gtrsim B$) for nonnegative quantities A and B to mean that there is a constant $c > 0$, depending only on indexes p, α, β, γ, such that $A \le cB$ (and $A \ge cB$, respectively). Finally, the notation $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2. Preliminaries

In this section we recall some basic notation, definitions and facts which will be used in the sequel.

2.1. Test Functions

For each $\sigma > 0$ and $w \in \mathbb{D}$ fixed, we define

$$k_w(z) := \frac{\left(1-|w|^2\right)^{\frac{\sigma}{p}}}{(1-\overline{w}z)^{\frac{\sigma+\alpha+2}{p}}}, \quad z \in \mathbb{D}.$$

These functions k_w play an important role in the study of Bergman spaces A^p_α and operators defined on them. From [21, Theorem 1.12] it follows that there is a constant c_0, depending only on p, α, σ, such that

$$c_0^{-1} \leq \|k_w\|_{p,\alpha} \leq c_0 \quad \text{for all } w \in \mathbb{D}. \tag{2.1}$$

2.2. Pseudo-Hyperbolic Distance

The *pseudo-hyperbolic distance* between z and ζ in $\mathbb{D}$ is given by

$$\rho(z,\zeta) := \left|\frac{z-\zeta}{1-\bar{z}\zeta}\right|.$$

The pseudo-hyperbolic disc with center $z \in \mathbb{D}$ and radius $r \in (0,1)$ is defined by $\Delta(z,r) := \{\zeta \in \mathbb{D} : \rho(z,\zeta) < r\}$. For simplicity, we write $\Delta(z)$ instead of $\Delta(z, \frac{1}{2})$.

By [21, Lemma 2.24], there exists a constant $c_1 = c_1(p,\alpha)$ such that

$$|f(z)|^p \leq \frac{c_1^p}{(1-|z|^2)^{\alpha+2}} \int_{\Delta(z)} |f(\zeta)|^p dA_\alpha(\zeta), \tag{2.2}$$

for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. This implies that

$$|f(z)| \leq \frac{c_1}{(1-|z|^2)^{\frac{\alpha+2}{p}}} \|f\|_{p,\alpha} \tag{2.3}$$

and, changing the constant c_1 if necessary,

$$|f'(z)| \leq \frac{c_1}{(1-|z|^2)^{\frac{\alpha+2}{p}+1}} \|f\|_{p,\alpha}, \tag{2.4}$$

for every $f \in A^p_\alpha$ and $z \in \mathbb{D}$.

Next, by [21, Lemma 2.20], for every $r \in (0,1)$, there is a constant $c_2 = c_2(r) > 0$ such that

$$c_2^{-1} \leq \frac{1-|z|^2}{1-|\zeta|^2} \leq c_2 \ \text{ for all } \zeta \in \Delta(z,r) \text{ and } z \in \mathbb{D}. \tag{2.5}$$

The next auxiliary result follows from [5, Lemma 3.2].

Lemma 2.1. *For every $r \in (0,1)$, there is a constant $c_3 = c_3(p,\alpha,r) > 0$ such that*

$$|f(z) - f(\zeta)|^p \leq c_3 \frac{\rho(z,\zeta)^p}{(1-|z|^2)^{\alpha+2}} \int_{\Delta\left(z,\frac{r+1}{2}\right)} |f'(\omega)|^p \left(1-|\omega|^2\right)^p dA_\alpha(\omega),$$

for all $f \in H(\mathbb{D})$ and $z, \zeta \in \mathbb{D}$ with $\zeta \in \Delta(z,r)$.

The following lemma is quite standard. It is originally noticed in [10] for the unit ball and given in [8, Lemma 3.2].

Lemma 2.2. *The following inequality holds*

$$\rho(z_t, z_s) \leq \frac{|t-s|}{1-(1-|t-s|)\,\rho(z,\zeta)} \rho(z,\zeta),$$

for every $z, \zeta \in \mathbb{D}$ and $t, s \in [0,1]$, where $z_t := (1-t)z + t\zeta$. In particular, $\rho(z_t, z_s) \leq \rho(z,\zeta)$ for the same z, ζ, z_t, and z_s.

2.3. Carleson Measure

A positive Borel measure μ on $\mathbb{D}$ is called an *α-Carleson measure*, if the embedding operator $I_\mu : A^p_\alpha \to L^p(\mathbb{D}, d\mu)$ is bounded, i.e. there exists a constant $c > 0$ such that

$$\left(\int_{\mathbb{D}} |f(z)|^p d\mu(z)\right)^{\frac{1}{p}} \leq c\|f\|_{p,\alpha} \quad \text{for all } f \in A^p_\alpha.$$

Moreover, if the operator I_μ is compact, then μ on $\mathbb{D}$ is called a *compact α-Carleson measure*. In this case we put $\|\mu\|_\alpha := \|I_\mu\|^p_{A^p_\alpha \to L^p(\mathbb{D},d\mu)}$. Here and below we omit the dependence on p of norms of measures and operators, since $p > 0$ is always an arbitrary fixed number.

By [21, Theorems 2.25 and 2.26], a positive Borel measure μ is an α-Carleson (respectively, a compact α-Carleson) one if and only if

$$\sup_{z\in\mathbb{D}} \frac{\mu(\Delta(z,r))}{(1-|z|^2)^{\alpha+2}} < \infty \left(\text{respectively, } \lim_{|z|\to 1^-} \frac{\mu(\Delta(z,r))}{(1-|z|^2)^{\alpha+2}} = 0\right),$$

for some number $r \in (0, 1)$. Note that these conditions are independent of p and r. Moreover, for each $r \in (0, 1)$, there is a constant $c_4 = c_4(\alpha, r)$ such that

$$c_4^{-1} \sup_{z\in\mathbb{D}} \frac{\mu(\Delta(z,r))}{(1-|z|^2)^{\alpha+2}} \leq \|\mu\|_\alpha \leq c_4 \sup_{z\in\mathbb{D}} \frac{\mu(\Delta(z,r))}{(1-|z|^2)^{\alpha+2}}. \tag{2.6}$$

On the other hand, in [16, Section 4], the α-Carleson measure was also characterized in terms of semidiscs $S(\zeta, \delta)$, where $S(\zeta, \delta) := \{z \in \mathbb{D} : |z-\zeta| < \delta\}$ with $\delta \in (0, 2]$ and $\zeta \in \partial\mathbb{D}$. By [16, Theorem 4.3], a positive Borel measure μ is an α-Carleson (respectively, a compact α-Carleson) one if and only if

$$\sup_{\delta\in(0,2],\zeta\in\partial\mathbb{D}} \frac{\mu(S(\zeta,\delta))}{\delta^{\alpha+2}} < \infty \left(\text{respectively, } \lim_{\delta\to 0} \sup_{\zeta\in\partial\mathbb{D}} \frac{\mu(S(\zeta,\delta))}{\delta^{\alpha+2}} = 0\right).$$

In addition,

$$\|\mu\|_\alpha \simeq \sup_{\delta\in(0,2],\zeta\in\partial\mathbb{D}} \frac{\mu(S(\zeta,\delta))}{\delta^{\alpha+2}}.$$

For a function $\varphi \in \mathcal{S}(\mathbb{D})$ and a Borel function $v : \mathbb{D} \to [0, \infty)$, we define the pull-back measure $(vA_\alpha) \circ \varphi^{-1}$ by

$$(vA_\alpha) \circ \varphi^{-1}(E) := \int_{\varphi^{-1}(E)} v(z) dA_\alpha(z),$$

for each Borel set $E \subset \mathbb{D}$. Then, for each $\varphi \in \mathcal{S}(\mathbb{D})$ and $f \in A^p_\alpha$,

$$\|C_\varphi f\|_{p,\alpha} = \left(\int_{\mathbb{D}} |f(\varphi(z))|^p dA_\alpha(z)\right)^{\frac{1}{p}} = \left(\int_{\mathbb{D}} |f(z)|^p d(A_\alpha \circ \varphi^{-1})(z)\right)^{\frac{1}{p}}.$$

From this and the boundedness of C_φ on A^p_α, it follows that $A_\alpha \circ \varphi^{-1}$ is always an α-Carleson measure and

$$\|A_\alpha \circ \varphi^{-1}\|_\alpha = \|C_\varphi\|^p_\alpha \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\alpha+2}. \tag{2.7}$$

Moreover, C_φ is compact on A^p_α if and only if $A_\alpha \circ \varphi^{-1}$ is a compact α-Carleson measure.

For each $\varphi \in \mathcal{S}(\mathbb{D})$, $\psi \in H(\mathbb{D})$, and $f \in A^p_\alpha$,

$$\begin{aligned}\|W_{\psi,\varphi} f\|_{p,\alpha} &= \left(\int_{\mathbb{D}} |\psi(z)|^p |f(\varphi(z))|^p dA_\alpha(z)\right)^{\frac{1}{p}}\\ &= \left(\int_{\mathbb{D}} |f(z)|^p d((|\psi|^p A_\alpha)\circ\varphi^{-1})(z)\right)^{\frac{1}{p}}.\end{aligned}$$

This implies that $W_{\psi,\varphi}$ is bounded (respectively, compact) on A^p_α if and only if $(|\psi|^p A_\alpha)\circ\varphi^{-1}$ is an α-Carleson measure (respectively, a compact α-Carleson measure). Moreover, we have

$$\|\,(|\psi|^p A_\alpha)\circ\varphi^{-1}\|_\alpha = \|W_{\psi,\varphi}\|^p_\alpha. \tag{2.8}$$

3. Path Components of the Space of Composition Operators

In this section we study the space $\mathcal{C}(A^p_\alpha)$ of all composition operators on A^p_α under the operator norm topology.

First, we establish a necessary and sufficient condition under which two composition operators C_φ and C_ϕ are *linearly connected* in $\mathcal{C}(A^p_\alpha)$, i.e., according to [8], the path C_{φ_t} with $\varphi_t(z) := (1-t)\varphi(z) + t\phi(z), t \in [0,1]$, is continuous in $\mathcal{C}(A^p_\alpha)$. Since $\varphi_t(z)$ lies on a straight-line path between $\varphi(z)$ and $\phi(z)$,

$$\frac{1}{1-|\varphi_t(z)|^s} \le \frac{1}{1-|\varphi(z)|^s} + \frac{1}{1-|\phi(z)|^s}, \tag{3.1}$$

for every $z \in \mathbb{D}$, $t \in [0,1]$, and $s > 0$.

We need the following auxiliary lemmas.

Lemma 3.1. *Let $p \in (0,\infty)$ and $\alpha, \gamma \in (-1,\infty)$ with $\beta = \alpha - \gamma \in (0,1]$. For every two functions φ, ϕ from $\mathcal{S}(\mathbb{D})$ and every Borel function $v : \mathbb{D} \to [0,1]$, the following inequality holds*

$$\|\,(vA_\alpha)\circ\varphi_t^{-1}\|_\alpha \lesssim M^\beta_{v,\varphi,\phi}\left(\frac{1}{1-|\varphi(0)|} + \frac{1}{1-|\phi(0)|}\right)^{\gamma+2},$$

for all $t \in [0,1]$, where, as above, $\varphi_t(z) := (1-t)\varphi(z) + t\phi(z)$, and

$$M_{v,\varphi,\phi} := \sup_{z\in\mathbb{D}} v(z)\left(\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\phi(z)|^2}\right).$$

Proof. Using (2.5) for $r = \frac{1}{2}$ and (3.1), we obtain

$$
\begin{aligned}
(vA_\alpha) \circ \varphi_t^{-1}(\Delta(z)) &= \int_{\varphi_t^{-1}(\Delta(z))} v(\omega) dA_\alpha(\omega) \\
&= \frac{\alpha+1}{\gamma+1} \int_{\varphi_t^{-1}(\Delta(z))} v(\omega)^{1-\beta} \left(v(\omega)(1-|\omega|^2)\right)^\beta dA_\gamma(\omega) \\
&\lesssim \sup_{\omega \in \varphi_t^{-1}(\Delta(z))} \left(v(\omega) \frac{1-|\omega|^2}{1-|\varphi_t(\omega)|^2}\right)^\beta \int_{\varphi_t^{-1}(\Delta(z))} (1-|\varphi_t(\omega)|^2)^\beta dA_\gamma(\omega) \\
&\lesssim \sup_{\omega \in \mathbb{D}} \left(v(\omega) \left(\frac{1-|\omega|^2}{1-|\varphi(\omega)|^2} + \frac{1-|\omega|^2}{1-|\phi(\omega)|^2}\right)\right)^\beta \left(1-|z|^2\right)^\beta \int_{\varphi_t^{-1}(\Delta(z))} dA_\gamma(\omega) \\
&= M_{v,\varphi,\phi}^\beta \left(1-|z|^2\right)^\beta A_\gamma \circ \varphi_t^{-1}(\Delta(z)),
\end{aligned}
$$

for every $t \in [0,1]$ and $z \in \mathbb{D}$. Then, using (2.6) for $r = \frac{1}{2}$, (2.7) and (3.1), we get

$$
\begin{aligned}
\|(vA_\alpha) \circ \varphi_t^{-1}\|_\alpha &\simeq \sup_{z \in \mathbb{D}} \frac{(vA_\alpha) \circ \varphi_t^{-1}(\Delta(z))}{(1-|z|^2)^{\alpha+2}} \\
&\lesssim M_{v,\varphi,\phi}^\beta \sup_{z \in \mathbb{D}} \frac{\left(1-|z|^2\right)^\beta A_\gamma \circ \varphi_t^{-1}(\Delta(z))}{(1-|z|^2)^{\alpha+2}} \\
&= M_{v,\varphi,\phi}^\beta \sup_{z \in \mathbb{D}} \frac{A_\gamma \circ \varphi_t^{-1}(\Delta(z))}{(1-|z|^2)^{\gamma+2}} \simeq M_{v,\varphi,\phi}^\beta \|A_\gamma \circ \varphi_t^{-1}\|_\gamma \\
&\leq M_{v,\varphi,\phi}^\beta \left(\frac{1+|\varphi_t(0)|}{1-|\varphi_t(0)|}\right)^{\gamma+2} \lesssim M_{v,\varphi,\phi}^\beta \left(\frac{1}{1-|\varphi_t(0)|}\right)^{\gamma+2} \\
&\lesssim M_{v,\varphi,\phi}^\beta \left(\frac{1}{1-|\varphi(0)|} + \frac{1}{1-|\phi(0)|}\right)^{\gamma+2},
\end{aligned}
$$

for every $t \in [0,1]$. □

Lemma 3.2. *For every functions φ and ϕ from $\mathcal{S}(\mathbb{D})$,*

$$
\|C_\varphi - C_\phi\|_\alpha \geq \frac{1}{2c_0c_1} \limsup_{\rho(\varphi(z),\phi(z))\to 1} \left[\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{\alpha+2}{p}} + \left(\frac{1-|z|^2}{1-|\phi(z)|^2}\right)^{\frac{\alpha+2}{p}}\right], \tag{3.2}
$$

where c_0, c_1 are the constants defined in (2.1) and (2.2), respectively, and, by definition, the limit on the right-hand side of (3.2) is zero if $\rho(\varphi(z),\phi(z)) \leq r_0$ for some $r_0 \in (0,1)$ and all $z \in \mathbb{D}$.

Proof. Obviously, it is enough to consider the case when

$$
\limsup_{\rho(\varphi(z),\phi(z))\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2} \geq \limsup_{\rho(\varphi(z),\phi(z))\to 1} \frac{1-|z|^2}{1-|\phi(z)|^2}
$$

and

$$
\limsup_{\rho(\varphi(z),\phi(z))\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2} > 0.
$$

Then taking a sequence $(z_n)_n \subset \mathbb{D}$ so that $\rho(\varphi(z_n), \phi(z_n)) \to 1$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{1-|z_n|^2}{1-|\varphi(z_n)|^2} = \limsup_{\rho(\varphi(z),\phi(z))\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2},$$

we easily verify that

$$\limsup_{n\to\infty} \frac{1-|\varphi(z_n)|^2}{1-|\phi(z_n)|^2} \leq 1.$$

Applying this and the well-known identity

$$1-\rho(\varphi(z_n), \phi(z_n))^2 = \frac{\left(1-|\varphi(z_n)|^2\right)\left(1-|\phi(z_n)|^2\right)}{\left|1-\overline{\varphi(z_n)}\phi(z_n)\right|^2},$$

we get

$$\limsup_{n\to\infty} \frac{\left(1-|\varphi(z_n)|^2\right)^2}{\left|1-\overline{\varphi(z_n)}\phi(z_n)\right|^2} \leq \lim_{n\to\infty}\left(1-\rho(\varphi(z_n), \phi(z_n))^2\right) = 0,$$

and hence,

$$\limsup_{n\to\infty} \frac{1-|\varphi(z_n)|^2}{\left|1-\overline{\varphi(z_n)}\phi(z_n)\right|} = 0. \tag{3.3}$$

Next, for every $n \in \mathbb{N}$, using (2.1) and (2.2), we have

$$\begin{aligned}
\|C_\varphi - C_\phi\|_\alpha \geq\ & \frac{1}{c_0}\|C_\varphi k_{\varphi(z_n)} - C_\phi k_{\varphi(z_n)}\|_{p,\alpha} \\
&= \frac{1}{c_0}\left(\int_{\mathbb{D}} |k_{\varphi(z_n)}(\varphi(z)) - k_{\varphi(z_n)}(\phi(z))|^p dA_\alpha(z)\right)^{\frac{1}{p}} \\
&\geq \frac{1}{c_0}\left(\int_{\Delta(z_n)} |k_{\varphi(z_n)}(\varphi(z)) - k_{\varphi(z_n)}(\phi(z))|^p dA_\alpha(z)\right)^{\frac{1}{p}} \\
&\geq \frac{1}{c_0 c_1}(1-|z_n|^2)^{\frac{\alpha+2}{p}} |k_{\varphi(z_n)}(\varphi(z_n)) - k_{\varphi(z_n)}(\phi(z_n))| \\
&= \frac{1}{c_0 c_1}(1-|z_n|^2)^{\frac{\alpha+2}{p}} \left| \frac{1}{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}} - \frac{\left(1-|\varphi(z_n)|^2\right)^{\frac{\sigma}{p}}}{\left(1-\overline{\varphi(z_n)}\phi(z_n)\right)^{\frac{\sigma+\alpha+2}{p}}} \right| \\
&\geq \frac{1}{c_0 c_1}\left(\frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}\right)^{\frac{\alpha+2}{p}} \left| 1 - \left(\frac{1-|\varphi(z_n)|^2}{\left|1-\overline{\varphi(z_n)}\phi(z_n)\right|}\right)^{\frac{\sigma+\alpha+2}{p}} \right|.
\end{aligned}$$

Letting $n \to \infty$ in the last inequality and using (3.3), we get

$$\|C_\varphi - C_\phi\|_\alpha \geq \frac{1}{c_0 c_1} \lim_{n\to\infty}\left(\frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}\right)^{\frac{\alpha+2}{p}},$$

which implies (3.2). □

Theorem 3.3. *For every functions φ and ϕ from $\mathcal{S}(\mathbb{D})$, the operators C_φ and C_ϕ are linearly connected in $\mathcal{C}(A^p_\alpha)$ if and only if*

$$\limsup_{\rho(\varphi(z),\phi(z))\to 1}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}+\frac{1-|z|^2}{1-|\phi(z)|^2}\right)=0. \tag{3.4}$$

Proof. As above, let $\varphi_t(z) := (1-t)\varphi(z)+t\phi(z), t\in[0,1]$.

(a) Necessity. Following the proof of [8, Theorem 3.2], assume (3.4) does not hold. Then, similarly to the proof of the previous Lemma 3.2, we can find a sequence $(z_n)_n$ in $\mathbb{D}$ so that (3.3) holds and

$$\limsup_{n\to\infty}\frac{1-|z_n|^2}{1-|\phi(z_n)|^2}\le\lim_{n\to\infty}\frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}=\varepsilon_0\in(0,\infty).$$

Then for every $t\in(0,1]$, by (3.3), we get

$$\begin{aligned}\frac{|1-\overline{\varphi_t(z_n)}\varphi(z_n)|}{1-|\varphi(z_n)|^2}&=\frac{|(1-t)(1-|\varphi(z_n)|^2)+t(1-\overline{\phi(z_n)}\varphi(z_n))|}{1-|\varphi(z_n)|^2}\\&\ge t\frac{|1-\overline{\varphi(z_n)}\phi(z_n)|}{1-|\varphi(z_n)|^2}-(1-t)\to\infty \text{ as } n\to\infty.\end{aligned}$$

Therefore, for every $t\in(0,1]$,

$$\begin{aligned}1-\rho(\varphi(z_n),\varphi_t(z_n))^2&=\frac{(1-|\varphi(z_n)|^2)(1-|\varphi_t(z_n)|^2)}{\left|1-\overline{\varphi(z_n)}\varphi_t(z_n)\right|^2}\\&\le 2\frac{1-|\varphi(z_n)|^2}{\left|1-\overline{\varphi(z_n)}\varphi_t(z_n)\right|}\to 0 \text{ as } n\to\infty.\end{aligned}$$

Consequently, by Lemma 3.2,

$$\begin{aligned}\|C_\varphi-C_{\varphi_t}\|_\alpha&\ge\frac{1}{2c_0c_1}\limsup_{n\to\infty}\left[\left(\frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}\right)^{\frac{\alpha+2}{p}}+\left(\frac{1-|z_n|^2}{1-|\varphi_t(z_n)|^2}\right)^{\frac{\alpha+2}{p}}\right]\\&\ge\frac{\varepsilon_0^{\frac{\alpha+2}{p}}}{2c_0c_1}\text{ for all } t\in(0,1].\end{aligned}$$

From this it follows that the path $C_{\varphi_t}, t\in[0,1]$, is not continuous at $t=0$, which is a contradiction.

(b) Sufficiency. Suppose that (3.4) holds. We prove that the map $t\mapsto C_{\varphi_t}$ is continuous in $\mathcal{C}(A^p_\alpha)$, i.e., $\lim_{s\to t}\|C_{\varphi_s}-C_{\varphi_t}\|_\alpha=0$ for each $t\in[0,1]$ fixed.

We take an arbitrary number $r\in(0,1)$ and put

$$E_r := \{z\in\mathbb{D}:\rho(\varphi(z),\phi(z))\le r\}\text{ and } E_r^c:=\mathbb{D}\setminus E_r.$$

For every $s\in[0,1]$ and $f\in A^p_\alpha$, we write

$$\begin{aligned}\|C_{\varphi_s}f-C_{\varphi_t}f\|^p_{p,\alpha}&=\int_{\mathbb{D}}|f(\varphi_s(z))-f(\varphi_t(z))|^p dA_\alpha(z)\\&=\left(\int_{E_r}+\int_{E_r^c}\right)|f(\varphi_s(z))-f(\varphi_t(z))|^p dA_\alpha(z),\end{aligned}$$

and estimate the integrals in the right-hand side separately.

First, we estimate the integral

$$\mathcal{I}(f,r,s) := \int_{E_r} |f(\varphi_s(z)) - f(\varphi_t(z))|^p dA_\alpha(z).$$

For each $z \in E_r$, by Lemma 2.2,

$$\rho(\varphi_s(z), \varphi_t(z)) \le \rho(\varphi(z), \phi(z)) \le r \text{ for every } s \in [0,1].$$

By this and Lemma 2.1, for some constant $c_3 = c_3(p,\alpha,r)$ and each $z \in E_r$, we have

$$|f(\varphi_s(z)) - f(\varphi_t(z))|^p \le c_3 \frac{\rho(\varphi_s(z), \varphi_t(z))^p}{(1-|\varphi_t(z)|^2)^{\alpha+2}} \int_\Delta |f'(\omega)|^p \left(1-|\omega|^2\right)^p dA_\alpha(\omega),$$

where, for simplicity, we write Δ instead of $\Delta(\varphi_t(z), \frac{r+1}{2})$. By Lemma 2.2, for each $z \in E_r$, we obtain

$$\begin{aligned}
|f(\varphi_s(z)) - f(\varphi_t(z))|^p &\le c_3|s-t|^p \frac{\rho(\varphi(z),\phi(z))^p}{(1-(1-|s-t|)\rho(\varphi(z),\phi(z)))^p} \\
&\quad \times \frac{1}{(1-|\varphi_t(z)|^2)^{\alpha+2}} \int_\Delta |f'(\omega)|^p(1-|\omega|^2)^p dA_\alpha(\omega) \\
&\le c_3|s-t|^p \frac{r^p}{(1-(1-|s-t|)r)^p} \\
&\quad \times \frac{1}{(1-|\varphi_t(z)|^2)^{\alpha+2}} \int_\Delta |f'(\omega)|^p \left(1-|\omega|^2\right)^p dA_\alpha(\omega).
\end{aligned}$$

Using this, Fubini's theorem, (2.5) and (2.6), for every $s \in [0,1]$ and $f \in A^p_\alpha$, we get

$$\begin{aligned}
\mathcal{I}(f,r,s) &= \int_{E_r} |f(\varphi_s(z)) - f(\varphi_t(z))|^p dA_\alpha(z) \\
&\le c_3|s-t|^p \frac{r^p}{(1-(1-|s-t|)r)^p} \\
&\quad \times \int_{\mathbb{D}} \frac{1}{(1-|\varphi_t(z)|^2)^{\alpha+2}} \left(\int_\Delta |f'(\omega)|^p(1-|\omega|^2)^p dA_\alpha(\omega)\right) dA_\alpha(z) \\
&= c_3|s-t|^p \frac{r^p}{(1-(1-|s-t|)r)^p} \\
&\quad \times \int_{\mathbb{D}} |f'(\omega)|^p(1-|\omega|^2)^p \left(\int_{\varphi_t^{-1}(\Delta(\omega,\frac{r+1}{2}))} \frac{1}{(1-|\varphi_t(z)|^2)^{\alpha+2}} dA_\alpha(z)\right) dA_\alpha(\omega) \\
&\le c_2^{\alpha+2} c_3|s-t|^p \frac{r^p}{(1-(1-|s-t|)r)^p} \\
&\quad \times \int_{\mathbb{D}} |f'(\omega)|^p(1-|\omega|^2)^p \frac{A_\alpha \circ \varphi_t^{-1}(\Delta(\omega,\frac{r+1}{2}))}{(1-|\omega|^2)^{\alpha+2}} dA_\alpha(\omega) \\
&\le c_2^{\alpha+2} c_3 c_4 |s-t|^p \frac{r^p}{(1-(1-|s-t|)\,r)^p} \|A_\alpha \circ \varphi_t^{-1}\|_\alpha \\
&\quad \times \int_{\mathbb{D}} |f'(\omega)|^p(1-|\omega|^2)^p dA_\alpha(\omega).
\end{aligned}$$

On the other hand, by (2.7) and (3.1),

$$\|A_\alpha \circ \varphi_t^{-1}\|_\alpha \leq \left(\frac{1+|\varphi_t(0)|}{1-|\varphi_t(0)|}\right)^{\alpha+2} \leq \left(\frac{2}{1-|\varphi(0)|}+\frac{2}{1-|\phi(0)|}\right)^{\alpha+2}.$$

Moreover, from [21, Theorem 2.16] and (2.3), it follows that

$$\begin{aligned}\int_{\mathbb{D}} |f'(\omega)|^p \left(1-|\omega|^2\right)^p dA_\alpha(\omega) &\simeq \int_{\mathbb{D}} |f(\omega)-f(0)|^p dA_\alpha(\omega) \\ &\lesssim \left(\|f\|_{p,\alpha}^p + |f(0)|^p\right) \lesssim \|f\|_{p,\alpha}^p.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{I}(f,r,s) \lesssim c_2^{\alpha+2} c_3 c_4 |s-t|^p &\frac{r^p}{(1-(1-|s-t|)r)^p} \\ &\times \left(\frac{2}{1-|\varphi(0)|}+\frac{2}{1-|\phi(0)|}\right)^{\alpha+2} \|f\|_{p,\alpha}^p,\end{aligned}$$

for every $s \in [0,1]$ and $f \in A_\alpha^p$.

Next, we estimate the integral

$$\mathcal{J}(f,r,s) := \int_{E_r^c} |f(\varphi_s(z)) - f(\varphi_t(z))|^p dA_\alpha(z).$$

We have

$$\begin{aligned}\mathcal{J}(f,r,s) &= \int_{E_r^c} |f(\varphi_s(z)) - f(\varphi_t(z))|^p dA_\alpha(z) \\ &\lesssim \int_{E_r^c} |f(\varphi_s(z))|^p dA_\alpha(z) + \int_{E_r^c} |f(\varphi_t(z))|^p dA_\alpha(z) \\ &= \int_{\mathbb{D}} |f(\varphi_s(z))|^p \chi_{E_r^c}(z) dA_\alpha(z) + \int_{\mathbb{D}} |f(\varphi_t(z))|^p \chi_{E_r^c}(z) dA_\alpha(z) \\ &= \int_{\mathbb{D}} |f(z)|^p d(\chi_{E_r^c} A_\alpha)\circ\varphi_s^{-1}(z) + \int_{\mathbb{D}} |f(z)|^p d\left(\chi_{E_r^c} A_\alpha\right)\circ\varphi_t^{-1}(z) \\ &\leq \|(\chi_{E_r^c} A_\alpha)\circ\varphi_s^{-1}\|_\alpha \|f\|_{p,\alpha}^p + \|\left(\chi_{E_r^c} A_\alpha\right)\circ\varphi_t^{-1}\|_\alpha \|f\|_{p,\alpha}^p,\end{aligned}$$

for every $s \in [0,1]$ and $f \in A_\alpha^p$, where χ_E denotes the characteristic function of a Borel subset $E \subset \mathbb{D}$. Applying Lemma 3.1 to two functions φ, ϕ from $\mathcal{S}(\mathbb{D})$, and the characteristic function $\chi_{E_r^c}$, we obtain

$$\|\left(\chi_{E_r^c} A_\alpha\right)\circ\varphi_s^{-1}\|_\alpha \lesssim M_{\chi_{E_r^c},\varphi,\phi}^\beta \left(\frac{1}{1-|\varphi(0)|}+\frac{1}{1-|\psi(0)|}\right)^{\gamma+2}$$

for all $s \in [0,1]$. Here, β and γ are the same as in Lemma 3.1.

Combining the above estimates for $\mathcal{I}(f,r,s)$ and $\mathcal{J}(f,r,s)$, yields

$$\begin{aligned}\|C_{\varphi_s} - C_{\varphi_t}\|_\alpha^p \lesssim c_2^{\alpha+2} c_3 c_4 |s-t|^p &\frac{r^p}{(1-(1-|s-t|)\,r)^p} \\ &\times \left(\frac{2}{1-|\varphi(0)|}+\frac{2}{1-|\phi(0)|}\right)^{\alpha+2} \\ &+ 2M_{\chi_{E_r^c},\varphi,\phi}^\beta \left(\frac{1}{1-|\varphi(0)|}+\frac{1}{1-|\phi(0)|}\right)^{\gamma+2},\end{aligned}$$

for every $s \in [0,1]$. Then, letting $s \to t$, we get

$$\lim_{s\to t} \|C_{\varphi_s} - C_{\varphi_t}\|_\alpha^p \lesssim M^\beta_{\chi_{E_r^c},\varphi,\phi} \left(\frac{1}{1-|\varphi(0)|} + \frac{1}{1-|\phi(0)|}\right)^{\gamma+2},$$

for every $r \in (0,1)$. Moreover, by (3.4),

$$M_{\chi_{E_r^c},\varphi,\phi} = \sup_{\rho(\varphi(z),\phi(z))\geq r} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\phi(z)|^2}\right) \to 0 \text{ as } r \to 1^-,$$

which implies that $\lim_{s\to t} \|C_{\varphi_s} - C_{\varphi_t}\|_\alpha = 0$ and completes the proof. □

From Theorem 3.3 we immediately get the following result.

Corollary 3.4. *Let two functions φ and ϕ from $\mathcal{S}(\mathbb{D})$ satisfy $\rho(\varphi(z),\phi(z)) \leq r_0$ for some number $r_0 \in (0,1)$ and every $z \in \mathbb{D}$. Then C_φ and C_ϕ are linearly connected in $\mathcal{C}(A^p_\alpha)$.*

Remark 3.5. Theorem 3.3 and Corollary 3.4 extend the results of [8, Theorem 3.3] and, respectively, [17, Theorem 8] on Hilbert Bergman spaces A^2_α to all Bergman spaces A^p_α with $p \in (0,\infty)$.

To describe path components of the space $\mathcal{C}(A^p_\alpha)$, we introduce the following notation. We say that two composition operators C_φ and C_ϕ are *equivalent* in $\mathcal{C}(A^p_\alpha)$, if their difference $C_\varphi - C_\phi$ is a compact operator on A^p_α. Obviously, this is an equivalence relation in $\mathcal{C}(A^p_\alpha)$. Let denote by $[C_\varphi]$ the equivalence class of all composition operators that are equivalent to the given one C_φ. Then the set $\mathcal{C}_0(A^p_\alpha)$ of all compact composition operators on A^p_α is the equivalence class $[C_0]$ of all operators from $\mathcal{C}(A^p_\alpha)$ that are equivalent to the operator $C_0 : f \mapsto f(0)$.

Recall, by [5, Theorem 1.1], [4, Proposition 4.1], and [17, Theorem 4], that for every functions φ and ϕ from $\mathcal{S}(\mathbb{D})$, the difference $C_\varphi - C_\phi$ is compact on A^p_α if and only if

$$\lim_{|z|\to 1^-} \rho(\varphi(z),\phi(z)) \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\phi(z)|^2}\right) = 0. \tag{3.5}$$

Theorem 3.6. *Each equivalence class $[C_\varphi]$ is path connected in the space $\mathcal{C}(A^p_\alpha)$.*

Proof. Let C_ϕ be an arbitrary operator in $[C_\varphi]$, i.e. $C_\varphi - C_\phi$ is compact on A^p_α and, hence, (3.5) holds. We can verify that all operators C_{φ_t}, where, as above, $\varphi_t(z) := (1-t)\varphi(z) + t\phi(z)$ for $t \in [0,1]$, belong to the class $[C_\varphi]$. Indeed, using (3.1) and Lemma 2.2, we get $\rho(\varphi(z),\varphi_t(z)) \leq \rho(\varphi(z),\phi(z))$ and

$$\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\varphi_t(z)|^2} \leq 2\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\phi(z)|^2},$$

for every $t \in [0,1]$ and $z \in \mathbb{D}$. Thus, by (3.5),

$$\lim_{|z|\to 1^-} \rho(\varphi(z),\varphi_t(z)) \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\varphi_t(z)|^2}\right) = 0,$$

for every $t \in [0,1]$. This and (3.5) imply that $C_\varphi - C_{\varphi_t}$ is compact on A^p_α for every $t \in [0,1]$.

On the other hand, it is easy to see that (3.5) implies (3.4). Then, by Theorem 3.3, C_φ and C_ϕ are in the same path component of the space $\mathcal{C}(A^p_\alpha)$ via the path $C_{\varphi_t}, t \in [0,1]$, in $[C_\varphi]$.

From this the assertion follows. □

Now we show that the set $\mathcal{C}_0(A^p_\alpha)$ forms a path component in the space $\mathcal{C}(A^p_\alpha)$. To do this we need some additional facts concerning angular derivatives of a function $\varphi \in \mathcal{S}(\mathbb{D})$ (for more information we refer the reader to [15]). Let $\zeta \in \partial\mathbb{D}$. The condition

$$\liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} < \infty \tag{3.6}$$

is necessary and sufficient for the existence of the finite angular derivative $\varphi'(\zeta)$ at ζ. Moreover, the limit in (3.6) is equal to $|\varphi'(\zeta)|$ and

$$|\varphi'(\zeta)| \geq \frac{1-|\varphi(0)|}{1+|\varphi(0)|} > 0.$$

If the limit if infinite, we put $\varphi'(\zeta) := \infty$. Following [15] we say that φ and ϕ from $\mathcal{S}(\mathbb{D})$ *have the same data at* ζ if they have radial limits of modulus 1 at ζ with $\varphi(\zeta) = \phi(\zeta)$ and $|\varphi'(\zeta)| = |\phi'(\zeta)|$. Obviously, in that case $\varphi'(\zeta) = \phi'(\zeta)$.

From [4, Theorem 3.5] it follows that there is a constant $c > 0$ depending only on α such that

$$\|C_\varphi - C_\phi\|_\alpha \geq c|\varphi'(\zeta)|^{-\frac{\alpha+2}{p}}, \tag{3.7}$$

for every point $\zeta \in \partial\mathbb{D}$, at which φ and ϕ do not have the same data.

The following result is an analog of [15, Theorem 2.4] and proved in the similar way by using inequality (3.7).

Proposition 3.7. *If the operator C_ϕ belongs to the path component of $\mathcal{C}(A^p_\alpha)$ containing C_φ, then φ and ϕ have the same data at every point $\zeta \in \partial\mathbb{D}$ for which $\varphi'(\zeta)$ is finite.*

Corollary 3.8. *The set $\mathcal{C}_0(A^p_\alpha)$ of all compact composition operators on A^p_α is a path component in $\mathcal{C}(A^p_\alpha)$.*

Proof. By Theorem 3.6, the set $\mathcal{C}_0(A^p_\alpha)$ is path connected in $\mathcal{C}(A^p_\alpha)$.

Recall, by [16, Theorem 3.5], that an operator C_φ is compact on A^p_α if and only if φ has no finite angular derivative at any point of $\partial\mathbb{D}$. Thus for any non-compact operator C_ϕ on A^p_α, there is at least one point $\zeta \in \partial\mathbb{D}$ with $\phi'(\zeta)$ finite. So a compact operator C_φ and a non-compact operator C_ϕ do not belong to the same path component of $\mathcal{C}(A^p_\alpha)$ by Proposition 3.7, since φ and ϕ do not have the same data at ζ. □

Remark 3.9. Corollary 3.8 extends the result on Hilbert Bergman spaces A^2_α in [6, Corollary 9.19] to Bergman spaces A^p_α for all $p \in (0,\infty)$. It should be noted that in the setting of Hardy spaces, the set of all compact composition operators on H^2 is path connected in the space $\mathcal{C}(H^2)$ (see, [15, Proposition 2.1]); however, it does not form a path component in this space (see, [9, Main Theorem]).

The next result shows that similarly to the Hardy space H^2 [3,18], the space H^∞ [14], and weighted Banach spaces with sup-norm [1], the well-known Shapiro-Sundberg conjecture (see [19, Page 149]) is also false for all Bergman spaces A^p_α. Since this example is constructed by the same reasons as in [1, Example 3.4] and [14, Examples 1 and 2], we only sketch its proof.

Example 3.10. Let $\varphi_0(z) := 1+a(z-1)$ with $0 < a < 1$. Then the class $[C_{\varphi_0}]$ is not a path component in $\mathcal{C}(A^p_\alpha)$.

Proof. Let $\delta := \frac{a(1-a)}{10}$. For each $t \in [-\delta, \delta]$, we put

$$\varphi_t(z) := \varphi_0(z) + t(z-1)^2.$$

Similarly to [1, Example 3.4] and [14, Example 1], we get $\varphi_t \in \mathcal{S}(\mathbb{D})$ and $C_{\varphi_t} \in \mathcal{C}(A^p_\alpha)$ for all $t \in [-\delta, \delta]$ and consider a sequence $(z_n) \subset \mathbb{D}$ such that $z_n \to 1$ along the arc $|1-z|^2 = 1-|z|^2$. Then, as in [1, Example 3.4], for all $n \geq 1$ and $t \in [-\delta, \delta]$,

$$\rho\left(\varphi_0\left(z_n\right), \varphi_t\left(z_n\right)\right) \geq \frac{|t|}{a(2-a)+|t|}$$

and hence,

$$\begin{aligned} &\rho\left(\varphi_0\left(z_n\right), \varphi_t\left(z_n\right)\right) \\ &\quad \left(\frac{1-|z_n|^2}{1-|\varphi_0(z_n)|^2} + \frac{1-|z_n|^2}{1-|\varphi_t(z_n)|^2}\right) \\ &\quad \geq \frac{|t|}{a(2-a)\left(a(2-a)+|t|\right)}. \end{aligned}$$

This and (3.5) imply that $C_{\varphi_t} - C_{\varphi_0}$ is not compact on A^p_α and so $C_{\varphi_t} \notin [C_{\varphi_0}]$ for all $0 < |t| \leq \delta$.

On the other hand, as in [14, Example 2], for every $z \in \mathbb{D}$,

$$\rho\left(\varphi_{-\delta}(z), \varphi_\delta(z)\right) \leq \frac{2\delta}{a(1-a)-2\delta-4\delta^2} < \frac{1}{2}.$$

From this and Corollary 3.4, it follows that the path C_{ϕ_s} with

$$\begin{aligned} \phi_s(z) &:= (1-s)\varphi_{-\delta}(z) + s\varphi_\delta(z) = \varphi_0(z) + \delta\left(2s-1\right)\left(z-1\right)^2 \\ &= \varphi_{\delta(2s-1)}(z), s \in [0,1], \end{aligned}$$

is a continuous path connecting $C_{\varphi_{-\delta}}$ and C_{φ_δ} in $\mathcal{C}(A^p_\alpha)$. Thus, the assertion follows. □

We end this section with the following characterizations for isolated and non-isolated points in the space $\mathcal{C}(A^p_\alpha)$.

Proposition 3.11. *Let φ be a function from $\mathcal{S}(\mathbb{D})$.*

(a) *If φ has a finite angular derivative on a set of positive measure, then C_φ is isolated in the space $\mathcal{C}(A^p_\alpha)$.*
(b) *If*

$$\int_0^{2\pi} \log(1-|\varphi(e^{i\theta})|)d\theta > -\infty, \tag{3.8}$$

then C_φ is not isolated in $\mathcal{C}(A^p_\alpha)$.

Proof. The part (a) follows from (3.7) by the arguments in the proof of [15, Corollary 2.3].

(b). We put

$$\phi(z) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(1-|\varphi(e^{i\theta})|)d\theta\right), z \in \mathbb{D}.$$

Then ϕ is a bounded outer function in $\mathbb{D}$ with $|\phi| \le 1-|\varphi|$ in $\mathbb{D}$ and $|\phi| = 1-|\varphi|$ almost everywhere on $\partial\mathbb{D}$.

For each $t \in (-1,1)$, we define $\phi_t(z) := \varphi(z) + t\phi(z)$. Obviously, $\phi_t \in \mathcal{S}(\mathbb{D})$ for every $t \in (-1,1)$. We claim that the path $C_{\phi_t}, t \in [-\delta, \delta]$ with $\delta = \frac{1}{6}$, is continuous in $\mathcal{C}(A^p_\alpha)$. Hence, C_φ is not isolated in $\mathcal{C}(A^p_\alpha)$.

It remains to prove the claim. For each $z \in \mathbb{D}$, we have

$$\begin{aligned}\rho\left(\phi_{-\delta}(z), \phi_\delta(z)\right) &= \left|\frac{\phi_{-\delta}(z)-\phi_\delta(z)}{1-\overline{\phi_{-\delta}(z)}\phi_\delta(z)}\right| \\ &\le \frac{2\delta|\phi(z)|}{1-|\varphi(z)|^2-2\delta|\varphi(z)||\phi(z)|-\delta^2|\phi(z)|^2} \\ &= \frac{2\delta}{\frac{1-|\varphi(z)|^2}{|\phi(z)|}-2\delta|\varphi(z)|-\delta^2|\phi(z)|} \\ &\le \frac{2\delta}{\frac{1-|\varphi(z)|}{|\phi(z)|}-2\delta-\delta^2} \le \frac{2\delta}{1-2\delta-\delta^2} \le \frac{2}{3}.\end{aligned}$$

From this and Corollary 3.4, it follows that the path C_{φ_s} with

$$\begin{aligned}\varphi_s(z) &:= (1-s)\phi_{-\delta}(z) + s\phi_\delta(z) \\ &= \varphi(z) + \delta(2s-1)\phi(z) = \phi_{\delta(2s-1)}(z), s \in [0,1],\end{aligned}$$

is a continuous path connecting $C_{\phi_{-\delta}}$ and C_{ϕ_δ} in $\mathcal{C}(A^p_\alpha)$. Thus, the claim follows. □

Remark 3.12. Part (a) of Proposition 3.11 is an extension of [15, Corollary 2.3], which was stated only for Hilbert Bergman spaces A^2_α.

Furthermore, (3.8) is a sufficient condition for the operator C_φ to be non-isolated in the space of composition operators on the Hardy space H^2 [19, Theorem 3.1], on the space H^∞ [14, Corollary 9], and on weighted Banach spaces with sup-norm [1, Proposition 3.6]. Part (b) of Proposition 3.11 extends this result to all Bergman spaces A^p_α.

4. Path Components of the Space of Weighted Composition Operators

In this section we study the topological structure of the space $\mathcal{C}_w(A^p_\alpha)$ of all nonzero bounded weighted composition operators on A^p_α under the operator norm topology. For simplicity, we write $W_{\psi,\varphi} \sim W_{\chi,\phi}$ in $\mathcal{C}_w(A^p_\alpha)$ if the operators $W_{\psi,\varphi}$ and $W_{\chi,\phi}$ are in the same path component of $\mathcal{C}_w(A^p_\alpha)$.

In our further considerations we use the following simple fact, which is proved similarly to [20, Lemma 4.8].

Lemma 4.1. *Every operator $W_{\psi,\varphi} \in \mathcal{C}_w(A^p_\alpha)$ is path connected with the operator C_φ in $\mathcal{C}_w(A^p_\alpha)$.*

The main result of this section is as follows.

Theorem 4.2. *The set $\mathcal{C}_{w,0}(A^p_\alpha)$ of all nonzero compact weighted composition operators on A^p_α is path connected in the space $\mathcal{C}_w(A^p_\alpha)$; but it is not a path component in this space.*

Proof. (a) To prove that the set $\mathcal{C}_{w,0}(A^p_\alpha)$ is path connected in the space $\mathcal{C}_w(A^p_\alpha)$, it suffices to show that every operator $W_{\psi,\varphi}$ in $\mathcal{C}_{w,0}(A^p_\alpha)$ and the operator $C_0 : f \mapsto f(0)$ belong to the same path component of $\mathcal{C}_w(A^p_\alpha)$ via a path in $\mathcal{C}_{w,0}(A^p_\alpha)$.

If $\psi(z) \equiv const$, then the assertion follows from Lemma 4.1 and Corollary 3.8.

Now suppose that ψ is non-constant. Obviously, $\psi = W_{\psi,\varphi}(\mathbf{1}) \in A^p_\alpha$. For each $t \in [0,1]$, we put

$$\psi_t(z) := 1 - t + t\psi(z) \text{ and } \varphi_t(z) := t\varphi(z), z \in \mathbb{D}.$$

Then, for every $t \in [0,1)$, ψ_t is a nonzero function in A^p_α and $\overline{\varphi_t(\mathbb{D})} \subset t\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$. From this it follows that all operators $W_{\psi_t,\varphi_t}, t \in [0,1)$, are compact on A^p_α. Indeed, for every bounded sequence $(f_n)_n$ in A^p_α converging to 0 uniformly on compact sets of $\mathbb{D}$, we get

$$\begin{aligned}\|W_{\psi_t,\varphi_t} f_n\|_{p,\alpha} &= \left(\int_{\mathbb{D}} |\psi_t(z)|^p |f_n(t\varphi(z))|^p dA_\alpha(z)\right)^{\frac{1}{p}} \\ &\le \|\psi_t\|_{p,\alpha} \sup_{|z|\le t} |f_n(z)| \to 0 \text{ as } n \to \infty.\end{aligned}$$

From this and a slight modification of [6, Proposition 3.11], the assertion follows. Thus $W_{\psi_t,\varphi_t} \in \mathcal{C}_{w,0}(A^p_\alpha)$ for all $t \in [0,1]$; moreover, $W_{\psi_0,\varphi_0} = C_0$ and $W_{\psi_1,\varphi_1} = W_{\psi,\varphi}$. We claim that the map

$$[0,1] \to \mathcal{C}_w(A^p_\alpha), t \mapsto W_{\psi_t,\varphi_t},$$

is continuous on $[0,1]$. Then $W_{\psi,\varphi} \sim C_0$ in $\mathcal{C}_w(A^p_\alpha)$ via a path $W_{\psi_t,\varphi_t}, t \in [0,1]$, in $\mathcal{C}_{w,0}(A^p_\alpha)$.

It remains to prove the claim. Obviously, $W_{\psi_t,\varphi_t} = (1-t)C_{t\varphi} + W_{t\psi,t\varphi}$, and hence,

$$\|W_{\psi_s,\varphi_s} - W_{\psi_t,\varphi_t}\|_\alpha \le \|(1-s)C_{s\varphi} - (1-t)C_{t\varphi}\|_\alpha + \|W_{s\psi,s\varphi} - W_{t\psi,t\varphi}\|_\alpha,$$

for every $t, s \in [0,1]$. Consequently, to prove the claim, it is enough to show that, for every $t \in [0,1]$ fixed,

(i) $\lim_{s\to t} \|(1-s)C_{s\varphi} - (1-t)C_{t\varphi}\|_\alpha = 0$ and **(ii)** $\lim_{s\to t} \|W_{s\psi,s\varphi} - W_{t\psi,t\varphi}\|_\alpha = 0$.

In our further demonstration we use the obvious inequality for functions $f \in H(\mathbb{D})$:

$$|f(sz) - f(tz)| \le |t-s||z| \max_{\tau\in[s,t]} |f'(\tau z)|, \ z \in \mathbb{D}, \ t, s \in [0,1], \tag{4.1}$$

where we briefly write $[s,t]$ for the interval between s and t.

First, we prove **(i)**. If $t = 1$, then, by (2.7),

$$\|(1-s)C_{s\varphi}\|_\alpha = (1-s)\|C_{s\varphi}\|_\alpha \leq (1-s)\left(\frac{1+s|\varphi(0)|}{1-s|\varphi(0)|}\right)^{\frac{\alpha+2}{p}} \to 0, s \to 1.$$

Let now $t \in [0,1)$ and $t_0 \in (t,1)$. For every $s \in [0,t_0)$ and $f \in A^p_\alpha$, using (2.4), (2.7) and (4.1), we get

$$\begin{aligned}
&\|(1-s)C_{s\varphi}f - (1-t)C_{t\varphi}f\|^p_{p,\alpha} \\
&\quad = \int_{\mathbb{D}} |(1-s)f(s\varphi(z)) - (1-t)f(t\varphi(z))|^p dA_\alpha(z) \\
&\quad \lesssim (1-s)^p \int_{\mathbb{D}} |f(s\varphi(z)) - f(t\varphi(z))|^p dA_\alpha(z) + |s-t|^p \int_{\mathbb{D}} |f(t\varphi(z))|^p dA_\alpha(z) \\
&\quad \leq |s-t|^p \int_{\mathbb{D}} |\varphi(z)|^p \max_{\tau\in[s,t]} |f'(\tau\varphi(z))|^p dA_\alpha(z) + |s-t|^p \|C_{t\varphi}f\|^p_{p,\alpha} \\
&\quad \leq c_1^p |s-t|^p \|f\|^p_{p,\alpha} \int_{\mathbb{D}} \max_{\tau\in[s,t]} \frac{1}{(1-|\tau\varphi(z)|^2)^{\alpha+2+p}} dA_\alpha(z) + |s-t|^p \|f\|^p_{p,\alpha} \|C_{t\varphi}\|^p_\alpha \\
&\quad \leq \frac{c_1^p}{(1-t_0^2)^{\alpha+p+2}} |s-t|^p \|f\|^p_{p,\alpha} + |s-t|^p \left(\frac{1+t|\varphi(0)|}{1-t|\varphi(0)|}\right)^{\alpha+2} \|f\|^p_{p,\alpha}.
\end{aligned}$$

Therefore, for every $s \in [0,t_0)$,

$$\|(1-s)C_{s\varphi} - (1-t)C_{t\varphi}\|^p_\alpha \lesssim \frac{c_1^p}{(1-t_0^2)^{\alpha+p+2}} |s-t|^p + |s-t|^p \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\alpha+2}.$$

Thus, $\|(1-s)C_{s\varphi} - (1-t)C_{t\varphi}\|_\alpha \to 0$ as $s \to t$, which completes the proof for **(i)**.

Next, we prove **(ii)**. For every $s,t \in [0,1]$ and $f \in A^p_\alpha$, we have

$$\begin{aligned}
\|W_{s\psi,s\varphi}f - W_{t\psi,t\varphi}f\|^p_{p,\alpha} &= \int_{\mathbb{D}} |s\psi(z)f(s\varphi(z)) - t\psi(z)f(t\varphi(z))|^p dA_\alpha(z) \\
&\lesssim |s|^p \int_{\mathbb{D}} |\psi(z)(f(s\varphi(z)) - f(t\varphi(z)))|^p dA_\alpha(z) \\
&\qquad + |s-t|^p \int_{\mathbb{D}} |\psi(z)f(t\varphi(z))|^p dA_\alpha(z).
\end{aligned}$$

To continue, we need several auxiliary estimates in the following cases.

Case 1. $t \in [0,1)$. We fix a number $t_0 \in (t,1)$.

Estimate 1.1. By (2.3), we have

$$\begin{aligned}
\int_{\mathbb{D}} |\psi(z)f(t\varphi(z))|^p dA_\alpha(z) &\leq c_1^p \|f\|^p_{p,\alpha} \int_{\mathbb{D}} \frac{1}{(1-|t\varphi(z)|^2)^{\alpha+2}} |\psi(z)|^p dA_\alpha(z) \\
&\leq \frac{c_1^p}{(1-t_0^2)^{\alpha+2}} \|f\|^p_{p,\alpha} \|\psi\|^p_{p,\alpha}.
\end{aligned}$$

Estimate 1.2. By (2.4) and (4.1), for every $s \in [0, t_0)$,

$$\begin{aligned}&\int_{\mathbb{D}} |\psi(z)(f(s\varphi(z)) - f(t\varphi(z)))|^p dA_\alpha(z)\\ &\quad\leq |s-t|^p \int_{\mathbb{D}} |\psi(z)\varphi(z)|^p \max_{\tau\in[s,t]} |f'(\tau\varphi(z))|^p dA_\alpha(z)\\ &\quad\leq c_1^p |s-t|^p \|f\|_{p,\alpha}^p \int_{\mathbb{D}} |\psi(z)|^p \max_{\tau\in[s,t]} \frac{1}{(1-|\tau\varphi(z)|^2)^{\alpha+2+p}} dA_\alpha(z)\\ &\quad\leq \frac{c_1^p}{(1-t_0^2)^{\alpha+2+p}} |s-t|^p \|f\|_{p,\alpha}^p \|\psi\|_{p,\alpha}^p.\end{aligned}$$

Using the above estimates, for every $s \in [0, t_0)$, we obtain

$$\begin{aligned}&\|W_{s\psi,s\varphi} - W_{t\psi,t\varphi}\|_\alpha^p\\ &\quad\lesssim \frac{c_1^p}{(1-t_0^2)^{\alpha+2+p}} |s-t|^p \|\psi\|_{p,\alpha}^p + \frac{c_1^p}{(1-t_0^2)^{\alpha+2}} |s-t|^p \|\psi\|_{p,\alpha}^p.\end{aligned}$$

This implies that

$$\lim_{s\to t} \|W_{s\psi,s\varphi} - W_{t\psi,t\varphi}\|_\alpha = 0.$$

Case 2. $t = 1$.

Estimate 2.1. We have

$$\int_{\mathbb{D}} |\psi(z) f(\varphi(z))|^p dA_\alpha(z) = \|W_{\psi,\varphi} f\|_{p,\alpha}^p \leq \|W_{\psi,\varphi}\|_\alpha^p \|f\|_{p,\alpha}^p.$$

Estimate 2.2. For each $r \in (0, 1)$, we put

$$E_r := \{z \in \mathbb{D} : |\varphi(z)| \leq r\} \text{ and } E_r^c := \mathbb{D} \setminus E_r.$$

By (2.4) and (4.1), for every $s \in [0, 1)$, $r \in (0, 1)$, and $f \in A_\alpha^p$, we have

$$\begin{aligned}&\int_{\mathbb{D}} |\psi(z)(f(s\varphi(z)) - f(\varphi(z)))|^p dA_\alpha(z)\\ &\quad= \left(\int_{E_r} + \int_{E_r^c}\right) |\psi(z)(f(s\varphi(z)) - f(\varphi(z)))|^p dA_\alpha(z)\\ &\quad\lesssim (1-s)^p \int_{E_r} |\psi(z)\varphi(z)|^p \max_{\tau\in[s,1]} |f'(\tau\varphi(z))|^p dA_\alpha(z)\\ &\qquad+ \int_{E_r^c} (|\psi(z) f(s\varphi(z))|^p + |\psi(z) f(\varphi(z))|^p)\, dA_\alpha(z)\\ &\quad\leq c_1^p (1-s)^p \|f\|_{p,\alpha}^p \int_{E_r} |\psi(z)|^p \max_{\tau\in[s,1]} \frac{1}{(1-|\tau\varphi(z)|^2)^{\alpha+2+p}} dA_\alpha(z)\\ &\qquad+ \|(\chi_{E_r^c} |\psi|^p A_\alpha) \circ (s\varphi)^{-1}\|_\alpha \|f\|_{p,\alpha}^p + \|(\chi_{E_r^c} |\psi|^p A_\alpha) \circ \varphi^{-1}\|_\alpha \|f\|_{p,\alpha}^p\\ &\quad\leq \frac{c_1^p}{(1-r^2)^{\alpha+2+p}} (1-s)^p \|f\|_{p,\alpha}^p \|\psi\|_{p,\alpha}^p\\ &\qquad+ \| \left(\chi_{E_r^c} |\psi|^p A_\alpha\right) \circ (s\varphi)^{-1}\|_\alpha \|f\|_{p,\alpha}^p + \| \left(\chi_{E_r^c} |\psi|^p A_\alpha\right) \circ \varphi^{-1}\|_\alpha \|f\|_{p,\alpha}^p.\end{aligned}$$

Thus, for every $s \in [0,1)$ and $r \in (0,1)$,

$$\|W_{s\psi,s\varphi} - W_{\psi,\varphi}\|_\alpha^p \lesssim (1-s)^p \|W_{\psi,\varphi}\|_\alpha^p + \frac{c_1^p}{(1-r^2)^{\alpha+2+p}}(1-s)^p\|\psi\|_{p,\alpha}^p$$
$$+ \| \left(\chi_{E_r^c}|\psi|^p A_\alpha\right) \circ (s\varphi)^{-1}\|_\alpha + \| \left(\chi_{E_r^c}|\psi|^p A_\alpha\right) \circ \varphi^{-1}\|_\alpha.$$

To complete the proof, we take an arbitrary number $\varepsilon > 0$. Since $W_{\psi,\varphi}$ is compact on A_α^p, $(|\psi|^p A_\alpha) \circ \varphi^{-1}$ is a compact α-Carleson measure. Then there is a number $\delta_0 = \delta_0(\varepsilon) \in (0,1)$ such that

$$(|\psi|^p A_\alpha) \circ \varphi^{-1}\left(S(\zeta,\delta)\right) < \varepsilon^p \left(\frac{\delta}{2}\right)^{\alpha+2}$$

for every $\delta < \delta_0$ and $\zeta \in \partial\mathbb{D}$.

For every $s \in [\frac{1}{2}, 1]$, $\delta \in (0,2]$, and $\zeta \in \partial\mathbb{D}$, by some geometric arguments, we can see that

$$\frac{1}{s}S(\zeta,\delta) \cap \mathbb{D} \subset S(\zeta, \frac{\delta}{s}),$$

Then, for every $r \in (0,1)$, $s \in [\frac{1}{2}, 1]$, $\delta < \frac{\delta_0}{2}$, and $\zeta \in \partial\mathbb{D}$, we get

$$\begin{aligned}
\left(\chi_{E_r^c}|\psi|^p A_\alpha\right) \circ (s\varphi)^{-1}(S(\zeta,\delta)) &= \int_{(s\varphi)^{-1}(S(\zeta,\delta))} \chi_{E_r^c}(\omega)|\psi(\omega)|^p dA_\alpha(\omega)\\
&\leq \int_{\varphi^{-1}(\frac{1}{s}S(\zeta,\delta)\cap\mathbb{D})} |\psi(\omega)|^p dA_\alpha(\omega) \leq \int_{\varphi^{-1}\left(S(\zeta,\frac{\delta}{s})\right)} |\psi(\omega)|^p dA_\alpha(\omega)\\
&= (|\psi|^p A_\alpha) \circ \varphi^{-1}\left(S(\zeta,\frac{\delta}{s})\right) < \varepsilon^p \left(\frac{\delta}{2s}\right)^{\alpha+2} \leq \varepsilon^p \delta^{\alpha+2}.
\end{aligned}$$

Since $(|\psi|^p A_\alpha) \circ \varphi^{-1}(B_r^c) \to 0$ as $r \to 1^-$ with $B_r^c := \{z \in \mathbb{D} : |z| > r\}$, there exists a number $r_0 \in (0,1)$ such that

$$(|\psi|^p A_\alpha) \circ \varphi^{-1}\left(B_r^c\right) < \varepsilon^p \left(\frac{\delta_0}{2}\right)^{\alpha+2} \quad \text{for every } r > r_0.$$

Then, for every $r > r_0$, $s \in [\frac{1}{2}, 1]$, $\delta \in [\frac{\delta_0}{2}, 2]$, and $\zeta \in \partial\mathbb{D}$, we have

$$\begin{aligned}
\left(\chi_{E_r^c}|\psi|^p A_\alpha\right) \circ (s\varphi)^{-1}(S(\zeta,\delta)) &= \int_{(s\varphi)^{-1}(S(\zeta,\delta))} \chi_{E_r^c}(\omega)|\psi(\omega)|^p dA_\alpha(\omega)\\
&= \int_{\varphi^{-1}\left(\frac{1}{s}S(\zeta,\delta)\cap\mathbb{D}\right)\cap E_r^c} |\psi(\omega)|^p dA_\alpha(\omega) \leq \int_{E_r^c} |\psi(\omega)|^p dA_\alpha(\omega)\\
&= (|\psi|^p A_\alpha) \circ \varphi^{-1}\left(B_r^c\right) < \varepsilon^p \left(\frac{\delta_0}{2}\right)^{\alpha+2} \leq \varepsilon^p \delta^{\alpha+2}.
\end{aligned}$$

Consequently, for every $r > r_0$ and $s \in [\frac{1}{2}, 1]$,

$$\| \left(\chi_{E_r^c}|\psi|^p A_\alpha\right) \circ (s\varphi)^{-1}\|_\alpha = \sup_{\delta\in(0,2],\zeta\in\partial\mathbb{D}} \frac{\left(\chi_{E_r^c}|\psi|^p A_\alpha\right) \circ (s\varphi)^{-1}(S(\zeta,\delta))}{\delta^{\alpha+2}} \leq \varepsilon^p.$$

Therefore, for every $r > r_0$ and $s \in [\frac{1}{2}, 1)$, we get

$$\|W_{s\psi,s\varphi} - W_{\psi,\varphi}\|_\alpha^p \lesssim (1-s)^p \|W_{\psi,\varphi}\|_\alpha^p + \frac{c_1^p}{(1-r^2)^{\alpha+2+p}}(1-s)^p\|\psi\|_{p,\alpha}^p + 2\varepsilon^p.$$

From this it follows that

$$\limsup_{s\to 1^-} \|W_{s\psi,s\varphi} - W_{\psi,\varphi}\|_\alpha^p \lesssim \varepsilon^p, \text{ hence, } \lim_{s\to 1^-} \|W_{s\psi,s\varphi} - W_{\psi,\varphi}\|_\alpha = 0.$$

Thus, **(ii)** is proved.

(b) Now we consider the operators W_{ψ_0,φ_0} and C_{φ_0}, where $\psi_0(z) := 1-z$ and $\varphi_0(z) := 1 + a(z-1)$ with $0 < a < 1$. Obviously, W_{ψ_0,φ_0} and C_{φ_0} belong to $\mathcal{C}_w(A^p_\alpha)$. However, it is easy to check that W_{ψ_0,φ_0} is compact, while C_{φ_0} is not compact on A^p_α.

Indeed, for all $r \in (0,1)$,

$$\frac{1-r^2}{1-|1+a(r-1)|^2} \geq 1.$$

Hence, by (1.1), C_{φ_0} is not compact on A^p_α.

Next, for any sequence $(z_n)_n$ in $\mathbb{D}$ with $|z_n| \to 1^-$ as $n \to \infty$, without loss of generality, we suppose that $z_n \to \eta \in \partial\mathbb{D}$. If $\eta \neq 1$, then $\varphi_0(z_n) \to 1 + a(\eta - 1) \in \mathbb{D}$ as $n \to \infty$, hence,

$$\frac{|\psi_0(z_n)|\left(1-|z_n|^2\right)}{1-|\varphi_0(z_n)|^2} \leq 2\frac{1-|z_n|^2}{1-|\varphi_0(z_n)|^2} \to 0 \text{ as } n \to \infty.$$

If $\eta = 1$, then $\psi_0(z_n) \to 0$ as $n \to \infty$, hence, using [6, Corollary 2.40], we get

$$\begin{aligned}\frac{|\psi_0(z_n)|\left(1-|z_n|^2\right)}{1-|\varphi_0(z_n)|^2} &\leq |\psi_0(z_n)| \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi_0(z)|^2}\\ &\leq \frac{2(2-a)}{a}|\psi_0(z_n)| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Consequently,

$$\lim_{|z|\to 1^-} \frac{|\psi_0(z)|\left(1-|z|^2\right)}{1-|\varphi_0(z)|^2} = 0,$$

which implies, by [17, Corollary 1], that W_{ψ_0,φ_0} is compact on A^p_α.

It remains to note that, by Lemma 4.1, $W_{\psi_0,\varphi_0} \sim C_{\varphi_0}$ in $\mathcal{C}_w(A^p_\alpha)$. From this it follows that the set $\mathcal{C}_{w,0}(A^p_\alpha)$ is not a path component of $\mathcal{C}_w(A^p_\alpha)$. □

From Lemma 4.1 and the results in Sect. 3, we get the following result for weighted composition operators.

Proposition 4.3. *Suppose that two functions φ and ϕ from $\mathcal{S}(\mathbb{D})$ satisfy either of the following conditions:*

(i) *the difference $C_\varphi - C_\phi$ is compact on A^p_α,*
(ii) *there is a number $r_0 \in (0,1)$ such that $\rho(\varphi(z),\phi(z)) \leq r_0$ for all $z \in \mathbb{D}$.*

Then all the operators $W_{\psi,\varphi}$ and $W_{\chi,\phi}$ in $\mathcal{C}_w(A^p_\alpha)$ belong to the same path component of $\mathcal{C}_w(A^p_\alpha)$.

Proof. By Lemma 4.1, $W_{\psi,\varphi} \sim C_\varphi$ and $W_{\chi,\phi} \sim C_\phi$ in $\mathcal{C}_w(A^p_\alpha)$. On the other hand, by Theorem 3.6 and Corollary 3.4, $C_\phi \sim C_\varphi$ in $\mathcal{C}(A^p_\alpha)$, and hence, in $\mathcal{C}_w(A^p_\alpha)$. Consequently, $W_{\chi,\phi} \sim W_{\psi,\varphi}$ in $\mathcal{C}_w(A^p_\alpha)$. □

In view of this proposition, for each function $\varphi \in \mathcal{S}(\mathbb{D})$, we denote by $\mathcal{W}([C_\varphi])$ the set of all weighted composition operators $W_{\psi,\phi} \in \mathcal{C}_w(A^p_\alpha)$ with $C_\phi \in [C_\varphi]$. The following result follows immediately from Proposition 4.3.

Corollary 4.4. *Each set $\mathcal{W}([C_\varphi])$ with $\varphi \in \mathcal{S}(\mathbb{D})$ is path connected in $\mathcal{C}_w(A^p_\alpha)$.*

Now we show that the sets $\mathcal{W}([C_\varphi])$ may be path components of the space $\mathcal{C}_w(A^p_\alpha)$ and may be not. To do this, we give the following examples.

Example 4.5. For $\varphi_0(z) := 1 + a(z-1)$ with $0 < a < 1$, the set $\mathcal{W}([C_{\varphi_0}])$ is not a path component of $\mathcal{C}_w(A^p_\alpha)$. More precisely, $\mathcal{W}([C_{\varphi_0}])$ is a proper subset of the path component of $\mathcal{C}_w(A^p_\alpha)$ containing $\mathcal{C}_{w,0}(A^p_\alpha)$.

Proof. By part (b) in the proof of Theorem 4.2, the operator W_{ψ_0,φ_0} with $\psi_0(z) := 1-z$ and $\varphi_0(z) := 1+a(z-1)$ is compact, while C_{φ_0} is not compact on A^p_α. Then, by Theorem 4.2 again, $W_{\psi_0,\varphi_0} \sim C_0$ in $\mathcal{C}_w(A^p_\alpha)$. But $C_{\varphi_0} - C_0$ is not compact on A^p_α, which implies that the operator C_0 does not belong to $\mathcal{W}([C_{\varphi_0}])$ and completes the proof. □

Example 4.6. For $\varphi_1(z) := z$, the set $\mathcal{W}([C_{\varphi_1}])$ is a path component of $\mathcal{C}_w(A^p_\alpha)$.

Proof. By Proposition 3.11(a), C_{φ_1} is isolated in $\mathcal{C}(A^p_\alpha)$, which, by Theorem 3.6, implies that $[C_{\varphi_1}] = \{C_{\varphi_1}\}$. Then

$$\mathcal{W}\left([C_{\varphi_1}]\right) = \left\{W_{\psi,\varphi_1} : 0 < \|\psi\|_\infty < \infty\right\},$$

where, as usual, $\|\psi\|_\infty := \sup_{z\in\mathbb{D}} |\psi(z)|$. Indeed, by (2.2), (2.6) and (2.8), for each $z \in \mathbb{D}$, we get

$$\begin{aligned}|\psi(z)|^p &\le \frac{c_1^p}{(1-|z|^2)^{\alpha+2}} \int_{\Delta(z)} |\psi(\omega)|^p dA_\alpha(\omega) \\ &= c_1^p \frac{(|\psi|^p A_\alpha) \circ \varphi_1^{-1}(\Delta(z))}{(1-|z|^2)^{\alpha+2}} \lesssim \|\left(|\psi|^p A_\alpha\right) \circ \varphi_1^{-1}\|_\alpha < \infty.\end{aligned}$$

Now we will prove that $\mathcal{W}([C_{\varphi_1}])$ is simultaneously open and closed in $\mathcal{C}_w(A^p_\alpha)$, from which the assertion follows.

Let $(W_{\psi_n,\varphi_1})_n$ be a sequence in $\mathcal{W}([C_{\varphi_1}])$ converging to some operator $W_{\chi,\phi}$ in $\mathcal{C}_w(A^p_\alpha)$. Then $W_{\psi_n,\varphi_1}(f) \to W_{\chi,\phi}(f)$ in A^p_α for all $f \in A^p_\alpha$. In particular, taking here $f(z) \equiv 1$ and $f(z) \equiv z$, we obtain that $\psi_n \to \chi$ and $\psi_n\varphi_1 \to \chi\phi$ in A^p_α as $n \to \infty$. Therefore,

$$\chi\left(\varphi_1 - \phi\right) = \left(\chi - \psi_n\right)\varphi_1 + \left(\psi_n\varphi_1 - \chi\phi\right) \to 0 \text{ in } A^p_\alpha.$$

Since $\chi \not\equiv 0$, this implies that $\phi = \varphi_1$. Thus, the set $\mathcal{W}([C_{\varphi_1}])$ is closed in $\mathcal{C}_w(A^p_\alpha)$. The fact that it is open in $\mathcal{C}_w(A^p_\alpha)$ follows immediately from the following auxiliary lemma. □

Lemma 4.7. *Let W_{ψ,φ_1} be an operator in $\mathcal{W}([C_{\varphi_1}])$. Then the inequality $\|W_{\psi,\varphi_1} - W_{\chi,\phi}\|_\alpha \gtrsim \|\psi\|_e$ holds for every operator $W_{\chi,\phi}$ in $\mathcal{C}_w(A^p_\alpha)$ with $\phi \neq \varphi_1$, where*

$$\|\psi\|_e := \inf\left\{\varepsilon > 0 : F(\psi,\varepsilon) \text{ has Lebesgue zero}\right\}$$

with

$$F\left(\psi,\varepsilon\right) := \left\{\zeta \in \partial\mathbb{D} : |\psi(\zeta)| \ge \varepsilon\right\}.$$

Proof. Using (2.1) and (2.2), for every $w, z \in \mathbb{D}$, we get

$$\begin{aligned}\|W_{\psi,\varphi_1} - W_{\chi,\phi}\|_\alpha &\geq \frac{1}{c_0}\|W_{\psi,\varphi_1}k_w - W_{\chi,\phi}k_w\|_{p,\alpha} \\ &= \frac{1}{c_0}\left(\int_{\mathbb{D}} |\psi(\zeta)k_w(\zeta) - \chi(\zeta)k_w(\phi(\zeta))|^p dA_\alpha(\zeta)\right)^{\frac{1}{p}} \\ &\geq \frac{1}{c_0}\left(\int_{\Delta(z)} |\psi(\zeta)k_w(\zeta) - \chi(\zeta)k_w(\phi(\zeta))|^p dA_\zeta(\omega)\right)^{\frac{1}{p}} \\ &\geq \frac{1}{c_0c_1}\left(1-|z|^2\right)^{\frac{\alpha+2}{p}} |\psi(z)k_w(z) - \chi(z)k_w\left(\phi(z)\right)|.\end{aligned}$$

In particular, with $w = z$, we have

$$\begin{aligned}\|W_{\psi,\varphi_1} - W_{\chi,\phi}\|_\alpha &\geq \frac{1}{c_0c_1}\left(1-|z|^2\right)^{\frac{\alpha+2}{p}} |\psi(z)k_z(z) - \chi(z)k_z(\phi(z))| \\ &\geq \frac{1}{c_0c_1}\left(|\psi(z)| - |\chi(z)|\left|\frac{1-|z|^2}{1-\overline{z}\phi(z)}\right|^{\frac{\sigma+\alpha+2}{p}}\right),\end{aligned}$$

for every $z \in \mathbb{D}$.

On the other hand, obviously, $\|\psi\|_e > 0$. Fix an arbitrary number $r \in (0, \|\psi\|_e)$. Then $F(\psi, r)$ has positive Lebesgue measure.

Since $\phi \neq \varphi_1$, the set $\{\zeta \in \partial\mathbb{D} : \phi(\zeta) = \zeta\}$ has Lebesgue measure zero. So there exist a point $\zeta \in F(\psi, r)$ and a sequence $(z_n)_n \subset \mathbb{D}$ such that $z_n \to \zeta$, $|\psi(z_n)| \to |\psi(\zeta)| \geq r$, and $\phi(z_n) \to \eta \neq \zeta$ as $n \to \infty$. Then for each $n \in \mathbb{N}$, using (2.3), we get

$$|\chi(z_n)|\left|\frac{1-|z_n|^2}{1-\overline{z_n}\phi(z_n)}\right|^{\frac{\sigma+\alpha+2}{p}} \leq c_1\|\chi\|_{p,\alpha}\frac{(1-|z_n|^2)^{\frac{\sigma}{p}}}{|1-\overline{z_n}\phi(z_n)|^{\frac{\sigma+\alpha+2}{p}}} \to 0 \text{ as } n \to \infty.$$

Thus,

$$\|W_{\psi,\varphi_1} - W_{\chi,\phi}\|_\alpha \geq \frac{1}{c_0c_1}\limsup_{n\to\infty}|\psi(z_n)| \geq \frac{r}{c_0c_1}$$

and so $\|W_{\psi,\varphi_1} - W_{\chi,\phi}\|_\alpha \geq \dfrac{\|\psi\|_e}{c_0c_1}$. From this the assertion follows. □

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Alexander V. Abanin
Southern Federal University
Rostov-on-Don
Russian Federation 344090
e-mail: avabanin@sfedu.ru

and

Southern Mathematical Institute
Vladikavkaz
Russian Federation 362027

Le Hai Khoi (✉)
Division of Mathematical Sciences, School of Physical and Mathematical Sciences
Nanyang Technological University (NTU)
Singapore 637371
Singapore
e-mail: lhkhoi@ntu.edu.sg

Pham Trong Tien
Faculty of Mathematics, Mechanics and Informatics
University of Science, Vietnam National University
Hanoi
Vietnam
e-mail: phamtien@vnu.edu.vn

and

TIMAS
Thang Long University
Nghiem Xuan Yem, Hoang Mai
Hanoi
Vietnam

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