Integral Equations
and Operator Theory

Schauder Estimates for Equations Associated with Lévy Generators

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Abstract. We study the regularity of solutions to the integro-differential equation $Af - \lambda f = g$ associated with the infinitesimal generator A of a Lévy process. We show that gradient estimates for the transition density can be used to derive Schauder estimates for f. Our main result allows us to establish Schauder estimates for a wide class of Lévy generators, including generators of stable Lévy processes and subordinated Brownian motions. Moreover, we obtain new insights on the (domain of the) infinitesimal generator of a Lévy process whose characteristic exponent ψ satisfies $\operatorname{Re} \psi(\xi) \asymp |\xi|^\alpha$ for large $|\xi|$. We discuss the optimality of our results by studying in detail the domain of the infinitesimal generator of the Cauchy process.

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1. Introduction

Let $(X_t)_{t \geq 0}$ be a Lévy process. By the Lévy–Khintchine formula, the infinitesimal generator $(A, \mathcal{D}(A))$ of $(X_t)_{t \geq 0}$ has the representation

$$Af(x) = b \cdot \nabla f(x) + \frac{1}{2} \operatorname{tr}(Q \cdot \nabla^2 f(x))$$

$$+ \int_{y \neq 0} \left(f(x+y) - f(x) - \nabla f(x) \cdot y \mathbb{1}_{(0,1)}(|y|) \right) \nu(dy)$$

for smooth compactly supported functions $f \in C_c^\infty(\mathbb{R}^d)$ where (b, Q, ν) is the Lévy triplet of $(X_t)_{t \geq 0}$, cf. Sect. 2. In this paper, we study the Hölder regularity of solutions $f \in \mathcal{D}(A)$ to the integro-differential equation

$$Af + \varrho f = g \tag{1}$$

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for fixed $\varrho \in \mathbb{R}$. We are interested in the following question: If g is β-Hölder continuous for some $\beta \geq 0$, then what can we say about the regularity of f? In particular: How regular is a function $f \in \mathcal{D}(A)$?

For the particular case that A is a second order differential operator, i. e. $\nu = 0$, the regularity of solutions to (1) is well understood, see e. g. [12], and therefore our focus is on non-local Lévy generators. An important example of a non-local Lévy generator is the fractional Laplacian

$$-(-\Delta)^{\alpha/2}f(x) = c_{d,\alpha} \int_{y \neq 0} \left(f(x+y) - f(x) - \nabla f(x) \cdot y \mathbb{1}_{(0,1)}(|y|)\right) \frac{1}{|y|^{d+\alpha}} \, dy$$

which is the infinitesimal generator of the isotropic α-stable Lévy process, $\alpha \in (0,2)$, and which plays an important role in analysis and probability theory, see e. g. the survey paper [27] for further information. Bass [3] showed that the solution to $-(-\Delta)^{\alpha/2}f = g$ satisfies the Schauder estimate

$$\|f\|_{C_b^{\alpha+\beta}(\mathbb{R}^d)} \leq L \left(\|f\|_\infty + \|g\|_{C_b^\beta(\mathbb{R}^d)}\right)$$

for $\beta \geq 0$ such that neither β nor $\alpha + \beta$ are integers. More recently, Ros-Oton and Serra [29] established Schauder estimates for solutions to (1) for generators of symmetric stable Lévy processes. Bae and Kassmann [1] introduced generalized Hölder space and studied, in particular, the regularity of solutions for Lévy operators of the form

$$Af(x) = \int_{y \neq 0} \left(f(x+y) - f(x) - \nabla f(x) \cdot y \mathbb{1}_{(0,1)}(|y|)\right) \frac{1}{|y|^d \varphi(|y|)} \, dy$$

where $\varphi : \mathbb{R}^d \to (0,\infty)$ is a "nice" function. Furthermore, it is known that the classical theory for pseudo-differential operators can be used to study the regularity of solutions to (1) if the characteristic exponent ψ of $(X_t)_{t \geq 0}$ is sufficiently smooth, see [17,37]. Since

$$\psi \in C^{2n}(\mathbb{R}^d) \iff \int_{|y|>1} |y|^{2n} \, \nu(dy) < \infty \iff \mathbb{E}(|X_1|^{2n}) < \infty, \quad n \in \mathbb{N},$$

cf. [20,30], this approach excludes many interesting examples of Lévy processes which do not have moments of sufficiently high order. Let us mention that the questions, which we discuss in this paper, are also related to the regularity of harmonic functions: If $g = 0$ and $\varrho = 0$ in (1), i. e. $Af = 0$, then f is harmonic for A, and there is an extensive literature on the regularity of functions which are harmonic for a Lévy generator, cf. [13,16,26,38] and the references therein. The regularity of solutions to elliptic integro-differential equations $Af = g$ has been studied, more generally, for classes of Lévy-type operators, see e. g. [1,3,11,17,22], and for non-linear integro-differential operators, greatly influenced by the works of Barles et al. [2] and Caffarelli and Silvestre [9].

The approach, which we follow in this paper, relies on regularizing properties of the resolvent R_λ associated with the Lévy process $(X_t)_{t \geq 0}$,

$$R_\lambda h(x) := \int_{(0,\infty)} e^{-\lambda t} \mathbb{E}h(x + X_t) \, dt, \qquad \lambda > 0, h \in \mathcal{B}_b(\mathbb{R}^d), x \in \mathbb{R}^d.$$

The main idea is to use gradient estimates for the transition density p_t of X_t to measure the regularizing effect of R_λ. More precisely, we will show that the gradient estimate

$$\int_{\mathbb{R}^d} |\nabla p_t(x)| \, dx \leq M t^{-1/\alpha}, \quad t \in (0, T],$$

implies that R_λ has a regularizing effect of order α, i.e.

$$h \in \mathcal{C}_b^\delta(\mathbb{R}^d) \implies R_\lambda h \in \mathcal{C}_b^{\alpha+\delta}(\mathbb{R}^d)$$

for any $\delta \geq 0$, cf. Sect. 2 for the definition of the Hölder–Zygmund spaces $\mathcal{C}_b^\gamma(\mathbb{R}^d)$. As $\mathcal{D}(A) = R_\lambda(C_\infty(\mathbb{R}^d))$ this gives, in particular, $\mathcal{D}(A) \subseteq \mathcal{C}_b^\alpha(\mathbb{R}^d)$. Our main result, Theorem 1.1, shows that, more generally, the implication

$$Af \in \mathcal{C}_b^\delta(\mathbb{R}^d) \implies f \in \mathcal{C}_b^{\alpha+\delta}(\mathbb{R}^d)$$

holds for any $\delta \geq 0$.

1.1. Theorem. *Let $(X_t)_{t\geq 0}$ be a Lévy process with infinitesimal generator $(A, \mathcal{D}(A))$ and characteristic exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ satisfying the Hartman–Wintner condition*

$$\lim_{|\xi|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\log(|\xi|)} = \infty. \tag{HW}$$

Assume that there exist constants $M > 0$, $T > 0$ and $\alpha \in (0, 2]$ such that the transition density p_t of $(X_t)_{t\geq 0}$ satisfies

$$\int_{\mathbb{R}^d} |\nabla p_t(x)| \, dx \leq M t^{-1/\alpha}, \qquad t \in (0, T]. \tag{2}$$

If $f \in \mathcal{D}(A)$ is such that

$$Af + \varrho f = g \in \mathcal{C}_b^\delta(\mathbb{R}^d)$$

for some $\delta \geq 0$ and $\varrho \in \mathbb{R}$, then $f \in \mathcal{C}_b^{\alpha+\delta}(\mathbb{R}^d)$ and the Schauder estimate

$$\|f\|_{\mathcal{C}_b^{\alpha+\delta}(\mathbb{R}^d)} \leq L\big(\|g\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + \|f\|_\infty\big) \tag{3}$$

holds for a constant $L = L(\varrho, \delta, \alpha, M, d, T)$. In particular, $\mathcal{D}(A) \subseteq \mathcal{C}_\infty^\alpha(\mathbb{R}^d)$.

1.2. *Remark.*　(i) From the proof of Theorem 1.1 it is possible to obtain an explicit expression for the constant L in terms of δ, α, M, d and T.

(ii) Condition (2) is equivalent to saying that the associated semigroup $P_t u(x) := \mathbb{E}u(x + X_t)$ satisfies the gradient estimate

$$\|\nabla P_t u\|_\infty \leq M' t^{-1/\alpha} \|u\|_\infty, \qquad t \in (0, T), \ u \in \mathcal{B}_b(\mathbb{R}^d),$$

cf. [24, Lemma 4.1] for details.

(iii) It is no restriction to assume that $\alpha \leq 2$. If (2) holds for some $\alpha \geq 0$, then $\alpha \leq 2$, cf. Remark 3.2(ii).

(iv) The Hartman–Wintner condition (HW) ensures that X_t has a smooth density p_t for all $t > 0$, see [19] for a thorough discussion of (HW).

Gradient estimates for Lévy processes have been intensively studied in the last years, e. g. [14,18,21,26,35] to mention but a few, and therefore Theorem 1.1 applies to a wide class of Lévy processes. If $(X_t)_{t\geq 0}$ is a subordinated Brownian motion, then it is possible to derive gradient estimates from heat kernel estimates for the transition density using the dimension walk formula, cf. [23, Corollary 3.2].

Theorem 1.1 will be proved in Sects. 3 and 4 we will illustrate Theorem 1.1 with some examples and applications. In particular, we will present Schauder estimates for elliptic equations $Af + \varrho f = g$ associated with generators of continuous Lévy processes, stable Lévy processes and subordinated Brownian motions. Moreover, we will study in detail the infinitesimal generator $(A, \mathcal{D}(A))$ of a Lévy process whose characteristic exponent ψ satisfies the sector condition, $|\operatorname{Im}\psi(\xi)| \leq c|\operatorname{Re}\psi(\xi)|$, and

$$\operatorname{Re}\psi(\xi) \asymp |\xi|^\alpha \quad \text{as } |\xi| \to \infty;$$

combining Theorem 1.1 with results from [25,35] we will show that

$$\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) := \bigcup_{\beta > \alpha} \mathcal{C}_\infty^\beta(\mathbb{R}^d) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_\infty^\alpha(\mathbb{R}^d), \tag{4}$$

and this, in turn, will allow us to prove that $\mathcal{D}(A)$ is an algebra, that is $f \cdot g \in \mathcal{D}(A)$ for any $f, g \in \mathcal{D}(A)$, and that

$$A(f \cdot g) = g \cdot Af + f \cdot Ag + \Gamma(f, g), \qquad f, g \in \mathcal{D}(A)$$

where Γ is the Carré du Champ operator, cf. Theorem 4.3. It is natural to ask whether the inclusions in (4) are strict and whether (4) is the optimal way to describe $\mathcal{D}(A)$ in terms of Hölder spaces. In Sect. 5 we will investigate these questions for the case $\alpha = 1$, which is of particular interest since there is no canonical way to define the Hölder space $\mathcal{C}^1(\mathbb{R}^d)$. We will show for the two-dimensional Cauchy process that (4) (with $\alpha = 1$) is indeed the best possible way to describe the domain in terms of Hölder spaces and, moreover, we will see that the inclusions are strict.

2. Basic Definitions and Notation

We consider the d-dimensional Euclidean space $\mathbb{R}^d$ with the canonical scalar product $x \cdot y := \sum_{j=1}^d x_j y_j$ and the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ generated by the open balls $B(x, r)$. For functions $f, g : \mathbb{R}^d \to (0, \infty)$ we write $f \asymp g$ as $|x| \to \infty$ if there exist constants $c > 0$ and $R > 0$ such that

$$\frac{1}{c}f(x) \leq g(x) \leq cf(x) \qquad \text{for all } |x| \geq R.$$

If f is a real-valued function, then $\operatorname{supp} f$ denotes its support, ∇f the gradient and $\nabla^2 f$ the Hessian of f. For $\alpha \geq 0$ we set

$$\lfloor \alpha \rfloor := \max\{k \in \mathbb{N}_0; k \leq \alpha\} \quad \text{and} \quad \llbracket \alpha \rrbracket := \max\{k \in \mathbb{N}_0; k < \alpha\}. \tag{5}$$

Function spaces $\mathcal{B}_b(\mathbb{R}^d)$ is the space of bounded Borel-measurable functions $f : \mathbb{R}^d \to \mathbb{R}$. The smooth functions with compact support are denoted by $C_c^\infty(\mathbb{R}^d)$, and $C_\infty(\mathbb{R}^d)$ is the space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$

vanishing at infinity. Superscripts $k \in \mathbb{N}$ are used to denote the order of differentiability, e.g. $f \in C_\infty^k(\mathbb{R}^d)$ means that f and its derivatives up to order k are $C_\infty(\mathbb{R}^d)$-functions. For $\alpha \geq 0$ we define Hölder–Zygmund spaces $\mathcal{C}_b^\alpha(\mathbb{R}^d)$ by

$$\mathcal{C}_b^\alpha(\mathbb{R}^d) := \left\{ f \in C_b(\mathbb{R}^d); \|f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} := \|f\|_\infty + \sup_{x \in \mathbb{R}^d, h \neq 0} \frac{|\Delta_h^{\lfloor \alpha \rfloor + 1} f(x)|}{|h|^\alpha} < \infty \right\} \tag{6}$$

where

$$\Delta_h f(x) := f(x+h) - f(x) \qquad \Delta_h^j f(x) := \Delta_h(\Delta_h^{j-1} f)(x), \quad j \geq 2$$

are iterated difference operators. Moreover, we set

$$\mathcal{C}_\infty^\alpha(\mathbb{R}^d) := \mathcal{C}_b^\alpha(\mathbb{R}^d) \cap C_\infty^{\llbracket \alpha \rrbracket}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) := \bigcup_{\varepsilon > 0} \mathcal{C}_\infty^{\alpha+\varepsilon}(\mathbb{R}^d). \tag{7}$$

For $\alpha \in (0, \infty) \backslash \mathbb{N}$ the Hölder space $\mathcal{C}_b^\alpha(\mathbb{R}^d)$ coincides with the "classical" Hölder space $C_b^\alpha(\mathbb{R}^d)$ equipped with norm

$$\|f\|_{C_b^\alpha(\mathbb{R}^d)} := \|f\|_\infty + \sum_{j=0}^{\lfloor \alpha \rfloor} \sum_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| = j}} \|\partial^\beta f\|_\infty + \max_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| = \lfloor \alpha \rfloor}} \sup_{x \neq y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x - y|^{\alpha - \lfloor \alpha \rfloor}}.$$

If $\alpha \in \mathbb{N}$ is an integer, then the Hölder–Zygmund space $\mathcal{C}_b^\alpha(\mathbb{R}^d)$ is strictly larger than $C_b^\alpha(\mathbb{R}^d)$. For $\alpha = 1$ it is possible to show that $\mathcal{C}_b^1(\mathbb{R}^d)$ is strictly larger than the space of bounded Lipschitz continuous functions, cf. [36, p. 148], which is, in turn, strictly larger than $C_b^1(\mathbb{R}^d)$. By [39, Theorem 2.7.2.2], it holds for all $\alpha > 0$ that

$$\|f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \asymp \|f\|_{C_b^\ell(\mathbb{R}^d)} + \sum_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| \leq \ell}} \sup_{\substack{x \in \mathbb{R}^d \\ 0 < |h| < \delta}} \frac{|\Delta_h^k \partial_x^\beta f(x)|}{|h|^{\alpha - \ell}} \tag{8}$$

for any $\delta \in (0, \infty]$ and $k, \ell \in \mathbb{N}_0$ such that $\ell < \alpha$ and $\ell + k > \alpha$. We will use the following result from interpolation theory. If $T : C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ is a linear operator such that

$$\|T\|_{C_b^{\alpha_1} \to \mathcal{C}_b^{\beta_1}} < \infty \quad \text{and} \quad \|T\|_{C_b^{\alpha_2} \to \mathcal{C}_b^{\beta_2}} < \infty$$

for some constants $\alpha_i \geq 0$, $\beta_i \geq 0$ where

$$\|T\|_{X \to Y} := \inf\{c > 0; \forall f \in Y, \|f\|_Y \leq 1 : \|Tf\|_X \leq c\},$$

then

$$\|T\|_{\mathcal{C}_b^{\lambda \alpha_1 + (1-\lambda)\alpha_2} \to \mathcal{C}_b^{\lambda \beta_1 + (1-\lambda)\beta_2}} \leq \|T\|_{C_b^{\alpha_1} \to \mathcal{C}_b^{\beta_1}}^\lambda \|T\|_{C_b^{\alpha_2} \to \mathcal{C}_b^{\beta_2}}^{1-\lambda} \tag{9}$$

for all $\lambda \in (0, 1)$; this inequality follows from the interpolation theorem, see e.g. [39, Sect. 1.3.3] or [28, Theorem 1.6], and the fact that $\mathcal{C}_b^{\lambda \alpha}(\mathbb{R}^d)$ is the real interpolation space $(C_b(\mathbb{R}^d), C_b^\alpha(\mathbb{R}^d))_{\lambda, \infty}$, cf. [39, Theorem 2.7.2.1].

Lévy processes Throughout, $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space. A stochastic process $X_t : \Omega \to \mathbb{R}^d$ is a (d-dimensional) Lévy process if $X_0 = 0$ almost

surely, $(X_t)_{t\geq 0}$ has independent and stationary increments and $t \mapsto X_t(\omega)$ is right-continuous with finite left-hand limits for almost all $\omega \in \Omega$. By the Lévy–Khintchine formula, any Lévy process is uniquely determined in distribution by its characteristic exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ through the relation

$$\mathbb{E} \exp(i\xi X_t) = \exp(-t\psi(\xi)), \qquad t \geq 0,\ \xi \in \mathbb{R}^d.$$

The characteristic exponent ψ has the Lévy–Khintchine representation

$$\psi(\xi) = -ib \cdot \xi + \frac{1}{2}\xi \cdot Q\xi + \int_{y\neq 0} \left(1 - e^{iy\cdot\xi} + iy\cdot\xi\mathbb{1}_{(0,1)}(|y|)\right)\nu(dy), \quad \xi \in \mathbb{R}^d,$$

where (b, Q, ν) is the Lévy triplet consisting of a vector $b \in \mathbb{R}^d$ (drift vector), a symmetric positive semi-definite matrix $Q \in \mathbb{R}^{d\times d}$ (diffusion matrix) and a measure ν on $\mathbb{R}^d\backslash\{0\}$ which satisfies the integrability condition $\int_{y\neq 0} \min\{1, |y|^2\}\,\nu(dy) < \infty$, the so-called Lévy measure. If the characteristic exponent ψ of a Lévy process $(X_t)_{t\geq 0}$ satisfies the Hartman–Wintner condition

$$\lim_{|\xi|\to\infty} \frac{\mathrm{Re}\,\psi(\xi)}{\log(|\xi|)} = \infty,$$

then X_t has a density p_t with respect to Lebesgue measure for any $t > 0$ and p_t has bounded derivatives of arbitrary order; we refer to [19] for a detailed discussion.

It follows from the independence and stationarity of the increments that any Lévy process $(X_t)_{t\geq 0}$ is a time-homogeneous Markov process, i.e. $P_t f(x) := \mathbb{E}f(x + X_t)$ defines a Markov semigroup. We denote by $(A, \mathcal{D}(A))$ the infinitesimal generator associated with $(X_t)_{t\geq 0}$,

$$\mathcal{D}(A) := \left\{ f \in C_\infty(\mathbb{R}^d); \exists g \in C_\infty(\mathbb{R}^d) : \lim_{t\to 0}\left\|\frac{P_t f - f}{t} - g\right\|_\infty = 0\right\},$$

$$Af := \lim_{t\to 0}\frac{P_t f - f}{t}, \quad f \in \mathcal{D}(A).$$

It is well-known that $C_c^\infty(\mathbb{R}^d)$ is contained in $\mathcal{D}(A)$ and that

$$Af(x) = b \cdot \nabla f(x) + \frac{1}{2}\mathrm{tr}(Q \cdot \nabla^2 f(x))$$
$$+ \int_{y\neq 0}\left(f(x+y) - f(x) - \nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(dy)$$

for any $f \in C_c^\infty(\mathbb{R}^d)$, see e.g. [30, Theorem 31.5]; here (b, Q, ν) denotes the Lévy triplet of $(X_t)_{t\geq 0}$. Moreover, the resolvent

$$R_\lambda f(x) := \int_{(0,\infty)} e^{-\lambda t}P_t f(x)\,dt, \qquad f \in \mathcal{B}_b(\mathbb{R}^d),\ \lambda > 0,\ x \in \mathbb{R}^d,$$

satisfies $R_\lambda(C_\infty(\mathbb{R}^d)) = \mathcal{D}(A)$ for any $\lambda > 0$. Our standard reference for Lévy processes is the monograph [30] by Sato.

3. Proof of Theorem 1.1

The first two results in this section prepare the proof of Theorem 1.1 but are of independent interest.

3.1. Proposition. *Let $(X_t)_{t\geq 0}$ be a Lévy process with resolvent $(R_\lambda)_{\lambda>0}$ and infinitesimal generator $(A, \mathcal{D}(A))$. Assume that the Hartman–Wintner condition* (HW) *holds. If the transition density p_t satisfies*

$$\int_{\mathbb{R}^d} |\partial_{x_j} p_t(x)|\, dx \leq M e^{mt} t^{-1/\alpha}, \qquad t > 0, \; j \in \{1, \ldots, d\}, \qquad (10)$$

for some constants $M > 0$, $m \geq 0$, and $\alpha \in (0, 2]$, then each of the following statements hold true.

(i) $R_\lambda(\mathcal{B}_b(\mathbb{R}^d)) \subseteq \mathcal{C}_b^\alpha(\mathbb{R}^d)$ *for any* $\lambda \geq 3m$ *and*

$$\|R_\lambda f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq K\|f\|_\infty, \qquad f \in \mathcal{B}_b(\mathbb{R}^d) \qquad (11)$$

for a constant $K = K(m, \alpha, d, \lambda, M)$.
(ii) *If* $\alpha > 1$ *then* $R_\lambda(C_\infty(\mathbb{R}^d)) \subseteq C_\infty^1(\mathbb{R}^d)$ *for any* $\lambda > m$.
(iii) $\mathcal{D}(A) \subseteq \mathcal{C}_\infty^\alpha(\mathbb{R}^d)$.

Proof of Proposition 3.1. (i) It was shown in [24, Lemma 4.1] that (10) implies

$$\int_{\mathbb{R}^d} |\partial_{x_i} \partial_{x_j} p_{2t}(x)|\, dx \leq c(t)^2 \qquad \text{for all } \; t > 0, \; i, j = 1, \ldots, d \qquad (12)$$

where $c(t) := M e^{mt} t^{-1/\alpha}$. For the readers' convenience we briefly explain the idea of the proof. By the Chapman–Kolmogorov equation, we have

$$p_{2t}(x) = \int_{\mathbb{R}^d} p_t(x - y) p_t(y)\, dy,$$

and so

$$\partial_{x_i} p_{2t}(x) = \int_{\mathbb{R}^d} p_t(y) \partial_{x_i} p_t(x - y)\, dy = \int_{\mathbb{R}^d} p_t(x - z) \partial_{x_i} p_t(z)\, dz \qquad (13)$$

which implies

$$\partial_{x_j} \partial_{x_i} p_{2t}(x) = \int_{\mathbb{R}^d} \big(\partial_{x_j} p_t(x - z)\big)\big(\partial_{x_i} p_t(z)\big)\, dz.$$

Applying Tonelli's theorem we conclude that

$$\int_{\mathbb{R}^d} |\partial_{x_i} \partial_{x_j} p_{2t}(x)|\, dx \leq \left(\int_{\mathbb{R}^d} |\partial_{x_i} p_t(z)|\, dz\right)\left(\int_{\mathbb{R}^d} |\partial_{x_j} p_t(z)|\, dz\right) \leq c(t)^2,$$

and this proves (12). Iterating the procedure, we get

$$\int_{\mathbb{R}^d} |\partial_x^\beta p_t(x)|\, dx \leq c(t)^{|\beta|} \qquad \text{for all } \; \beta \in \mathbb{N}_0^d. \qquad (14)$$

Now fix $f \in \mathcal{B}_b(\mathbb{R}^d)$, $\lambda \geq 3m$ and $x, h \in \mathbb{R}^d$. Since

$$R_\lambda f(z) = \int_{(0,\infty)} \int_{\mathbb{R}^d} e^{-\lambda t} f(y) p_t(y - z)\, dy\, dt$$

we have

$$|R_\lambda f(x+3h) + 3R_\lambda f(x+h) - 3R_\lambda f(x+2h) - R_\lambda f(x)| \le I_1 + I_2$$

where

$$I_1 := \left| \int_{t \le |h|^\alpha} \int_{\mathbb{R}^d} e^{-\lambda t} f(y) q(t,y) \, dy \, dt \right|$$

$$I_2 := \left| \int_{t > |h|^\alpha} \int_{\mathbb{R}^d} e^{-\lambda t} f(y) q(t,y) \, dy \, dt \right|$$

with

$$q(t,y) := (p_t(y - x - 3h) + 3p_t(y - x - h) - 3p_t(y - x - 2h) - p_t(y - x)).$$

Using $\int_{\mathbb{R}^d} |p_t(z+y)| \, dy = 1$ it follows from the triangle inequality that

$$I_1 \le 8\|f\|_\infty \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} |p_t(z+y)| \, dy \int_0^{|h|^\alpha} dt = 8|h|^\alpha \|f\|_\infty.$$

To estimate I_2 we note that, by the multivariate version of Taylor's theorem,

$$|q(t,y)| \le C|h|^3 \sum_{i=1}^3 \sum_{|\beta|=3} \int_0^1 |\partial_x^\beta p_t(y - x - irh)| \, dr$$

for an absolute constant $C > 0$. Applying Tonelli's theorem and using (14) we get

$$I_2 \le Cd^3|h|^3\|f\|_\infty \max_{|\beta|=3} \sum_{i=1}^3 \int_{t>|h|^\alpha} e^{-\lambda t} \int_0^1 \int_{\mathbb{R}^d} |\partial_x^\beta p_t(y - x - irh)| \, dy \, dr \, dt$$

$$\le Cd^3 M^3 |h|^3 \|f\|_\infty \int_{t>|h|^\alpha} e^{(3m-\lambda)t} t^{-3/\alpha} \, dt$$

As $\lambda \ge 3m$ and $\alpha \in (0,2]$ this implies

$$I_2 \le Cd^3 M^3 |h|^3 \|f\|_\infty \int_{t>|h|^\alpha} t^{-3/\alpha} \, dt = Cd^3 M^3 \frac{\alpha}{3-\alpha} \|f\|_\infty |h|^\alpha.$$

Consequently, we have shown that

$$|R_\lambda f(x+3h) + 3R_\lambda f(x+h) - 3R_\lambda f(x+2h) - R_\lambda f(x)| \le C'|h|^\alpha$$

for all $x, h \in \mathbb{R}^d$, and by (8) this proves (11).

(ii) If $\alpha > 1$ then a straight-forward application of the differentiation lemma for parametrized integrals, see e. g. [32, Theorem 12.5] or [24, Proposition A.1], shows that

$$\partial_{x_j} R_\lambda f(x) = - \int_{(0,\infty)} \int_{\mathbb{R}^d} e^{-\lambda t} f(y) \partial_{x_j} p_t(y-x) \, dy \, dt$$

for $f \in \mathcal{B}_b(\mathbb{R}^d)$, $\lambda > m$ and $j \in \{1, \ldots, d\}$. Since this clearly implies that

$$\partial_{x_j} R_\lambda f(x) = - \int_{(0,\infty)} \int_{\mathbb{R}^d} e^{-\lambda t} f(x+y) \partial_{x_j} p_t(y) \, dy \, dt,$$

the dominated convergence theorem entails that $\lim_{|x|\to\infty} |\partial_{x_j} R_\lambda f(x)| = 0$ for any $f \in C_\infty(\mathbb{R}^d)$.

(iii) Since $R_\lambda(C_\infty(\mathbb{R}^d)) = \mathcal{D}(A)$ for any $\lambda > 0$, the assertion is obvious from (i) and (ii). $\qquad \square$

3.2. Remark. (i) If there are constants $M > 0$, $T > 0$ and $\alpha \in (0,2]$ such that

$$\int_{\mathbb{R}^d} |\partial_{x_i} p_t(x)| \, dx \le Mt^{-1/\alpha}, \qquad t \in (0,T], \, i \in \{1,\ldots,d\}, \tag{15}$$

then there exists $m \ge 0$ such that (10) holds. *Indeed:* Fix $t \in (0,T)$ and $i \in \{1,\ldots,d\}$. It follows from (13) that

$$\int_{\mathbb{R}^d} |\partial_{x_i} p_{2t}(x)| \, dx \le \int_{\mathbb{R}^d} |\partial_{x_i} p_t(z)| \left(\int_{\mathbb{R}^d} |p_t(x-z)| \, dx \right) dz$$

$$= \int_{\mathbb{R}^d} |\partial_{x_i} p_t(z)| \, dz \le c(t) := Mt^{-1/\alpha},$$

which gives

$$\int_{\mathbb{R}^d} |\partial_{x_i} p_s(x)| \, dx \le c(s/2) = M 2^{1/\alpha} s^{-1/\alpha} \qquad \text{for all } \ s \in [T, 2T).$$

By iteration we find that

$$\int_{\mathbb{R}^d} |\partial_{x_i} p_s(x)| \, dx \le M(2^{1/\alpha})^k s^{-1/\alpha} \qquad \text{for all } \ s \in [2^{k-1}T, 2^k T), \, k \in \mathbb{N}.$$

Hence, (10) holds for $m := \log(2^{1/(\alpha T)})$.

(ii) If (15) holds for some $\alpha \ge 0$, then $\alpha \le 2$. Indeed: The Fourier transform of $x \mapsto \partial_{x_j} p_t(x)$ equals $i\xi_j e^{-t\psi(\xi)}$, and therefore

$$\sup_{\xi \in \mathbb{R}^d} |\xi_j e^{-t\psi(\xi)}| \le \|\partial_{x_j} p_t\|_{L^1} = \int_{\mathbb{R}^d} |\partial_{x_j} p_t(x)| \, dx.$$

Since the characteristic exponent ψ satisfies $|\psi(\xi)| \le c(1+|\xi|^2)$, $\xi \in \mathbb{R}^d$, for some constant $c > 0$ this gives

$$\int_{\mathbb{R}^d} |\partial_{x_j} p_t(x)| \, dx \ge \sup_{\xi \in \mathbb{R}^d} |\xi_j e^{-ct(1+|\xi|^2)} \ge c' t^{-1/2}.$$

In Proposition 3.1 we have seen that $R_\lambda f \in \mathcal{C}_b^\alpha(\mathbb{R}^d)$ for $f \in C_b(\mathbb{R}^d)$. Our next result, Corollary 3.3, shows that, more generally,

$$f \in \mathcal{C}_b^\beta(\mathbb{R}^d) \implies R_\lambda f \in \mathcal{C}_b^{\alpha+\beta}(\mathbb{R}^d)$$

for any $\beta \ge 0$.

3.3. Corollary. *Let $(X_t)_{t\ge0}$ be a Lévy process with resolvent $(R_\lambda)_{\lambda>0}$ and infinitesimal generator $(A, \mathcal{D}(A))$ such that its characteristic exponent ψ satisfies the Hartman–Wintner condition (HW). If there exist constants $M > 0$, $m \ge 0$ and $\alpha \in (0,2]$ such that the transition density p_t satisfies*

$$\int_{\mathbb{R}^d} |\partial_{x_i} p_t(x)| \, dx \le Me^{mt} t^{-1/\alpha}, \qquad t > 0, \, i \in \{1,\ldots,d\},$$

then there exists for any $k \in \mathbb{N}$ a constant $K = K(d, M, \alpha, \lambda, k) > 0$ such that

$$\|R_\lambda f\|_{\mathcal{C}_b^{\alpha+\beta}(\mathbb{R}^d)} \leq K\|f\|_{\mathcal{C}_b^\beta(\mathbb{R}^d)} \qquad \text{for all } f \in \mathcal{C}_b^\beta(\mathbb{R}^d), \, \beta \in (0, k), \, \lambda \geq 3m. \tag{16}$$

Proof. Fix $\lambda \geq 3m$ and $k \in \mathbb{N}$, and let $f \in C_b^k(\mathbb{R}^d)$. Since

$$R_\lambda f(x) = \int_{(0,\infty)} \int_{\mathbb{R}^d} e^{-\lambda t} f(x+y) p_t(y) \, dy \, dt, \qquad x \in \mathbb{R}^d$$

it follows from an application of the differentiation lemma for parametrized integrals that

$$\partial_x^\gamma R_\lambda f(x) = \int_{(0,\infty)} \int_{\mathbb{R}^d} e^{-\lambda t} \partial_x^\gamma f(x+y) p_t(y) \, dy \, dt = R_\lambda(\partial_x^\gamma f)(x) \tag{17}$$

for any multi-index $\gamma \in \mathbb{N}_0^d$ with $|\gamma| := \sum_{i=1}^d \gamma_i \leq k$. By Proposition 3.1, there exists a constant $K > 0$ such that

$$\|\partial^\gamma R_\lambda f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} = \|R_\lambda(\partial^\gamma f)(x)\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq K\|\partial^\gamma f\|_\infty,$$

and so, by (8),

$$\|R_\lambda f\|_{\mathcal{C}_b^{\alpha+k}(\mathbb{R}^d)} \leq cK\|f\|_{C_b^k(\mathbb{R}^d)} \qquad \text{for all } f \in C_b^k(\mathbb{R}^d)$$

for some constant $c = c(k) \geq 1$. On the other hand, Proposition 3.1 shows that

$$\|R_\lambda h\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq K\|h\|_{C_b(\mathbb{R}^d)} \qquad \text{for all } h \in C_b(\mathbb{R}^d).$$

Applying the interpolation theorem, cf. (9), we thus find that

$$\|R_\lambda h\|_{\mathcal{C}_b^{\alpha+\beta}(\mathbb{R}^d)} \leq cK\|h\|_{\mathcal{C}_b^\beta(\mathbb{R}^d)}, \qquad h \in \mathcal{C}_b^\beta(\mathbb{R}^d)$$

for any $\beta \in (0, k)$. $\qquad\qquad\square$

A close look at the proof of Corollary 3.3 shows that $R_\lambda f \in C_\infty^{[\![\beta]\!]+[\![\alpha]\!]}(\mathbb{R}^d)$ for any $f \in C_\infty^\beta(\mathbb{R}^d)$, cf. (5) for the definition of $[\![\alpha]\!]$ and $[\![\beta]\!]$; this is a consequence of (17) and Proposition 3.1(ii).

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $f \in \mathcal{D}(A)$ be such that $Af + \varrho f = g \in \mathcal{C}_b^\delta(\mathbb{R}^d)$ for some $\delta \geq 0$ and $\varrho \in \mathbb{R}$. It follows from (2) and Remark 3.2(i) that p_t satisfies (10) for some $m \geq 0$, and we set $\lambda := 3m+1$. Since $\mathcal{D}(A) = R_\lambda(C_\infty(\mathbb{R}^d))$ there is $h \in C_\infty(\mathbb{R}^d)$ such that $f = R_\lambda h$. As $Af + \varrho f = g$ we have $AR_\lambda h + \varrho R_\lambda h = g$, and using that $(\lambda \, \mathrm{id} - A) R_\lambda h = h$ this gives

$$h = (\lambda + \varrho) R_\lambda h - g. \tag{18}$$

We claim that for any $k \in \mathbb{N}_0$ there exists a constant $c_k > 0$ (not depending on f, g) such that

$$h \in \mathcal{C}_b^{\delta \wedge (k\alpha)}(\mathbb{R}^d) \quad \text{and} \quad \|h\|_{\mathcal{C}_b^{\delta \wedge (k\alpha)}(\mathbb{R}^d)} \leq c_k \left(\|g\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + \|h\|_\infty \right); \tag{19}$$

we prove (19) by induction. For $k = 0$ the assertion is obvious as $h \in C_b(\mathbb{R}^d)$ and $\|\lambda R_\lambda h\|_\infty \leq \|h\|_\infty$. Now suppose that (19) holds for some $k \in \mathbb{N}_0$. It

follows from Corollary 3.3 (with $\beta = \delta \wedge (\alpha k)$) that $R_\lambda h \in \mathcal{C}_b^{\alpha + (\delta \wedge (\alpha k))}(\mathbb{R}^d)$ and

$$\|R_\lambda h\|_{\mathcal{C}_b^{\alpha + (\delta \wedge (\alpha k))}(\mathbb{R}^d)} \leq K \|h\|_{\mathcal{C}_b^{\delta \wedge (k\alpha)}(\mathbb{R}^d)}.$$

Since, by assumption, $g \in \mathcal{C}_b^\delta(\mathbb{R}^d)$ we find from (18) that $h \in \mathcal{C}_b^{((k+1)\alpha) \wedge \delta}(\mathbb{R}^d)$ and

$$\begin{aligned}
\|h\|_{\mathcal{C}_b^{\delta \wedge ((k+1)\alpha)}(\mathbb{R}^d)} &\leq \|g\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + |\lambda + \varrho| \, \|R_\lambda h\|_{\mathcal{C}_b^{\alpha + (\delta \wedge (\alpha k))}(\mathbb{R}^d)} \\
&\leq \|g\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + K|\lambda + \varrho| \, \|h\|_{\mathcal{C}_b^{\delta \wedge (k\alpha)}(\mathbb{R}^d)} \\
&\leq \|g\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + K|\lambda + \varrho| c_k \left(\|g\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + \|h\|_\infty \right),
\end{aligned}$$

i.e. (19) holds for $k + 1$. We conclude that (19) holds for any $k \in \mathbb{N}_0$. If we choose $k \in \mathbb{N}$ sufficiently large such that $k\alpha \geq \delta$, then we find in particular $h \in \mathcal{C}_b^\delta(\mathbb{R}^d)$. Applying once more Corollary 3.3 we obtain that

$$\|R_\lambda h\|_{\mathcal{C}_b^{\alpha + \delta}(\mathbb{R}^d)} \leq K \|h\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} \leq K c_k \big(\|g\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + \|h\|_\infty \big).$$

Finally we note that $h = (\lambda \operatorname{id} - A)f$ implies

$$\|h\|_\infty \leq \lambda \|f\|_\infty + \|Af\|_\infty = \lambda \|f\|_\infty + \|g\|_\infty,$$

and therefore we conclude that

$$\|f\|_{\mathcal{C}_b^{\alpha + \delta}(\mathbb{R}^d)} = \|R_\lambda h\|_{\mathcal{C}_b^{\alpha + \delta}(\mathbb{R}^d)} \leq L \big(\|g\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + \|f\|_\infty \big). \qquad \square$$

4. Examples

In this section we illustrate Theorem 1.1 with some examples and applications.

Applying Theorem 1.1 to Lévy processes with continuous sample paths, we recover a classical result, see e.g. [12], on the regularity of the solutions to the second order elliptic differential equation

$$\varrho f(x) + \sum_{j=1}^d b_j \partial_{x_j} f(x) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d q_{ij} \partial_{x_i} \partial_{x_j} f(x) = g(x).$$

4.1. *Example.* Let $(B_t)_{t \geq 0}$ be a d-dimensional Brownian motion, $b \in \mathbb{R}^d$, and let $Q \in \mathbb{R}^{d \times d}$ be a symmetric positive definite matrix. The infinitesimal generator $(A, \mathcal{D}(A))$ of the Lévy process $X_t := bt + Q \cdot B_t$ satisfies

$$Af(x) = b \cdot \nabla f(x) + \frac{1}{2} \operatorname{tr}(Q \cdot \nabla^2 f(x)), \qquad f \in C_\infty^2(\mathbb{R}^d),$$

and has the following properties:

(i) $\mathcal{D}(A) \subseteq \mathcal{C}_\infty^2(\mathbb{R}^d)$,
(ii) If $Af + \varrho f = g \in \mathcal{C}_b^\delta(\mathbb{R}^d)$ for some $\delta \geq 0$ and $\varrho \in \mathbb{R}$, then $f \in \mathcal{C}_b^{2+\delta}(\mathbb{R}^d)$. Moreover, there exists a finite constant $L = L(d, \delta, \varrho) > 0$ such that

$$\|f\|_{\mathcal{C}_b^{\delta+2}(\mathbb{R}^d)} \leq L \left(\|Af\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + \|f\|_\infty \right), \qquad f \in \mathcal{D}(A).$$

For the definition of the Hölder spaces $\mathcal{C}_\infty^\alpha(\mathbb{R}^d)$ and $\mathcal{C}_b^\alpha(\mathbb{R}^d)$ we refer the reader to Sect. 2. Since there is a closed formula for the transition density p_t of X_t – which is of course of Gaussian type – it can be easily verified that the assumptions of Theorem 1.1 are satisfied for $\alpha = 2$, and this proves the assertion of Example 4.1.

Our next result applies to a large class of Lévy processes, including stable Lévy processes.

4.2. *Example.* Let $(L_t)_{t\geq 0}$ be a pure-jump Lévy process with infinitesimal generator $(A, \mathcal{D}(A))$. Assume that its Lévy measure ν satisfies

$$\nu(A) \geq \int_0^{r_0} \int_{\mathbb{S}^{d-1}} \mathbb{1}_A(r\theta) r^{-1-\alpha} \, \mu(d\theta) \, dr$$
$$+ \int_{r_0}^\infty \int_{\mathbb{S}^{d-1}} \mathbb{1}_A(r\theta) r^{-1-\beta} \, \mu(d\theta) \, dr, \quad A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$$

for some constants $\alpha \in (0,2)$, $\beta \in (0,\infty]$ and a finite measure μ on the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ which is non-degenerate, in the sense that its support is not contained in $\mathbb{S}^{d-1} \cap V$ where $V \subseteq \mathbb{R}^d$ is a lower-dimensional subspace. Then:

(i) $\mathcal{D}(A) \subseteq \mathcal{C}_\infty^\alpha(\mathbb{R}^d)$,
(ii) If $f \in \mathcal{D}(A)$ is such that $Af + \varrho f = g \in \mathcal{C}_b^\delta(\mathbb{R}^d)$ for some $\delta \geq 0$ and $\varrho \in \mathbb{R}$, then $f \in \mathcal{C}_b^{\alpha+\delta}(\mathbb{R}^d)$. Moreover, there exists for any $\delta \geq 0$ a finite constant $L = L(\alpha, \beta, \mu, d, \delta, \varrho)$ such that

$$\|f\|_{\mathcal{C}_b^{\alpha+\delta}(\mathbb{R}^d)} \leq L \left(\|Af\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + \|f\|_\infty \right), \qquad f \in \mathcal{D}(A).$$

Example 4.2 is a direct consequence of Theorem 1.1, Remark 1.2(ii) and [35, Example 1.5].

The remaining part of this section is devoted to Lévy processes whose characteristic exponent ψ satisfies

$$\operatorname{Re} \psi(\xi) \asymp |\xi|^\alpha \quad \text{as } |\xi| \to \infty$$

for some $\alpha \in (0,2)$. This class of Lévy processes covers many important and interesting examples, e. g.

- isotropic stable, relativistic stable and tempered stable Lévy processes,
- subordinated Brownian motions with characteristic exponent of the form $\psi(\xi) = f(|\xi|^2)$ for a Bernstein function f satisfying $f(\lambda) \asymp \lambda^{\alpha/2}$ for large λ, cf. [34] for details.
- Lévy processes with symbol of the form

$$\psi(\xi) = |\xi|^\alpha + |\xi|^\beta, \qquad \xi \in \mathbb{R}^d,$$

for $\beta \in (0, \alpha)$.

4.3. Theorem. *Let $(X_t)_{t\geq 0}$ be a Lévy process with infinitesimal generator $(A, \mathcal{D}(A))$. If the characteristic exponent ψ satisfies the sector condition, $|\operatorname{Im} \psi(\xi)| \leq c \operatorname{Re} \psi(\xi)$, and*

$$\operatorname{Re} \psi(\xi) \asymp |\xi|^\alpha \quad \text{as } |\xi| \to \infty \tag{20}$$

for some $\alpha \in (0,2)$, then:

(i) $\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) := \bigcup_{\beta > \alpha} \mathcal{C}_\infty^\beta(\mathbb{R}^d) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_\infty^\alpha(\mathbb{R}^d)$.

(ii) If $f \in \mathcal{D}(A)$ is such that $Af + \varrho f \in \mathcal{C}_b^\delta(\mathbb{R}^d)$ for some $\delta \geq 0$ and $\varrho \in \mathbb{R}$, then $f \in \mathcal{C}_b^{\alpha+\delta}(\mathbb{R}^d)$ and

$$\|f\|_{\mathcal{C}_b^{\alpha+\delta}(\mathbb{R}^d)} \leq L \left(\|Af\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} + \|f\|_\infty \right), \qquad f \in \mathcal{D}(A).$$

for some constant $L = L(\delta, \varrho, \alpha)$.

(iii) $\mathcal{D}(A)$ is an algebra, i. e. $f, g \in \mathcal{D}(A)$ implies $f \cdot g \in \mathcal{D}(A)$, and

$$A(f \cdot g)(x) = f(x)Ag(x) + g(x)Af(x) + \Gamma(f, g)(x) \tag{21}$$

for any $f, g \in \mathcal{D}(A)$ where

$$\Gamma(f, g)(x) := \int_{y \neq 0} \big(f(x+y) - f(x) \big) \big(g(x+y) - g(x) \big) \nu(dy) \tag{22}$$

is the Carré du Champ operator, cf. Remark (iii); here ν denotes the Lévy measure of $(X_t)_{t \geq 0}$.

Part (i) and (ii) of Theorem 4.3 follow by combining Theorem 1.1 with results obtained in [25, 35]. The main part of the proof is concerned with part (iii); they key point is to prove that

$$\lim_{t \to 0} \frac{\mathbb{E} \left([f(x + X_t) - f(x)] \cdot [g(x + X_t) - g(x)] \right)}{t} = \Gamma(f, g)(x), \quad f, g \in \mathcal{D}(A),$$

and for this the information on the regularity of $f, g \in \mathcal{D}(A)$ (i.e. part (i)) plays the crucial role; in particular, it allows us to show that the integral on the right-hand side of (22) is well-defined.

4.4. *Remark.* (i) The proof of Theorem 4.3(iii) shows the following slightly more general statement: Let $(X_t)_{t \geq 0}$ be a Lévy process with generator $(A, \mathcal{D}(A))$ and characteristic exponent ψ satisfying

$$\limsup_{|\xi| \to \infty} \frac{|\psi(\xi)|}{|\xi|^\alpha} < \infty$$

for some $\alpha \in (0, 2)$. Let $f, g \in \mathcal{D}(A)$ be such that

$$|f(x+y) - f(x)| \leq C_1 |y|^{\beta_1} \quad \text{and} \quad |g(x+y) - g(x)| \leq C_2 |y|^{\beta_2}$$

for all $x \in \mathbb{R}^d$ and $|y| \leq 1$. If $\beta_1 + \beta_2 > \alpha$ then $f \cdot g \in \mathcal{D}(A)$ and (21) holds.

(ii) Theorem 4.3 can be used to prove inclusions of the form $\mathcal{D}(A) \subseteq \mathcal{D}(L)$ for Lévy generators A and L. More precisely, if $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are Lévy processes with characteristic exponent ψ and $\tilde{\psi}$, respectively, which both satisfy the sector condition and

$$\operatorname{Re}\psi(\xi) \asymp |\xi|^\alpha \quad \text{and} \quad \operatorname{Re}\tilde{\psi}(\xi) \asymp |\xi|^\beta \quad \text{as } |\xi| \to \infty$$

for $\alpha < \beta$, then Theorem 4.3 shows that the domain of the generator of $(Y_t)_{t \geq 0}$ is contained in the domain of the generator of $(X_t)_{t \geq 0}$. For instance, the domain $\mathcal{D}(A^{(\alpha)})$ of the generator associated with the isotropic α-stable Lévy process, $\alpha \in (0, 2]$, satisfies $\mathcal{D}(A^{(\beta)}) \subseteq \mathcal{D}(A^{(\alpha)})$ for $\alpha < \beta$; this is a well-known result which can be, for instance, also proved using subordination, cf. [34, Theorem 13.6].

(iii) In contrast to other authors, we consider the Carré du champ operator Γ as an operator on $C_\infty(\mathbb{R}^d)$ and not on $L^2(dx)$. For further information on the Carré du champ operator we refer the reader to [8,10].

Proof of Theorem 4.3. Under the growth condition (20) it is shown in [35] that the semigroup $P_t f(x) := \mathbb{E}f(x + X_t)$ satisfies the gradient estimate

$$\|\nabla P_t f\|_\infty \le ct^{-1/\alpha}\|f\|_\infty, \qquad t \in (0,1], \ f \in \mathcal{B}_b(\mathbb{R}^d)$$

for some absolute constant $c > 0$. Since this implies $\int |\nabla p_t(x)|\, dx \le c't^{-1/\alpha}$, cf. Remark 1.2(ii), Theorem 1.1 gives (ii) and $\mathcal{D}(A) \subseteq \mathcal{C}_\infty^\alpha(\mathbb{R}^d)$. To prove $\mathcal{C}_\infty^\beta(\mathbb{R}^d) \subseteq \mathcal{D}(A)$, $\beta > \alpha$, we need some properties of the Lévy triplet (b, Q, ν) which are consequences of the growth condition (20) and the sector condition. As $\alpha < 2$ it follows from [25, Lemma A.3] that $Q = 0$ and [25, Lemma A.3] also shows $b = \int_{|y|<1} y\, \nu(dy)$ if $\alpha < 1$. Moreover,

$$\int_{0<|y|<1} |y|^\beta\, \nu(dy) < \infty \qquad \text{for all } \ \beta > \alpha, \tag{23}$$

see e.g. [5,31] or [25, Lemma A.2] for a detailed proof. By [25, Theorem 4.1], these properties of the Lévy triplet imply that $\mathcal{C}_\infty^\beta(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ for $\beta > \alpha$. It remains to prove (iii). Let $f, g \in \mathcal{D}(A)$ and fix $x \in \mathbb{R}^d$. We will first show that

$$\lim_{t\to 0} \frac{\mathbb{E}([f(x + X_t) - f(x)] \cdot [g(x + X_t) - g(x)])}{t} = \Gamma(f, g)(x) \tag{24}$$

with $\Gamma(f, g)(x)$ defined in (22). Pick a truncation function $\chi \in C_c^\infty(\mathbb{R}^d)$, $\mathbb{1}_{B(0,1)} \le \chi \le \mathbb{1}_{B(0,2)}$ and set $\chi_\varepsilon(y) := \chi(\varepsilon^{-1}y)$. Since the function

$$y \mapsto (1 - \chi_\varepsilon(y))(f(x + y) - f(x))(g(x + y) - g(x))$$

is continuous and equal to zero in a neighbourhood of x, the weak convergence $t^{-1}\mathbb{P}(X_t \in \cdot) \to \nu(\cdot)$ as $t \to 0$, cf. [30, Corollary 8.9] or [25, Corollary 3.3], yields

$$\lim_{t\to 0} \frac{\mathbb{E}\big((1 - \chi_\varepsilon(X_t))(f(x + X_t) - f(x))(g(x + X_t) - g(x))\big)}{t}$$
$$= \int_{y\neq 0} (1 - \chi_\varepsilon(y))(f(x + y) - f(x))(g(x + y) - g(x))\, \nu(dy).$$

By (i), we have $f, g \in \mathcal{D}(A) \subseteq \mathcal{C}_\infty^\alpha(\mathbb{R}^d)$ and so

$$|(f(x + y) - f(x))(g(x + y) - g(x))| \le C \min\{|y|^{2(\alpha\wedge 1)}, 1\}, \quad x, y \in \mathbb{R}^d; \tag{$\star$}$$

using (23) a straight-forward application of the dominated convergence theorem now shows that the right-hand side of the previous equation converges to $\Gamma(f, g)(x)$ as $\varepsilon \to 0$. On the other hand, $\operatorname{supp}\chi_\varepsilon \subseteq B(0, 2\varepsilon)$ and ($\star$) give

$$\Big|\mathbb{E}\big(\chi_\varepsilon(X_t)(f(x + X_t) - f(x))(g(x + X_t) - g(x))\big)\Big|$$
$$\le \mathbb{E}\left(|X_t|^{2(\alpha\wedge 1)}\mathbb{1}_{\{|X_t|\le 2\varepsilon\}}\right) = \int_{(0,2\varepsilon)} \mathbb{P}\left(|X_t|^{2(\alpha\wedge 1)} \ge r\right)\, dr.$$

Applying the maximal inequality, see e.g. [7, Corollary 5.2], and invoking the growth condition (20) we thus find

$$\left|\mathbb{E}\big(\chi_\varepsilon(X_t)(f(X_t) - f(x))(g(X_t) - g(x))\big)\right| \leq ct \int_0^{2\varepsilon} \sup_{|\xi| \leq r^{-1/2(\alpha \wedge 1)}} |\psi(\xi)|\, dr$$

$$\leq c't \int_0^{2\varepsilon} r^{-\alpha/2(\alpha \wedge 1)}\, dr$$

for absolute constants $c, c' > 0$. As $\int_0^1 r^{-\alpha/2(\alpha \wedge 1)}\, dr < \infty$ an application of the monotone convergence theorem yields

$$\limsup_{\varepsilon \to 0} \limsup_{t \to 0} \frac{\left|\mathbb{E}\big(\chi_\varepsilon(X_t)(f(X_t) - f(x))(g(X_t) - g(x))\big)\right|}{t} = 0,$$

and combining this with the earlier consideration, this proves (24). Now let $f, g \in \mathcal{D}(A)$ and fix $x \in \mathbb{R}^d$. Clearly,

$$\mathbb{E}\left(f(x + X_t)g(x + X_t)\right) - f(x)g(x)$$
$$= f(x)\mathbb{E}(g(x + X_t) - g(x)) + g(x)\mathbb{E}(f(x + X_t) - f(x))$$
$$+ \mathbb{E}\left((f(x + X_t) - f(x))(g(x + X_t) - g(x))\right).$$

Dividing both sides by t and letting t to 0 we obtain from (24) and the very definition of the generator A that

$$L(f \cdot g)(x) := \lim_{t \to 0} \frac{1}{t} \left[\mathbb{E}\left(f(x + X_t)g(x + X_t)\right) - f(x)g(x)\right]$$
$$= f(x)Ag(x) + g(x)Af(x) + \Gamma(f, g)(x).$$

Using the estimate $(\star)$ it follows from the dominated convergence theorem that $\Gamma(f, g) \in C_\infty(\mathbb{R}^d)$, and, hence, $L(f \cdot g) \in C_\infty(\mathbb{R}^d)$. This implies that $f \cdot g \in \mathcal{D}(A)$ and $A(f \cdot g) = L(f \cdot g)$, see e.g. [7, Theorem 1.33]. $\qquad\square$

5. Domain of the Infinitesimal Generator of Two-Dimensional Cauchy Process

Let $(X_t)_{t \geq 0}$ be an isotropic α-stable Lévy process, $\alpha \in (0, 2)$. It follows from Theorem 4.3(i) that the domain $\mathcal{D}(A)$ of the infinitesimal generator of $(X_t)_{t \geq 0}$ satisfies

$$\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) := \bigcup_{\beta > \alpha} \mathcal{C}_\infty^\beta(\mathbb{R}^d) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_\infty^\alpha(\mathbb{R}^d).$$

In this section we investigate whether this is the optimal way to describe $\mathcal{D}(A)$ in terms of Hölder spaces and whether the inclusions are strict. The case $\alpha = 1$ is particularly interesting since there are several functions spaces which are possible candidates to describe the domain:

- the space of Lipschitz continuous functions $\mathrm{Lip}(\mathbb{R}^d)$ vanishing at infinity,
- the space $C_\infty^1(\mathbb{R}^d)$ of differentiable functions vanishing at infinity,

- the Zygmund space $\mathcal{C}^1_\infty(\mathbb{R}^d)$ of functions f vanishing at infinity and satisfying

$$|\Delta^2_h f(x)| = |f(x+h) + f(x-h) - 2f(x)| \leq C|h|, \qquad x, h \in \mathbb{R}^d,$$

for some constant $C > 0$, see (6).

We will show that the domain $\mathcal{D}(A)$ of the generator of the two-dimensional Cauchy process has the following properties:

- There exists a function $f \in C^1_\infty(\mathbb{R}^2)$ which is not in $\mathcal{D}(A)$, cf. Proposition 5.1.
- There exists a function $f \in \mathcal{D}(A)$ which is not Lipschitz continuous, cf. Theorem 5.2.

This implies that

$$\mathcal{D}(A) \not\subseteq \mathrm{Lip}(\mathbb{R}^2) \cap C_\infty(\mathbb{R}^2) \qquad\qquad \mathrm{Lip}(\mathbb{R}^2) \cap C_\infty(\mathbb{R}^2) \not\subseteq \mathcal{D}(A)$$
$$\mathcal{D}(A) \not\subseteq C^1_\infty(\mathbb{R}^2) \qquad\qquad\qquad\qquad\qquad C^1_\infty(\mathbb{R}^2) \not\subseteq \mathcal{D}(A)$$

which clearly shows that the function spaces $\mathrm{Lip}(\mathbb{R}^2) \cap C_\infty(\mathbb{R}^2)$ and $C^1_\infty(\mathbb{R}^2)$ are not well suited for describing $\mathcal{D}(A)$. We conclude that

$$\mathcal{C}^{1+}_\infty(\mathbb{R}^2) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}^1_\infty(\mathbb{R}^2)$$

is the best possible way to describe $\mathcal{D}(A)$ in terms of Hölder spaces and, moreover, the inclusions are strict.

5.1. Proposition. *Let $(X_t)_{t \geq 0}$ be a d-dimensional Cauchy process with generator $(A, \mathcal{D}(A))$. Then there exists a function $f \in C^1_\infty(\mathbb{R}^d)$ which is not in $\mathcal{D}(A)$.*

Proof. Let $\chi \in C^\infty_c(\mathbb{R}^d)$ be such that $\mathbb{1}_{B(0,1/4)} \leq \chi \leq \mathbb{1}_{B(0,1/2)}$, and define

$$f(x) := \mathbb{1}_{\mathbb{R}^d \setminus \{0\}}(x) \frac{|x|}{|\log|x||} \chi(x), \qquad x \in \mathbb{R}^d.$$

If we set $\varphi(r) := |\log r|^{-1}$, then $\varphi(r) \to 0$ as $r \to 0$ and

$$f(x) = 0 + 0 \cdot x + |x|\varphi(|x|), \qquad |x| < \frac{1}{2}$$

which shows that f is differentiable at $x = 0$ and $\nabla f(0) = 0$. For $x \neq 0$ the differentiability is obvious. Clearly, f and its derivatives are vanishing at infinity, and so $f \in C^1_\infty(\mathbb{R}^d)$. Since the transition density p_t of X_t satisfies

$$p_t(y) \geq c\frac{t}{|y|^{d+1}}, \qquad |y| \geq t,$$

for some constant $c > 0$, we find from

$$\frac{\mathbb{E}f(X_t) - f(0)}{t} \geq \frac{1}{t}\mathbb{E}\left(\frac{|X_t|}{|\log(|X_t|)|} \mathbb{1}_{\{0 < |X_t| < 1/4\}}\right)$$

that

$$\frac{\mathbb{E}f(X_t) - f(0)}{t} \geq \frac{1}{t}\int_{0 < |y| < 1/4} \frac{|y|}{|\log(|y|)|} p_t(y)\,dy$$

$$\geq c\int_{t < |y| < 1/4} \frac{1}{|y|^d} \frac{1}{|\log(|y|)|}\,dy \xrightarrow{t \to 0} \infty,$$

and so $f \notin \mathcal{D}(A)$.   □

5.2. Theorem. *Let $(X_t)_{t\geq 0}$ be a 2-dimensional Cauchy process with generator $(A, \mathcal{D}(A))$. Then there exists a function $f \in \mathcal{D}(A)$ which is not Lipschitz continuous.*

   Let us mention that the proof of Theorem 5.2 has been inspired by Günter [15] who constructed a function $f \in C_\infty(\mathbb{R}^3)$ which is in the domain of the generator of three-dimensional Brownian motion but which is not twice differentiable, see [33, Example 7.25] for a modern account.

   For the proof of Theorem 5.2 we need an auxiliary result concerning the potential operator of an isotropic α-stable Lévy process $(X_t)_{t\geq 0}$. Recall that the potential operator $(R_0, \mathcal{D}(R_0))$ (in the sense of Yoshida) associated with a Lévy process $(X_t)_{t\geq 0}$ and resolvent $(R_\lambda)_{\lambda > 0}$ is defined by

$$\mathcal{D}(R_0) := \{f \in C_\infty(\mathbb{R}^d); \exists g \in C_\infty(\mathbb{R}^d) : \lim_{\lambda \to 0} \|R_\lambda f - g\|_\infty = 0\},$$

$$R_0 f := \lim_{\lambda \to 0} R_\lambda f, \quad f \in \mathcal{D}(R_0),$$

see [4, Sect. 11] for a thorough discussion.

5.3. Lemma. *Let $(X_t)_{t\geq 0}$ be a d-dimensional isotropic α-stable Lévy process with resolvent $(R_\lambda)_{\lambda > 0}$. If $\alpha < d$ then there exists a finite constant $c_{d,\alpha} > 0$ such that*

$$\sup_{\lambda > 0} R_\lambda u(x) = c_{d,\alpha} \int_{\mathbb{R}^d} |z|^{-d+\alpha} u(x+z)\, dz \tag{25}$$

for any non-negative $u \in C_\infty(\mathbb{R}^d)$. In particular, any non-negative function $u \in C_\infty(\mathbb{R}^d) \cap L^1(dx)$ is in the domain $\mathcal{D}(R_0)$ of the potential operator R_0.

Proof of Lemma 5.3. Identity (25) is a direct consequence of the scaling property of the transition density of $(X_t)_{t\geq 0}$; it is a classical result in potential theory, see e.g. [6] for a proof. For the second assertion, we note that $\int_{\mathbb{R}^d \cap B(0,1)} |y|^{\alpha-d}\, dy < \infty$ implies, by dominated convergence, that $\sup_{\lambda > 0} R_\lambda u \in C_\infty(\mathbb{R}^d)$ for any non-negative function $u \in C_\infty(\mathbb{R}^d) \cap L^1(dx)$. By [33, Theorem 7.24(d)] this entails that $u \in \mathcal{D}(R_0)$ for any such function u.   □

Proof of Theorem 5.2. As $R_0(\mathcal{D}(R_0)) \subseteq \mathcal{D}(A)$, cf. [4, Lemma 11.13(vi)], it suffices to find $u \in \mathcal{D}(R_0)$ such that $R_0 u$ is not Lipschitz continuous. It follows from Lemma 5.3 and the linearity of R_0 that

$$R_0 u(x) = c \int_{\mathbb{R}^2} |z|^{-1} u(x-z)\, dz, \qquad x \in \mathbb{R}^2$$

for any function $u \in C_\infty(\mathbb{R}^2) \cap L^1(dx)$. Pick a function $f \in C_c([0,1))$ such that $f \geq 0$ and $f(0) = 0$. If we define

$$u(x_1, x_2) := \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \frac{|x_2|}{\sqrt{x_1^2 + x_2^2}} f\left(\sqrt{x_1^2 + x_2^2}\right), \qquad x = (x_1, x_2) \in \mathbb{R}^2,$$

then $u \in C_\infty(\mathbb{R}^2) \cap L^1(dx) \subseteq \mathcal{D}(R_0)$. We will show that f can be chosen in such a way that $x \mapsto R_0 u(x)$ is not Lipschitz continuous at $x = 0$. Introducing

polar coordinates we find

$$R_0 u(0, x_2) = c \int_0^1 r f(r) \left(\int_0^{2\pi} \frac{|\sin\varphi| \cos\varphi}{\sqrt{x_2^2 + r^2 - 2rx_2 \cos\varphi}} \, d\varphi \right) dr.$$

Writing

$$I := \int_0^{2\pi} \frac{|\sin\varphi| \cos\varphi}{\sqrt{x_2^2 + r^2 - 2rx_2 \cos\varphi}} \, d\varphi = \left(\int_0^\pi + \int_\pi^{2\pi} \right) \frac{|\sin\varphi| \cos\varphi}{\sqrt{x_2^2 + r^2 - 2rx_2 \cos\varphi}} \, d\varphi$$

and performing a change of variables, $t := \sqrt{x_2^2 + r^2 - 2rx_2 \cos\varphi}$, we get for $x_2 > 0$

$$I = \frac{1}{rx_2} \int_{|x_2-r|}^{x_2+r} \cos\varphi(t) \, dt - \frac{1}{rx_2} \int_{x_2+r}^{|x_2-r|} \cos\varphi(t) \, dt$$

$$= \frac{2}{rx_2} \int_{|x_2-r|}^{x_2+r} \frac{x_2^2 + r^2 - t^2}{rx_2} \, dt$$

and so

$$I = \frac{4}{3} \frac{r}{x_2^2} \mathbb{1}_{\{r \le x_2\}} + \frac{4}{3} \frac{x_2}{r^2} \mathbb{1}_{\{r > x_2\}}.$$

Hence,

$$R_0 u(0, x_2) = \frac{4}{3} c \left(\frac{1}{x_2^2} \int_0^{x_2} r^2 f(r) \, dr + x_2 \int_{x_2}^1 \frac{f(r)}{r} \, dr \right).$$

As $u(x) = -u(-x)$ we have

$$\int_{\mathbb{R}^2} |z|^{-1} u(z) \, dz = \int_{\mathbb{R}^2} |z|^{-1} u(-z) \, dz = - \int_{\mathbb{R}^2} |z|^{-1} u(z) \, dz$$

which implies $R_0 u(0) = 0$. Consequently, we have shown that

$$\frac{R_0 u(0, x_2) - R_0 u(0)}{x_2} = \frac{4}{3} c \left(\frac{1}{x_2^3} \int_0^{x_2} r^2 f(r) \, dr + \int_{x_2}^1 \frac{f(r)}{r} \, dr \right).$$

If we choose $f(r) = |\log(r)|^{-1} \chi(r) 1_{(0,\infty)}(r)$ for a cut-off function χ satisfying $\mathbb{1}_{[0,1/2]} \le \chi \le \mathbb{1}_{[0,1)}$, then $\lim_{x_2 \downarrow 0} \int_{x_2}^1 \frac{f(r)}{r} \, dr = \infty$ and

$$\left| \frac{1}{x_2^3} \int_0^{x_2} r^2 f(r) \, dr \right| = \left| \frac{1}{x_2^3} \int_0^{x_2} \frac{r^2}{|\log r|} \, dr \right| \le \frac{1}{|\log |x_2||} \frac{1}{x_2^3} \int_0^{x_2} r^2 \, dr \xrightarrow{x_2 \downarrow 0} 0.$$

Thus,

$$\lim_{x_2 \downarrow 0} \frac{R_0 u(0, x_2) - R_0 u(0)}{x_2} = \infty,$$

i.e. $x \mapsto R_0 u(x)$ is not Lipschitz continuous at $x = 0$. □

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