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Compactness of Hankel Operators with Symbols Continuous on the Closure of Pseudoconvex Domains

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Abstract. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with Lipschitz boundary or a bounded convex domain in $\mathbb{C}^n$ and $\phi \in C(\overline{\Omega})$ such that the Hankel operator H_ϕ is compact on the Bergman space $A^2(\Omega)$. Then $\phi \circ f$ is holomorphic for any holomorphic $f : \mathbb{D} \to b\Omega$.

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Let Ω be a domain in $\mathbb{C}^n$ and $A^2(\Omega)$ denote the Bergman space of Ω, the space of square integrable holomorphic functions on Ω. Since $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, the space of square integrable functions on Ω, there exists an orthogonal projection $P : L^2(\Omega) \to A^2(\Omega)$, called the Bergman projection. The Hankel operator $H_\phi : A^2(\Omega) \to L^2(\Omega)$ with symbol $\phi \in L^\infty(\Omega)$ is defined as $H_\phi f = (I - P)(\phi f)$ where I denotes the identity operator. Hankel operators have been well studied on the Bergman space of the unit disc. Sheldon Axler in [2] proved the following interesting theorem.

Theorem (Axler). *Let $\phi \in A^2(\mathbb{D})$. Then $H_{\overline{\phi}}$ is compact if and only if $(1 - |z|^2)\phi'(z) \to 0$ as $|z| \to 1$.*

The space of holomorphic functions satisfying the condition in the theorem is called little Bloch space. One can check that $\phi(z) = \exp((z+1)/(z-1))$ is bounded on $\mathbb{D}$ but it does not belong to the little Bloch space. Hence not every bounded symbol that is smooth on the domain produces compact Hankel operator on the disc. However, Hankel operators with symbols continuous on the closure are compact for bounded domains in $\mathbb{C}$ (see, for instance, [22, Proposition 1]). We refer the reader to [26] for more information on the theory of Hankel operators (as well as Toeplitz operators) on the Bergman space of the unit disc. We note that Sheldon Axler's result has been extended to a small class of domains in $\mathbb{C}^n$, such as strongly pseudoconvex domains, by Peloso [20] and Li [17].

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The situation in $\mathbb{C}^n$ for $n \geq 2$ is radically different. For instance, $H_{\overline{z}_1}$ is not compact when Ω is the bidisc (see, for instance, [4,5,8,16]). Hence in higher dimensions compactness of Hankel operators is not guaranteed even if the symbol is smooth up to the boundary. We refer the reader to [13,23] for more information about Hankel operators in higher dimensions and their relations to $\overline{\partial}$-Neumann problem.

We are interested in studying compactness of Hankel operators on Bergman spaces defined on domains in $\mathbb{C}^n$. We would like to understand compactness of Hankel operators in terms of the interaction of the symbol with the boundary geometry. This interaction does not surface for domains in $\mathbb{C}$ as the boundary has no complex geometry. However, to relate the symbol to the boundary geometry we will restrict ourselves to symbols that are at least continuous up to the boundary. The first results in this direction are due to Željko Čučković and the third author in [6]. They obtain results about compactness of Hankel operators in terms of the behavior of the symbols along analytic discs in the boundary, on smooth bounded pseudoconvex domains (with a restriction on the Levi form) and on smooth bounded convex domains in $\mathbb{C}^n$. Moreover, for convex domains in $\mathbb{C}^2$ they obtain a characterization for compactness (see [6, Corollary 2]). We note that even though they state their results for C^∞-smooth domains and symbols, observation of the proofs shows that only C^1-smoothness is sufficient. One of their results, stated with C^1 regularity, is the following theorem.

Theorem (Čučković-Şahutoğlu). *Let Ω be a C^1-smooth bounded convex domain in $\mathbb{C}^2$ and $\phi \in C^1(\overline{\Omega})$. Then H_ϕ is compact if and only if $\phi \circ f$ is holomorphic for any holomorphic $f : \mathbb{D} \to b\Omega$.*

The theorem above can be interpreted as follows: H_ϕ is compact if and only if ϕ is "holomorphic along" every non-trivial analytic disc in the boundary.

The situation for symbols that are only continuous up to the boundary is less understood. When Ω is a bounded convex domain in $\mathbb{C}^n$ with no non-trivial discs in $b\Omega$ (that is, any holomorphic mapping $f : \mathbb{D} \to b\Omega$ is constant) all of the Hankel operators with symbols continuous on $\overline{\Omega}$ are compact. This follows from the following two facts: on such domains the $\overline{\partial}$-Neumann operator is compact (see [11]); compactness of the $\overline{\partial}$-Neumann operator implies that Hankel operators with symbols continuous on closure are compact (see [23, Proposition 4.1]).

In case of the polydisc Trieu Le in [16] proved the following characterization.

Theorem (Le). *Let ϕ be continuous on $\overline{\mathbb{D}^n}$ for $n \geq 2$. Then H_ϕ is compact if and only if there exist $\phi_1, \phi_2 \in C(\overline{\mathbb{D}^n})$ such that ϕ_1 is holomorphic on $\mathbb{D}^n, \phi_2 = 0$ on $b\mathbb{D}^n$, and $\phi = \phi_1 + \phi_2$.*

A domain $\Omega \subset \mathbb{C}^n$ is called Reinhardt if $(z_1, \ldots, z_n) \in \Omega$ implies that $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in \Omega$ for any $\theta_1, \ldots, \theta_n \in \mathbb{R}$. That is, Reinhardt domains are invariant under rotation in each variable. These are generalizations of the ball and the polydisc. Reinhardt domains are useful in describing domain of convergence for power series centered at the origin (see, for instance, [15,19,21]).

Motivated by the previous results mentioned above, recently, the first and the last authors proved the following result on convex Reinhardt domains in $\mathbb{C}^2$ (see [8]), generalizing the results in [6] (in terms of regularity of the symbol but on a small class of domains) and [16] (in terms of the domain in $\mathbb{C}^2$).

Theorem (Clos-Şahutoğlu). *Let Ω be a bounded convex Reinhardt domain in $\mathbb{C}^2$ and $\phi \in C(\overline{\Omega})$. Then H_ϕ is compact if and only if $\phi \circ f$ is holomorphic for any holomorphic $f : \mathbb{D} \to b\Omega$.*

We note that on piecewise smooth bounded convex Reinhardt domains in $\mathbb{C}^2$, the first author studied compactness of Hankel operators with conjugate holomorphic square integrable functions in [4]. Furthermore, compactness of products of two Hankel operators with symbols continuous up to the boundary was studied by Željko Čučković and the last author in [7].

In this paper we are able to partially generalize the result of Clos-Şahutoğlu to more general domains. In case the domain is in $\mathbb{C}^2$ we have the following result.

Theorem 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with Lipschitz boundary and $\phi \in C(\overline{\Omega})$ such that H_ϕ is compact on $A^2(\Omega)$. Then $\phi \circ f$ is holomorphic for any holomorphic $f : \mathbb{D} \to b\Omega$.*

However, for convex domains we can prove the following result in $\mathbb{C}^n$.

Theorem 2. *Let Ω be a bounded convex domain in $\mathbb{C}^n$ and $\phi \in C(\overline{\Omega})$ such that H_ϕ is compact on $A^2(\Omega)$. Then $\phi \circ f$ is holomorphic for any holomorphic $f : \mathbb{D} \to b\Omega$.*

As a corollary of Theorem 2 we obtain the following result for locally convexifiable domains in $\mathbb{C}^n$.

Corollary 1. *Let Ω be a bounded locally convexifiable domain in $\mathbb{C}^n$ and $\phi \in C(\overline{\Omega})$ such that H_ϕ is compact on $A^2(\Omega)$. Then $\phi \circ f$ is holomorphic for any holomorphic $f : \mathbb{D} \to b\Omega$.*

A domain $\Omega \subset \mathbb{C}^n$ is called complete Reinhardt if $(z_1, \ldots, z_n) \in \Omega$ and $\xi_1, \ldots, \xi_n \in \mathbb{C}$ with $|\xi_j| \leq 1$ for all j then $(\xi_1 z_1, \ldots, \xi_n z_n) \in \Omega$. We note that convex Reinhardt domains are complete Reinhardt but the converse is not true.

As a second corollary we obtain the following result for pseudoconvex complete Reinhardt domains in $\mathbb{C}^2$.

Corollary 2. *Let Ω be a bounded pseudoconvex complete Reinhardt domain in $\mathbb{C}^2$ and $\phi \in C(\overline{\Omega})$ such that H_ϕ is compact on $A^2(\Omega)$. Then $\phi \circ f$ is holomorphic for any holomorphic $f : \mathbb{D} \to b\Omega$.*

Remark 1. Peter Matheos, in his thesis [18] (see also [12, Theorem 10] and [23, Theorem 4.25]), constructed a smooth bounded pseudoconvex complete

Hartogs domain in $\mathbb{C}^2$ that has no analytic disc in its boundary, yet the $\overline{\partial}$-Neumann operator on the domain is not compact. Furthermore, Zeytuncu and the third author [24, Theorem 1] proved that on smooth bounded pseudoconvex Hartogs domains in $\mathbb{C}^2$, compactness of the $\overline{\partial}$-Neumann operator is equivalent to compactness of all Hankel operators with symbols smooth up to the boundary. Therefore, on Matheos' example the condition of Theorem 1 is trivially satisfied, yet there exists a non-compact Hankel operator with a symbol smooth on the closure of the domain. Namely, the converse of Theorem 1 is not true. On the other hand, the converse of Theorem 2 is open.

The plan of the paper is as follows: First we will prove a localization result for compactness of Hankel operators with bounded (not necessarily continuous) symbols. Then we concentrate on the proof of Theorem 1. Finally, we prove Theorem 2 and the corollaries.

Localization of Compactness

We note that H^U_ϕ denotes the Hankel operator on $A^2(U)$ with symbol ϕ which is an essentially bounded function on a domain U. Furthermore, we will use the following notation: $A \lesssim B$ means that there exists $c > 0$ that does not depend on quantities of interest such that $A \leq cB$. Also the constant c might change at every appearance. In the following lemma and the rest of the paper, $B(p, r)$ denotes the open ball centered at p with radius r.

Lemma 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $\phi \in L^\infty(\Omega)$, $p \in b\Omega, 0 < r_1 < r_2$, and $R_{r_2,r_1} : A^2(B(p, r_2) \cap \Omega) \to A^2(B(p, r_1) \cap \Omega)$ be the restriction operator defined as $R_{r_2,r_1} f = f|_{B(p,r_1)\cap\Omega}$. Assume that H^Ω_ϕ is compact on $A^2(\Omega)$. Then $H^{B(p,r_1)\cap\Omega}_\phi R_{r_2,r_1}$ is compact on $A^2(B(p, r_2) \cap \Omega)$.*

Proof. First we will simplify the notation and define the necessary operators. Let $U_j = B(p, r_j) \cap \Omega, Q^{U_j} = I - P^{U_j} : L^2(U_j) \to L^2(U_j)$ for $j = 1, 2$, and $Q^\Omega = I - P^\Omega : L^2(\Omega) \to L^2(\Omega)$. Also, in the following calculations $\|.\|_{U_j}$ and $\|.\|_\Omega$ denote the L^2 norms on U_j and Ω, respectively.

By [22, Lemma 3] we have a bounded operator $E_\varepsilon : A^2(U_2) \to A^2(\Omega)$ with the following estimate

$$\|R_{U_1}(f - E_\varepsilon f)\|_{U_1} \leq \varepsilon \|R_{U_1} f\|_{U_1}$$

for $f \in A^2(U_2)$ where R_{U_1} denotes the restriction onto U_1. Then, $H^{U_1}_\phi R_{U_1} g = Q^{U_1} R_{U_1} H^\Omega_\phi g$ for any $g \in A^2(\Omega)$. Then for $f \in A^2(U_2)$ we have

$$\begin{aligned}
\left\| H^{U_1}_\phi R_{r_2,r_1} f \right\|^2_{U_1} &= \langle R_{U_1}(\phi(f - E_\varepsilon f)), Q^{U_1} R_{U_1}(\phi f)\rangle_{U_1} \\
&\quad + \langle R_{U_1}(\phi E_\varepsilon f), Q^{U_1} R_{U_1}(\phi f)\rangle_{U_1} \\
&\lesssim \varepsilon \|R_{U_1} f\|^2_{U_1} + \left|\langle R_{U_1}(\phi E_\varepsilon f), Q^{U_1} R_{U_1}(\phi(f - E_\varepsilon f))\rangle_{U_1}\right| \\
&\quad + \left\| H^{U_1}_\phi R_{U_1}(E_\varepsilon f)\right\|^2_{U_1}
\end{aligned}$$

$$\lesssim (\varepsilon + \varepsilon(1+\varepsilon))\|R_{U_1} f\|^2_{U_1} + \left\|Q^{U_1} R_{U_1} H^{\Omega}_{\phi} E_\varepsilon f\right\|^2_{U_1}$$
$$\lesssim \varepsilon(2+\varepsilon)\|R_{U_1} f\|^2_{U_1} + \left\|R_{U_1} H^{\Omega}_{\phi} E_\varepsilon f\right\|^2_{U_1}.$$

Next we will use the compactness characterization of operators in [23, Lemma 4.3] (see also [9, Proposition V.2.3]). Since H^{Ω}_{ϕ} is compact, for every $\varepsilon' > 0$ there exists a compact operator $K_{\varepsilon'} : A^2(\Omega) \to L^2(\Omega)$ such that

$$\left\|R_{U_1} H^{\Omega}_{\phi} E_\varepsilon f\right\|^2_{U_1} \le \left\|H^{\Omega}_{\phi} E_\varepsilon f\right\|^2_{\Omega} \le \varepsilon' \|E_\varepsilon f\|^2_{\Omega} + \|K_{\varepsilon'} E_\varepsilon f\|^2_{\Omega}.$$

Therefore, we have

$$\left\|R_{U_1} H^{\Omega}_{\phi} E_\varepsilon f\right\|^2_{U_1} \le \varepsilon' \|E_\varepsilon\|^2 \|f\|^2_{U_1} + \|K_{\varepsilon'} E_\varepsilon f\|^2_{\Omega}.$$

We note that $K_{\varepsilon'} E_\varepsilon : A^2(U_2) \to L^2(\Omega)$ is compact for any ε and ε'. Now we choose ε' sufficiently small so that $\varepsilon' \|E_\varepsilon\|^2 < \varepsilon$. Hence, there exists $C > 0$ (independent of $\varepsilon, \varepsilon'$ and f) such that

$$\left\|H^{U_1}_{\phi} R_{r_2,r_1} f\right\|^2_{U_1} \le C\varepsilon(3+\varepsilon)\|f\|^2_{U_2} + \|K_{\varepsilon'} E_\varepsilon f\|^2_{\Omega}.$$

Finally, [23, Lemma 4.3] implies that $H^{U_1}_{\phi} R_{r_2,r_1}$ is compact on $A^2(U_2)$. □

Remark 2. We note that the third author proved a localization result previously in [22]. In [22, Theorem 1] the domain may be very irregular but the symbol was assumed to be C^1-smooth up to the boundary. In Lemma 1, however, we assume that the symbol is only bounded on the domain.

Proof of Theorem 1

Lemma 2. *Let U be a domain in $\mathbb{C}$ and $\phi \in C(U)$ that is not holomorphic. Assume that $\{\phi_k\} \subset C^1(U)$ such that $\langle \phi_k, h\rangle \to \langle \phi, h\rangle$ as $k \to \infty$ for all $h \in C^\infty_0(U)$. Then there exists a subsequence $\{\phi_{k_j}\}, \delta > 0$, and $h \in C^\infty_0(U)$ such that*

$$\left|\left\langle \frac{\partial \phi_{k_j}}{\partial \overline{z}}, h\right\rangle\right| \ge \delta$$

for all j.

Proof. We want to show that there exists $h \in C^\infty_0(U)$ such that $\langle (\phi_k)_{\overline{z}}, h\rangle$ does not converge to 0 as $k \to \infty$. Suppose that $\langle (\phi_k)_{\overline{z}}, h\rangle \to 0$ as $k \to \infty$ for all $h \in C^\infty_0(U)$. Then,

$$\langle \phi_k, h_z\rangle \to \langle \phi, h_z\rangle \text{ as } k \to \infty.$$

Hence, in the limit we have $\langle \phi, h_z\rangle = 0$ for all $h \in C^\infty_0(U)$. This implies that ϕ is in the kernel of the $\overline{\partial}$ operator (in the distribution sense) on U. In particular, ϕ is harmonic. Then ϕ is C^∞-smooth (see, for instance, [10, Corollary 2.20]) and, in turn, it is holomorphic. This contradicts with the assumption that ϕ is not holomorphic.

Therefore, there exists $\delta > 0, h \in C^\infty_0(U)$, and a subsequence ϕ_{k_j} such that

$$|\langle (\phi_{k_j})_{\overline{z}}, h\rangle| \ge \delta$$

for all j. □

Lemma 3. *Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{C}^n$, $F:\Omega_1\to\Omega_2$ be a biholomorphism, and $\phi\in L^\infty(\Omega_2)$. Furthermore, let U_1 is an open set in $\Omega_1, F(U_1)=U_2$, and $R_j: A^2(\Omega_j)\to A^2(U_j)$ be the restriction operators for $j=1,2$. Assume that $H^{U_2}_{\phi}R_2$ is compact. Then $H^{U_1}_{\phi\circ F}R_1$ is compact.*

Proof. First, we mention the following formula about Bergman projections. Let J_F denote the determinant of (complex) Jacobian of F and $g\in L^2(U_2)$. Then by [3, Theorem 1] we have

$$P_{U_1}(J_F\cdot(g\circ F)) = J_F\cdot P_{U_2}(g)\circ F$$

(see also [14, Proof of Theorem 12.1.11]).

Next, using the equality above, we will get an equality between Hankel operators on U_1 and U_2. To that end let h be a square integrable holomorphic function on U_2. Then

$$\begin{aligned} H^{U_1}_{\phi\circ F}(h\circ F) &= \phi\circ F\cdot h\circ F - P_{U_1}(\phi\circ F\cdot h\circ F)\\ &= J_F\cdot\phi\circ F\cdot\frac{h\circ F}{J_F} - J_F\cdot P_{U_2}\left(\frac{\phi\circ F\cdot h\circ F}{J_F}\circ F^{-1}\right)\circ F\\ &= J_F\cdot\left(\phi\cdot\frac{h}{J_F\circ F^{-1}} - P_{U_2}\left(\phi\cdot\frac{h}{J_F\circ F^{-1}}\right)\right)\circ F\\ &= J_F\cdot H^{U_2}_{\phi}\left(\frac{h}{J_F\circ F^{-1}}\right)\circ F. \end{aligned}$$

We need to make sure that $\frac{f\circ F^{-1}}{J_F\circ F^{-1}}\in A^2(\Omega_2)$ for any $f\in A^2(\Omega_1)$. In fact,

$$\begin{aligned} \left\|\frac{f\circ F^{-1}}{J_F\circ F^{-1}}\right\|^2_{\Omega_2} &= \int_{\Omega_2}\left|\frac{f\circ F^{-1}(w)}{J_F\circ F^{-1}(w)}\right|^2 dV(w)\\ &= \int_{\Omega_1}\left|\frac{f\circ F^{-1}(F(z))}{J_F\circ F^{-1}(F(z))}\right|^2 |J_F(z)|^2 dV(z)\\ &= \int_{\Omega_1}\left|\frac{f(z)}{J_F(z)}\right|^2 |J_F(z)|^2 dV(z)\\ &= \int_{\Omega_1}|f(z)|^2 dV(z)\\ &= \|f\|^2_{\Omega_1}. \end{aligned}$$

So far we have shown that if $\{f_j\}$ is a bounded sequence in $A^2(\Omega_1)$ then $\{f_j\circ F^{-1}/(J_F\circ F^{-1})\}$ is a bounded sequence in $A^2(\Omega_2)$ and

$$H^{U_1}_{\phi\circ F}R_1(f_j) = J_F\cdot H^{U_2}_{\phi}R_2\left(\frac{f_j\circ F^{-1}}{J_F\circ F^{-1}}\right)\circ F. \tag{1}$$

Then compactness of $H^{U_2}_{\phi}R_2$ implies that $\left\{H^{U_2}_{\phi}R_2\left(\frac{f_j\circ F^{-1}}{J_F\circ F^{-1}}\right)\right\}$ has a convergent subsequence in $L^2(U_2)$. Using the fact that $\|h\|_{U_2}=\|J_F\cdot h\circ F\|_{U_1}$

for any $h \in L^2(U_2)$ together with (1), we conclude that $\{H^{U_1}_{\phi\circ F} R_1(f_j)\}$ has a convergent subsequence in $L^2(U_1)$. Therefore, $H^{U_1}_{\phi\circ F} R_1$ is compact. □

Let $\chi \in C_0^\infty(B(0,1))$ such that $\int_{B(0,1)} \chi(z)dV(z) = 1$. We define

$$\chi_k(z) = k^{2n}\chi(kz)$$

for $k = 1, 2, 3, \ldots$

Proof of Theorem 1. We assume that H_ϕ is compact and there is a holomorphic map $f : \mathbb{D} \to b\Omega$ such that $\phi \circ f$ is not holomorphic. Then f is a non-constant mapping. We can use Lemma 1 to localize the compactness of H_ϕ near a boundary point $f(\xi_0) = p \in b\Omega$ such that $\phi \circ f$ is not holomorphic near ξ_0. That is, we choose $0 < r_1 < r_2$ such that $H_\phi^{\Omega\cap B(p,r_1)} R_{r_2,r_1}$ is compact on $A^2(\Omega\cap B(p, r_2))$ and $\phi\circ f$ is not holomorphic on $f^{-1}(B(p, r_1))$. To simplify the geometry we want to straighten the disc near p yet keep compactness of the Hankel operator locally. So, shrinking r_1, r_2 if necessary, we use a local holomorphic change of coordinates

$$F : \Omega \cap B(p, r_2) \to \mathbb{C}^2$$

so that $F\circ f$ maps $f^{-1}(B(p, r_2))$ onto an open set on z_1-axis and $F\circ f(\xi_0) = 0$.

To simplify the notation, let us denote $\Omega_1 = F(\Omega \cap B(p, r_1))$ and $\Omega_2 = F(\Omega\cap B(p, r_2))$. Lemma 3 implies that $H^{\Omega_1}_{\phi\circ F^{-1}} R$ is compact on $A^2(\Omega_2)$ where $R : A^2(\Omega_2) \to A^2(\Omega_1)$ is the restriction operator. Therefore, without loss of generality, we may assume that

i. $\phi \in C(\mathbb{C}^2)$, using Tietze extension theorem,
ii. $(0, 0) \in \Gamma_1 \times \{0\} \subset b\Omega_2$ is a non-trivial affine disc where $\Gamma_1 = \{z \in \mathbb{C} : |z| < s_1\}$,
iii. $\Omega_2 \subset \left\{(z_1, z_2) \in \mathbb{C}^2 : |\arg(z_2)| < \theta_1\right\}$ for some $0 < \theta_1 < \pi$,
iv. $H^{\Omega_1}_\phi R$ is compact on $A^2(\Omega_2)$.

Next we will use mollifiers (approximations to the identity) and trivial extensions to approximate ϕ on z_1-axis by suitable smooth functions ϕ_k. We define $\widetilde{\phi} = \phi|_{\{(z_1,z_2)\in\mathbb{C}^2 : z_2=0\}}$ and

$$\phi_k = E(\widetilde{\phi} * \chi_k)$$

where $*$ and E denote the convolution and the trivial extension from $\{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$ to $\mathbb{C}^2$, respectively. Then $\phi_k \to \phi$ uniformly on compact subsets in $\{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$ as $k \to \infty$ (see, for instance, [1, 2.29 Theorem]). We note that, since ϕ_ks are extended trivially in z_2-variable, the sequence $\{\phi_k\}$ is uniformly convergent on compact sets in $\mathbb{C}^2$. Hence, $\{\phi_k\}$ is uniformly bounded on $\overline{\Omega}_1$.

Lemma 2 implies that there exist $\delta > 0, h \in C_0^\infty(\Gamma_1)$, and a subsequence $\{\phi_{k_j}\}$ such that

$$\left|\langle(\phi_{k_j})_{\overline{z}_1}, h\rangle_{\Gamma_1}\right| \geq \delta > 0$$

for all $j = 1, 2, 3, \ldots$. By passing to a subsequence, if necessary, we can assume that

$$|\langle (\phi_k)_{\overline{z}_1}, h\rangle_{\Gamma_1}| \geq \delta > 0$$

for all $k = 1, 2, 3, \ldots$.

Since Ω_2 has Lipschitz boundary there exist $s_2 > s_1$, $0 < t_1 < t_2$, and $0 < \theta_1 < \pi/2 < \theta_2 < \pi$ such that

$$\Gamma_1 \times W_{t_1,\theta_1} \subset \Omega_1 \subset \Omega_2 \subset \Gamma_2 \times W_{t_2,\theta_2}$$

where $\Gamma_2 = \{z \in \mathbb{C} : |z| < s_2\}$ and

$$W_{t_j,\theta_j} = \left\{\rho e^{i\theta} \in \mathbb{C} : 0 < \rho < t_j, |\theta| < \theta_j\right\}$$

for $j = 1, 2$.

We define a sequence of functions on Ω_2 as

$$f_j(z_1, z_2) = \frac{\alpha_j}{z_2^{\beta_j}}$$

where $\beta_j = 1 - 1/j$ and $\alpha_j \to 0$ such that $\|f_j\|_{L^2(W_{t_1,\theta_1})} = 1$ for all j. One can show that $\{f_j\}$ is a bounded sequence in $A^2(\Omega_2)$ as $\|f_j\|_{L^2(W_{t_2,\theta_2})}$ are uniformly bounded. Furthermore, the sequence $\{f_j\}$ converges to zero uniformly on compact subsets that are away from $\{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$. Then $f_j \to 0$ weakly in $A^2(\Omega_2)$ as $j \to \infty$. Later on we will reach a contradiction by showing that $\|H^{\Omega_1}_{\phi} Rf_j\|_{\Omega_1}$ stays away from zero.

We remind the reader that for any $f \in A^2(\Omega_2)$ and k we have

$$(H^{\Omega_1}_{\phi_k} Rf)_{\overline{z}_1} = (Rf\phi_k)_{\overline{z}_1} - (P_{\Omega_1} R(f\phi_k))_{\overline{z}_1} = R\left(f(\phi_k)_{\overline{z}_1}\right).$$

Using the identity above (when we pass from second to third line below) and the Cauchy-Schwarz inequality (in z_1 on Γ_1 on the second inequality below) we get

$$\begin{aligned}
\delta^2 = \delta^2\|f_j\|^2_{L^2(W_{t_1,\theta_1})} &\leq \int_{W_{t_1,\theta_1}} |\langle (\phi_k)_{\overline{z}_1}, h\rangle_{\Gamma_1}|^2 f_j(., z_2)\overline{f_j(., z_2)} dV(z_2) \\
&= \int_{W_{t_1,\theta_1}} \langle (\phi_k f_j)_{\overline{z}_1}, h\rangle_{\Gamma_1} \overline{\langle (\phi_k f_j)_{\overline{z}_1}, h\rangle_{\Gamma_1}} dV(z_2) \\
&= \int_{W_{t_1,\theta_1}} \left|\langle (H^{\Omega_1}_{\phi_k} Rf_j)_{\overline{z}_1}, h\rangle_{\Gamma_1}\right|^2 dV(z_2) \\
&= \int_{W_{t_1,\theta_1}} \left|\langle H^{\Omega_1}_{\phi_k} Rf_j, h_{z_1}\rangle_{\Gamma_1}\right|^2 dV(z_2) \\
&\leq \int_{W_{t_1,\theta_1}} \|H^{\Omega_1}_{\phi_k} Rf_j(., z_2)\|^2_{\Gamma_1} \|h_{z_1}\|^2_{\Gamma_1} dV(z_2) \\
&= \|H^{\Omega_1}_{\phi_k} Rf_j\|^2_{\Gamma_1 \times W_{t_1,\theta_1}} \|h_{z_1}\|^2_{\Gamma_1}.
\end{aligned}$$

Then, for all j and k, we have

$$\frac{\delta}{\|h_{z_1}\|_{\Gamma_1}} \leq \|H^{\Omega_1}_{\phi_k} Rf_j\|_{\Omega_1}.$$

Using the facts that $\phi_k \to \phi$ uniformly on Γ_1, the sequence $\{\phi_k\}$ is uniformly bounded on $\overline{\Omega}_1$, and $f_j \to 0$ uniformly on compact subsets away from z_1-axis, one can show that

$$\|H^{\Omega_1}_{\phi_k-\phi} Rf_j\|_{\Omega_1} \le \|(\phi_k - \phi)Rf_j\|_{\Omega_1} \to 0 \text{ as } j,k \to \infty.$$

Then we have

$$\frac{\delta}{\|h_{z_1}\|_{\Gamma_1}} \le \|H^{\Omega_1}_{\phi_k} Rf_j\|_{\Omega_1} \le \|H^{\Omega_1}_{\phi_k-\phi} Rf_j\|_{\Omega_1} + \|H^{\Omega_1}_{\phi} Rf_j\|_{\Omega_1}.$$

Then if we let $j,k \to \infty$ we get

$$0 < \frac{\delta}{\|h_{z_1}\|_{\Gamma_1}} \le \liminf_{j\to\infty} \|H^{\Omega_1}_{\phi} Rf_j\|_{\Omega_1}.$$

Finally, we conclude that $H^{\Omega_1}_{\phi} R$ is not compact on $A^2(\Omega_2)$ because if it were, the sequence $\{H^{\Omega_1}_{\phi} Rf_j\}$ would converge to zero in norm. Therefore, using Lemma 1, we reach a contradiction with the assumption that H_ϕ is compact. □

Proof of Theorem 2 and Corollaries

In Lemma 4 below we will use the following notation: $L_{z_0,z_1} : \mathbb{D} \to b\Omega$ is defined as $L_{z_0,z_1}(\xi) = z_0 + \xi z_1$ where $z_0, z_1 \in \mathbb{C}^n$.

Lemma 4. *Let Ω be a bounded convex domain in $\mathbb{C}^n$ and $\phi \in C(\overline{\Omega})$. Assume that there exists a holomorphic function $f : \mathbb{D} \to b\Omega$ so that $\phi \circ f$ is not holomorphic. Then there exist $z_0 \in b\Omega, z_1 \in \mathbb{C}^n$ such that $L_{z_0,z_1}(\mathbb{D}) \subset b\Omega$ and $\phi \circ L_{z_0,z_1}$ is not holomorphic.*

Proof. We first use [6, Lemma 2] (see also [11, Section 2]) to conclude that the convex hull of $f(\mathbb{D})$ is contained in an affine variety $V \subset b\Omega$. So $\phi|_V$ is not holomorphic. Next we use the following fact: a continuous function is holomorphic on an open set U if and only if it is holomorphic on every complex line in U. Therefore, we conclude that there is a complex line $L_{z_0,z_1}(\mathbb{D}) \subset V$ such that $\phi \circ L_{z_0,z_1}$ is not holomorphic on $\mathbb{D}$. □

We will need the following lemma in the proof of Theorem 2.

Lemma 5. *Let Ω be a domain in $\mathbb{C}^n$ with Lipschitz boundary such that $0 \in b\Omega$. Then the function $f(z) = |z_n|^{-p}$ is not square integrable on Ω for $p \ge n$.*

Proof. We can use rotation to assume that positive y_n-axis is transversal to $b\Omega$ and there exists $\alpha, \varepsilon > 0$ such that

$$W_{\varepsilon,\alpha} = \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + x_n^2 < \alpha^2 y_n^2, -\varepsilon < y_n < 0\} \subset \Omega$$

where $z_n = x_n + iy_n$. In the following calculation $w_{\varepsilon,\alpha} = \{x_n + iy_n \in \mathbb{C} : |x_n| + \alpha y_n < 0, -\varepsilon < y_n < 0\}$ is a wedge in z_n-axis.

$$\begin{aligned}\int_\Omega |z_n|^{-2p} dV(z) &\ge \int_{W_{\varepsilon,\alpha}} |z_n|^{-2p} dV(z', z_n) \\ &= \int_{z_n \in w_{\varepsilon,\alpha}} \int_{|z'|^2 < \alpha^2 y_n^2 - x_n^2} |z_n|^{-2p} dV(z') dV(z_n)\end{aligned}$$

$$\gtrsim \int_{z_n \in w_{\varepsilon,\alpha}} (\alpha^2 y_n^2 - x_n^2)^{n-1} |z_n|^{-2p} dV(z_n)$$
$$\gtrsim \int_0^{\varepsilon} \frac{1}{r^{1+2(p-n)}} dr.$$

Therefore, if $p \geq n$ the function $f(z) = |z_n|^{-p}$ is not square integrable on Ω as the last integral above is infinite. □

Proof of Theorem 2. Using holomorphic linear translation, if necessary, we may assume that $\Omega \subset \{y_n < 0\}$ and the origin is in the boundary of Ω. Furthermore, by Lemma 4 we may assume that $0 \in \Gamma = \{z \in \mathbb{C} : (z, 0, \dots, 0) \in b\Omega\}$ is a non-trivial affine analytic disc such that $\phi(., 0, \dots, 0)$ is not holomorphic. Finally, since convex domains have Lipschitz boundary (see, for instance, [25]), we may also assume that positive y_n-axis is transversal to $b\Omega$ on Γ.

Let $\Omega^{z_1} = \{z'' \in \mathbb{C}^{n-1} : (z_1, z'') \in \Omega\}$ be the slice of Ω perpendicular to Γ at $z_1 \in \Gamma$. Convexity of Ω and the fact that $0 \in \Gamma \times \{0\} \subset b\Omega$ imply that

$$\left(\frac{z_1}{2}, \frac{z''}{2}\right) = \frac{1}{2}(z_1, 0) + \frac{1}{2}(0, z'') \in \Omega$$

for $z_1 \in \Gamma$ and $z'' \in \Omega^0$. That is, $\Omega^0 \subset 2\Omega^{z_1/2}$ for $z_1 \in \Gamma$. Equivalently, $\Omega^0 \subset 2\Omega^{z_1}$ for $z_1 \in \frac{1}{2}\Gamma$. Hence,

$$\frac{1}{2}(\Gamma \times \Omega^0) \subset \Omega. \tag{2}$$

To get another inclusion, let $0 < r_1$ such that $\{z_1 \in \mathbb{C} : |z_1| < r_1\} \subset \Gamma$ and $z'' \in \Omega^{z_1}$. Then, we have $(z_1, z'') \in \Omega$ and $(-z_1, 0) \in \Gamma$. Hence

$$\left(0, \frac{z''}{2}\right) = \frac{1}{2}(-z_1, 0) + \frac{1}{2}(z_1, z'') \in \Omega.$$

That is, $\frac{1}{2}\Omega^{z_1} \subset \Omega^0$ for $|z_1| < r_1$. Namely, $\Omega^{z_1} \subset 2\Omega^0$ for $|z_1| < r_1$. Hence,

$$\Omega \cap B(0, r_1) \subset 2(\Gamma \times \Omega^0).$$

Therefore, combining the previous inclusion with (2) we get

$$\frac{1}{2}(\Gamma \times \Omega^0) \cap B(0, r_1) \subset \Omega \cap B(0, r_1) \subset 2(\Gamma \times \Omega^0).$$

Next we will use Lemma 5 to produce a bounded sequence $\{f_j\}$ in $A^2(\Omega)$ that is convergent to zero weakly but its image under a "local" Hankel operator does not converge to zero.

Since, by Lemma 5, the function $f(z) = z_n^{-n+1}$ is not square integrable on Ω^0 (an $(n-1)$-dimensional slice of Ω) and the L^2-norm of $(z_n - i\delta)^{-n+1}$ on Ω^0 continuously depends on $\delta > 0$, we can choose a positive sequence $\{\delta_j\}$ such that $\delta_j \to 0$ as $j \to \infty$ and $\|f_j\|_{\frac{1}{2}\Omega^0} = 1$ where

$$f_j(z) = \frac{1}{j(z_n - i\delta_j)^{n-1}}. \tag{3}$$

Furthermore, $|f_j(4z)| \le |f_j(z)|$ for all $z \in \frac{1}{2}\Omega^0$ as $\Omega \subset \{y_n < 0\}$ and $\delta_j > 0$. Then,

$$\begin{aligned}\int_{2\Omega^0} |f_j(\xi)|^2 dV(\xi) &= 16^{n-1} \int_{\frac{1}{2}\Omega^0} |f_j(4\eta)|^2 dV(\eta)\\ &\le 16^{n-1} \int_{\frac{1}{2}\Omega^0} |f_j(\eta)|^2 dV(\eta)\\ &= 16^{n-1}.\end{aligned}$$

Hence $\{f_j\}$ is a bounded sequence in $A^2(\Omega)$ (as $\|f_j\|_{2\Omega^0}$ is uniformly bounded) and $f_j \to 0$ weakly in $A^2(\Omega)$ as $j \to \infty$.

The rest of the proof follows the proof of Theorem 1. Namely, we define $\Gamma_1 = \{z \in \mathbb{C} : |z| < \frac{r_1}{2}\}$ and $\widetilde{\phi} = \phi|_{\Gamma \times \{0\}}$. Without loss of generality, we may assume that $\phi \in C(\mathbb{C}^n)$. We define

$$\phi_k = E(\widetilde{\phi} * \chi_k)$$

where E denotes trivial extension from $\{(z_1, z'') \in \mathbb{C}^2 : z'' = 0\}$ to $\mathbb{C}^n$, respectively. Using Lemma 2 we can choose $\delta > 0$ and $h \in C_0^\infty(\Gamma_1)$ so that, by passing to a subsequence if necessary, we can assume that

$$|\langle (\phi_k)_{\overline{z}_1}, h \rangle_{\Gamma_1}| \ge \delta > 0$$

for all $k = 1, 2, 3, \ldots$. Then for $\Omega_1 = \Omega \cap B(p, r_1)$ we get

$$\frac{\delta}{\|h_{z_1}\|_{\Gamma_1}} \le \|H^{\Omega_1}_{\phi_k} R f_j\|_{\Omega_1}$$

for all j, k where $R : A^2(\Omega) \to A^2(\Omega_1)$ is the restriction operator. Then letting $j, k \to \infty$ we get

$$0 < \frac{\delta}{\|h_{z_1}\|_{\Gamma_1}} \le \liminf_{j \to \infty} \|H^{\Omega_1}_{\phi} R f_j\|_{\Omega_1}. \tag{4}$$

Hence, $H^{\Omega_1}_{\phi} R$ is not compact and we reach a contradiction with the assumption that H_ϕ is compact. Therefore, the proof of Theorem 2 is complete. □

Proof of Corollary 1. Suppose $\Omega \subset \mathbb{C}^n$ is a bounded locally convexifiable domain, $\phi \in C(\overline{\Omega})$ is such that H_ϕ is compact on $A^2(\Omega)$, and $f : \mathbb{D} \to b\Omega$ is a holomorphic function. Let $p \in f(\mathbb{D})$ and choose $r > 0$ such that $B(p, r) \cap \Omega$ is convexifiable. Furthermore, without loss of generality, we may assume that the range of f is contained in $B(p, r/2)$. Then using Lemma 1 and Lemma 3 (and shrinking r is necessary) we may assume that $U = B(p, r) \cap \Omega$ is convex and $H^V_\phi R$ is compact on $A^2(U)$ where $V = B(p, r/2) \cap \Omega$ and $R : A^2(U) \to A^2(V)$ is the restriction from U onto V. Then the proof of Theorem 2 implies that $\phi \circ f$ is holomorphic. □

Proof of Corollary 2. Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex complete Reinhardt domain, $\phi \in C(\overline{\Omega})$, and H_ϕ is compact on $A^2(\Omega)$. By [21, Theorem 3.28], Ω is locally convexifiable away from the coordinate axes under the map $(z_1, z_2) \to (\log z_1, \log z_2)$.

Assume that there is a non-trivial analytic disc in the boundary away from the coordinate axes. Then there exists a non-constant holomorphic function $f : \mathbb{D} \to b\Omega$ such that $f(\mathbb{D}) \subset \{(z_1, z_2) \in \mathbb{C}^2 : z_1 \neq 0 \text{ and } z_2 \neq 0\}$. Using an argument similar to the one in the proof of Corollary 1 we conclude that $\phi \circ f$ is holomorphic. Therefore, ϕ is holomorphic along any disc away from the coordinate axis.

Next, if the disc intersects one of the coordinate axis, without loss of generality, we assume that $f(\mathbb{D}) \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0\} \neq \emptyset$. Let $f = (f_1, f_2)$. Then $f_1 : \mathbb{D} \to \mathbb{C}$ has a zero. Since zeroes of a holomorphic function on a planar domain are isolated, we can choose f so that $f_1(z) = 0$ if and only if $z = 0$. Therefore, we may assume that $f(z)$ is on a coordinate axis if and only if $z = 0$. Then, similarly as in the previous paragraph, we conclude that $\phi \circ f$ is holomorphic on $\mathbb{D} \setminus \{0\}$. Furthermore, 0 is a removable singularity for $\phi \circ f$ as $\phi \circ f$ is continuous on $\mathbb{D}$. That is, $\phi \circ f$ is holomorphic on $\mathbb{D}$. □

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