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Integral Equations and Operator Theory

Some Convergence Theorems for Operator Sequences

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Abstract. Let A, T, and B be bounded linear operators on a Banach space. This paper is concerned mainly with finding some necessary and sufficient conditions for convergence in operator norm of the sequences $\{A^n TB^n\}$ and $\left\{\frac{1}{n}\sum_{i=0}^{n-1} A^i TB^i\right\}$. These results are applied to the Toeplitz, composition, and model operators. Some related problems are also discussed.

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1. Introduction

Throughout the paper, H will denote a complex separable infinite dimensional Hilbert space and $B(H)$, the algebra of all bounded linear operators on H. The ideal of compact operators on H will be denoted by $K(H)$. The quotient algebra $B(H) \diagup K(H)$ is a C^*-algebra and called the *Calkin algebra*. By $\|T\|_{\text{ess}}$ we will denote the essential norm of $T \in B(H)$. As usual, H^2 will denote the classical Hardy space on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and H^∞, the space of all bounded analytic functions on $\mathbb{D}$.

Let $\mathbb{T} := \partial\mathbb{D}$ be the unit circle and let m be the normalized Lebesgue measure on $\mathbb{T}$. Recall that for a given symbol $\varphi \in L^\infty := L^\infty(\mathbb{T}, m)$, the *Toeplitz operator* $\mathcal{T}_\varphi$ on H^2 is defined by

$$\mathcal{T}_\varphi f = P_+(\varphi f), \quad f \in H^2,$$

where P_+ is the orthogonal projection from $L^2(\mathbb{T}, m)$ onto H^2. Let

$$Sf(z) = zf(z)$$

be the unilateral shift operator on H^2. According to a theorem of Brown and Halmos [4], $\mathcal{T} \in B(H^2)$ is a Toeplitz operator if and only if

$$S^*\mathcal{T}S = \mathcal{T}.$$

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Barria and Halmos [2] examined the so-called *strongly asymptotically Toeplitz operators* T on H^2 for which the sequence $\{S^{*n}TS^n\}$ converges strongly. This class includes the Hankel algebra, the operator norm-closed algebra generated by all Toeplitz and Hankel operators together [2].

An operator $T \in B(H^2)$ is said to be *uniformly asymptotically Toeplitz* if the sequence $\{S^{*n}TS^n\}$ converges in the uniform operator topology. This class of operators is closed in operator norm and under adjoints. It contains both Toeplitz operators and the compact ones. Feintuch [9, Theorem 4.1] proved that an operator $T \in B(H^2)$ is uniformly asymptotically Toeplitz if and only if it has the decomposition

$$T = \mathcal{T} + K,$$

where $\mathcal{T}$ is a Toeplitz and K is a compact operator.

Recall that each holomorphic function $\phi : \mathbb{D} \to \mathbb{D}$ induces a composition operator (bounded and linear) $\mathcal{C}_\phi$ on H^2 by $\mathcal{C}_\phi f = f \circ \phi$ (for instance, see [18, Ch. 5]). Nazarov and Shapiro [23, Theorem 1.1] proved that a composition operator on H^2 is uniformly asymptotically Toeplitz if and only if it is either compact or the identity operator (it follows that the only composition operator which is also Toeplitz is the identity operator).

Throughout, X will denote a complex Banach space and $B(X)$, the algebra of all bounded linear operators on X. Let A, T, and B be in $B(X)$. The main purpose of this paper is to find necessary and sufficient conditions for convergence in operator norm of the sequences $\{A^nTB^n\}$ and $\left\{\frac{1}{n}\sum_{i=0}^{n-1} A^iTB^i\right\}$.

2. The Sequence $\{A^nTB^n\}$

In this section, we give some results concerning convergence in operator norm of the sequence $\{A^nTB^n\}$ for Hilbert space operators. Recall that an operator $T \in B(H)$ is said to be *essentially isometric* (resp. *essentially unitary*) if $I - T^*T \in K(H)$ (resp. $I - T^*T \in K(H)$ and $I - TT^* \in K(H)$).

We have the following:

Theorem 2.1. *Let A and B^* be two essentially isometric operators on H such that $\|A^nx\| \to 0$ and $\|B^{*n}x\| \to 0$ for all $x \in H$. If $T \in B(H)$, then the sequence $\{A^nTB^n\}$ converges in operator norm if and only if we have the decomposition*

$$T = T_0 + K,$$

where $AT_0B = T_0$ and $K \in K(H)$.

For the proof, we need some preliminary results.

As is well known (for instance, see [5, Ch.III, § 7]), there is a bounded linear functional ϕ on the Banach space l^∞ of all bounded complex-valued sequences $c = \{c_n\}$ with the properties: (1) $\phi(c) \geq 0$ for all c with $c_n \geq 0$ $(\forall n \in \mathbb{N})$; (2) $\phi(c) = \phi(Dc)$, where D is the shift operator defined by $(Dc)_{(n)} = c_{n+1}$; (3) $\phi(c) = \lim_{n\to\infty} c_n$ if c is a convergent sequence. The functional ϕ is said to be a *Banach limit* (there is not necessary a unique Banach limit). For convenience, $\phi(c)$ will be denoted by $\text{l.i.m.}_n c_n$.

Let H_0 be the linear space of all weakly null sequences $\{x_n\}$ in H. Let us define a semi-inner product in H_0 by

$$\langle\{x_n\},\{y_n\}\rangle = \text{l.i.m.}_n\langle x_n, y_n\rangle,$$

where l.i.m. is a fixed Banach limit. If

$$E := \left\{\{x_n\} \in H_0 :\ \text{l.i.m.}_n \|x_n\|^2 = 0\right\},$$

then H_0/E becomes a pre-Hilbert space with respect to the inner product defined by

$$\langle\{x_n\} + E,\ \{y_n\} + E\rangle = \text{l.i.m.}_n\langle\{x_n\},\{y_n\}\rangle.$$

Let $\widehat{H}$ be the Hilbert space defined by the completion of H_0/E with respect to the induced norm

$$\|\{x_n\} + E\| = \left(\text{l.i.m.}_n \|x_n\|^2\right)^{\frac{1}{2}}.$$

Now, for a given $T \in B(H)$ we can define an operator $\widehat{T}$ on H_0/E by

$$\widehat{T} : \{x_n\} + E \mapsto \{Tx_n\} + E.$$

Consequently, we have

$$\left\|\widehat{T}(\{x_n\} + E)\right\| = \left(\text{l.i.m.}_n \|Tx_n\|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

It follows that

$$\left\|\widehat{T}(\{x_n\} + E)\right\| \leq \|T\| \, \|\{x_n\} + E\|.$$

Since H_0/E is dense in $\widehat{H}$, the operator $\widehat{T}$ can be extended to the whole $\widehat{H}$ which we also denote by $\widehat{T}$. Clearly, $\left\|\widehat{T}\right\| \leq \|T\|$. The operator $\widehat{T}$ will be called *limit operator* associated with T.

Proposition 2.2. *If $\widehat{T}$ is the limit operator associated with $T \in B(H)$, then:*

(a) *The map $T \mapsto \widehat{T}$ is a contractive $*-$ homomorphism.*
(b) *T is a compact operator if and only if $\widehat{T} = 0$.*
(c) *T is an essentially isometry (resp. essentially unitary) if and only if $\widehat{T}$ is an isometry (resp. unitary).*
(d) *For an arbitrary $T \in B(H)$, we have $\left\|\widehat{T}\right\| = \|T\|_{ess}$.*

Proof. The proof of (a) being very easy is omitted.

(b) Let $\{x_n\}$ be a weakly null sequence in H. If $T \in K(H)$, then as $\|Tx_n\| \to 0$, by (2.1) we have $\widehat{T} = 0$. Now, assume that $\widehat{T} = 0$. We must show that $\|Tx_n\| \to 0$. As $\underline{\lim}_{n\to\infty} \|Tx_n\| \geq 0$, it suffices to show that $\overline{\lim}_{n\to\infty} \|Tx_n\| = 0$. Since $\widehat{T} = 0$ by (2.1), $\text{l.i.m.}_n \|Ty_n\|^2 = 0$ for all weakly null sequences $\{y_n\}$. Observe that

$$\overline{\lim_{n\to\infty}} \|Tx_n\| = \lim_{k\to\infty} \|Tx_{n_k}\|,$$

for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Since $x_{n_k} \to 0$ weakly,

$$\left(\overline{\lim_{n\to\infty}} \|Tx_n\|\right)^2 = \lim_{k\to\infty} \|Tx_{n_k}\|^2 = \text{l.i.m.}_k \|Tx_{n_k}\|^2 = 0.$$

So we have $\overline{\lim}_{n\to\infty} \|Tx_n\| = 0$.

(c) is an immediate consequence of (b).

(d) Let $\widehat{K}$ be the limit operator associated with $K \in K(H)$. Since $\widehat{K} = 0$, we get

$$\left\|\widehat{T}\right\| = \left\|\widehat{T} + \widehat{K}\right\| \le \|T + K\| \text{ for all } K \in K(H).$$

This implies $\left\|\widehat{T}\right\| \le \|T\|_{\text{ess}}$. For the reverse inequality, recall [3, p. 94] that

$$\|T\|_{\text{ess}} = \sup\left\{\overline{\lim_{n\to\infty}} \|Tx_n\| : \|x_n\| = 1,\ \forall n \in \mathbb{N},\ \text{and } x_n \to 0 \text{ weakly}\right\}.$$

It follows that for an arbitrary $\varepsilon > 0$, there exists a sequence $\{x_n\}$ in H such that $\|x_n\| = 1$ $(\forall n \in \mathbb{N})$, $x_n \to 0$ weakly, and

$$\overline{\lim_{n\to\infty}} \|Tx_n\| \ge \|T\|_{\text{ess}} - \varepsilon.$$

Consequently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} \|Tx_{n_k}\| \ge \|T\|_{\text{ess}} - \varepsilon.$$

On the other hand,

$$\left\|\widehat{T}\right\| = \sup\left\{\left(\text{l.i.m.}_n \|Tx_n\|^2\right)^{\frac{1}{2}} : \text{l.i.m.}_n \|x_n\|^2 = 1 \text{ and } x_n \to 0 \text{ weakly}\right\}.$$

As $\text{l.i.m.}_k \|x_{n_k}\|^2 = 1$ and $x_{n_k} \to 0$ $(k \to \infty)$ weakly, by the preceding identity we can write

$$\left\|\widehat{T}\right\| \ge \lim_{k\to\infty} \|Tx_{n_k}\| \ge \|T\|_{\text{ess}} - \varepsilon.$$

Since ε is arbitrary, we have $\left\|\widehat{T}\right\| \ge \|T\|_{\text{ess}}$. □

Lemma 2.3. (a) *Let A, B be in $B(H)$ and assume that $\|A^n x\| \to 0$ and $\|B^{*n} x\| \to 0$ for all $x \in H$. Then, for an arbitrary $K \in K(H)$ we have*

$$\lim_{n\to\infty} \|A^n K B^n\| = \lim_{n\to\infty} \|A^n K\| = \lim_{n\to\infty} \|K B^n\| = 0.$$

(b) *Let A and B^* be two essentially isometric operators on H. Assume that either*

$$\lim_{n\to\infty} \|A^n T B^n\| = 0,\ \ \lim_{n\to\infty} \|A^n T\| = 0,\ \ \textit{or}\ \ \lim_{n\to\infty} \|T B^n\| = 0.$$

Then, T is a compact operator.

Proof. (a) Let us prove the identity $\lim_{n\to\infty} \|A^n K B^n\| = 0$. The proofs of other identities are similar. For $x, y \in H$, let $x \otimes y$ be the rank-one operator on H defined by

$$x \otimes y : w \mapsto \langle w, y\rangle x, \quad w \in H.$$

Since finite rank operators are dense (in operator norm) in $K(H)$, we may assume that K is a finite rank operator, say, $K = \sum_{i=1}^{N} x_i \otimes y_i$, where $x_i, y_i \in H$ $(i = 1, ..., N)$. Consequently, we can write

$$\begin{aligned}\|A^n K B^n\| &= \left\| \sum_{i=1}^{N} A^n x_i \otimes B^{*n} y_i \right\| \\ &\leq \sum_{i=1}^{N} \|A^n x_i\| \, \|B^{*n} y_i\| \to 0 \quad (n \to \infty).\end{aligned}$$

(b) Let $\widehat{A}$, $\widehat{T}$, and $\widehat{B}$ be the limit operators associated with A, T, and B, respectively. By Proposition 2.2 , $\widehat{A}$ and $\widehat{B}^*$ are isometries. Since the map $T \mapsto \widehat{T}$ is a contractive homomorphism, we get

$$\left\| \widehat{T} \right\| = \left\| \widehat{A}^n \widehat{T} \widehat{B}^n \right\| \leq \|A^n T B^n\| \to 0 \quad (n \to \infty).$$

Hence $\widehat{T} = 0$. By Proposition 2.2, T is a compact operator. In the same way, we can see that if either $\|A^n T\| \to 0$ or $\|T B^n\| \to 0$, then T is a compact operator. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1 If $T = T_0 + K$, where $AT_0B = T_0$ and $K \in K(H)$, then

$$A^n T B^n = T_0 + A^n K B^n \text{ for all } n \in \mathbb{N}.$$

By Lemma 2.3, $\|A^n K B^n\| \to 0$ and therefore $\|A^n T B^n - T_0\| \to 0$. Now, assume that there exists $T_0 \in B(H)$ such that $\|A^n T B^n - T_0\| \to 0$. Since

$$\left\| A^{n+1} T B^{n+1} - AT_0 B \right\| \to 0,$$

we have $AT_0B = T_0$ which implies $A^n T_0 B^n = T_0$ for all $n \in \mathbb{N}$. Further, since

$$\|A^n (T - T_0) B^n\| \to 0,$$

by Lemma 2.3, $T - T_0$ is a compact operator. So we have $T = T_0 + K$, where $K \in K(H)$. □

As an immediate consequence of Theorem 2.1 we have the following:

Corollary 2.4. *Let $A \in B(H)$ and assume that $I - AA^* \in K(H)$ and $\|A^{*n} x\| \to 0$ for all $x \in H$. If $T \in B(H)$, then the sequence $\{A^{*n} T A^n\}$ converges in operator norm if and only if we have the decomposition*

$$T = T_0 + K,$$

where $A^* T_0 A = T_0$ and $K \in K(H)$.

Notice that the operator $I - SS^*$ is one dimensional and $\|S^{*n} f\| \to 0$ for all $f \in H^2$. By taking $A = S$ in Corollary 2.4, we obtain Feintuch's result mentioned above.

Let $\phi, \psi \in L^\infty$ and assume that one of the functions ϕ, ψ is a trigonometric polynomial, say, $\psi = \sum_{-N}^{N} c_k e^{ik\theta}$. Then as

$$\mathcal{T}_\psi = \sum_{k=1}^{N} c_{-k} S^{*k} + \sum_{k=0}^{N} c_k S^k,$$

$S^{*n}\mathcal{T}_\phi S^{*k}S^n = S^{*k}\mathcal{T}_\phi$ $(\forall n \geq k)$, and $S^{*n}\mathcal{T}_\phi S^k S^n = \mathcal{T}_\phi S^k$ $(k = 0, 1, ...)$, we have

$$S^{*n}\mathcal{T}_\phi \mathcal{T}_\psi S^n = \sum_{k=1}^{N} c_{-k} S^{*k}\mathcal{T}_\phi + \sum_{k=0}^{N} c_k \mathcal{T}_\phi S^k \quad \text{for all } n \geq N.$$

If $\phi = \sum_{-N}^{N} c_k e^{ik\theta}$, then as $S^{*n}S^{*k}\mathcal{T}_\psi S^n = S^{*k}\mathcal{T}_\psi$ $(k = 0, 1, ...)$ and $S^{*n}S^k\mathcal{T}_\psi S^n = \mathcal{T}_\psi S^k$ $(\forall n \geq k)$, we have

$$S^{*n}\mathcal{T}_\phi \mathcal{T}_\psi S^n = \sum_{k=1}^{N} c_{-k} S^{*k}\mathcal{T}_\psi + \sum_{k=0}^{N} c_k \mathcal{T}_\psi S^k \quad \text{for all } n \geq N.$$

Therefore, if one of the functions ϕ, ψ is continuous, then $\mathcal{T}_\phi \mathcal{T}_\psi$ is a uniformly asymptotically Toeplitz operator. Further, if $\psi = h + f$, where $h \in H^\infty$ and $f \in C(\mathbb{T})$, then as $\mathcal{T}_\phi \mathcal{T}_h = \mathcal{T}_{\phi h}$ we get

$$\begin{aligned} S^{*n}\mathcal{T}_\phi \mathcal{T}_\psi S^n &= S^{*n}\mathcal{T}_\phi (\mathcal{T}_h + \mathcal{T}_f) S^n \\ &= S^{*n}\mathcal{T}_{\phi h} S^n + S^{*n}\mathcal{T}_\phi \mathcal{T}_f S^n \\ &= \mathcal{T}_{\phi h} + S^{*n}\mathcal{T}_\phi \mathcal{T}_f S^n. \end{aligned}$$

It follows that $\mathcal{T}_\phi \mathcal{T}_\psi$ is a uniformly asymptotically Toeplitz operator for all $\phi \in L^\infty$ and $\psi \in H^\infty + C(\mathbb{T})$ (recall that the algebraic sum $H^\infty + C(\mathbb{T})$ is a uniformly closed subalgebra of L^∞ and sometimes called a *Douglas algebra*). Similarly, we can see that if $\phi = \overline{h} + f$, where $h \in H^\infty$ and $f \in C(\mathbb{T})$, then $\mathcal{T}_\phi \mathcal{T}_\psi$ is also a uniformly asymptotically Toeplitz operator.

It is known [2, Theorem 4] that if $\phi, \psi \in L^\infty$, then $S^{*n}\mathcal{T}_\phi \mathcal{T}_\psi S^n \to \mathcal{T}_{\phi\psi}$ strongly. From this and from Feintuch's result (or from Corollary 2.4) it follows that the operator $\mathcal{T}_\phi \mathcal{T}_\psi$ is uniformly asymptotically Toeplitz if and only if $\mathcal{T}_\phi \mathcal{T}_\psi - \mathcal{T}_{\phi\psi}$ is a compact operator. By the Axler–Chang–Sarason–Volberg theorem [1,26], this is the case if and only if

$$H^\infty\left[\overline{\phi}\right] \cap H^\infty[\psi] \subseteq H^\infty + C(\mathbb{T}),$$

where $H^\infty[\varphi]$ denotes the uniformly closed subalgebra of L^∞ generated by $\varphi \in L^\infty$ and H^∞.

For a given symbol $\varphi \in L^\infty$, the *Hankel operator* $\mathcal{H}_\varphi$ on H^2 is defined by

$$\mathcal{H}_\varphi f = P_+ J(\varphi f),$$

where J is a *flip* operator on L^2, that is, $Jh(z) = h(\overline{z})$. It is well known that $\mathcal{H} \in B(H^2)$ is a Hankel operator if and only if $S^*\mathcal{H} = \mathcal{H}S$. Hartman's theorem [25, Theorem 2.2.5] characterizes those $\varphi \in L^\infty$ for which $\mathcal{H}_\varphi$ is compact. This is the case if and only if $\varphi \in H^\infty + C(\mathbb{T})$.

Proposition 2.5. *A Hankel operator is uniformly asymptotically Toeplitz if and only if it is compact.*

Proof. Assume that a Hankel operator $\mathcal{H}$ is uniformly asymptotically Toeplitz. By Feintuch's result (or by Corollary 2.4), $\mathcal{H} = \mathcal{T}_\varphi + K$, where $\mathcal{T}_\varphi$ is a Teoplitz operator with symbol $\varphi \in L^\infty$ and $K \in K(H^2)$. We have $S^*\mathcal{H} = \mathcal{T}_{\overline{z}\varphi} + S^*K$ and $\mathcal{H}S = \mathcal{T}_{z\varphi} + KS$. Since $S^*\mathcal{H} = \mathcal{H}S$, the operator $\mathcal{T}_{\overline{z}\varphi - z\varphi}$ is compact. It is well known that the only compact Teoplitz operator is 0. It follows that $\varphi = 0$. □

In Corollary 2.4, compactness condition of the operator $I - AA^*$ is essential. To see this, let $A = V$ be the Volterra integral operator on $H = L^2[0,1]$. Then $I - VV^* \notin K(H)$ and as $\|V^n\| \to 0$, we have $\|V^{*n}x\| \to 0$ for all $x \in H$. Since $\|V^{*n}TV^n\| \to 0$ for all $T \in B(H)$, the equation $V^*T_0V = T_0$ has only zero solution. If the conclusion of Corollary 2.4 were true, we would get $B(H) \subseteq K(H)$ which is a contradiction.

Note that if T is a contraction on H, then Corollary 2.4 can be applied to the model operator of T [21,22] in the case when the operator T satisfies the following conditions: (1) $\|T^{*n}x\| \to 0$ for all $x \in H$; (2) The defect operator $D_{T^*} := (I - TT^*)^{\frac{1}{2}}$ is compact.

Let X be a Banach space. Recall that an operator $T \in B(X)$ is said to be *almost periodic* if for every $x \in X$, the orbit $\{T^n x : n \in \mathbb{N}\}$ is relatively compact. Clearly, an almost periodic operator is power bounded, that is, $\sup_{n\geq 0} \|T^n\| < \infty$. If $T \in B(X)$ is an almost periodic operator, then by the Jacobs-Glicksberg-de Leeuw decomposition theorem [8, Ch.I, Theorem 1.15], every $x \in X$ can be written as $x = x_0 + x_1$, where

$$\|T^n x_0\| \to 0 \text{ and } x_1 \in \overline{\text{span}}\{y \in X : \exists \xi \in \mathbb{T},\ Ty = \xi y\}.$$

From now on, for a given $T \in B(X)$, the left and the right multiplication operators on $B(X)$ will be denoted by L_T and R_T, respectively.

Proposition 2.6. *Let $A \in B(H)$ and assume that $I - AA^* \in K(H)$, $\|A^n x\| \to 0$, and $\|A^{*n}x\| \to 0$ for all $x \in H$. For an arbitrary $T \in B(H)$, the following assertions are equivalent:*

(a) *$\{A^{*n}TA^n : n \in \mathbb{N}\}$ is relatively compact in the operator norm topology.*
(b) $\lim_{n\to\infty} \|A^{*n}TA^n\| = 0.$
(c) *T is a compact operator.*

Proof. (a) ⇒ (b) Let E be the set of all $Q \in B(H)$ such that

$$\{(L_{A^*}R_A)^n Q : n \in \mathbb{N}\}$$

is relatively compact in the operator norm topology. Since the operator $L_{A^*}R_A$ is power bounded, E is a closed (in operator norm) $L_{A^*}R_A$-invariant subspace. Consequently, $L_{A^*}R_A\mid_E$, the restriction of $L_{A^*}R_A$ to E is an almost periodic operator. Since $T \in E$, by the Jacobs-Glicksberg-de Leeuw decomposition theorem, $T = T_0 + T_1$, where

$$\lim_{n\to\infty} \|A^{*n}T_0A^n\| = 0 \text{ and } T_1 \in \overline{\text{span}}^{\|\cdot\|}\{Q \in E : \exists \xi \in \mathbb{T},\ A^*QA = \xi Q\}.$$

We must show that $T_1 = 0$. For this, it suffices to show that the identity $A^*QA = \xi Q$ ($\xi \in \mathbb{T}$) implies $Q = 0$. Indeed, since

$$A^{*n}QA^n = \xi^n Q \quad (\forall n \in \mathbb{N}),$$

we get

$$|\langle Qx, y\rangle| = |\langle QA^n x, A^n y\rangle| \leq \|Q\|\,\|A^n x\|\,\|A^n y\| \to 0, \quad \forall x, y \in H.$$

Hence $Q = 0$.

(b) ⇒ (c) ⇒ (a) are obtained from Lemma 2.3. □

Next, we have the following:

Theorem 2.7. *Let A and B^* be two essentially isometric contractions on H and assume that $\|A^n x\| \to 0$ and $\|B^{*n} x\| \to 0$ for all $x \in H$. Then, for an arbitrary $T \in B(H)$ we have*

$$\|T\|_{ess} = \lim_{n\to\infty} \|A^n T B^n\| = \lim_{n\to\infty} \|A^n T\| = \lim_{n\to\infty} \|T B^n\|.$$

Proof. If $K \in K(H)$, then by Lemma 2.3, $\|A^n K B^n\| \to 0$. Since

$$\|A^n (T+K) B^n\| \leq \|T+K\|,$$

we have

$$\lim_{n\to\infty} \|A^n T B^n\| \leq \|T\|_{\text{ess}}.$$

For the reverse inequality, let $\widehat{A}$, $\widehat{T}$, and $\widehat{B}$ be the limit operators associated with A, T, and B, respectively. By Proposition 2.2, $\widehat{A}$ and $\widehat{B}^*$ are isometries. By using the same proposition again, we can write

$$\|T\|_{\text{ess}} = \left\|\widehat{T}\right\| = \left\|\widehat{A}^n \widehat{T} \widehat{B}^n\right\| \leq \|A^n T B^n\| \quad \text{for all } n \in \mathbb{N}.$$

So we have

$$\|T\|_{\text{ess}} \leq \lim_{n\to\infty} \|A^n T B^n\|.$$

In the same way, we can see that

$$\|T\|_{\text{ess}} = \lim_{n\to\infty} \|A^n T\| = \lim_{n\to\infty} \|T B^n\|. \qquad \square$$

As an immediate consequence of Theorem 2.7, we have the following:

Corollary 2.8. *Let $A \in B(H)$ be a contraction and assume that $I - AA^* \in K(H)$ and $\|A^{*n} x\| \to 0$ for all $x \in H$. Then, for an arbitrary $T \in B(H)$,*

$$\|T\|_{ess} = \lim_{n\to\infty} \|A^{*n} T A^n\| = \lim_{n\to\infty} \|A^{*n} T\| = \lim_{n\to\infty} \|T A^n\|.$$

In particular, for an arbitrary $T \in B\left(H^2\right)$,

$$\|T\|_{ess} = \lim_{n\to\infty} \|S^{*n} T S^n\| = \lim_{n\to\infty} \|S^{*n} T\| = \lim_{n\to\infty} \|T S^n\|.$$

Recall that a contraction T on H is said to be *completely non-unitary* (c.n.u.) if it has no proper reducing subspace on which it acts as a unitary operator. If T is a c.n.u. contraction, then $f(T)$ ($f \in H^\infty$) can be defined by the Nagy-Foiaş functional calculus [21, Ch.III]. A contraction T on H is a C_0-*contraction* if it is c.n.u. and there exists a nonzero function $f \in H^\infty$ such that $f(T) = 0$. If T is a C_0-contraction, then $\|T^n x\| \to 0$ and $\|T^{*n} x\| \to 0$ for all $x \in H$ [21, Ch.III, Proposition 4.2]. In [13, Theorem 3.4], it was proved that if T is an essentially isometric C_0-contraction (an essentially isometric C_0-contraction is essentially unitary), then $f(T)$ is a compact operator if and only if $\lim_{n\to\infty} \|T^n f(T)\| = 0$. By Corollary 2.8, we have the following generalization of this result.

Corollary 2.9. *Let T be an essentially isometric contraction on H and assume that $\|T^n x\| \to 0$ for all $x \in H$. Then, for an arbitrary $f \in H^\infty$ one has*

$$\|f(T)\|_{ess} = \lim_{n\to\infty} \|T^n f(T)\|.$$

It is well known [6, Corollary 7.13] that every Toeplitz operator $\mathcal{T}_\varphi$ with symbol $\varphi \in L^\infty$ satisfies

$$\|\mathcal{T}_\varphi\|_{\text{ess}} = \|\mathcal{T}_\varphi\|.$$

Notice that Corollary 2.8 contains this fact.

For A, B in $B(H)$, we put

$$\mathfrak{T}_{A,B} = \{T \in B(H) : ATB = T\}.$$

Corollary 2.10. *Assume that the operators A, B in $B(H)$ satisfy the hypotheses of Theorem 2.7. Then, for an arbitrary $K \in K(H)$ we have*

$$\|K + \mathfrak{T}_{A,B}\| \geq \frac{1}{2}\|K\|.$$

In the case $AB = I$, this estimate is the best possible.

Proof. Assume that there exists $K \in K(H)$ such that

$$\|K + \mathfrak{T}_{A,B}\| < \frac{1}{2}\|K\|.$$

Then, there exists $T \in \mathfrak{T}_{A,B}$ such that

$$\|K + T\| < \frac{1}{2}\|K\|.$$

By Theorem 2.7, $\|T\| = \|T\|_{\text{ess}}$ which implies $\|T\| \leq \|K + T\|$. Consequently, we can write

$$\|K\| \leq \|K + T\| + \|T\| \leq 2\|K + T\| < \|K\|,$$

which is a contradiction.

In the case $AB = I$ we have $I \in \mathfrak{T}_{A,B}$. Let $K = x \otimes x$, where $x \in H$ and $\|x\| = 1$. Then $\|K\| = 1$ and for $T = -\frac{1}{2}I$ we have $\|K + T\| = \frac{1}{2}$. □

Let $\mathfrak{T}$ be the space of all Toeplitz operators. By taking $A = S^*$ and $B = S$ in Corollary 2.10, we have

$$\|K + \mathfrak{T}\| \geq \frac{1}{2}\|K\| \text{ for all } K \in K\left(H^2\right),$$

where this estimate is the best possible.

3. The Essential Norm

In this section, we present some results related to the essential norm of some class of operators.

Let T be a contraction on H and assume that

$$\lim_{n\to\infty}\|T^n x\| = \lim_{n\to\infty}\|T^{*n}x\| = 0 \text{ for all } x \in H.$$

In addition, if

$$\dim(I - TT^*)H = \dim(I - T^*T)H = 1,$$

then by the Model Theorem of Nagy–Foiaş [21, Ch.VI], T is unitary equivalent to its model operator

$$S_\theta = P_\theta S\mid_{H^2_\theta}$$

acting on the *model space*

$$H_\theta^2 = H^2 \ominus \theta H^2,$$

where θ is an inner function and P_θ is the orthogonal projection from H^2 onto H_θ^2. Beurling's theorem (for instance, see [6,24]) says that these spaces are generic invariant subspaces for the backward shift operator

$$(S^* f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in H^2.$$

Notice also that

$$S_\theta = \left(S^* |_{H_\theta^2}\right)^*.$$

Let θ be an inner function and let S_θ be the model operator on the model space H_θ^2. For an arbitrary $f \in H^\infty$, we can define the operator

$$f(S_\theta) = P_\theta f(S) |_{H_\theta^2}$$

which is unitary equivalent to $f(T)$. The map $f \mapsto f(S_\theta)$ is linear, multiplicative, and by the Nehari formula [24, Lecture VIII],

$$\|f(S_\theta)\| = \operatorname{dist}\left(\overline{\theta} f, H^\infty\right).$$

Let us mention Sarason's theorem [24, Lecture VIII] which asserts that an operator $Q \in B\left(H_\theta^2\right)$ is a commutant of S_θ if and only if $Q = f(S_\theta)$ for some $f \in H^\infty$. Let us also mention that the classical theorem of Hartman and Sarason [24, Lecture VIII] classifies compactness of the operators $f(S_\theta)$. The operator $f(S_\theta)$ $(f \in H^\infty)$ is compact if and only if $\overline{\theta} f \in H^\infty + C(\mathbb{T})$.

We have the following quantitative generalization of the Hartman-Sarason theorem.

Theorem 3.1. *Let θ be an inner function and let S_θ be the model operator on the model space H_θ^2. Then, for an arbitrary $f \in H^\infty$ we have*

$$\|f(S_\theta)\|_{ess} = dist\left(\overline{\theta} f, H^\infty + C(\mathbb{T})\right).$$

For the proof, we need the following two lemmas.

Lemma 3.2. *Let $\{E_n\}$ be an increasing sequence of closed subspaces of a Banach space X. Then, for an arbitrary $x \in X$ we have*

$$\lim_{n\to\infty} dist(x, E_n) = dist\left(x, \overline{\bigcup_{n=1}^{\infty} E_n}\right).$$

Proof. If $x \in X$, then the sequence $\{\operatorname{dist}(x, E_n)\}$ is decreasing. Let

$$\alpha := \lim_{n\to\infty} \operatorname{dist}(x, E_n) = \inf_n \operatorname{dist}(x, E_n).$$

Since

$$\operatorname{dist}\left(x, \overline{\bigcup_{n=1}^{\infty} E_n}\right) \le \operatorname{dist}(x, E_n),$$

we have

$$\operatorname{dist}\left(x, \overline{\bigcup_{n=1}^{\infty} E_n}\right) \le \alpha.$$

If

$$\operatorname{dist}\left(x, \overline{\bigcup_{n=1}^{\infty} E_n}\right) < \alpha,$$

then $\|x - x_0\| < \alpha$ for some $x_0 \in \bigcup_{n=1}^{\infty} E_n$. Consequently, $x_0 \in E_{n_0}$ for some n_0. Hence $\operatorname{dist}(x, E_{n_0}) < \alpha$. This contradicts dist $(x, E_{n_0}) \geq \alpha$. □

Lemma 3.3. *For an arbitrary $\varphi \in L^\infty$ we have*

$$\lim_{n\to\infty} dist\left(\varphi, \overline{z}^n H^\infty\right) = dist\left(\varphi, H^\infty + C\left(\mathbb{T}\right)\right).$$

Proof. We know that $H^\infty + C(\mathbb{T})$ is a uniformly closed subalgebra of L^∞ generated by $\overline{z}$ and H^∞. If $E_n := \overline{z}^n H^\infty$ $(n = 0, 1, \ldots)$, then $\{E_n\}$ is an increasing sequence of closed subspaces of L^∞. Since

$$H^\infty + C(\mathbb{T}) = \overline{\operatorname{span}}_{L^\infty}\{\overline{z}^n H^\infty : n \geq 0\}$$

and

$$\begin{aligned}\overline{z}^n f_1 + \overline{z}^m f_2 &= \overline{z}^{n+m}\left(z^m f_1 + z^n f_2\right)\\ &\in \overline{z}^{n+m} H^\infty = E_{n+m}, \quad \forall f_1, f_2 \in H^\infty,\end{aligned}$$

we have

$$\overline{\bigcup_{n=1}^{\infty} E_n} = H^\infty + C(\mathbb{T}).$$

Applying Lemma 3.2 to the subspaces $\{E_n\}$, we obtain our result. □

Now, we can prove Theorem 3.1.

Proof of Theorem 3.1 As we have noted above, the model operator S_θ is an essentially unitary contraction on H_θ^2. Moreover, $\|S_\theta^n h\| \to 0$ and $\|S_\theta^{*n} h\| \to 0$ for all $h \in H_\theta^2$. By Corollary 2.8,

$$\|T\|_{\text{ess}} = \lim_{n\to\infty} \|T S_\theta^n\| \ \text{ for all } T \in B\left(H_\theta^2\right).$$

By taking $T = f(S_\theta)$ we have

$$\|f(S_\theta)\|_{\text{ess}} = \lim_{n\to\infty} \|f(S_\theta) S_\theta^n\|.$$

On the other hand, by the Nehari formula mentioned above and by Lemma 3.3, we can write

$$\begin{aligned}\lim_{n\to\infty} \|f(S_\theta) S_\theta^n\| &= \lim_{n\to\infty} \operatorname{dist}\left(\overline{\theta} f z^n, H^\infty\right)\\ &= \lim_{n\to\infty} \operatorname{dist}\left(\overline{\theta} f, \overline{z}^n H^\infty\right)\\ &= \operatorname{dist}\left(\overline{\theta} f, H^\infty + C(\mathbb{T})\right).\end{aligned}$$

So we have $\|f(S_\theta)\|_{\text{ess}} = \operatorname{dist}\big(\overline{\theta} f, H^\infty + C(\mathbb{T})\big)$. □

Let $\mathcal{H}_\varphi$ be the Hankel operator with symbol $\varphi \in L^\infty$. A classical result of Nehari [24, Lecture VIII] gives distance from φ to H^∞ as the norm of $\mathcal{H}_\varphi$;

$$\|\mathcal{H}_\varphi\| = \operatorname{dist}(\varphi, H^\infty). \tag{3.1}$$

The Adamyan–Arov–Krein formula [25, p.213] gives distance from φ to $H^\infty + C(\mathbb{T})$ as the essential norm of $\mathcal{H}_\varphi$;

$$\|\mathcal{H}_\varphi\|_{\text{ess}} = \operatorname{dist}(\varphi, H^\infty + C(\mathbb{T})).$$

This formula is a quantitative generalization of the Hartman theorem mentioned above. Note that the Adamyan–Arov–Krein formula can be obtained from the above mentioned results. Indeed, taking into account Corollary 2.8, Lemma 3.3, formula (3.1), and the fact that $S^{*n}\mathcal{H}_\varphi = \mathcal{H}_{z^n\varphi}$ $(\forall n \in \mathbb{N})$, we get

$$\begin{aligned}\|\mathcal{H}_\varphi\|_{\text{ess}} &= \lim_{n\to\infty} \|S^{*n}\mathcal{H}_\varphi\| \\ &= \lim_{n\to\infty} \|\mathcal{H}_{z^n\varphi}\| \\ &= \lim_{n\to\infty} \operatorname{dist}(z^n\varphi, H^\infty) \\ &= \lim_{n\to\infty} \operatorname{dist}(\varphi, \overline{z}^n H^\infty) \\ &= \operatorname{dist}(\varphi, H^\infty + C(\mathbb{T})).\end{aligned}$$

Let $\mathcal{C}_\phi$ be the composition operator on H^2. Since $\mathcal{C}_\phi S^n = \phi^n \mathcal{C}_\phi$ $(\forall n \in \mathbb{N})$, by Corollary 2.8 we have the following:

Corollary 3.4. *For an arbitrary composition operator $\mathcal{C}_\phi$ on H^2,*

$$\|\mathcal{C}_\phi\|_{ess} = \lim_{n\to\infty} \|\phi^n \mathcal{C}_\phi\|.$$

It follows from Corollary 3.4 that if $\|\phi\|_\infty < 1$, then $\mathcal{C}_\phi$ is a compact operator [18, Theorem 5.1.16]. Moreover, if $\mathcal{C}_\phi$ is a compact operator, then $\left|\widetilde{\phi}\left(e^{it}\right)\right| < 1$ a.e. [18, Theorem 5.1.17], where $\widetilde{\phi}$ is the boundary value of ϕ.

As usual, $\sigma(T)$ will denote the spectrum of $T \in B(H)$. Given $T \in B(H)$, we let A_T denote the closure in the uniform operator topology of all polynomials in T. Then, A_T is a commutative unital Banach algebra. The Gelfand space of A_T can be identified with $\sigma_{A_T}(T)$, the spectrum of T with respect to the algebra A_T. Since $\sigma(T)$ is a (closed) subset of $\sigma_{A_T}(T)$, for every $\lambda \in \sigma(T)$ there is a multiplicative functional ϕ_λ on A_T such that $\phi_\lambda(T) = \lambda$. By $\widehat{Q}$ we will denote the Gelfand transform of $Q \in A_T$. Instead of $\widehat{Q}(\phi_\lambda)\,(= \phi_\lambda(Q))$, where $\lambda \in \sigma(T)$, we will use the notation $\widehat{Q}(\lambda)$. It follows from the Shilov Theorem [6, Theorem 2.54] that if T is a contraction, then

$$\sigma_{A_T}(T) \cap \mathbb{T} = \sigma(T) \cap \mathbb{T}.$$

The following result was obtained in [19] (see also, [27]).

Theorem 3.5. *If T is a contraction on a Hilbert space, then for an arbitrary $Q \in A_T$ we have*

$$\lim_{n\to\infty} \|T^n Q\| = \sup_{\xi\in\sigma(T)\cap\mathbb{T}} \left|\widehat{Q}(\xi)\right|.$$

For a non-empty closed subset Γ of $\mathbb{T}$, by H^∞_Γ we will denote the set of all those functions f in H^∞ that have a continuous extension $\widetilde{f}$ to $\mathbb{D} \cup \Gamma$. Clearly, H^∞_Γ is a closed subspace of H^∞. It follows from the general theory of H^p spaces that if Γ has positive Lebesgue measure and $f \in H^\infty_\Gamma$ is not identically zero, then $\widetilde{f}$ cannot vanish identically on Γ.

If T is a contraction on a Hilbert space H, then there is a canonical decomposition of H into two T-reducing subspaces H_0 and H_u such that

$H = H_0 \oplus H_u$, $T_0 := T\mid_{H_0}$ is c.n.u. and $T_u := T\mid_{H_u}$ is unitary [21, Ch.I, Theorem 3.2]. It can be seen that

$$\sigma(T_u) \subseteq \sigma(T) \cap \mathbb{T}.$$

Let f be in $H^\infty_{\sigma(T)\cap\mathbb{T}}$ with continuous extension $\widetilde{f}$ to $\mathbb{D} \cup (\sigma(T) \cap \mathbb{T})$. As in [11], we can define $f(T) \in B(H)$ by

$$f(T) = f(T_0) \oplus \widetilde{f}(T_u),$$

where $f(T_0)$ is given by the Nagy–Foiaş functional calculus and

$$\widetilde{f}(T_u) = \left(\widetilde{f}\mid_{\sigma(T)\cap\mathbb{T}}\right)(T_u).$$

It can be seen that

$$\|f(T)\| \le \|f\|_\infty \text{ for all } f \in H^\infty_{\sigma(T)\cap\mathbb{T}}.$$

Further, by the Gamelin–Garnett theorem [10], there exists a sequence $\{f_n\}$ in H^∞ such that each f_n has an analytic extension g_n to a neighborhood O_n of $\mathbb{D} \cup (\sigma(T) \cap \mathbb{T})$ and

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0.$$

Then, $g_n(T)$ can be defined by the Riesz–Dunford functional calculus. Since $f_n(T) = g_n(T) \in A_T$ and

$$\|f_n(T) - f(T)\| \le \|f_n - f\|_\infty \to 0,$$

we have that $f(T) \in A_T$. Moreover,

$$\widehat{f(T)}(\xi) = \widetilde{f}(\xi) \text{ for all } \xi \in \sigma(T) \cap \mathbb{T}.$$

As a consequence of Theorem 3.5, we have the following:

Corollary 3.6. *Let T be a contraction on a Hilbert space. If $f \in H^\infty_{\sigma(T)\cap\mathbb{T}}$ with continuous extension $\widetilde{f}$ to $\mathbb{D} \cup (\sigma(T) \cap \mathbb{T})$, then*

$$\lim_{n\to\infty} \|T^n f(T)\| = \sup_{\xi\in\sigma(T)\cap\mathbb{T}} \left|\widetilde{f}(\xi)\right|.$$

Corollary 3.6 combined with Corollary 2.9 yields the next result.

Corollary 3.7. *Let T be an essentially isometric contraction on H and assume that $\|T^n x\| \to 0$ for all $x \in H$. If $f \in H^\infty_{\sigma(T)\cap\mathbb{T}}$ with continuous extension $\widetilde{f}$ to $\mathbb{D} \cup (\sigma(T) \cap \mathbb{T})$, then*

$$\|f(T)\|_{ess} = \sup_{\xi\in\sigma(T)\cap\mathbb{T}} \left|\widetilde{f}(\xi)\right|.$$

In particular, if f is a function from the disc-algebra, then

$$\|f(T)\|_{ess} = \sup_{\xi\in\sigma(T)\cap\mathbb{T}} |f(\xi)|.$$

Corollary 3.7 generalizes [13, Corollary 2.3].

Let θ be an inner function and let S_θ be the model operator on the model space H^2_θ. If $f \in H^\infty_{\sigma(S_\theta)\cap\mathbb{T}}$ with continuous extension $\widetilde{f}$ to $\mathbb{D} \cup (\sigma(S_\theta) \cap \mathbb{T})$, then by Corollary 3.6,

$$\lim_{n\to\infty} \|S^n_\theta f(S_\theta)\| = \sup_{\xi\in\sigma(S_\theta)\cap\mathbb{T}} \left|\widetilde{f}(\xi)\right|.$$

On the other hand, by Corollary 2.8,

$$\|f(S_\theta)\|_{\text{ess}} = \lim_{n\to\infty} \|f(S_\theta) S^n_\theta\|.$$

So we have

$$\|f(S_\theta)\|_{\text{ess}} = \sup_{\xi\in\sigma(S_\theta)\cap\mathbb{T}} \left|\widetilde{f}(\xi)\right|.$$

Corollary 3.8. *Let θ be an inner function and let S_θ be the model operator on the model space H^2_θ. For an arbitrary $f \in H^\infty_{\sigma(S_\theta)\cap\mathbb{T}}$ with continuous extension $\widetilde{f}$ to $\mathbb{D} \cup (\sigma(S_\theta) \cap \mathbb{T})$, we have*

$$\|f(S_\theta)\|_{ess} = \sup_{\xi\in\sigma(S_\theta)\cap\mathbb{T}} \left|\widetilde{f}(\xi)\right|.$$

In particular, if f is a function from the disc-algebra, then

$$\|f(S_\theta)\|_{ess} = \sup_{\xi\in\sigma(S_\theta)\cap\mathbb{T}} |f(\xi)|.$$

4. The Sequence $\left\{\frac{1}{n}\sum_{i=0}^{n-1} A^i T B^i\right\}$

In this section, we give some results concerning convergence in operator norm of the sequence $\left\{\frac{1}{n}\sum_{i=0}^{n-1} A^i T B^i\right\}$ for Hilbert space operators.

Let X be a Banach space. It is easy to check that if $T \in B(X)$ is power bounded, then

$$\overline{(T-I)X} = \left\{x \in X : \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=0}^{n-1} T^i x\right\| = 0\right\}.$$

The following result is well known (for instance, see [14, Ch.2]).

Proposition 4.1. *Let $T \in B(X)$ be power bounded and let E be the set of all $x \in X$ such that the sequence $\left\{\frac{1}{n}\sum_{i=0}^{n-1} T^i x\right\}$ converges in norm. Then, we have the decomposition*

$$E = \ker(T-I) \oplus \overline{(T-I)X}.$$

If X is reflexive, then $E = X$.

Lemma 4.2. *Let $T \in B(X)$ be power bounded and assume that*

$$\lim_{n\to\infty}\left\|T^{n+1}x - T^n x\right\| = 0 \text{ for some } x \in X.$$

(a) *If the sequence $\left\{\frac{1}{n}\sum_{i=0}^{n-1} T^i x\right\}$ converges in norm, then the sequence $\{T^n x\}$ converges in norm (to same element), too.*

(b) *If X is reflexive, then the sequence $\{T^n x\}$ converges in norm.*

Proof. (a) Notice that

$$F := \left\{ y \in X : \lim_{n\to\infty} \left\| T^{n+1}y - T^n y \right\| = 0 \right\}$$

is a closed T-invariant subspace and $x \in F$. Since T is power bounded and

$$\| T^n (T - I) y \| = \left\| T^{n+1} y - T^n y \right\| \to 0 \quad (\forall y \in F),$$

we have $\| T^n y \| \to 0$ for all $y \in \overline{(T - I)\, F}$. Now, let E be the set of all $y \in F$ such that the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i y \right\}$ converges in norm. Since $x \in E$, by Proposition 4.1, $x = x_0 + y_0$, where $Tx_0 = x_0$ and $y_0 \in \overline{(T - I)\, F}$. As $T^n x = x_0 + T^n y_0$ and $\| T^n y_0 \| \to 0$, we have $\| T^n x - x_0 \| \to 0$.

(b) If X is reflexive, then by Proposition 4.1 the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i x \right\}$ converges in norm for every $x \in X$. By (a), the sequence $\{T^n x\}$ converges in norm. □

Next, we have the following:

Theorem 4.3. *Let A and B^* be two essentially isometric operators on H and let $T \in B(H)$. Assume that:*

(i) $\| A^n x \| \to 0$ *and* $\| B^{*n} x \| \to 0$ *for all* $x \in H$.
(ii) $ATB - T \in K(H)$.

Then, the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} A^i T B^i \right\}$ converges in operator norm if and only if we have the decomposition $T = T_0 + K$, where $AT_0 B = T_0$ and $K \in K(H)$.

Proof. Assume that the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} A^i T B^i \right\}$ converges in operator norm. As $A^i T B^i = (L_A R_B)^i T$, the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} (L_A R_B)^i T \right\}$ converges in operator norm. Since $ATB - T \in K(H)$, by Lemma 2.3,

$$\lim_{n\to\infty} \left\| (L_A R_B)^{n+1} T - (L_A R_B)^n T \right\| = \lim_{n\to\infty} \| A^n (ATB - T) B^n \| = 0.$$

Notice also that the operator $L_A R_B$ is power bounded. Applying Lemma 4.2 to the operator $L_A R_B$ on the space $B(X)$, we obtain that the sequence $\{A^n T B^n\}$ converges in operator norm. By Theorem 2.1, $T = T_0 + K$, where $AT_0 B = T_0$ and $K \in K(H)$.

If $T = T_0 + K$, where $AT_0 B = T_0$ and $K \in K(H)$, then

$$\frac{1}{n} \sum_{i=0}^{n-1} A^i T B^i = T_0 + \frac{1}{n} \sum_{i=0}^{n-1} A^i K B^i.$$

By Lemma 2.3, $\| A^n K B^n \| \to 0$ and therefore $\left\| \frac{1}{n} \sum_{i=0}^{n-1} A^i K B^i \right\| \to 0$. Hence

$$\frac{1}{n} \sum_{i=0}^{n-1} A^i T B^i \to T_0 \text{ in operator norm.}$$

□

Theorem 4.3 has several corollaries.

Corollary 4.4. *Assume that the operators A, T in $B(H)$ satisfy the following conditions:*

(i) $I - AA^* \in K(H)$.
(ii) $\|A^{*n}x\| \to 0$ *for all* $x \in H$.
(iii) $A^*TA - T \in K(H)$.

Then, the sequence $\left\{\frac{1}{n}\sum_{i=0}^{n-1} A^{*i}TA^i\right\}$ *converges in operator norm if and only if we have the decomposition* $T = T_0 + K$, *where* $A^*T_0A = T_0$ *and* $K \in K(H)$.

Recall that $T \in B\left(H^2\right)$ is an *essentially Toeplitz operator* if

$$S^*TS - T \in K\left(H^2\right).$$

It is easy to see that the operator T is an essentially Toeplitz if and only if it is an essential commutant of the unilateral shift S. On the other hand, essential commutant of the unilateral shift is a C^*-algebra. Consequently, the set of all essentially Toeplitz operators is a C^*-algebra and therefore contains the C^*-algebra generated by all Toeplitz operators (for instance, see [17]).

Corollary 4.5. *An essentially Toeplitz operator T is a compact perturbation of a Toeplitz operator if and only if the sequence* $\left\{\frac{1}{n}\sum_{i=0}^{n-1} S^{*i}TS^i\right\}$ *converges in operator norm.*

In [23, Theorem 1.1], it was proved that if the composition operator $\mathcal{C}_\phi$ on H^2 is neither compact nor the identity, then $\mathcal{C}_\phi$ cannot be compact perturbation of a Toeplitz operator.

Corollary 4.6. *Assume that the composition operator $\mathcal{C}_\phi$ on H^2 is essentially Toeplitz. Then, the sequence* $\left\{\frac{1}{n}\sum_{i=0}^{n-1} S^{*i}\mathcal{C}_\phi S^i\right\}$ *converges in operator norm if and only if either $\mathcal{C}_\phi$ is compact or the identity operator.*

5. Banach Space Operators

In this section, we present some convergence theorems in Banach spaces.

Let X be a Banach space. For $T \in B(X)$ and $x \in X$, we define $\rho_T(x)$ to be the set of all $\lambda \in \mathbb{C}$ for which there exists a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in X such that

$$(zI - T)\,u(z) = x \text{ for all } z \in U_\lambda.$$

This set is open and contains the resolvent set of T. By definition, the *local spectrum* of T at $x \in X$, denoted by $\sigma_T(x)$, is the complement of $\rho_T(x)$, so it is a compact subset of $\sigma(T)$, the spectrum of T. This object is the most tractable if the operator T has the *single-valued extension property* (SVEP), i.e. for every open set U in $\mathbb{C}$, the only analytic function $u : U \to X$ for which the equation $(zI - T)\,u(z) = 0$ holds is the constant function $u \equiv 0$. If T has SVEP, then $\sigma_T(x) \neq \emptyset$, whenever $x \in X \setminus \{0\}$ [15, Proposition 1.2.16].

If T is power bounded, then clearly, $\sigma(T) \subset \overline{\mathbb{D}}$ and $\sigma_T(x) \cap \mathbb{T}$ consists of all $\xi \in \mathbb{T}$ such that the function $z \to (zI - T)^{-1} x$ $(|z| > 1)$ has no analytic extension to a neighborhood of ξ.

We mention the following classical result of Katznelson and Tzafriri [12, Theorem 1]. If $T \in B(X)$ is power bounded, then $\lim_{n\to\infty} \left\| T^{n+1} - T^n \right\| = 0$ if and only if $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$.

Notice that local spectrum of T may be "very small" with respect to its usual spectrum. To see this, let σ be a "small" clopen part of $\sigma(T)$, P_σ be the spectral projection associated with σ, and let $X_\sigma := P_\sigma X$. Then, X_σ is a closed T-invariant subspace of X and $\sigma(T|_{X_\sigma}) = \sigma$. Clearly, $\sigma_T(x) \subseteq \sigma$ for every $x \in X_\sigma$.

We have the following local version of the Katznelson–Tzafriri theorem [20, Theorem 4.2].

Theorem 5.1. *Let $T \in B(X)$ be power bounded and let $x \in X$. If $\sigma_T(x) \cap \mathbb{T} \subseteq \{1\}$, then*

$$\lim_{n\to\infty} \left\| T^{n+1}x - T^n x \right\| = 0.$$

In contrast with the Katznelson–Tzafriri theorem, the converse of Theorem 5.1 does not hold, in general. Indeed, if S^* is the backward shift operator on H^2, then as $\|S^{*n} f\| \to 0$, we have

$$\lim_{n\to\infty} \left\| S^{*(n+1)} f - S^{*n} f \right\| = 0 \text{ for all } f \in H^2.$$

On the other hand, since

$$(\lambda I - S^*)^{-1} f(z) = \frac{\lambda^{-1} f\left(\lambda^{-1}\right) - z f(z)}{1 - \lambda z} \quad (|\lambda| > 1),$$

$\sigma_{S^*}(f) \cap \mathbb{T}$ consists of all $\xi \in \mathbb{T}$ for which the function f has no analytic extension to a neighborhood of ξ (see, [7, p.24]).

Theorem 5.1 combined with Lemma 4.2 yields the next result.

Theorem 5.2. *Let $T \in B(X)$ be power bounded and assume that $\sigma_T(x) \cap \mathbb{T} \subseteq \{1\}$ for some $x \in X$. If the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i x \right\}$ converges in norm to $y \in X$, then $T^n x \to y$ in norm.*

As a consequence of Theorem 5.2, we have the following:

Corollary 5.3. *Let $T \in B(X)$ be power bounded and let $S := \frac{1}{N} \sum_{i=0}^{N-1} T^i$ ($N > 1$ is a fixed integer). If the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} S^i x \right\}$ converges in norm to $y \in X$, then $S^n x \to y$ in norm.*

Proof. It is easy to check that S is power bounded, that is,

$$\sup_{n\geq 0} \|S^n\| \leq \sup_{n\geq 0} \|T^n\| < \infty.$$

Notice also that if

$$f(z) = \frac{1}{N} \sum_{i=0}^{N-1} z^i \ \ (z \in \mathbb{C}),$$

then $f(1) = 1$ and $|f(z)| < 1$ for all $z \in \overline{\mathbb{D}} \setminus \{1\}$. On the other hand, by [15, Theorem 3.3.8],

$$\sigma_S(x) = \sigma_{f(T)}(x) = f(\sigma_T(x)).$$

Since $\sigma_T(x) \subseteq \overline{\mathbb{D}}$, we have $\sigma_S(x) \cap \mathbb{T} \subseteq \{1\}$. By Theorem 5.2, $S^n x \to y$ in norm. □

We put

$$D_+ = \{z \in \mathbb{C} : \mathrm{Re} z \geq 1,\ \mathrm{Im} z \geq 0\}$$

and

$$D_- = \{z \in \mathbb{C} : \mathrm{Re} z \geq 1,\ \mathrm{Im} z \leq 0\}.$$

Corollary 5.4. *Assume that the operators A, B in $B(X)$ satisfy the following conditions:*

(i) $\sup_{n\geq 0} \|A^n T B^n\| < \infty$ *for every* $T \in B(X)$.
(ii) *Either* $\sigma(A) \subset D_+$ *and* $\sigma(B) \subset D_-$ *or* $\sigma(A) \subset D_-$ *and* $\sigma(B) \subset D_+$.

If for some $T \in B(X)$, the sequence $\left\{\frac{1}{n}\sum_{i=0}^{n-1} A^i T B^i\right\}$ converges in operator norm to $Q \in B(X)$, then $A^n T B^n \to Q$ in operator norm.

Proof. Notice that the operator $L_A R_B$ is power bounded and therefore $\sigma(L_A R_B) \subseteq \overline{\mathbb{D}}$. On the other hand, by the Lumer-Rosenblum theorem [16, Theorem 10],

$$\sigma(L_A R_B) = \{\lambda\mu : \lambda \in \sigma(A),\ \mu \in \sigma(B)\}$$

which implies

$$\sigma(L_A R_B) \subset \{z \in \mathbb{C} : \mathrm{Re} z \geq 1\}.$$

So we have

$$\sigma(L_A R_B) \subseteq \overline{\mathbb{D}} \cap \{z \in \mathbb{C} : \mathrm{Re} z \geq 1\} = \{1\}.$$

It follows that $\sigma_{L_A R_B}(T) \subseteq \{1\}$ for all $T \in B(X)$. By Theorem 5.2,

$$A^n T B^n = (L_A R_B)^n T \to Q \text{ in operator norm.}$$ □

Next, we will show that the hypothesis $\sigma_T(x) \cap \mathbb{T} \subseteq \{1\}$ in Theorem 5.2 is the best possible, in general.

Let N be a normal operator on a Hilbert space H with the spectral measure P and let $x \in H$. Define a measure μ_x on $\sigma(N)$ by

$$\mu_x(\Delta) = \langle P(\Delta) x, x\rangle = \|P(\Delta) x\|^2, \tag{5.1}$$

where Δ is a Borel subset of $\sigma(N)$. It follows from the Spectral Theorem that $\sigma(N) = \mathrm{supp} P$ and $\sigma_N(x) = \mathrm{supp} \mu_x$. It is easy to check that if N is a contraction (a normal operator is power bounded if and only if it is a contraction), then

$$\frac{1}{n}\sum_{i=0}^{n-1} N^i x \to P(\{1\}) x \text{ in norm for all } x \in H. \tag{5.2}$$

Proposition 5.5. *Let N be a normal contraction operator on H with the spectral measure P and let $x \in H$. The sequence $\{N^n x\}$ converges in norm if and only if*

$$P\left(\sigma_N\left(x\right) \cap \mathbb{T} \setminus \{1\}\right) x = 0.$$

In this case, $N^n x \to P\left(\{1\}\right) x$ in norm.

Proof. Let μ_x be the measure on $\sigma\left(N\right)$ defined by (5.1). Then, we can write

$$\begin{aligned}
\lim_{n\to\infty} \left\|N^{n+1}x - N^n x\right\|^2 &= \lim_{n\to\infty} \int_{\sigma_N(x)} \left|z^{n+1} - z^n\right|^2 d\mu_x\left(z\right) \\
&= \lim_{n\to\infty} \int_{\sigma_N(x)\setminus(\sigma_N(x)\cap\mathbb{T})} |z|^{2n} |z-1|^2 d\mu_x\left(z\right) \\
&\quad + \lim_{n\to\infty} \int_{\sigma_N(x)\cap\mathbb{T}} |z|^{2n} |z-1|^2 d\mu_x\left(z\right) \\
&= \int_{\sigma_N(x)\cap\mathbb{T}} |z-1|^2 d\mu_x\left(z\right) \\
&= \int_{\sigma_N(x)\cap\mathbb{T}\setminus\{1\}} |z-1|^2 d\mu_x\left(z\right).
\end{aligned}$$

It follows that $\left\|N^{n+1}x - N^n x\right\| \to 0$ if and only if

$$\mu_x\left(\sigma_N\left(x\right) \cap \mathbb{T} \setminus \{1\}\right) = 0.$$

By Lemma 4.2, the sequence $\{N^n x\}$ converges in norm if and only if

$$P\left(\sigma_N\left(x\right) \cap \mathbb{T} \setminus \{1\}\right) x = 0.$$

Under this condition, by (5.2) we have

$$\lim_{n\to\infty} N^n x = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} N^i x = P\left(\{1\}\right) x.$$

□

Let $W^*\left(N\right)$ be the von Neumann algebra generated by N. Recall that $x \in H$ is a *separating vector* for N if the only operator A in $W^*\left(N\right)$ such that $Ax = 0$ is $A = 0$. As is known [5, Ch.IX, Sect.8.1], each normal operator has a separating vector. If x is a separating vector for N, then the spectral measure of N and the measure μ_x are mutually absolutely continuous [5, Ch.IX, Proposition 8.3], where μ_x is defined by (5.1).

Corollary 5.6. *Let N be a normal contraction operator on H with the spectral measure P. If x is a separating vector for N, then the sequence $\{N^n x\}$ converges in norm if and only if*

$$P\left(\sigma_N\left(x\right) \cap \mathbb{T} \setminus \{1\}\right) = 0. \tag{5.3}$$

Let K be a compact subset of $\overline{\mathbb{D}}$ such that $1 \in K$ and let ν be a regular positive Borel measure in $\mathbb{C}$ with support K. Define the operator N on $L^2\left(K, \nu\right)$ by $Nf = zf$. Then, N is a normal contraction on $L^2\left(K, \nu\right)$ and $\sigma\left(N\right) = K$. Moreover,

$$P\left(\Delta\right) f = \chi_\Delta f, \;\; \forall f \in L^2\left(K, \nu\right),$$

where χ_{Δ} is the characteristic function of Δ. It can be seen that the identity one function $\mathbf{1}$ on K is a separating vector for N and $\sigma(N) = \sigma_N(\mathbf{1})$. By (5.3), the sequence $\{N^n \mathbf{1}\}$ converges in norm if and only if $\chi_{\sigma_N(\mathbf{1}) \cap \mathbb{T}} = \chi_{\{1\}}$ or $\sigma_N(\mathbf{1}) \cap \mathbb{T} = \{1\}$.

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