Integr. Equ. Oper. Theory (2020) 92:31
https://doi.org/10.1007/s00020-020-02589-2
Published online July 9, 2020

Integral Equations and Operator Theory

Strongly Mixing Convolution Operators on Fréchet Spaces of Entire Functions of a Given Type and Order

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Abstract. We show that convolution operators on certain spaces of entire functions of a given type and order on Banach spaces are strongly mixing with respect to an invariant Borel probability measure with full support (a stronger property than frequent hypercyclicity). Based on results of S. Muro, D. Pinasco and M. Savransky we also show the existence of frequently hypercyclic entire functions of exponential growth, and the existence of frequently hypercyclic subspaces for such convolution operators.

Mathematics Subject Classification. Primary 47A16, 47B38; Secondary 30D15, 46G20.

Keywords. Convolution operators, Entire functions, Frequent hypercyclicity, Strongly mixing operators.

1. Introduction

Motivated by classical results of Malgrange [38] for convolution equations on the space $\mathcal{H}(\mathbb{C}^n)$ of all complex-valued entire functions on $\mathbb{C}^n$, Martineau [39] in 1967 proved existence and approximation theorems for solutions of convolution equations on spaces of entire functions on $\mathbb{C}^n$ of a given type and order. These spaces encompass the classical notion of function of exponential type, which has been extensively studied in the last century, both for their applications and for their own sake. Recall that an entire function $f\colon \mathbb{C} \to \mathbb{C}$ is said to be of exponential type if there are non-negative constants c and A such that

Blas M. Caraballo was supported in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior- Brasil (CAPES) - Finance Code 001 and in part by CNPq. Vinícius V. Fávaro is supported by FAPEMIG Grant PPM-00217-18; and CNPq Grant 310500/2017-6.

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$$|f(z)| < Ae^{c|z|},$$

for all $z \in \mathbb{C}$.

A natural step in this line of investigation is to consider convolution operators on spaces of entire functions on a complex Banach space. References [38,39] can be regarded as starting points of a series of related results for convolution operators on spaces of holomorphic functions on complex Banach spaces (see Gupta [35] 1969 and [36] 1970, Dineen [17] 1971, Dwyer III [21] 1971 and [20] 1976, Colombeau-Matos [15] 1980, Colombeau-Perrot [16] 1980, Matos-Nachbin [43] 1981, Matos [40] 1980, [41] 1984 and [42] 1986, Fávaro [22] 2008 and [23] 2009, Fávaro and Jatobá [24] 2010 and [25] 2012, and Fávaro-Mujica [28] 2018).

On the other hand, in the last 30 years the study of dynamics of convolution operators on spaces of entire functions has also been explored by several authors (see for instance [1,9,14,27–31,46,49,50]). The main notion studied in linear dynamics is hypercyclicity. Recall that, for a topological vector space X, a continuous linear operator $T: X \longrightarrow X$ is *hypercyclic* if the *orbit of* x, given by $\{x, T(x), T^2(x), \ldots\}$ is dense in X for some $x \in X$. In this case, x is said to be a *hypercyclic vector for* T. There are several other versions of hypercyclicity explored in linear dynamics. In this work we are mainly interested in exploring the notion of frequent hypercyclicity. Roughly speaking, a continuous linear operator $T: X \longrightarrow X$ is *frequently hypercyclic* if there is a vector $x \in X$, called a *frequently hypercyclic vector*, whose orbit intersects each nonempty open set along a set of integers having positive lower density. This notion was introduced by Bayart and Grivaux [2,4] and explored by many authors in the last decade, see for instance [4,5,7,8,12,13,32,45,46,52]. A recent criterion of Bayart and Matheron [7] gives sufficient conditions for a continuous linear operator T on a Fréchet space to be *strongly mixing in the gaussian sense,* a stronger property than frequent hypercyclicity (details in Section 3).

In this article, using this aforementioned criterion, we will show that under suitable conditions, nontrivial convolution operators on the space $Exp^k_{\Theta,0}$ (E) of complex-valued entire functions of type zero and a given order $k \in (1, \infty]$ on a complex Banach space E are strongly mixing in the gaussian sense, in particular frequently hypercyclic. By a *nontrivial* convolution operator we mean a convolution operator which is not a scalar multiple of the identity. As very particular cases, we recover results of the same type obtained in [12,28,46]. This result also generalizes some hypercyclicity results obtained in [9,10,14,27,31,37]. An important investigation in linear dynamics is the growth of hypercyclic entire functions of differentiation or translation operators at infinity (see [10,19,33,37,51], for instance). Blasco, Bonilla and Grosse-Erdmann [11], Bonilla and Grosse-Erdmann [12] and Muro, Pinasco and Savranski [46] study the growth at infinity of frequently hypercyclic entire functions associated to these operators. Inspired by results and techniques of [12,46] we obtain the existence of frequently hypercyclic entire functions of exponential growth associated to convolution operators. Moreover, we show the existence of closed infinite-dimensional vector subspaces of $Exp^k_{\Theta,0}(E)$

formed, excepted by the null function, by frequently hypercyclic functions. These results also extend results of the same type obtained in [12,13,46].

It is worth mentioning that the spaces introduced in [46, Definition 3.2] to prove the result about existence of frequently hypercyclic entire functions of exponential growth (see [46, Theorem 3.11]) are particular cases of the spaces $Exp^1_{\Theta,0,A}(E)$ of all complex-valued entire functions of a given type $A > 0$ and order one explored in [25,26].

Throughout this paper $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$. By $\mathbb{D}$ and $\partial\mathbb{D}$ we mean the open unit disk and the unit circle in the complex field $\mathbb{C}$, respectively. E and F are always complex Banach spaces and E' denotes the topological dual of E. The Banach space of all continuous$j$-homogeneous polynomials from E into F endowed with its usual sup norm is denoted by $\mathcal{P}(^jE;F)$. The subspace of $\mathcal{P}(^jE;F)$ of all polynomials of finite type, that is the linear span of $\{\varphi^j \cdot b : \varphi \in E' \text{ and } b \in F\}$, is represented by $\mathcal{P}_f(^jE;F)$. $\mathcal{H}(E;F)$ denotes the vector space of all holomorphic mappings from E into F. In all these cases, when $F = \mathbb{C}$ we write $\mathcal{P}(^jE)$, $\mathcal{P}_f(^jE)$ and $\mathcal{H}(E)$ instead of $\mathcal{P}(^jE;\mathbb{C})$, $\mathcal{P}_f(^jE;\mathbb{C})$ and $\mathcal{H}(E;\mathbb{C})$, respectively. For the general theory of homogeneous polynomials and holomorphic functions or any unexplained notation we refer to Dineen [18], Mujica [44] and Nachbin [48].

2. Preliminaires

We start recalling several concepts and results involving holomorphy on infinite dimensional spaces.

Definition 2.1. Let U be an open subset of E. A function $f: U \longrightarrow F$ is said to be *holomorphic on* U if for every $a \in U$ there exists a sequence $(P_j)_{j=0}^\infty$, where each $P_j \in \mathcal{P}(^jE;F)$ ($\mathcal{P}(^0E;F) = F$), such that $f(x) = \sum_{j=0}^\infty P_j(x-a)$ uniformly on some open ball with center a. The j-homogeneous polynomial $j!P_j$ is called the *j-th derivative of f at a* and it is denoted by $\hat{d}^jf(a)$. In particular, if $P \in \mathcal{P}(^jE;F)$, $a \in E$ and $k \in \{0,1,\ldots,j\}$, then

$$\hat{d}^kP(a)(x) = \frac{j!}{(j-k)!}\check{P}(\underbrace{x,\ldots,x}_{k\, times},a,\ldots,a)$$

for every $x \in E$, where $\check{P}$ is the unique symmetric j-linear mapping associated to P.

Definition 2.2. (Nachbin [48]) A *holomorphy type* Θ *from* E *to* F is a sequence of Banach spaces $(\mathcal{P}_\Theta(^jE;F))_{j=0}^\infty$, the norm on each of them being denoted by $\|\cdot\|_\Theta$, such that the following conditions hold true:

(1) Each $\mathcal{P}_\Theta(^jE;F)$ is a vector subspace of $\mathcal{P}(^jE;F)$ and $\mathcal{P}_\Theta(^0E;F)$ coincides with F as a normed vector space;
(2) There is a real number $\sigma \geq 1$ for which the following is true: given any $k \in \mathbb{N}_0$, $j \in \mathbb{N}_0$, $k \leq j$, $a \in E$, and $P \in \mathcal{P}_\Theta(^jE;F)$, we have $\hat{d}^kP(a) \in \mathcal{P}_\Theta(^kE;F)$ and

$$\left\|\frac{1}{k!}\hat{d}^k P(a)\right\|_\Theta \le \sigma^j \|P\|_\Theta \|a\|^{j-k}.$$

A holomorphy type from E to F shall be denoted by either Θ or $(\mathcal{P}_\Theta(^j E;F))_{j=0}^\infty$. When $F=\mathbb{C}$ we write $\mathcal{P}_\Theta(^j E)$ instead of $\mathcal{P}_\Theta(^j E;\mathbb{C})$, for every $j \in \mathbb{N}_0$.

It is obvious that each inclusion $\mathcal{P}_\Theta(^j E;F) \subset \mathcal{P}(^j E;F)$ is continuous and $\|P\| \le \sigma^j \|P\|_\Theta$.

Now we recall the definition of the spaces $\mathcal{B}^k_{\Theta,\rho}(E)$ that will be used in Definitions 2.4 and 2.6 .

Definition 2.3. ([25, Definition 2.2]) Let $(\mathcal{P}_\Theta(^j E))_{j=0}^\infty$ be a holomorphy type from E to $\mathbb{C}$. For $\rho > 0$ and $k \ge 1$, we denote by $\mathcal{B}^k_{\Theta,\rho}(E)$ the complex Banach space of all $f \in \mathcal{H}(E)$ such that $\hat{d}^j f(0) \in \mathcal{P}_\Theta\left(^j E\right)$, for all $j \in \mathbb{N}_0$ and

$$\|f\|_{\Theta,k,\rho} = \sum_{j=0}^\infty \rho^{-j} \left(\frac{j}{ke}\right)^{\frac{j}{k}} \left\|\frac{1}{j!}\hat{d}^j f(0)\right\|_\Theta < \infty,$$

with the norm given by $\|\cdot\|_{\Theta,k,\rho}$.

Now we are in the position to introduce the spaces of entire functions of a given type A and a given order k on a Banach space E. These spaces and their notation were inspired by the finite dimensional case $E=\mathbb{C}^n$ studied by Martineau [39], who introduced these spaces to generalize results on the growth of functions of exponential type to the so-called functions of *infinite order*, and also to obtain results for convolution equations.

Definition 2.4. ([25, Definition 2.4]) Let $\left(\mathcal{P}_\Theta(^j E)\right)_{j=0}^\infty$ be a holomorphy type from E to $\mathbb{C}$, $A \in [0,\infty)$ and $k \ge 1$. We denote by $Exp^k_{\Theta,0,A}(E)$ the complex vector space $\bigcap_{\rho > A} \mathcal{B}^k_{\Theta,\rho}(E)$ with the locally convex projective limit topology.

In case $A=0$ we denote

$$Exp^k_{\Theta,0}(E) := Exp^k_{\Theta,0,0}(E) = \bigcap_{\rho>0} \mathcal{B}^k_{\Theta,\rho}(E).$$

By [25, Proposition 2.7] $Exp^k_{\Theta,0,A}(E)$ is a Fréchet space.

Proposition 2.5. ([25, Proposition 2.5]) Let $\left(\mathcal{P}_\Theta(^j E)\right)_{j=0}^\infty$ be a holomorphy type from E to $\mathbb{C}$ and $k \in [1,\infty)$. If $f \in \mathcal{H}(E)$ is such that $\hat{d}^j f(0) \in \mathcal{P}_\Theta\left(^j E\right)$, for any $j \in \mathbb{N}_0$, then for each $A \in [0,\infty)$, $f \in Exp^k_{\Theta,0,A}(E)$ if, and only if,

$$\limsup_{j\to\infty} \left(\frac{j}{ke}\right)^{\frac{1}{k}} \left\|\frac{1}{j!}\hat{d}^j f(0)\right\|_\Theta^{\frac{1}{j}} \le A.$$

Now we recall the definition of the space of holomorphic functions of type A and infinite order.

Definition 2.6. ([25, Definition 2.8]) Let $\left(\mathcal{P}_\Theta({}^jE)\right)_{j=0}^{\infty}$ be a holomorphy type from E to $\mathbb{C}$. If $A \in [0,\infty)$, we denote by $Exp_{\Theta,0,A}^{\infty}(E)$ the Fréchet space of all $f \in \mathcal{H}\left(B_{\frac{1}{A}}(0)\right)$ such that $\hat{d}^j f(0) \in \mathcal{P}_\Theta\left({}^jE\right)$, for all $j \in \mathbb{N}_0$ and

$$\limsup_{j\to\infty}\left\|\frac{1}{j!}\hat{d}^j f(0)\right\|_\Theta^{\frac{1}{j}} \le A,$$

endowed with the locally convex topology generated by the family of seminorms $\left(p_{\Theta,\rho}^{\infty}\right)_{\rho>A}$, where

$$p_{\Theta,\rho}^{\infty}(f) = \sum_{j=0}^{\infty}\rho^{-j}\left\|\frac{1}{j!}\hat{d}^j f(0)\right\|_\Theta.$$

Here $B\left(0;\frac{1}{A}\right)$ denotes the open ball in E with center 0 and radius $\frac{1}{A}$, and by convention $B\left(0;\frac{1}{0}\right) := E$. Usually we write $Exp_{\Theta,0}^{\infty}(E)$ instead of $Exp_{\Theta,0,0}^{\infty}(E)$.

Remark 2.7. Note that the space $Exp_{\Theta,0}^{\infty}(E)$ coincides with the space $\mathcal{H}_{\Theta b}(E)$ explored in [9,24,28].

The aim of this work is the study of the notion of linear dynamics called frequent hypercyclicity for convolution operators on $Exp_{\Theta,0}^{k}(E)$, $k \in (1,\infty]$. Our results are closely related with the results obtained by Muro *et al* [46]. Since $Exp_{\Theta,0}^{\infty}(E) = \mathcal{H}_{\Theta b}(E)$ the results obtained in this work extend the corresponding results obtained in [46]. If $k \neq \infty$, $E = \mathbb{C}^n$ and Θ is the current holomorphy type, then the space $Exp_{\Theta,0}^{k}(\mathbb{C}^n)$ coincides with the so-called *space of entire functions of finite order* (see Martineau [39, page 112]).

For our purpose, the injectivity of the Fourier-Borel transform obtained in [25, Theorems 4.6] plays a central hole. First we recall the concepts of π_1 and $\pi_{2,k}$-holomorphy types. The definitions of π_1 and π_2 were introduced in [24] as tool to obtain a general method to prove existence and approximation results for convolution operators on $\mathcal{H}_{\Theta b}(E)$. In [9, Definition 2.5] these concepts were refined to obtain hypercyclicity results for convolution operators on $\mathcal{H}_{\Theta b}(E)$. Using these refinements Fávaro and Mujica [28] obtained frequent hypercyclicity results for convolution operators on $\mathcal{H}_{\Theta b}(E)$. Recently, with the aim of obtaining a general method to prove existence and approximation results for convolution operators on $Exp_{\Theta,0}^{k}(E)$, Fávaro and Jatobá [26, Definition 3.7] introduced the notion of $\pi_{2,k}$-holomorphy type, which is a variation of the notion of π_2-holomorphy type. In [26] the authors also obtained hypercyclicity results for convolution operators on $Exp_{\Theta,0}^{k}(E)$.

Definition 2.8. A holomorphy type $(\mathcal{P}_\Theta({}^jE;F))_{j=0}^{\infty}$ is said to be a π_1-*holomorphy type* if the following conditions hold:

(1) $\mathcal{P}_f({}^jE;F) \subset \mathcal{P}_\Theta({}^jE;F)$ and there exists $K > 0$ such that $\|\varphi^j \cdot b\|_\Theta \le K^j\|\varphi\|^j\|b\|$, for all $\varphi \in E'$, $b \in F$ and $j \in \mathbb{N}_0$.
(2) For $j \in \mathbb{N}_0$, $\mathcal{P}_f({}^jE;F)$ is dense in $\left(\mathcal{P}_\Theta({}^jE;F), \|\cdot\|_\Theta\right)$.

In the sequel, if $P \in \mathcal{P}(^j E)$, let $\check{P}$ denote the unique symmetric j-linear mapping associated to P, and for $0 \leq m \leq j$ write

$$\check{P}(x)^m (y)^{j-m} := \check{P}(\underbrace{x, \ldots, x}_{m \text{ times}}, y, \ldots, y), \quad x, y \in E.$$

Definition 2.9. Let $k \in [1, \infty]$ and $A \in [0, \infty)$. A holomorphy type $(\mathcal{P}_\Theta(^j E))_{j=0}^\infty$ from E to $\mathbb{C}$ is said to be a $\pi_{2,k}$-*holomorphy type* if, for each $T \in \left[Exp^k_{\Theta,0,A}(E)\right]'$, the following conditions hold:

(1) For $j \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$, $m \leq j$, if $P \in \mathcal{P}_\Theta\left(^j E\right)$ then the polynomial

$$T\left(\widehat{\check{P}(\cdot)^m}\right) : E \longrightarrow \mathbb{C}$$
$$y \longmapsto T\left(\check{P}(\cdot)^m y^{j-m}\right)$$

belongs to $\mathcal{P}_\Theta\left(^{j-m} E\right)$.

(2) For any constants $C > 0$ and $\rho > A$ such that

$$|T(f)| \leq C \|f\|_{\Theta,k,\rho}, \quad \text{if } k \in [1, \infty),$$

$$|T(f)| \leq C p^\infty_{\Theta,\rho}(f), \quad \text{if } k = \infty,$$

for all $f \in Exp^k_{\Theta,0,A}(E)$, and for each $\varepsilon > 0$, there is $D(\varepsilon) > 0$ such that

$$\left\|T\left(\widehat{\check{P}(\cdot)^m}\right)\right\|_\Theta \leq C D(\varepsilon)(1+\varepsilon)^j \rho^{-m} \left(\frac{m}{ke}\right)^{\frac{m}{k}} \|P\|_\Theta, \quad \text{if } k \in [1, \infty),$$

$$\left\|T\left(\widehat{\check{P}(\cdot)^m}\right)\right\|_\Theta \leq C D(\varepsilon)(1+\varepsilon)^j \rho^{-m} \|P\|_\Theta, \quad \text{if } k = \infty.$$

for every $P \in \mathcal{P}_\Theta\left(^j E\right)$, $j \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$, $m \leq j$.

Remark 2.10. (i) Note that the constants C and ρ of Definition 2.9 (2) exist because $T \in \left[Exp^k_{\Theta,0,A}(E)\right]'$.

(ii) When $k = \infty$ and $A = 0$ the concepts of $\pi_{2,\infty}$-holomorphy type and π_2-holomorphy type coincide (see [9, Definition 2.5]). So in this case we write π_2 instead of $\pi_{2,\infty}$.

(iii) The condition (2) of the definition of $\pi_{2,k}$-holomorphy type is slightly different from the original definition introduced in [26, Definition 3.7]. In the original definition, condition (2) is replaced by: (2′) For any constants $C > 0$ and $\rho > A$ such that

$$|T(f)| \leq C \|f\|_{\Theta,k,\rho}, \quad \text{if } k \in [1, \infty),$$
$$|T(f)| \leq C p^\infty_{\Theta,\rho}(f), \quad \text{if } k = \infty,$$

for all $f \in Exp^k_{\Theta,0,A}(E)$, there is a constant $M > 0$ such that

$$\left\|T\left(\widehat{\check{P}(\cdot)^m}\right)\right\|_\Theta \leq C M^j \rho^{-m} \left(\frac{m}{ke}\right)^{\frac{m}{k}} \|P\|_\Theta, \quad \text{if } k \in [1, \infty),$$
$$\left\|T\left(\widehat{\check{P}(\cdot)^m}\right)\right\|_\Theta \leq C M^j \rho^{-m} \|P\|_\Theta, \quad \text{if } k = \infty.$$

for every $P \in \mathcal{P}_\Theta\left(^j E\right)$, $j \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$, $m \leq j$.

However, all the results in the present paper where the hypothesis of $\pi_{2,k}$ is made work equally well with both conditions (2) and (2′), except Proposition 3.13 and its consequences, that is, Theorem 3.8 and Corollaries 3.9 and 3.16 , where we need to use (2). It is worth to mention that all examples of holomorphy types explored in [9,14,17,24,26–29,46] satisfy (2).

The following result is an important tool for our purpose.

Theorem 2.11. ([25, Theorem 4.6]) Let $k \in (1,\infty]$, $A \in [0,\infty)$ and $(\mathcal{P}_\Theta(^jE))_{j=0}^\infty$ be a π_1-holomorphy type from E to $\mathbb{C}$.

Then *the Fourier-Borel transform*

$$\mathcal{F}\colon \left[Exp^k_{\Theta,0,A}(E)\right]' \longrightarrow \mathcal{H}(E'),$$

given by $\mathcal{F}T(\varphi) = T(e^\varphi)$, for all $T \in \left[Exp^k_{\Theta,0,A}(E)\right]'$ and $\varphi \in E'$, is an injective linear transformation.

Proposition 2.12. ([26, Proposition 3.1]) Let $\left(\mathcal{P}_\Theta(^jE)\right)_{j=0}^\infty$ be a holomorphy type from E to $\mathbb{C}$, $a \in E$, $k \in [1,\infty]$, $A \in [0,\infty)$ and $f \in Exp^k_{\Theta,0,A}(E)$. Then $\hat{d}^n f(\cdot)\, a \in Exp^k_{\Theta,0,\sigma A}(E)$, for any constant σ satisfying condition (2) of Definition 2.2. Besides

$$\hat{d}^n f(\cdot)\, a = \sum_{j=0}^\infty \frac{1}{j!} \widehat{(d^{j+n} f(0))(\cdot)^j}(a),$$

in the topology of $Exp^k_{\Theta,0,\sigma A}(E)$.

We recall that $\tau_a f = f(\cdot - a)$ denotes the translation operator by a. The following proposition ensures that the translation operators on $Exp^k_{\Theta,0}(E)$ are well-defined and continuous.

Proposition 2.13. ([26, Proposition 3.3]) Let $\left(\mathcal{P}_\Theta(^jE)\right)_{j=0}^\infty$ be a holomorphy type from E to $\mathbb{C}$ and $k \in [1,\infty]$. If $f \in Exp^k_{\Theta,0}(E)$ and $a \in E$, then $\tau_{-a} f \in Exp^k_{\Theta,0}(E)$ and

$$\tau_{-a} f = \sum_{n=0}^\infty \frac{1}{n!} \hat{d}^n f(\cdot)\, a,$$

in the topology of $Exp^k_{\Theta,0}(E)$.

Now we are able to define convolution operators on $Exp^k_{\Theta,0}(E)$.

Definition 2.14. Let $\left(\mathcal{P}_\Theta(^jE)\right)_{j=0}^\infty$ be a holomorphy type from E to $\mathbb{C}$ and $k \in [1,\infty]$.

(a) A *convolution operator on* $Exp^k_{\Theta,0}(E)$ is a continuous linear operator

$$L\colon Exp^k_{\Theta,0}(E) \longrightarrow Exp^k_{\Theta,0}(E)$$

such that $L(\tau_a f) = \tau_a(Lf)$ for all $a \in E$ and $f \in Exp^k_{\Theta,0}(E)$. By Proposition 2.13, the convolution operators on $Exp^k_{\Theta,0}(E)$ are well-defined. We denote the set of all convolution operators on $Exp^k_{\Theta,0}(E)$ by $\mathcal{A}^k_{\Theta,0}$.

(b) We denote by $\gamma_{\Theta,0}^k$ the linear transformation

$$\gamma_{\Theta,0}^k\colon \mathcal{A}_{\Theta,0}^k \longrightarrow \left[Exp_{\Theta,0}^k(E)\right]'$$

given by $\gamma_{\Theta,0}^k(L)(f) = (Lf)(0)$, for $f \in Exp_{\Theta,0}^k(E)$ and $L \in \mathcal{A}_{\Theta,0}^k$.

3. Frequently Hypercyclic Convolution Operators on $\boldsymbol{Exp_{\Theta,0}^k(E)}$

We begin this section recalling the definition of frequently hypercyclic operators, which was recently introduced by Bayart and Grivaux in [3,4] and it is a stronger condition than hypercyclicity.

Definition 3.1. Let X be a separable Fréchet space. A continuous linear operator $T\colon X \to X$ is said to be *frequently hypercyclic* if there exists $x \in X$ such that, for every non-empty open set $U \subset X$ we have that

$$\liminf_{N\to\infty} \frac{\operatorname{card}\{0 \leq n \leq N-1:\ T^n x \in U\}}{N} > 0.$$

In this case x is said to be a *frequently hypercyclic vector*, and the set of all frequently hypercyclic vectors is denoted by $FCH(T)$.

Definition 3.2. ([6, Chapter 5]) Let X be a topological vector space, let $T : X \to X$ be a continuous linear operator, and let μ be a Borel probability measure on X.

(a) μ is said to be T-*invariant* if $\mu(T^{-1}(A)) = \mu(A)$ for every Borel set $A \subset X$.
(b) μ is said to have *full support* if $\mu(U) > 0$ for every non-empty open set $U \subset X$.
(c) T is said to be *strongly mixing with respect to* μ if for any two Borel sets $A, B \subset X$ we have that

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$

The main result of this section is the next theorem. First, recall that if X is a topological vector space, then a continuous linear operator $T\colon X \to X$ is said to be *(topologically) mixing* if for any two non-empty open sets $U, V \subset X$, there is $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$, for all $n \geq n_0$.

Theorem 3.3. Let $k \in (1, \infty]$, E be a Banach space with separable dual and $(\mathcal{P}_\Theta(^jE))_{j=0}^\infty$ be a π_1-holomorphy type from E to $\mathbb{C}$. If L is a nontrivial convolution operator on $Exp_{\Theta,0}^k(E)$, then L is strongly mixing with respect to an L-invariant Borel probability measure μ, on $Exp_{\Theta,0}^k(E)$, with full support. In particular L is (topologically) mixing and μ-almost every $f \in Exp_{\Theta,0}^k(E)$ is a frequently hypercyclic function for L.

The proof of this theorem has several ingredients. It rests on the following criterion which is a corollary of [7, Theorem 1.1].

Theorem 3.4. ([7, Theorem 1.1]) Let X be a separable Fréchet space and let $L\colon X \to X$ be a continuous linear operator. Assume that for any $D \subset \partial\mathbb{D}$ such that $\partial\mathbb{D} \setminus D$ is dense in $\partial\mathbb{D}$, the linear span of $\bigcup_{\lambda \in \partial\mathbb{D}\setminus D} \ker(L-\lambda)$ is dense in X. Then L is strongly mixing with respect to an L-invariant Borel probability measure μ, on X, with full support. In particular L is (topologically) mixing and μ-almost every $x \in X$ is frequently hypercyclic for L.

We also need the following results:

Lemma 3.5. Let $k \in (1, \infty]$, E be a Banach space with separable dual and $(\mathcal{P}_\Theta(^jE))_{j=0}^\infty$ be a π_1-holomorphy type from E to $\mathbb{C}$. Let $f \in \mathcal{H}(E')$ be non constant and $B \subset \mathbb{C}$. Suppose that there is $\varphi_0 \in E'$ such that $f(\varphi_0)$ is an accumulation point of B. Then $\text{span}\{e^\varphi : \ \varphi \in E', \ f(\varphi) \in B\}$ is dense in $Exp_{\Theta,0}^k(E)$.

Proof. By the Hahn-Banach theorem it suffices to prove that if a functional $T \in [Exp_{\Theta,0}^k(E)]'$ vanishes on $\text{span}\{e^\varphi : f(\varphi) \in B\}$, then T is identically zero. Since the Fourier-Borel transform $\mathcal{F} : [Exp_{\Theta,0}^k(E)]' \to \mathcal{H}(E')$ is injective, it is enough to show that $\mathcal{F}T = 0$. Let $\{U_i : i \in I\}$ be a basis for the open sets of E'. Since f is a nonconstant entire function, for each $i \in I$ there is $\varphi_i \in U_i$ such that $f(\varphi_0) \neq f(\varphi_i)$. Now, for $i \in I$, we denote by L_i the complex line through φ_0 that intersects U_i in φ_i, that is

$$L_i := \{\varphi_0 + z(\varphi_i - \varphi_0) : z \in \mathbb{C}\}.$$

Then f is non constant on L_i and $\bigcup_{i\in I} L_i$ is dense in E'. By considering the restriction

$$f|_{L_i} : z \in \mathbb{C} \to f(\varphi_0 + z(\varphi_i - \varphi_0)) \in \mathbb{C},$$

since $f|_{L_i}$ is a non constant holomorphic function, it follows from the classical open mapping theorem that $f|_{L_i}$ is an open function. Since $(f|_{L_i})(0) = f(\varphi_0)$ is an accumulation point of B, then 0 is an accumulation point of $(f|_{L_i})^{-1}(B)$. By the hypothesis, $(\mathcal{F}T)(\varphi) = 0$ for every $\varphi \in f^{-1}(B)$, in particular $(\mathcal{F}T)(\varphi_0 + z(\varphi_i - \varphi_0)) = 0$ for every $z \in (f|_{L_i})^{-1}(B)$. So, the set of zeros of the entire function $z \in \mathbb{C} \to (\mathcal{F}T)(\varphi_0 + z(\varphi_i - \varphi_0)) \in \mathbb{C}$ has an accumulation point, which implies that $\mathcal{F}T = 0$ on each L_i, $i \in I$. Therefore $\mathcal{F}T$ vanishes on $\bigcup_{i\in I} L_i$ and by the density of $\bigcup_{i\in I} L_i$ in E', we have $\mathcal{F}T = 0$ on E'. □

The next lemma was proved in [26] and it will be very useful in the proof of Theorem 3.3.

Lemma 3.6. ([26, Lemma 4.4]) Let $k \in (1, \infty]$, E be a Banach space with separable dual, $(\mathcal{P}_\Theta(^jE))_{j=0}^\infty$ be a π_1-holomorphy type from E to $\mathbb{C}$ and L be convolution operator on $Exp_{\Theta,0}^k(E)$. Then:

- (a) $L(e^\varphi) = \mathcal{F}[\gamma_{\Theta,0}^k(L)](\varphi)e^\varphi$ for every $\varphi \in E'$.
- (b) L is a scalar multiple of the indentity if and only if the entire function $\mathcal{F}[\gamma_{\Theta,0}^k(L)] : E' \to \mathbb{C}$ is constant.

With these results we are able to prove Theorem 3.3.

Proof of Theorem 3.3. Since L is a convolution operator on $Exp^k_{\Theta,0}(E)$ which is not a scalar multiple of the identity, it follows from Lemma 3.6 (b) that the entire function $\mathcal{F}[\gamma^k_{\Theta,0}(L)]$ is not constant. Consider the sets

$$V = \{\varphi \in E' : |\mathcal{F}[\gamma^k_{\Theta,0}(L)](\varphi)| < 1\} = [\mathcal{F}[\gamma^k_{\Theta,0}(L)]]^{-1}(\mathbb{D})$$

and

$$W = \{\varphi \in E' : |\mathcal{F}[\gamma^k_{\Theta,0}(L)](\varphi)| > 1\} = [\mathcal{F}[\gamma^k_{\Theta,0}(L)]]^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}).$$

By Liouville's Theorem, both sets V and W are disjoint non-empty open subsets of E'. By considering a path in E' from a point in V to a point in W, we can find a point $\varphi_0 \in E'$ such that $|\mathcal{F}[\gamma^k_{\Theta,0}(L)](\varphi_0)| = 1$. Let $D \subset \partial\mathbb{D}$ such that $\partial\mathbb{D} \setminus D$ is dense in $\partial\mathbb{D}$. Thus $\mathcal{F}[\gamma^k_{\Theta,0}(L)](\varphi_0)$ is an accumulation point of $\partial\mathbb{D} \setminus D$, since $\partial\mathbb{D}$ is a metric space without isolated points. Therefore, it follows from Lemma 3.5 that

$$\text{span}\{e^{\varphi} : \mathcal{F}[\gamma^k_{\Theta,0}(L)](\varphi) \in \partial\mathbb{D} \setminus D\}$$

is dense in $Exp^k_{\Theta,0}(E)$. By Lemma 3.6 (a)

$$\{e^{\varphi} : \mathcal{F}[\gamma^k_{\Theta,0}(L)](\varphi) \in \partial\mathbb{D} \setminus D\} \subset \bigcup_{\mathcal{F}[\gamma^k_{\Theta,0}(L)](\varphi) \in \partial\mathbb{D}\setminus D} \ker(L - \mathcal{F}[\gamma^k_{\Theta,0}(L)](\varphi))$$

$$\subset \bigcup_{\lambda \in \partial\mathbb{D}\setminus D} \ker(L - \lambda),$$

and then the linear span of $\cup_{\lambda \in \partial\mathbb{D}\setminus D} \ker(L - \lambda)$ is also dense in $Exp^k_{\Theta,0}(E)$. By Theorem 3.4 L is strongly mixing with respect to an L-invariant Borel probability measure μ, on $Exp^k_{\Theta,0}(E)$, with full support. □

3.1. Frequently Hypercyclic Functions in $Exp^1_{\Theta,0,A}(E)$

As mentioned in the introduction, the problem of determining possible rates of growth of frequently hypercyclic entire functions for convolution operators on $\mathcal{H}(\mathbb{C}^n)$ was studied for the first time in [11,12]. For some spaces of entire functions on infinite-dimensional Banach spaces the same kind of growth was firstly explored by Muro *et al* [46].

In this section we study the rate of growth of frequently hypercyclic entire functions for convolution operators on $Exp^k_{\Theta,0}(E)$.

Definition 3.7. (Nachbin [47, p. 226, Remark 1]) Let E be a Banach space and let $A \in [0, \infty)$. A function $f \in \mathcal{H}(E)$ is said to be of *exponential type less than* A if, for each $\varepsilon > 0$, there is $c > 0$ such that

$$|f(x)| \le ce^{(A+\varepsilon)\|x\|}$$

for every $x \in E$.

Nachbin proved that, $f \in \mathcal{H}(E)$ is of exponential type less than A if, and only if,

$$\limsup_{j\to\infty} \left\|\hat{d}^j f(0)\right\|^{\frac{1}{j}} \le A.$$

Using Proposition 2.5 it is easy to see that every function in $Exp^1_{\Theta,0,A}(E)$ satisfies

$$\limsup_{j\to\infty} \left\|\hat{d}^j f(0)\right\|_{\Theta}^{\frac{1}{j}} \leq A. \tag{3.1}$$

Our main result in this section is the following:

Theorem 3.8. Let $k \in (1, \infty]$ and E be a Banach space with separable dual, let $(\mathcal{P}_{\Theta}(^j E))_{j=0}^{\infty}$ be a π_1-$\pi_{2,k}$-holomorphy type from E to $\mathbb{C}$, and let L be a nontrivial convolution operator on $Exp^k_{\Theta,0}(E)$. Then, there exists $\alpha_L \geq 0$ such that for each $\varepsilon > 0$, L has a frequently hypercyclic function f in $Exp^1_{\Theta,0,K(\alpha_L+\varepsilon)}(E)$, where $K > 0$ comes from condition in Definition 2.8 (1). Besides, the function f satisfies the following growth condition: given any $\delta > 0$ there is $C_\delta > 0$ such that

$$|f(x)| \leq C_\delta e^{(K(\alpha_L+\varepsilon)+\delta)\|x\|} \quad \text{for every} \quad x \in E.$$

Since $Exp^{\infty}_{\Theta,0}(E) = \mathcal{H}_{\Theta b}(E)$ we obtain, as particular case of Theorem 3.8, the result of growth of frequently hypercyclic entire functions for convolution operators due to Muro *et al* [46, Theorem 3.11] and consequently the corresponding result due to Bonilla and Grosse-Erdmann [12, Theorem 3.4]. More precisely, using the nomenclature considered in this paper, we obtain [46, Theorem 3.11] as the following corollary:

Corollary 3.9. Let E be a Banach space with separable dual, let $(\mathcal{P}_{\Theta}(^j E))_{j=0}^{\infty}$ be a π_1-π_2-holomorphy type from E to $\mathbb{C}$, and let L be a nontrivial convolution operator on $\mathcal{H}_{\Theta b}(E)$. Then, there exists $\alpha_L \geq 0$ such that for each $\varepsilon > 0$, L admits a frequently hypercyclic function $f \in Exp^1_{\Theta,0,K(\alpha_L+\varepsilon)}(E)$, where $K > 0$ comes from condition in Definition 2.8 (1).

In order to prove Theorem 3.8 we need several preparatory results. We start studying the relationship among the spaces $Exp^1_{\Theta,0,A}(E)$ and $Exp^k_{\Theta,0}(E)$, for each $A > 0$ and $k \in (1, \infty]$.

First we recall that the topology of $Exp^k_{\Theta,0,A}(E)$ coincides with the topology generated by the family of norms

$$\|f\|_{\Theta,k,\rho} = \sum_{j=0}^{\infty} \rho^{-j} \left(\frac{j}{ke}\right)^{\frac{j}{k}} \left\|\tfrac{1}{j!}\hat{d}^j f(0)\right\|_{\Theta}, \quad \text{if} \quad 1 \leq k < \infty$$

and

$$p^{\infty}_{\Theta,\rho}(f) = \sum_{j=0}^{\infty} \rho^{-j} \left\|\tfrac{1}{j!}\hat{d}^j f(0)\right\|_{\Theta}, \quad \text{if} \quad k = \infty.$$

Note also that for $\rho > A$, (3.1) ensures that the mapping

$$f \in Exp^1_{\Theta,0,A}(E) \longmapsto \|f\|_{\Theta,\rho} := \sum_{j=0}^{\infty} \rho^{-j} \left\|\hat{d}^j f(0)\right\|_{\Theta}$$

is a seminorm well-defined on $Exp^1_{\Theta,0,A}(E)$. Since $\lim_{j\to\infty} \frac{j}{e\sqrt[j]{j!}} = 1$, for each $\rho > A$ there exist $\rho', \rho^+ > A$ and $c, C > 0$ such that

$$\|f\|_{\Theta,1,\rho} \leq c\|f\|_{\Theta,\rho'} \quad \text{and} \quad \|f\|_{\Theta,\rho} \leq C\|f\|_{\Theta,1,\rho+} \tag{3.2}$$

for every $f \in Exp^1_{\Theta,0,A}(E)$. In fact, since $\lim_{j\to\infty} \frac{j}{e\sqrt[j]{j!}} = 1$, for each $\varepsilon > 0$ there exists $c > 0$ such that

$$\left(\frac{j}{e}\right)^j \frac{1}{j!} < c(1+\varepsilon)^j, \quad \text{for all } \ j \in \mathbb{N}_0. \tag{3.3}$$

So, let $\rho > A$. Choosing $\rho > \rho' > A$, it follows from (3.3) that there exists $c > 0$ such that

$$\left(\frac{j}{e}\right)^j \frac{1}{j!} < c(\rho/\rho')^j, \quad \text{for all } \ j \in \mathbb{N}_0.$$

Hence,

$$\|f\|_{\Theta,1,\rho} = \sum_{j=0}^{\infty} \rho^{-j}\left(\frac{j}{e}\right)^j \left\|\tfrac{1}{j!}\hat{d}^j f(0)\right\|_{\Theta} \leq c\sum_{j=0}^{\infty} \rho^{-j}(\rho/\rho')^j \left\|\hat{d}^j f(0)\right\|_{\Theta} = c\|f\|_{\Theta,\rho'}.$$

On the other hand, since we may write

$$\|f\|_{\Theta,\rho} = \|f(0)\|_{\Theta} + \sum_{j=1}^{\infty} \rho^{-j}\left(\frac{e}{j}\right)^j j!\left(\frac{j}{e}\right)^j \left\|\frac{1}{j!}\hat{d}^j f(0)\right\|_{\Theta},$$

a similar argument shows the second estimate in (3.2).

Proposition 3.10. Let $k \in (1,\infty]$, $A > 0$ and $\left(\mathcal{P}_{\Theta}(^jE)\right)_{j=0}^{\infty}$ be a holomorphy type from E to $\mathbb{C}$. Then:

- (a) $Exp^1_{\Theta,0,A}(E) \subset Exp^k_{\Theta,0}(E)$ and the inclusion $Exp^1_{\Theta,0,A}(E) \hookrightarrow Exp^k_{\Theta,0}(E)$ is continuous.
- (b) If Θ is a π_1-holomorphy type, then $Exp^1_{\Theta,0,A}(E)$ is a dense subspace of $Exp^k_{\Theta,0}(E)$.

Proof. (a) First let $k \in (1,\infty)$. Since

$$\lim_{j\to\infty}\left(\frac{j}{ke}\right)^{\frac{1}{k}}\left(\frac{e}{j}\right) = 0$$

whenever $k > 1$, for each $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\left(\frac{j}{ke}\right)^{\frac{j}{k}}\left(\frac{e}{j}\right)^j \leq C(\varepsilon)\varepsilon^j,$$

for every $j \in \mathbb{N}_0$. Let $\rho > 0$. Choosing $\varepsilon > 0$ such that $\frac{\varepsilon}{\rho} < \frac{1}{A}$, we can find $\gamma > A$ such that $\frac{\varepsilon}{\rho} < \frac{1}{\gamma} < \frac{1}{A}$. So

$$\begin{aligned}\|f\|_{\Theta,k,\rho} &= \sum_{j=0}^{\infty} \rho^{-j} \left(\frac{j}{ke}\right)^{\frac{j}{k}} \left\| \tfrac{1}{j!} \hat{d}^j f(0) \right\|_\Theta \\ &\leq \sum_{j=0}^{\infty} C(\varepsilon) \left(\frac{\varepsilon}{\rho}\right)^j \left(\frac{j}{e}\right)^j \left\| \tfrac{1}{j!} \hat{d}^j f(0) \right\|_\Theta \\ &\leq C(\varepsilon) \sum_{j=0}^{\infty} \gamma^{-j} \left(\frac{j}{e}\right)^j \left\| \tfrac{1}{j!} \hat{d}^j f(0) \right\|_\Theta = C(\varepsilon) \|f\|_{\Theta,1,\gamma},\end{aligned}$$

for every $f \in Exp^1_{\Theta,0,A}(E)$. Therefore the inclusion is continuous.

Now, let $k = \infty$. Since

$$\lim_{j\to\infty} \frac{e}{j} = 0,$$

for each $\rho > 0$ we may choose $\varepsilon > 0$, $C(\varepsilon) > 0$ and $\gamma > A$ such that

$$\frac{\varepsilon}{\rho} < \frac{1}{\gamma} < \frac{1}{A} \quad \text{and} \quad \left(\frac{e}{j}\right)^j \leq C(\varepsilon)\varepsilon^j,$$

for every $j \in \mathbb{N}_0$. Thus

$$\begin{aligned}p^{\infty}_{\Theta,\rho}(f) &= \sum_{j=0}^{\infty} \rho^{-j} \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_\Theta = \sum_{j=0}^{\infty} \rho^{-j} \left(\frac{e}{j}\right)^j \left(\frac{j}{e}\right)^j \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_\Theta \\ &\leq C(\varepsilon) \sum_{j=0}^{\infty} \gamma^{-j} \left(\frac{j}{e}\right)^j \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_\Theta = C(\varepsilon) \|f\|_{\theta,1,\gamma},\end{aligned}$$

for every $f \in Exp^1_{\Theta,0,A}(E)$, proving that the inclusion is continuous.

(b) For $\varphi \in E'$, we have that $\hat{d}^j(e^\varphi)(0) = \varphi^j$ and $\|\varphi^j\|_\Theta \leq K^j \|\varphi\|^j$ for every $j \in \mathbb{N}_0$, where $K > 0$ is the constant of Definition 2.8(1). Thus

$$\limsup_{j\to\infty} \left\| \hat{d}^j(e^\varphi)(0) \right\|_\Theta^{\frac{1}{j}} \leq K\|\varphi\|$$

and it follows from Proposition 2.5 that $e^\varphi \in Exp^1_{\Theta,0,A}(E)$ if and only if $\|\varphi\| \leq A/K$. Since

$$\text{span}\{e^\varphi : \varphi \in E' \text{ with } \|\varphi\| \leq A/K\}$$

is a dense subspace of $Exp^k_{\Theta,0}(E)$ (see [26, Proposition 4.3]), the result follows. □

Proposition 3.10 assures that $Exp^1_{\Theta,0,A}(E)$ is a Fréchet space which is continuously and densely embedded in $Exp^k_{\Theta,0}(E)$. Now we will prove that, under suitable conditions, the restriction of a convolution operator on $Exp^k_{\Theta,0}(E)$ to $Exp^1_{\Theta,0,A}(E)$ is also a convolution operator. Before that, we recall the concept of convolution product and a result of [26].

Definition 3.11. Let $\left(\mathcal{P}_\Theta({}^jE)\right)_{j=0}^\infty$ be a holomorphy type from E to $\mathbb{C}$, $k \in [1,\infty]$, $T \in \left[Exp_{\Theta,0}^k(E)\right]'$ and $f \in Exp_{\Theta,0}^k(E)$. We define *the convolution product of* T *and* f by $(T * f)(x) = T(\tau_{-x}f)$, for all $x \in E$.

Theorem 3.12. ([26, Theorem 3.11]) If $k \in [1,\infty]$ and $(\mathcal{P}_\Theta({}^jE))_{j=0}^\infty$ is a $\pi_{2,k}$-holomorphy type from E to $\mathbb{C}$, then the mapping $\gamma_{\Theta,0}^k$ is a linear bijection and its inverse is the mapping

$$\Gamma_{\Theta,0}^k\colon \left[Exp_{\Theta,0}^k(E)\right]' \longrightarrow \mathcal{A}_{\Theta,0}^k$$

given by $\Gamma_{\Theta,0}^k(T)(f) = T * f$, for $T \in \left[Exp_{\Theta,0}^k(E)\right]'$ and $f \in Exp_{\Theta,0}^k(E)$.

Proposition 3.13. Let $A > 0$, $k \in (1,\infty]$ and $(\mathcal{P}_\Theta({}^jE))_{j=0}^\infty$ be a $\pi_{2,k}$-holomorphy type from E to $\mathbb{C}$. If

$$L\colon Exp_{\Theta,0}^k(E) \to Exp_{\Theta,0}^k(E)$$

is a convolution operator, then the restriction

$$L\Big|_{Exp_{\Theta,0,A}^1(E)} : Exp_{\Theta,0,A}^1(E) \to Exp_{\Theta,0,A}^1(E)$$

is a well-defined convolution operator.

Proof. By Theorem 3.12 there is $T \in [Exp_{\Theta,0}^k(E)]'$ such that $Lf = T * f$, for every $f \in Exp_{\Theta,0}^k(E)$. To prove that $L\Big|_{Exp_{\Theta,0,A}^1(E)}$ is a well-defined continuous linear operator, it is sufficient to prove that for $\gamma > A$ there exist $\gamma^+ > A$ and $D > 0$ such that

$$\|Lf\|_{\Theta,1,\gamma} \leq D\|f\|_{\Theta,1,\gamma^+}$$

for every $f \in Exp_{\Theta,0,A}^1(E)$. By Propositions 2.12 and 2.13 and by the linearity and continuity of T, we have that

$$\begin{aligned} T * f(x) = T(\tau_{-x}f) &= \sum_{n=0}^\infty \frac{1}{n!} T\left(\hat{d}^n f(\cdot)(x)\right) \\ &= \sum_{n=0}^\infty \frac{1}{n!} \sum_{j=0}^\infty \frac{1}{j!} T\left(\widehat{(d^{j+n}f(0))(\cdot)^j}\right)(x) \\ &= \sum_{n=0}^\infty \frac{1}{n!} \sum_{j=n}^\infty \frac{1}{(j-n)!} T\left(\widehat{(d^jf(0))(\cdot)^{j-n}}\right)(x) \end{aligned}$$

for every $f \in Exp_{\Theta,0,A}^1(E)$ and $x \in E$. Since $(\mathcal{P}_\Theta({}^jE))_{j=0}^\infty$ is a $\pi_{2,k}$-holomorphy type, $k \in (1,\infty]$, we have that the mappings

$$x \in E \mapsto T\left(\widehat{(d^{j+n}f(0))(\cdot)^j}\right)(x) \in \mathbb{C} \quad \text{and}$$

$$x \in E \mapsto \frac{1}{(j-n)!} T\left(\widehat{(d^jf(0))(\cdot)^{j-n}}\right)(x) \in \mathbb{C}$$

belong to $\mathcal{P}_\Theta({}^nE)$ and satisfy the condition (2) of Definition 2.9. It is not difficult to see that

$$\sum_{j=0}^{\infty}\left\|\frac{1}{j!}T\left(\widehat{(d^{j+n}f(0))(\cdot)^j}\right)\right\|<\infty \quad\text{and}\quad \sum_{j=n}^{\infty}\left\|\frac{1}{(j-n)!}T\left(\widehat{(d^{j}f(0))(\cdot)^{j-n}}\right)\right\|<\infty,$$

and since $\mathcal{P}({}^nE)$ is complete, it follows that

$$\sum_{j=0}^{\infty}\frac{1}{j!}T\left(\widehat{(d^{j+n}f(0))(\cdot)^j}\right) \quad\text{and}\quad \sum_{j=n}^{\infty}\frac{1}{(j-n)!}T\left(\widehat{(d^{j}f(0))(\cdot)^{j-n}}\right)$$

belong to $\mathcal{P}({}^nE)$. On the other hand, it is not difficult to show that the power series

$$\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{\infty}\frac{1}{j!}T\left(\widehat{(d^{j+n}f(0))(\cdot)^j}\right)(x) \quad\text{and}\quad \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=n}^{\infty}\frac{1}{(j-n)!}T\left(\widehat{(d^{j}f(0))(\cdot)^{j-n}}\right)(x)$$

have infinite radius of convergence. Thus, it follows from [44, Example 5.4] that

$$\begin{aligned}\hat{d}^n(T*f)(0) &= \sum_{j=0}^{\infty}\frac{1}{j!}T\left(\widehat{(d^{j+n}f(0))(\cdot)^j}\right)\\ &= \sum_{j=n}^{\infty}\frac{1}{(j-n)!}T\left(\widehat{(d^{j}f(0))(\cdot)^{j-n}}\right)\end{aligned}$$

for every $f\in Exp^1_{\Theta,0,A}(E)$ and $n\in\mathbb{N}_0$. Since Θ is a $\pi_{2,k}$-holomorphy type, for each $j,n\in\mathbb{N}_0$ with $j\le n$ and $f\in Exp^k_{\Theta,0}(E)$ the mapping

$$x\in E\to T\left(\widehat{(d^{j}f(0))(\cdot)^{j-n}}\right)(x)\in\mathbb{C}$$

belongs to $\mathcal{P}_\Theta({}^nE)$, and for all $\varepsilon>0$ there exists $M(\varepsilon)>0$ such that

$$\left\|T\left(\widehat{(d^{j}f(0))(\cdot)^{j-n}}\right)\right\|_\Theta \le CM(\varepsilon)(1+\varepsilon)^j\rho^{n-j}\left\|\hat{d}^jf(0)\right\|_\Theta,$$

if $k=\infty$ and

$$\left\|T\left(\widehat{(d^{j+n}f(0))(\cdot)^{j}}\right)\right\|_\Theta \le CM(\varepsilon)(1+\varepsilon)^{j+n}\rho^{-j}\left(\frac{j}{ke}\right)^{\frac{j}{k}}\left\|\hat{d}^{j+n}f(0)\right\|_\Theta,$$

if $k>1$, for some positive constants C and ρ as in Definition 2.9.

We divide the proof into two cases.

Case $k=\infty$: Let $f \in Exp^1_{\Theta,0,A}(E)$ and $\gamma > A$. By (3.2) there are constants $B_\gamma > 0$ and $\gamma' > A$ such that

$$\begin{aligned}
\|Lf\|_{\Theta,1,\gamma} &= \sum_{n=0}^{\infty} \gamma^{-n} \left(\frac{n}{e}\right)^n \left\| \frac{1}{n!} \hat{d}^n (T * f)(0) \right\|_\Theta \overset{(3.2)}{\leq} B_\gamma \sum_{n=0}^{\infty} (\gamma')^{-n} \left\| \hat{d}^n (T * f)(0) \right\|_\Theta \\
&\leq B_\gamma C M(\varepsilon) \sum_{n=0}^{\infty} (\gamma')^{-n} \sum_{j=n}^{\infty} \frac{1}{(j-n)!} (1+\varepsilon)^j \rho^{n-j} \left\| \hat{d}^j f(0) \right\|_\Theta \\
&= B_\gamma C M(\varepsilon) \sum_{j=0}^{\infty} (1+\varepsilon)^j \left\| \hat{d}^j f(0) \right\|_\Theta \sum_{n=0}^{j} (\gamma')^{-n} \frac{\rho^{n-j}}{(j-n)!} \\
&\leq B_\gamma C M(\varepsilon) \sum_{j=0}^{\infty} \left(\frac{1+\varepsilon}{\gamma'}\right)^j \left\| \hat{d}^j f(0) \right\|_\Theta \sum_{n=0}^{j} \frac{\left(\frac{\gamma'}{\rho}\right)^{j-n}}{(j-n)!} \\
&\leq B_\gamma C M(\varepsilon) e^{(\gamma'/\rho)} \sum_{j=0}^{\infty} \left(\frac{1+\varepsilon}{\gamma'}\right)^j \left\| \hat{d}^j f(0) \right\|_\Theta .
\end{aligned}$$

Choosing $0 < \varepsilon < \frac{\gamma'}{A} - 1$ we obtain

$$\|Lf\|_{\Theta,1,\gamma} \leq B_\gamma C M(\varepsilon) e^{(\gamma'/\rho)} \|f\|_{\Theta,\frac{\gamma'}{1+\varepsilon}} .$$

Using again (3.2), it follows that there exist constants $K_\gamma > 0$ and $\gamma^+ > A$ such that

$$\|Lf\|_{\Theta,1,\gamma} \leq K_\gamma B_\gamma C M(\varepsilon) e^{(\gamma'/\rho)} \|f\|_{\Theta,1,\gamma^+} .$$

Hence $L\Big|_{Exp^1_{\Theta,0,A}(E)}$ is a well-defined continuous linear operator.

Case $k > 1$: Let $f \in Exp^1_{\Theta,0,A}(E)$ and $\gamma > A$. Then

$$\begin{aligned}
\|Lf\|_{\Theta,1,\gamma} &= \sum_{n=0}^{\infty} \gamma^{-n} \left(\frac{n}{e}\right)^n \left\| \frac{1}{n!} \hat{d}^n (T * f)(0) \right\|_\Theta \overset{(3.2)}{\leq} B_\gamma \sum_{n=0}^{\infty} (\gamma')^{-n} \left\| \hat{d}^n (T * f)(0) \right\|_\Theta \\
&\leq B_\gamma C M(\varepsilon) \sum_{n=0}^{\infty} (\gamma')^{-n} \sum_{j=0}^{\infty} \frac{1}{j!} (1+\varepsilon)^{j+n} \rho^{-j} \left(\frac{j}{ke}\right)^{\frac{j}{k}} \left\| \hat{d}^{j+n} f(0) \right\|_\Theta \\
&\leq B_\gamma C M(\varepsilon) c_\delta \sum_{n=0}^{\infty} (\gamma')^{-n} \sum_{j=0}^{\infty} (1+\varepsilon)^{j+n} \left(\frac{\delta^{\frac{1}{k}}}{\rho}\right)^j \left\| \hat{d}^{j+n} f(0) \right\|_\Theta \\
&= B_\gamma C M(\varepsilon) c_\delta \sum_{n=0}^{\infty} (\gamma')^{-n} \sum_{j=n}^{\infty} (1+\varepsilon)^{j} \left(\frac{\delta^{\frac{1}{k}}}{\rho}\right)^{j-n} \left\| \hat{d}^{j} f(0) \right\|_\Theta \\
&= B_\gamma C M(\varepsilon) c_\delta \sum_{j=0}^{\infty} (1+\varepsilon)^{j} \left\| \hat{d}^{j} f(0) \right\|_\Theta \sum_{n=0}^{j} (\gamma')^{-n} \left(\frac{\delta^{\frac{1}{k}}}{\rho}\right)^{j-n} \\
&= B_\gamma C M(\varepsilon) c_\delta \sum_{j=0}^{\infty} \left(\frac{1+\varepsilon}{\gamma'}\right)^{j} \left\| \hat{d}^{j} f(0) \right\|_\Theta \sum_{n=0}^{j} \left(\frac{\gamma' \delta^{\frac{1}{k}}}{\rho}\right)^{j-n} .
\end{aligned}$$

Taking $0 < \varepsilon < \frac{\gamma'}{A} - 1$ and $0 < \delta < \left(\frac{\rho}{\gamma'}\right)^k$ we obtain

$$\|Lf\|_{\Theta,1,\gamma} \leq K_\gamma B_\gamma C M(\varepsilon) c_\delta \frac{\rho}{\rho - \gamma' \delta^{\frac{1}{k}}} \|f\|_{\Theta,1,\gamma^+} . \tag{3.4}$$

for some constants $K_\gamma > 0$ and $\gamma^+ > A$. $\square$

Remark 3.14. Since $e^\varphi \in Exp^1_{\Theta,0,A}(E)$ whenever $\varphi \in E'$ and $\|\varphi\| \leq A/K$ where K comes from condition in Definition 2.8(1), the natural way to define the Fourier-Borel transform on $[Exp^1_{\Theta,0,A}(E)]'$ is $\mathcal{F}T(\varphi) = T(e^\varphi)$, for every $T \in [Exp^1_{\Theta,0,A}(E)]'$ and every $\varphi \in E'$ with $\|\varphi\| < A/K$. With this definition $\mathcal{F}$ is an injective linear operator from $[Exp^1_{\Theta,0,A}(E)]'$ into $\mathcal{H}(B_{E'}(0; A/K))$, where $B_{E'}(0; A/K)$ denotes the open ball in E' with center 0 and radius A/K.

The proof of the following lemma is very similar to the proof of Lemma 3.5 and thus we omit it.

Lemma 3.15. Let $A > 0$, E be a Banach space with separable dual, and $(\mathcal{P}_\Theta(^jE))_{j=0}^\infty$ be a π_1-holomorphy type from E to $\mathbb{C}$. Let $f \in \mathcal{H}(B_{E'}(0; A))$ be non constant and $B \subset \mathbb{C}$. Suppose that there is $\varphi_0 \in B_{E'}(0; A)$ such that $f(\varphi_0)$ is an accumulation point of B. Then $\text{span}\{e^\varphi : \varphi \in E',\ f(\varphi) \in B,\ \|\varphi\| < A\}$ is dense in $Exp^1_{\Theta,0,KA}(E)$, where $K > 0$ comes from Definition 2.8(1).

Now we are able to prove the main result of this section.

Proof of Theorem 3.8. Let $k \in (1, \infty]$ and L be a nontrivial convolution operator on $Exp^k_{\Theta,0}(E)$. Defining

$$\alpha_L := \inf\{\|\varphi\| : \varphi \in E' \text{ with } |L(e^\varphi)(0)| = 1\}$$

it follows from the proof of Theorem 3.3 that α_L is finite. Given $\varepsilon > 0$ let $\varphi_0 \in E'$ be such that

$$\alpha_L \leq \|\varphi_0\| < \alpha_L + \varepsilon \text{ and } |L(e^{\varphi_0})(0)| = 1.$$

Let $A = \alpha_L + \varepsilon$. Since $Exp^1_{\Theta,0,KA}(E)$ is continuously and densely embedded in $Exp^k_{\Theta,0}(E)$, by applying the hypercyclic comparison principle [34, p. 338] to show Theorem 3.8, it suffices to prove that the restriction $L|_{Exp^1_{\Theta,0,KA}(E)} : Exp^1_{\Theta,0,KA}(E) \to Exp^1_{\Theta,0,KA}(E)$ is frequently hypercyclic. Since e^φ is an eigenvector of L associated to the eigenvalue $L(e^\varphi)(0)$, by Theorem 3.4 we only need to show that for each $D \subset \partial\mathbb{D}$ such that $\partial\mathbb{D} \setminus D$ is dense in $\partial\mathbb{D}$, the linear space

$$\text{span}\{e^\varphi : \varphi \in E',\ \|\varphi\| < A,\ L(e^\varphi)(0) \in \partial\mathbb{D} \setminus D\}$$

is also dense in $Exp^1_{\Theta,0,KA}(E)$.

Let $T \in [Exp^1_{\Theta,0,KA}(E)]'$ be such that $Lf = T * f$ for every $f \in Exp^1_{\Theta,0,KA}(E)$. Note that the functional T is the restriction to $Exp^1_{\Theta,0,KA}(E)$ of the functional S defined on $Exp^k_{\Theta,0}(E)$ and such that $Lf = S * f$ for every $f \in Exp^k_{\Theta,0}(E)$ (see Theorem 3.12). Moreover, Proposition 3.13 ensures that the convolution product $T * f$ is well-defined. In particular,

$$L(e^\varphi)(0) = (T * e^\varphi)(0) = T(\tau_0 e^\varphi) = T(e^\varphi) = \mathcal{F}T(\varphi)$$

for every $\varphi \in E'$ with $\|\varphi\| < A$. Since L is a nontrivial convolution operator, it follows from Lemma 3.6 that the holomorphic function $\mathcal{F}T$ defined on the ball $B_{E'}(0; A)$ is non constant.

Finally, let D be a subset of $\partial\mathbb{D}$, such that $\partial\mathbb{D} \setminus D$ is dense in $\partial\mathbb{D}$. Since $\mathcal{F}T(\varphi_0) = L(e^{\varphi_0})(0) \in \partial\mathbb{D}$, we have that $\mathcal{F}T(\varphi_0)$ is an accumulation point of $\partial\mathbb{D} \setminus D$. Therefore, it follows from Lemma 3.15 that

$$\operatorname{span}\{e^{\varphi} : \varphi \in E',\ \|\varphi\| < A,\ \mathcal{F}T(\varphi) \in \partial\mathbb{D} \setminus D\}$$

is dense in $Exp^1_{\Theta,0,KA}(E)$. □

As an immediate application of this result, in the particular case of translation operators on $Exp^k_{\Theta,0}(E)$, the growth condition of Theorem 3.8 can be rewritten according to Corollary 3.16 (a). The proof is similar to the proof of [46, Proposition 4.5].

Corollary 3.16. Let $k \in (1, \infty]$, E be a Banach space with separable dual, let $(\mathcal{P}_\Theta(^jE))_{j=0}^\infty$ be a π_1-$\pi_{2,k}$-holomorphy type from E to $\mathbb{C}$ and τ_a be the translation operator by a non-zero vector a on $Exp^k_{\Theta,0}(E)$. Then

(a) given any $\varepsilon > 0$ there are $C > 0$ and a entire function $f \in Exp^k_{\Theta,0}(E)$, which is frequently hypercyclic for τ_a, satisfying

$$|f(x)| \leq Ce^{\varepsilon\|x\|} \quad \text{for every} \quad x \in E,$$

(b) let $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\liminf_{r\to\infty} \varepsilon(r) = 0$ and $C > 0$. Then there is no frequently hypercyclic entire function $f \in Exp^k_{\Theta,0}(E)$ for τ_a satisfying

$$|f(x)| \leq Ce^{\varepsilon(\|x\|)\|x\|} \quad \text{for every} \quad x \in E.$$

Proof. (a) Observe that $\tau_a(e^{\varphi})(0) = e^{\varphi(0-a)} = e^{-\varphi(a)}$, for every $\varphi \in E'$. Thus $\alpha_{\tau_a} = 0$, since $|\tau_a(e^{\varphi})(0)| = 1$ when $\varphi = 0$. On the other hand, since τ_a is a nontrivial convolution operator, it follows from Theorem 3.8 that, for each $\varepsilon > 0$ there is a frequently hypercyclic function $f \in Exp^k_{\Theta,0}(E)$ for τ_a which satisfies the following growth condition: given $\delta > 0$ there is $C_\delta > 0$ such that

$$|f(x)| \leq C_\delta e^{(K(\alpha_{\tau_a} + \frac{\varepsilon}{2K}) + \delta)\|x\|}$$

for every $x \in E$. Taking $\delta = \frac{\varepsilon}{2}$ the result follows.

(b) Consider the complex line $L := \{za : z \in \mathbb{C}\}$ and the restriction operator $R_L : Exp^k_{\Theta,0}(E) \to \mathcal{H}(\mathbb{C})$ given by

$$R_L(g)(z) := g\,|_L(z) = g(za),$$

for every $g \in Exp^k_{\Theta,0}(E)$ and $z \in \mathbb{C}$. In particular,

$$R_L(\tau_a g)(z) = \tau_a g(za) = g((z-1)a) = R_L(g)(z-1) = \tau_1(R_L(g))(z)$$

for every $g \in Exp^k_{\Theta,0}(E)$ and $z \in \mathbb{C}$. Thus, the following diagram is commutative

$$\begin{array}{ccc} Exp^k_{\Theta,0}(E) & \xrightarrow{\tau_a} & Exp^k_{\Theta,0}(E) \\ {\scriptstyle R_L}\downarrow & & \downarrow{\scriptstyle R_L} \\ \mathcal{H}(\mathbb{C}) & \xrightarrow[\tau_1]{} & \mathcal{H}(\mathbb{C}) \end{array}$$

Since $R_L(\varphi^n)(z) = \varphi^n(za) = \varphi^n(a)z^n$ for every $\varphi \in E'$, $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$, it follows that $R_L(\varphi^n) = \varphi^n(a)(\cdot)^n \in \mathcal{H}(\mathbb{C})$ for every $n \in \mathbb{N}_0$. Therefore, R_L has dense range. Now, assume that the result is false, that is, suppose that there exists a frequently hypercyclic entire function f for τ_a such that $|f(x)| \leq Ce^{\varepsilon(\|x\|)\|x\|}$ for all $x \in E$. Applying the hypercyclic comparison principle, $R_L(f)$ is a frequently hypercyclic entire function for τ_1 such that

$$|R_L(f)(z)| = |f(za)| \leq Ce^{\varepsilon(\|za\|)\|za\|}, \tag{3.5}$$

for all $z \in \mathbb{C}$. This, however, contradicts [34, Theorem 9.26], since it was proved that there is no frequently hypercyclic function for the translation operators on $\mathcal{H}(\mathbb{C})$ satisfying (3.5). □

4. Frequently Hypercyclic Subspaces

In this section we will show the existence of a *frequently hypercyclic subspace* of $Exp^k_{\Theta,0}(E)$ for a given nontrivial convolution operator, that means, a closed infinite-dimensional subspace of $Exp^k_{\Theta,0}(E)$ formed, excepted by the null function, by frequently hypercyclic functions of this given convolution operator. Our main result is the following theorem:

Theorem 4.1. Let $k \in (1, \infty]$, E be a Banach space with separable dual, with $\dim E > 1$, and $(\mathcal{P}_\Theta(^jE))_{j=0}^\infty$ be a π_1-holomorphy type from E to $\mathbb{C}$. Then, every nontrivial convolution operator L on $Exp^k_{\Theta,0}(E)$ has a frequently hypercyclic subspace.

In 2016, Bayart *et al* [2] showed that the differentiation operator D on $\mathcal{H}(\mathbb{C})$ (or more generally the operator $P(D)$, where P is a non-constant polynomial) does not have a frequently hypercyclic subspace. Since $Exp^\infty_{\Theta,0}(\mathbb{C}) = \mathcal{H}(\mathbb{C})$, Theorem 4.1 is not valid for $k = \infty$ and $\dim E = 1$.

The proof of Theorem 4.1 is a slightly modified version of [46, Theorem 4.4]. We need some preliminary results.

Proposition 4.2. ([34, Remark 9.10]) Let T be a continuous linear operator on a separable F-space X. Suppose that there exists a dense subset X_0 of X and for any $x \in X_0$ there is a sequence $(u_n(x))_{n=0}^\infty$ in X such that,

1. $\sum_{n=0}^\infty T^n x$ converges unconditionally,
2. $\sum_{n=0}^\infty u_n(x)$ converges unconditionally,
3. $u_0(x) = x$ and $T^m u_n(x) = u_{n-m}(x)$, for every $m \leq n$.

Then T is frequently hypercyclic.

Theorem 4.3. ([46, Theorem 4.3]) Let X be a separable F-space with a continuous norm, I be the identity operator on X and let T be a continuous linear operator on X that satisfies the hypothesis of the previous proposition. If $\dim\ker(T-\lambda I)=\infty$ for some scalar λ with $|\lambda|<1$, then T has a frequently hypercyclic subspace.

The following lemma is very useful to show the main result of this section, and its proof is based on arguments of [34, Theorems 9.22 and 9.25] and of the second proof of [12, Theorem 1.3].

Lemma 4.4. Let $k\in(1,\infty]$, E be a Banach space with separable dual, $(\mathcal{P}_\Theta(^jE))_{j=0}^\infty$ be a π_1-holomorphy type from E to $\mathbb{C}$, and let L be a nontrivial convolution operator on $Exp_{\Theta,0}^k(E)$. Then:

(a) there are C^2-functions $E_n:\partial\mathbb{D}\to Exp_{\Theta,0}^k(E)$, $n\in\mathbb{N}$, such that

$$L(E_n(\lambda))=\lambda E_n(\lambda)\quad\text{for every}\quad\lambda\in\partial\mathbb{D};$$

(b) if $B\subset\partial\mathbb{D}$ is a Borel subset of $\partial\mathbb{D}$ of full Lebesgue measure in $\partial\mathbb{D}$, then

$$\operatorname{span}\{E_n(\lambda):\lambda\in B,n\in\mathbb{N}\}$$

is dense in $Exp_{\Theta,0}^k(E)$;

(c) if for each $j\in\mathbb{Z}$ and each $n\in\mathbb{N}$, we define

$$x_{n,j}=\int_{\partial\mathbb{D}}\lambda^jE_n(\lambda)d\lambda$$

where the integral is in the sense of Riemann, then the space

$$X_0:=\operatorname{span}\{x_{n,j}:j\in\mathbb{Z},n\in\mathbb{N}\}$$

is dense in $Exp_{\Theta,0}^k(E)$;

(d) if for each $x=\sum_{l=1}^p a_lx_{n_l,j_l}\in X_0$ and each $m\in\mathbb{N}_0$ we define

$$u_m(x)=\sum_{l=1}^p a_lx_{n_l,j_l-m},$$

then the series

$$\sum_{m=0}^\infty L^mx\quad\text{and}\quad\sum_{m=0}^\infty u_m(x)$$

converge unconditionally for every $x\in X_0$.

Proof. (a) Let $\varphi_0\in E'$ with $|\mathcal{F}[\gamma_{\Theta,0}^k(L)](\varphi_0)|=1$. Since $\mathcal{F}[\gamma_{\Theta,0}^k(L)]$ is a non constant entire function on E' and since E' is separable, we can find a sequence L_n, $n\in\mathbb{N}$, of complex lines through φ_0 on which $\mathcal{F}[\gamma_{\Theta,0}^k(L)]$ is non constant and such that $\bigcup_{n\in\mathbb{N}}L_n$ is a dense subset of E'. Thus, for each $n\in\mathbb{N}$, the restriction $\mathcal{F}[\gamma_{\Theta,0}^k(L)]|_{L_n}$ belongs to $\mathcal{H}(\mathbb{C})$ and it is an open function. So we can choose non-empty open subarcs β_n of $\partial\mathbb{D}$ and C^2-functions $\psi_n:\beta_n\to L_n$ such that

$$\mathcal{F}[\gamma_{\Theta,0}^k(L)](\psi_n(\lambda))=\lambda\quad\text{for every}\quad\lambda\in\beta_n.$$

Now, for each $n \in \mathbb{N}$, consider a non-empty open subarc $\tilde{\beta}_n$ of β_n and a C^2-function $f_n : \partial\mathbb{D} \to \mathbb{C}$ being non-zero in $\tilde{\beta}_n$ and zero outside of $\tilde{\beta}_n$. We now define $E_n : \partial\mathbb{D} \to Exp^k_{\Theta,0}(E)$ by

$$E_n(\lambda) = f_n(\lambda) e^{\psi_n(\lambda)} \quad \text{if} \quad \lambda \in \beta_n$$

and

$$E_n(\lambda) = 0 \quad \text{if} \quad \lambda \notin \beta_n.$$

To show that E_n is a C^2-function, it is enough to prove that the mapping

$$e^{\psi_n} : \beta_n \to L_n \to Exp^k_{\Theta,0}(E)$$

is a C^2-function. To do this, note that it is enough to show that the mapping

$$\lambda \in \mathbb{C} \to e^{\lambda\varphi} \in Exp^k_{\Theta,0}(E),$$

with $\varphi \in E'$ fixed, is holomorphic. Since $e^{\varphi} \in Exp^k_{\Theta,0}(E)$ and $\hat{d}^j(e^{\varphi})(0) = \varphi^j$, for every $j \in \mathbb{N}_0$, we have that

$$\sum_{j=0}^{m} \frac{1}{j!}\varphi^j \to e^{\varphi} \ \text{ in } Exp^k_{\Theta,0}(E) \tag{4.1}$$

and

$$\limsup_{j\to\infty} \left(\frac{j}{ke}\right)^{\frac{1}{k}} \left\| \frac{1}{j!}\varphi^j \right\|_{\Theta}^{\frac{1}{j}} = 0. \tag{4.2}$$

By (4.1) we have

$$\frac{e^{\lambda\varphi} - e^{\lambda_0\varphi}}{\lambda - \lambda_0} - \varphi e^{\lambda_0\varphi} = \sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{\lambda^j - \lambda_0^j}{\lambda - \lambda_0} - j\lambda^{j-1}\right)\varphi^j \ \text{ in } Exp^k_{\Theta,0}(E),$$

for every $\lambda, \lambda_0 \in \mathbb{C}$, with $\lambda \neq \lambda_0$. Since

$$\frac{\lambda^j - \lambda_0^j}{\lambda - \lambda_0} - j\lambda^{j-1} = \left(\lambda^{j-1} + \lambda^{j-2}\lambda_0 + \lambda^{j-3}\lambda_0^2 + \cdots + \lambda\lambda_0^{j-2} + \lambda_0^{j-1}\right) - j\lambda^{j-1},$$

we have

$$\lim_{\lambda\to\lambda_0}\left(\frac{\lambda^j - \lambda_0^j}{\lambda - \lambda_0} - j\lambda^{j-1}\right) = j\lambda_0^{j-1} - j\lambda_0^{j-1} = 0.$$

Thus, for $\rho > 0$ we have

$$\begin{aligned}\left\|\frac{e^{\lambda\varphi}-e^{\lambda_0\varphi}}{\lambda-\lambda_0}-\varphi e^{\lambda_0\varphi}\right\|_{\Theta,k,\rho} &= \left\|\sum_{j=1}^{\infty}\frac{1}{j!}\left(\frac{\lambda^j-\lambda_0^j}{\lambda-\lambda_0}-j\lambda^{j-1}\right)\varphi^j\right\|_{\Theta,k,\rho}\\ &= \sum_{j=1}^{\infty}\rho^{-j}\left(\frac{j}{ke}\right)^{\frac{j}{k}}\left\|\frac{1}{j!}\left(\frac{\lambda^j-\lambda_0^j}{\lambda-\lambda_0}-j\lambda^{j-1}\right)\varphi^j\right\|_{\Theta}\\ &= \sum_{j=1}^{\infty}\rho^{-j}\left(\frac{j}{ke}\right)^{\frac{j}{k}}\left\|\frac{1}{j!}\varphi^j\right\|_{\Theta}\left|\frac{\lambda^j-\lambda_0^j}{\lambda-\lambda_0}-j\lambda^{j-1}\right|.\end{aligned}$$

Making $\lambda \to \lambda_0$ we get

$$\frac{e^{\lambda\varphi}-e^{\lambda_0\varphi}}{\lambda-\lambda_0}\to\varphi e^{\lambda_0\varphi}\quad\text{in}\quad Exp_{\Theta,0}^k(E).$$

Hence, the function $\lambda\in\mathbb{C}\to e^{\lambda\varphi}\in Exp_{\Theta,0}^k(E)$ is holomorphic and so each E_n is a C^2-function. Now, if $\lambda\in\beta_n$ then

$$\begin{aligned}L(E_n(\lambda)) &= L(f_n(\lambda)e^{\psi_n(\lambda)}) = f_n(\lambda)L(e^{\psi_n(\lambda)})\\ &= f_n(\lambda)\mathcal{F}[\gamma_{\Theta,0}^k(L)](\psi_n(\lambda))e^{\psi_n(\lambda)} = \lambda E_n(\lambda),\end{aligned}$$

and if $\lambda\notin\beta_n$ then $L(E_n(\lambda)) = 0 = \lambda E_n(\lambda)$. Therefore

$$L(E_n(\lambda)) = \lambda E_n(\lambda)\quad\text{for every}\quad\lambda\in\partial\mathbb{D}.$$

(b) It is not difficult to prove that

$$\operatorname{span}\{E_n(\lambda):\lambda\in B, n\in\mathbb{N}\} = \operatorname{span}\{e^{\varphi}:\varphi\in\psi_n(\tilde{\beta}_n\cap B), n\in\mathbb{N}\}.$$

To show that

$$\operatorname{span}\{E_n(\lambda):\lambda\in B, n\in\mathbb{N}\}$$

is dense in $Exp_{\Theta,0}^k(E)$, using the Hahn-Banach theorem, it suffices to prove that if a functional $T\in[Exp_{\Theta,0}^k(E)]'$ vanishes on $\operatorname{span}\{e^{\varphi}:\varphi\in\psi_n(\tilde{\beta}_n\cap B), n\in\mathbb{N}\}$, then T is identically zero. Assume that T vanishes on $\operatorname{span}\{e^{\varphi}:\varphi\in\psi_n(\tilde{\beta}_n\cap B), n\in\mathbb{N}\}$, then $\mathcal{F}T(\varphi) = T(e^{\varphi}) = 0$ for every $\varphi\in\psi_n(\tilde{\beta}_n\cap B)$ and all $n\in\mathbb{N}$. Note that $\psi_n(\tilde{\beta}_n\cap B)$ has an accumulation point in L_n , since B has full Lebesgue measure. It follows that the set of the zeros of the entire function

$$[\mathcal{F}T]|_{L_n}: z\in\mathbb{C}\to[\mathcal{F}T](\varphi_0+z\varphi_n)\in\mathbb{C}\quad(\text{for some}\quad\varphi_n\in E')$$

has an accumulation point in $\mathbb{C}$. Hence $[\mathcal{F}T]|_{L_n} = 0$ for every $n\in\mathbb{N}$. Since $\bigcup_{n\in\mathbb{N}}L_n$ is dense in E', $\mathcal{F}T = 0$, and then $T = 0$.

(c) Again, we will use the Hahn-Banach theorem. Let $T\in[Exp_{\Theta,0}^k(E)]'$ such that $Tx_{n,j} = 0$ for every $n\in\mathbb{N}$ and $j\in\mathbb{Z}$. By the linearity and continuity of T, we have

$$0 = T(x_{n,j}) = \int_{\partial\mathbb{D}}\lambda^j T(E_n(\lambda))d\lambda = \int_0^{2\pi} ie^{it(j+1)}T(E_n(e^{it}))dt. \quad (4.3)$$

On the other hand, for each $n \in \mathbb{N}$, the function $t \in [0, 2\pi] \to T(E_n(e^{it})) \in \mathbb{C}$ is continuous and therefore it belongs to the Hilbert space $L^2[0, 2\pi]$. Since the set $\left\{\frac{1}{\sqrt{2\pi}} e^{itj} : j \in \mathbb{Z}\right\}$ is an orthonormal basis for $L^2[0, 2\pi]$, it follows from (4.3) that the functions $t \in [0, 2\pi] \to T(E_n(e^{it})) \in \mathbb{C}$, $n \in \mathbb{N}$, are identically zero, that is, $T(E_n(e^{it})) = 0$ for every $t \in [0, 2\pi]$ and all $n \in \mathbb{N}$. Since $\mathrm{span}\{E_n(\lambda) : \lambda \in \partial\mathbb{D}, n \in \mathbb{N}\}$ is dense in $Exp^k_{\Theta,0}(E)$, it follows that T is identically zero on $Exp^k_{\Theta,0}(E)$. Thus X_0 is dense in $Exp^k_{\Theta,0}(E)$.

(d) By the Riemann-Lebesgue lemma (see [34, Lemma 9.23]) the series

$$\sum_{m=0}^{\infty} \int_0^{2\pi} ie^{it(j+m+1)} E_n(e^{it}) dt \quad \text{and} \quad \sum_{m=0}^{\infty} \int_0^{2\pi} ie^{it(j-m+1)} E_n(e^{it}) dt$$

converge unconditionally for every $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. Since

$$L^m x_{n,j} = \int_0^{2\pi} ie^{it(j+m+1)} E_n(e^{it}) dt \quad \text{and}$$
$$u_m(x_{n,j}) = x_{n,j-m} = \int_0^{2\pi} ie^{it(j-m+1)} E_n(e^{it}) dt,$$

it follows that the series

$$\sum_{m=0}^{\infty} L^m x_{n,j} \quad \text{and} \quad \sum_{m=0}^{\infty} u_m(x_{n,j})$$

converge unconditionally for every $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. By the linearity of L^m and by definition of u_m, we have that the series $\sum_{m=0}^{\infty} L^m x$ and $\sum_{m=0}^{\infty} u_m(x)$ converge unconditionally for every $x \in X_0$.

□

Proof of Theorem 4.1. We will apply Theorem 4.3. By Lemma 4.4, every nontrivial convolution operator L on $Exp^k_{\Theta,0}(E)$ satisfies the first two conditions of Proposition 4.2. To complete the proof we only need to prove that L satisfies the following two conditions:

(i) $u_0(x) = x$ and $L^m u_n(x) = u_{n-m}(x)$, for every $m \leq n$ and $x \in X_0$.
(ii) $\dim \ker(L - \lambda I) = \infty$ for some scalar λ with $|\lambda| < 1$.

Here, X_0 and $u_n(x)$ are as in Lemma 4.4. Note that (i) follows immediately from the fact that

$$L^m x_{n,j} = x_{n,j+m} \quad \text{for every } j \in \mathbb{Z} \text{ and all } m, n \in \mathbb{N}.$$

On the other hand, given $\lambda \in \mathbb{C}$, let

$$Z(\mathcal{F}[\gamma^k_{\Theta,0}(L)] - \lambda) = \{\varphi \in E' : \mathcal{F}[\gamma^k_{\Theta,0}(L)](\varphi) - \lambda = 0\},$$

that is, the set of zeros of the complex-valued entire function $\mathcal{F}[\gamma^k_{\Theta,0}(L)] - \lambda$. Since $L(e^{\varphi}) = \mathcal{F}[\gamma^k_{\Theta,0}(L)](\varphi) e^{\varphi}$ for every $\varphi \in E'$, it follows that

$$\ker(L - \lambda I) \supset \{e^{\varphi} : \varphi \in Z(\mathcal{F}[\gamma^k_{\Theta,0}(L)] - \lambda)\}.$$

Since $\{e^{\varphi} : \varphi \in E'\}$ is a linearly independent set in $Exp^{k}_{\Theta,0}(E)$, to show that $\dim\ker(L-\lambda I)=\infty$ for some scalar λ with $|\lambda|<1$, it suffices to prove that $Z(\mathcal{F}[\gamma^{k}_{\Theta,0}(L)]-\lambda)$ is infinite, for some scalar λ with $|\lambda|<1$. Since

$$[\mathcal{F}[\gamma^{k}_{\Theta,0}(L)]]^{-1}(\mathbb{D})=\{\varphi\in E' : |\mathcal{F}[\gamma^{k}_{\Theta,0}(L)](\varphi)|<1\}$$

is non-empty, there exists $\varphi_0\in E'$ such that $|\mathcal{F}[\gamma^{k}_{\Theta,0}(L)](\varphi_0)|<1$. Denoting $\lambda_0 := \mathcal{F}[\gamma^{k}_{\Theta,0}(L)](\varphi_0)$, since $\dim E>1$ and $Z(\mathcal{F}[\gamma^{k}_{\Theta,0}(L)]-\lambda_0)\neq\emptyset$, it follows that $Z(\mathcal{F}[\gamma^{k}_{\Theta,0}(L)]-\lambda_0)$ is an infinite set. □

Acknowledgements

We are very much indebted to the referee for his/her careful reading and many valuable suggestions and comments.

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Received: October 24, 2019.
Revised: June 13, 2020.