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Heat Content in Non-compact Riemannian Manifolds

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Abstract. Let Ω be an open set in a complete, smooth, non-compact, m-dimensional Riemannian manifold M without boundary, where M satisfies a two-sided Li-Yau gaussian heat kernel bound. It is shown that if Ω has infinite measure, and if Ω has finite heat content $H_\Omega(T)$ for some $T > 0$, then $H_\Omega(t) < \infty$ for all $t > 0$. Comparable two-sided bounds for $H_\Omega(t)$ are obtained for such Ω.

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1. Introduction

Let (M, g) be a complete, smooth, non-compact, m-dimensional Riemannian manifold without boundary, and let Δ be the Laplace–Beltrami operator acting in $L^2(M)$. It is well known (see [1–4]) that the heat equation

$$\Delta u(x;t) = \frac{\partial u(x;t)}{\partial t}, \quad x \in M, \quad t > 0, \tag{1.1}$$

has a unique, minimal, positive fundamental solution $p_M(x, y; t)$ where $x \in M$, $y \in M$, $t > 0$. This solution, the heat kernel for M, is symmetric in x, y, strictly positive, jointly smooth in $x, y \in M$ and $t > 0$, and it satisfies the semigroup property

$$p_M(x, y; s + t) = \int_M dz\, p_M(x, z; s) p_M(z, y; t), \tag{1.2}$$

for all $x, y \in M$ and all $t, s > 0$, where dz is the Riemannian measure on M.

We define the heat content of an open set Ω in M with boundary $\partial\Omega$ at t by

$$H_\Omega(t) = \int_\Omega dx \int_\Omega dy\, p_M(x, y; t).$$

It was shown ([5]) that if Ω is non-empty, bounded, and $\partial\Omega$ is of class C^∞, and if (M, g) satisfies exactly one of the following three conditions: (i)

M is compact and without boundary, (ii) $(M, g) = (\mathbb{R}^m, g_e)$ where g_e is the usual Euclidean metric on $\mathbb{R}^m$, (iii) M is a compact submanifold of $\mathbb{R}^m$ with smooth boundary and $g = g_e|_M$, then there exists a complete asymptotic series such that

$$H_\Omega(t) = \sum_{j=0}^{J-1} \beta_j t^{j/2} + O(t^{J/2}),\ t \downarrow 0, \tag{1.3}$$

where $J \in \mathbb{N}$ is arbitrary, and where the $\beta_j : j = 0, 1, 2, \dots$ are locally computable geometric invariants. In particular, we have that

$$\beta_0 = |\Omega|,\ \beta_1 = -\pi^{-1/2}\mathrm{Per}(\Omega),\ \beta_2 = 0,$$

where $|\Omega|$ is the measure of Ω, and $\mathrm{Per}(\Omega)$ is the perimeter of Ω.

For earlier results in the Euclidean setting we refer to [6–8], and subsequently to [9,10], and [11].

Define $u_\Omega : \Omega \times (0, \infty) \mapsto \mathbb{R}$ by

$$u_\Omega(x; t) = \int_\Omega dy\, p_M(x, y; t). \tag{1.4}$$

Then u_Ω is a solution of the heat equation (1.1) and satisfies

$$\lim_{t\downarrow 0} u_\Omega(x; t) = \mathbf{1}_\Omega(x),\ x \in M - \partial\Omega, \tag{1.5}$$

where $\mathbf{1}_\Omega : M \mapsto \{0, 1\}$ is the characteristic function of Ω, and where the convergence in (1.5) is locally uniform. It can be shown that if $|\Omega| < \infty$, then the convergence is also in $L^1(M)$. If Ω has infinite measure and $|\partial\Omega| = 0$, then the convergence is also in $L^1_{\mathrm{loc}}(M)$ (Section 7.4 in [12]).

In this paper we obtain bounds for the heat content in the case where Ω has possibly infinite measure or infinite perimeter, and where M satisfies the following condition.

There exists $C \in [2, \infty)$ such that for all $x \in M$, $y \in M$, $t > 0$, $R > 0$,

$$\frac{e^{-Cd(x,y)^2/t}}{C\sqrt{|B(x; t^{1/2})||B(y; t^{1/2})|}} \le p_M(x, y; t) \le \frac{Ce^{-d(x,y)^2/(Ct)}}{\sqrt{|B(x; t^{1/2})||B(y; t^{1/2})|}}, \tag{1.6}$$

and

$$|B(x; 2R)| \le C|B(x; R)|, \tag{1.7}$$

where $B(x; R) = \{y \in M : d(x, y) < R\}$, and $d(x, y)$ denotes the geodesic distance between x and y.

It was shown independently in [13] and [14] that M satisfying a volume doubling property and a Poincaré inequality is equivalent to M satisfying a parabolic Harnack principle, and is also equivalent to the Li-Yau bound (1.6) above. See for example Theorem 5.4.12 in [3]. We included (1.7) in the definition of the constant C, even though the volume doubling property is implied by (1.6).

We recall a few basic facts

(i) Volume doubling implies that for $x \in M, r_0 > 0$,

$$\int_{r_0}^\infty dr\, r(\log|B(x; r)|)^{-1} = +\infty.$$

Hence u_Ω, defined by (1.4), is the unique, bounded solution of (1.1) with initial condition (1.5) in the sense of $L^1_{\mathrm{loc}}(M)$. Moreover stochastic completeness holds. That is for all $x \in M$, $t > 0$,

$$\int_M dy\, p_M(x, y; t) = 1. \tag{1.8}$$

We refer to Chapter 9 in [2].

(ii) If $H_\Omega(t) < \infty$ for all $t > 0$, then for all $t > 0$, $s > 0$ we have by Cauchy–Schwarz's inequality, (1.2) and (1.4) that

$$\begin{aligned} H_\Omega((t+s)/2) &= \int_M dz \int_\Omega dy \int_\Omega dx\, p_M(x, z; t/2) p_M(y, z; s/2) \\ &= \int_M dx\, u_\Omega(x; t/2) u_\Omega(x; s/2) \\ &\le \left(\int_M dx\, u_\Omega^2(x; t/2)\right)^{1/2} \left(\int_M dx\, u_\Omega^2(x; s/2)\right)^{1/2} \\ &= (H_\Omega(t) H_\Omega(s))^{1/2}. \end{aligned}$$

Hence $t \mapsto H_\Omega(t)$ is mid-point log-convex, log-convex, convex, and hence continuous on $(0, \infty)$.

(iii) If (1.7) holds for all $x \in M$, $R > 0$ then

$$\frac{|B(x; r_2)|}{|B(x; r_1)|} \le C \left(\frac{r_2}{r_1}\right)^{(\log C)/\log 2}, r_2 \ge r_1. \tag{1.9}$$

We refer to (2.2) in [14].

(iv) $t \mapsto H_\Omega(t)$ is decreasing: if $H_\Omega(t) < \infty$ for some $t > 0$, then for $s > 0$,

$$\begin{aligned} H_\Omega(t+s) &= \int_M dx\, u_\Omega^2(x; (t+s)/2) \\ &= \int_M dx \int_M dy_1\, p_M(x, y_1; s/2) u_\Omega(y_1; t/2) \\ &\quad \times \int_M dy_2\, p_M(x, y_2; s/2) u_\Omega(y_2; t/2) \\ &= \int_M dy_1 \int_M dy_2\, p_M(y_1, y_2; s) u_\Omega(y_1; t/2) u_\Omega(y_2; t/2) \\ &\le \frac{1}{2} \int_M dy_1 \int_M dy_2\, p_M(y_1, y_2; s)(u_\Omega^2(y_1; t/2) + u_\Omega^2(y_2; t/2)) \\ &= \int_M dy_1 \int_M dy_2\, p_M(y_1, y_2; s) u_\Omega^2(y_1; t/2) \\ &\le \int_M dy\, u_\Omega^2(y; t/2) \\ &= H_\Omega(t). \end{aligned} \tag{1.10}$$

We make the following.

Definition 1.1. For $x \in M$, $\Omega \subset M$, and $R > 0$,

$$\mu_\Omega(x;R) = |B(x;R) \cap \Omega|,$$
$$\nu_\Omega(x;R) = |B(x;R) - \Omega|.$$

Our main result is the following.

Theorem 1.2. *Let M be a complete, smooth, non-compact, m-dimensional Riemannian manifold without boundary, and let $\Omega \subset M$ be open. Suppose that (1.6), (1.7) holds for some $C \in [2,\infty)$. Then*

(i) *If $H_\Omega(T) < \infty$ for some $T > 0$, then*

$$\int_\Omega dx \frac{\mu_\Omega(x;t^{1/2})}{|B(x;t^{1/2})|} < \infty, \tag{1.11}$$

for all $t > 0$.

(ii) *If (1.11) holds for some $t = T > 0$, then*

$$K_1 \int_\Omega dx \frac{\mu_\Omega(x;t^{1/2})}{|B(x;t^{1/2})|} \le H_\Omega(t) \le K_2 \int_\Omega dx \frac{\mu_\Omega(x;t^{1/2})}{|B(x;t^{1/2})|}, \tag{1.12}$$

for all $t > 0$, where

$$K_1 = C^{-2}e^{-C},$$
$$K_2 = 2C^{15/4}\left(C\log\left(2C^{(7/2)+((\log C)/(\log 2))}\right)\right)^{(3\log C)/(4\log 2)}. \tag{1.13}$$

If Ω has finite Lebesgue measure, then we define the heat loss of Ω in M at t by

$$F_\Omega(t) = |\Omega| - H_\Omega(t). \tag{1.14}$$

We have that the heat loss $t \mapsto F_\Omega(t)$ of Ω in M is increasing, concave, subadditive, and continuous. If Ω is bounded and $\partial\Omega$ is smooth, then, by (1.3), there exists an asymptotic series of which the first few coefficients are known explicitly. Theorem 1.3 below concerns the general situation $|\Omega| < \infty$, and gives bounds in non-classical geometries where e.g. either Ω has infinite perimeter, and/or $\partial\Omega$ is not smooth.

Theorem 1.3. *Let M be a complete, smooth, non-compact, m-dimensional Riemannian manifold without boundary, and let $\Omega \subset M$ be open with finite Lebesgue measure. Suppose that (1.6), (1.7) holds for some $C \in [2,\infty)$. Then*

$$L_1 \int_\Omega dx \frac{\nu_\Omega(x;t^{1/2})}{|B(x;t^{1/2})|} \le F_\Omega(t) \le L_2 \int_\Omega dx \frac{\nu_\Omega(x;t^{1/2})}{|B(x;t^{1/2})|}, \quad \text{for all } t > 0, \tag{1.15}$$

where

$$L_1 = C^{-2}e^{-C},$$
$$L_2 = 4C^{15/4}\left(C\log\left(2C^{7+((\log C)/\log 2)}\right)\right)^{1+((3\log C)/(4\log 2))}. \tag{1.16}$$

This paper is organised as follows. In Sect. 2 we give the proofs of Theorems 1.2 and 1.3. In Sect. 3 we analyse an example of Ω in $\mathbb{R}^m$ where precise analysis of $H_\Omega(t)$ is possible.

2. Proofs

The main idea in the proof of Theorem 1.2 is to use the Li-Yau bound (1.6), and (1.9) to bootstrap $\int_{\{x\in M-\Omega:\inf_{y\in\Omega} d(x,y)\ge ct^{1/2}\}} dx\, u_\Omega(x;t)$ in terms of $H_\Omega(t)$. This is possible for c sufficiently large (in terms of C). A similar bootstrap argument features in the proof of Theorem 1.3. There, the stochastic completeness of M, (1.8), is also exploited.

Proof of Theorem 1.2. (i) Let $t \ge T > 0$, and suppose that $H_\Omega(T) < \infty$. Let $R > 0$. By (1.6) and (1.10) we have that

$$\begin{aligned} H_\Omega(T) &\ge H_\Omega(t) \\ &\ge \int_\Omega dx \int_{\Omega\cap B(x;R)} dy\, p_M(x,y;t) \\ &\ge C^{-1}e^{-CR^2/t}\int_\Omega dx \int_{\Omega\cap B(x;R)} dy\, (|B(x;t^{1/2})||B(y;t^{1/2})|)^{-1/2}. \end{aligned} \tag{2.1}$$

For $d(x,y) < R$, $B(y;t^{1/2}) \subset B(x;R+t^{1/2})$, so that by (1.9),

$$|B(y;t^{1/2})| \le C\left(\frac{R+t^{1/2}}{t^{1/2}}\right)^{(\log C)/\log 2} |B(x;t^{1/2})|. \tag{2.2}$$

The choice $R = t^{1/2}$ implies, by (2.1) and (2.2), that

$$H_\Omega(T) \ge H_\Omega(t) \ge K_1 \int_\Omega dx \frac{\mu_\Omega(x;t^{1/2})}{|B(x;t^{1/2})|},\ t \ge T, \tag{2.3}$$

with K_1 given in (1.13).

Next suppose that $0 < t \le T$. By (1.9), and (2.3) for $t = T$, we have that

$$\begin{aligned} \int_\Omega dx \frac{\mu_\Omega(x;t^{1/2})}{|B(x;t^{1/2})|} &\le C\left(\frac{T}{t}\right)^{(\log C)/\log 4} \int_\Omega dx \frac{\mu_\Omega(x;T^{1/2})}{|B(x;T^{1/2})|} \\ &\le \frac{C}{K_1}\left(\frac{T}{t}\right)^{(\log C)/\log 4} H_\Omega(T). \end{aligned}$$

This completes the proof of the assertion in part (i).

(ii) Let $n > 0$, $p \in \Omega, R > 0$, and $\Omega_n = \Omega \cap B(p;n)$, and suppose that (1.11) holds for some $t = T > 0$. Then $|\Omega_n| \le |B(p;n)| < \infty$. Reversing the roles of x and y in (2.2) we have that for $d(x,y) < R$,

$$|B(y;t^{1/2})| \ge C^{-1}\left(\frac{t^{1/2}}{R+t^{1/2}}\right)^{(\log C)/\log 2} |B(x;t^{1/2})|. \tag{2.4}$$

We have that

$$\begin{aligned} &\int_{\Omega_n} dx \int_{\Omega_n} dy\, p_M(x,y;t) \\ &\quad = \int_{\Omega_n} dx \int_{\Omega_n\cap B(x;R)} dy\, p_M(x,y;t) + \int_{\Omega_n} dx \int_{\Omega_n - B(x;R)} dy\, p_M(x,y;t). \end{aligned} \tag{2.5}$$

Using (1.6) and (2.4), we see that

$$\begin{aligned}
&\int_{\Omega_n} dx \int_{\Omega_n \cap B(x;R)} dy\, p_M(x,y;t)\\
&\le C \int_{\Omega_n} dx |B(x;t^{1/2})|^{-1} \int_{\Omega_n \cap B(x;R)} dy \left(\frac{|B(x;t^{1/2})|}{|B(y;t^{1/2})|}\right)^{1/2} e^{-d(x,y)^2/(Ct)}\\
&\le C^{3/2} \left(\frac{R+t^{1/2}}{t^{1/2}}\right)^{(\log C)/\log 4} \int_{\Omega_n} dx \frac{\mu_{\Omega_n}(x;R)}{|B(x;t^{1/2})|}. \qquad (2.6)
\end{aligned}$$

To bound the second term in the right-hand side of (2.5), we note that

$$d(x,y)^2/(Ct) \ge R^2/(2Ct) + d(x,y)^2/(2Ct)\,, y \in \Omega_n - B(x;R). \qquad (2.7)$$

Hence,

$$\begin{aligned}
&\int_{\Omega_n} dx \int_{\Omega_n - B(x;R)} dy\, p_M(x,y;t)\\
&\le C \int_{\Omega_n} dx \int_{\Omega_n} dy\, (|B(x;t^{1/2})||B(y;t^{1/2})|)^{-1/2} e^{-(d(x,y)^2+R^2)/(2Ct)}\\
&\le C^{(5/2)+((\log C)/\log 2)} \int_{\Omega_n} dx \int_{\Omega_n} dy \left(|B(x;(2C^2t)^{1/2})||B(y;(2C^2t)^{1/2})|\right)^{-1/2}\\
&\quad \times e^{-(d(x,y)^2+R^2)/(2Ct)}\\
&\le C^{(7/2)+((\log C)/\log 2)} e^{-R^2/(2Ct)} \int_{\Omega_n} dx \int_{\Omega_n} dy\, p_M(x,y;2C^2t)\\
&\le C^{(7/2)+((\log C)/\log 2)} e^{-R^2/(2Ct)} H_{\Omega_n}(2C^2t)\\
&\le C^{(7/2)+((\log C)/\log 2)} e^{-R^2/(2Ct)} H_{\Omega_n}(t), \qquad (2.8)
\end{aligned}$$

where we have used (2.7), (1.9), the lower bound in (1.6), and (1.10). We now choose R^2 such that the coefficient of $H_{\Omega_n}(t)$ in the right-hand side of (2.8) is equal to $\frac{1}{2}$. That is

$$R_*^2 = 2Ct \log\left(2C^{(7/2)+((\log C)/\log 2)}\right). \qquad (2.9)$$

Rearranging and bootstrapping gives, by (2.5)–(2.9), and the fact that $t^{1/2} \le R_*$, that

$$\begin{aligned}
H_{\Omega_n}(t) &\le 2C^{3/2} \left(\frac{R_*+t^{1/2}}{t^{1/2}}\right)^{(\log C)/\log 4} \int_{\Omega} dx \frac{\mu_{\Omega_n}(x;R_*)}{|B(x;t^{1/2})|}\\
&\le 2C^{5/2} \left(\frac{R_*+t^{1/2}}{t^{1/2}}\right)^{(\log C)/\log 4} \left(\frac{R_*}{t^{1/2}}\right)^{(\log C)/\log 2} \int_{\Omega_n} dx \frac{\mu_{\Omega}(x;R_*)}{|B(x;R_*)|}\\
&\le 2C^{3} \left(\frac{R_*^2}{t}\right)^{(3\log C)/(4\log 2)} \int_{\Omega} dx \frac{\mu_{\Omega_n}(x;R_*)}{|B(x;R_*)|}\\
&\le 2C^{3} \left(2C \log\left(2C^{(7/2)+((\log C)/\log 2)}\right)\right)^{(3\log C)/(4\log 2)} \int_{\Omega} dx \frac{\mu_{\Omega_n}(x;R_*)}{|B(x;R_*)|}. \qquad (2.10)
\end{aligned}$$

We choose $t = t_T$ such that $R_* = T$, and take the limit $n \to \infty$ in the right-hand side of (2.10). This limit is finite by the hypothesis at the beginning of the proof. We conclude that

$$H_{\Omega_n}(t_T) \le 2C^3 \left(2C \log\left(2C^{(7/2)+((\log C)/\log 2)}\right)\right)^{(3\log C)/(4\log 2)} \times \int_\Omega dx \frac{\mu_\Omega(x;T)}{|B(x;T)|}. \tag{2.11}$$

By monotone convergence,

$$H_{\Omega}(t_T) \le 2C^3 \left(2C \log\left(2C^{(7/2)+((\log C)/\log 2)}\right)\right)^{(3\log C)/(4\log 2)} \times \int_\Omega dx \frac{\mu_\Omega(x;T)}{|B(x;T)|}. \tag{2.12}$$

By (i) we obtain that

$$H_{\Omega}(t) \le 2C^3 \left(2C \log\left(2C^{(7/2)+((\log C)/\log 2)}\right)\right)^{(3\log C)/(4\log 2)} \times \int_\Omega dx \frac{\mu_\Omega(x;R_*)}{|B(x;R_*)|} \tag{2.13}$$

for all $t > 0$ with R_* given by (2.9). Since $H_\Omega(t)$ is decreasing in t, and since $R_* \ge t^{1/2}$ we conclude from (2.10) that

$$H_{\Omega}(R_*^2) \le 2C^{15/4} \left(C \log\left(2C^{(7/2)+((\log C)/(\log 2))}\right)\right)^{(3\log C)/(4\log 2)} \times \int_\Omega dx \frac{\mu_\Omega(x;R_*)}{|B(x;R_*)|}. \tag{2.14}$$

Rescaling t gives the upper bound in (1.12) with K_2 given in (1.13). This completes the proof of the assertion in part (ii). □

Proof of Theorem 1.3. To prove the lower bound in (1.15), we have by definition of $F_\Omega(t)$ in (1.14), and by (1.8) that

$$\begin{aligned} F_\Omega(t) &= \int_\Omega dx \int_M dy\, p_M(x,y;t) - \int_\Omega dx \int_\Omega dy\, p_M(x,y;t) \\ &= \int_\Omega dx \int_{M-\Omega} dy\, p_M(x,y;t). \end{aligned} \tag{2.15}$$

Hence by (1.6) we have for $R > 0$ that

$$\begin{aligned} F_\Omega(t) &\ge \int_\Omega dx \int_{B(x;R)-\Omega} dy\, p_M(x,y;t) \\ &\ge C^{-1} e^{-CR^2/t} \int_\Omega dx \int_{B(x;R)-\Omega} dy \left(|B(x;t^{1/2})||B(y;t^{1/2})|\right)^{-1/2}. \end{aligned}$$

Since $B(y;t^{1/2}) \subset B(x;R+t^{1/2})$, for $y \in B(x;R)$, we have by (2.2) that

$$F_\Omega(t) \ge C^{-3/2} \left(\frac{t^{1/2}}{R+t^{1/2}}\right)^{(\log C)/\log 4} e^{-CR^2/t} \int_\Omega dx\, \frac{\nu_\Omega(x;R)}{|B(x;t^{1/2})|}.$$

The choice $R = t^{1/2}$ gives the lower bound in (1.15), with L_1 given in (1.16).

To prove the upper bound in (1.15), we let $R > 0$, and write (2.15) as

$$F_\Omega(t) = \int_\Omega dx \int_{(M-\Omega)\cap B(x;R)} dy\, p_M(x,y;t) + \int_\Omega dx \int_{(M-\Omega)\cap(M-B(x;R))} dy\, p_M(x,y;t). \tag{2.16}$$

By (1.6) and (2.4),

$$\begin{aligned}\int_\Omega dx \int_{(M-\Omega)\cap B(x;R)} dy\, p_M(x,y;t) &\le C\int_\Omega dx \int_{(M-\Omega)\cap B(x;R)} dy\, \Big(|B(x;t^{1/2})||B(y;t^{1/2})|\Big)^{-1/2}\\ &\le C^{3/2}\left(\frac{R+t^{1/2}}{t^{1/2}}\right)^{(\log C)/\log 4}\int_\Omega dx \frac{\nu_\Omega(x;R)}{|B(x;t^{1/2})|}.\end{aligned} \tag{2.17}$$

Furthermore, by (1.6),

$$\begin{aligned}&\int_\Omega dx \int_{(M-\Omega)\cap(M-B(x;R))} dy\, p_M(x,y;t)\\ &\le C\int_\Omega dx \int_{M-\Omega} dy \frac{e^{-(d(x,y)^2+R^2)/(2Ct)}}{(|B(x;t^{1/2})||B(y;t^{1/2})|)^{1/2}}\\ &\le C^{(5/2)+((\log C)/\log 2)}\int_\Omega dx \int_{M-\Omega} dy \frac{e^{-(d(x,y)^2+R^2)/(2Ct)}}{(|B(x;(2C^2t)^{1/2})||B(y;(2C^2t)^{1/2})|)^{1/2}}\\ &\le C^{(7/2)+((\log C)/\log 2)}e^{-R^2/(2Ct)}\int_\Omega dx \int_{M-\Omega} dy\, p_M(x,y;2C^2t)\\ &\le C^{(7/2)+((\log C)/\log 2)}e^{-R^2/(2Ct)}F_\Omega(2C^2t).\end{aligned} \tag{2.18}$$

Since F is subadditive with $F(0) = 0$ and $C \ge 2$, we have that

$$F_\Omega(2C^2t) \le F_\Omega(([2C^2]+1)t) \le ([2C^2]+1)F_\Omega(t) \le C^{7/2}F_\Omega(t).$$

Hence, by (2.18),

$$\int_\Omega dx \int_{(M-\Omega)\cap(M-B(x;R))} dy\, p_M(x,y;t) \le C^{7+((\log C)/\log 2)}e^{-R^2/(2Ct)}F_\Omega(t). \tag{2.19}$$

We choose R^2 such that the coefficient of $F_\Omega(t)$ in (2.19) is equal to $\frac{1}{2}$. That is

$$R_*^2 = 2Ct\log\left(2C^{7+((\log C)/\log 2)}\right). \tag{2.20}$$

Rearranging and bootstrapping gives, by (2.16)–(2.19), that

$$\begin{aligned}F_\Omega(t) &\le 2C^{3/2}\left(\frac{R_*+t^{1/2}}{t^{1/2}}\right)^{(\log C)/\log 4}\int_\Omega dx \frac{\nu_\Omega(x;R_*)}{|B(x;t^{1/2})|}\\ &\le 2C^{5/2}\left(\frac{R_*+t^{1/2}}{t^{1/2}}\right)^{(\log C)/\log 4}\left(\frac{R_*}{t^{1/2}}\right)^{(\log C)/\log 2}\int_\Omega dx \frac{\nu_\Omega(x;R_*)}{|B(x;R_*)|}\\ &\le 2C^3\left(\frac{R_*^2}{t}\right)^{(3\log C)/(4\log 2)}\int_\Omega dx \frac{\nu_\Omega(x;R_*)}{|B(x;R_*)|}.\end{aligned} \tag{2.21}$$

Since $t \mapsto F_\Omega(t)$ is concave, with $F_\Omega(t) \geq 0$, we see that

$$F_\Omega(t) \geq \frac{t}{R_*^2} F_\Omega(R_*^2). \tag{2.22}$$

Combining (2.20)–(2.22), gives that

$$F_\Omega(R_*^2) \leq 4C^{15/4}\left(C\log\left(2C^{7+((\log C)/\log 2)}\right)\right)^{1+((3\log C)/(4\log 2))} \times \int_\Omega dx\, \frac{\nu_\Omega(x;R_*)}{|B(x;R_*)|}.$$

This gives, after rescaling t, the upper bound in (1.15) with L_2 given in (1.16). This completes the proof of Theorem 1.3. □

3. Analysis of an Example

In this section we present the asymptotic analysis of $H_\Omega(t)$ as $t \downarrow 0$, of an open set Ω in $M = \mathbb{R}^m$ consisting of disjoint balls with centres in $\mathbb{Z}^m$, and decreasing radii. Recall that $p_{\mathbb{R}^m}(x,y;t) = (4\pi t)^{-m/2}e^{-|x-y|^2/(4t)}$. Let

$$\Omega = \cup_{i\in\mathbb{N}} B(z_i;r_i), \tag{3.1}$$

where $(z_i)_{i\in\mathbb{N}}$ is an enumeration of $\mathbb{Z}^m$, and where $r_1 \geq r_2 \geq \dots$. Furthermore, let

$$\delta = 1 - 2r_1 > 0. \tag{3.2}$$

Theorem 3.1 (ii) below asserts that if $H_\Omega(t) < \infty$ for all $t > 0$, and if (3.2) holds then the balls loose heat independently as $t \downarrow 0$ up to a term exponentially small in t.

Theorem 3.1. (i) *If $\delta > 0$, then $H_\Omega(t) < \infty$ for all $t > 0$ if and only if*

$$\sum_{i=1}^{\infty} r_i^{2m} < \infty. \tag{3.3}$$

(ii) *If $\delta > 0$ and (3.3) holds, then*

$$\left| H_\Omega(t) - \sum_{i=1}^{\infty} H_{B(z_i;r_i)}(t) \right| \leq \omega_m^2 e^{-\delta^2/(8t)} \left(\frac{\sqrt{2}}{\delta} + \frac{1}{(4\pi t)^{1/2}}\right)^m \sum_{i=1}^{\infty} r_i^{2m}, \tag{3.4}$$

where $\omega_m = |B(0;1)|$.

Below we consider four main regimes: $\frac{1}{2m} < \alpha < \frac{1}{m}$, $\frac{1}{m} < \alpha < \frac{1}{m-1}$, $\frac{1}{m-1} < \alpha < \frac{1}{m-2}$, and $\frac{1}{m-2} < \alpha$. The latter regime is absent for $m = 2$. In the first regime Ω has infinite measure, and Theorem 1.2 (iii) gives the order of magnitude as $t \downarrow 0$. This has been refined in (3.5)–(3.6) below. In the second regime Ω has infinite perimeter, and Theorem 1.3 gives the order of magnitude as $t \downarrow 0$. This has been refined in (3.12)–(3.13) below. In the third and fourth regimes Ω has finite perimeter. Theorem 1.3 gives two-sided bounds of order $t^{1/2}$. In (3.9) and (3.11) below we show that the perimeter term appears with the usual numerical constant. The remainder estimates depend on whether $\sum_{i\in\mathbb{N}} r_i^{m-2}$ is infinite or finite. Furthermore there are several borderline cases: $\alpha = \frac{1}{m}, \frac{1}{m-1}, \frac{1}{m-2}$. They all involve logarithmic corrections in the heat content. We only analyse, as an example, the case $\alpha = \frac{1}{m-2}$. The latter case is again absent for $m = 2$.

Theorem 3.2. *Let $0 < a \le \frac{1}{4}, m \ge 2,$ and let $r_i = ai^{-\alpha},\ i \in \mathbb{N}$.*

If $\frac{1}{2m} < \alpha < \frac{1}{m}$ then

$$H_\Omega(t) = c_{\alpha,m} t^{(m\alpha-1)/(2\alpha)} + O(1),\ t \downarrow 0, \tag{3.5}$$

where

$$c_{\alpha,m} = 2^{m-1-\frac{1}{\alpha}} \pi^{-m/2} \alpha^{-1} \Gamma((2m\alpha - 1)/(2\alpha)) a^{1/\alpha} \times \int_{B(0;1)} dx \int_{B(0;1)} dy\, |x-y|^{(1-2m\alpha)/\alpha}. \tag{3.6}$$

If $\frac{1}{m} < \alpha < \frac{1}{m-1}$ then

$$F_\Omega(t) = d_{\alpha,m} t^{(m\alpha-1)/(2\alpha)} + O(t^{1/2}),\ t \downarrow 0, \tag{3.7}$$

where

$$d_{\alpha,m} = 2^{m-1-\frac{1}{\alpha}} \pi^{-m/2} \alpha^{-1} \Gamma((2m\alpha - 1)/(2\alpha)) a^{1/\alpha} \times \int_{B(0;1)} dx \int_{\mathbb{R}^m - B(0;1)} dy\, |x-y|^{(1-2m\alpha)/\alpha}. \tag{3.8}$$

If $m > 2$ and $\frac{1}{m-1} < \alpha < \frac{1}{m-2}$ or if $m = 2$ and $\frac{1}{m-1} < \alpha$, then

$$F_\Omega(t) = \pi^{-1/2} \mathrm{Per}(\Omega) t^{1/2} + O(t^{(m\alpha-1)/(2\alpha)}),\ t \downarrow 0. \tag{3.9}$$

If $m > 2$ and $\alpha = \frac{1}{m-2}$ then

$$F_\Omega(t) = \pi^{-1/2} \mathrm{Per}(\Omega) t^{1/2} + O\left(t \log \frac{1}{t}\right),\ t \downarrow 0. \tag{3.10}$$

If $m > 2$ and $\frac{1}{m-2} < \alpha$ then

$$F_\Omega(t) = \pi^{-1/2} \mathrm{Per}(\Omega) t^{1/2} + O(t),\ t \downarrow 0. \tag{3.11}$$

Proof of Theorem 3.1. To prove part (i) we first suppose that $H_\Omega(t) < \infty$ for some $t > 0$. Then

$$\begin{aligned} H_\Omega(t) &= \sum_{i=1}^\infty \sum_{j=1}^\infty \int_{B(z_i;r_i)} dx \int_{B(z_j;r_j)} dy\, p_{\mathbb{R}^m}(x,y;t) \\ &\ge \sum_{i=1}^\infty \int_{B(z_i;r_i)} dx \int_{B(z_i;r_i)} dy\, p_{\mathbb{R}^m}(x,y;t) \\ &\ge (4\pi t)^{-m/2} e^{-r_1^2/t} \omega_m^2 \sum_{i=1}^\infty r_i^{2m}. \end{aligned} \tag{3.12}$$

Next suppose that $\sum_{i\in\mathbb{N}} r_i^{2m} < \infty$. Then

$$\begin{aligned}\int_\Omega dx \frac{\mu_\Omega(x;\delta/2)}{|B(x;\delta/2)|} &\le \sum_{\{i:r_i\ge\delta/4\}} \int_{B(z_i;r_i)} dx + \sum_{\{i:r_i<\delta/4\}} \int_{B(z_i;r_i)} dx\ \frac{\mu_\Omega(x;\delta/2)}{|B(x;\delta/2)|} \\ &\le \sum_{\{i:r_i\ge\delta/4\}} \int_{B(z_i;r_i)} dx (4r_i/\delta)^m + \omega_m \left(\frac{2}{\delta}\right)^m \sum_{\{i:r_i<\delta/4\}} r_i^{2m} \\ &\le \omega_m \left(\frac{4}{\delta}\right)^m \sum_{i\in\mathbb{N}} r_i^{2m}, \qquad (3.13)\end{aligned}$$

which implies the reverse implication by Theorem 1.2 (ii). This proves the assertion under (i).

To prove part (ii) we note that the lower bound in (3.4) follows from the first inequality in (3.12). To prove the upper bound we observe that if $x \in B(z_i;r_i), y \in B(z_j;r_j), i \ne j$, then

$$|x-y| \ge |z_i - z_j| - |z_i - x| - |y - z_j| \ge |z_i - z_j| + \delta - 1 \ge \delta|z_i - z_j|.$$

Hence

$$\begin{aligned}&\sum_{i=1}^\infty \sum_{\{j\in\mathbb{N}:j\ne i\}} \int_{B(z_i;r_i)} dx \int_{B(z_j;r_j)} dy\, p_{\mathbb{R}^m}(x,y;t) \\ &\le \omega_m^2 (4\pi t)^{-m/2} e^{-\delta^2/(8t)} \sum_{i=1}^\infty \sum_{\{j\in\mathbb{N}:j\ne i\}} r_i^m r_j^m e^{-|z_i-z_j|^2\delta^2/(8t)} \\ &\le \frac{1}{2}\omega_m^2 (4\pi t)^{-m/2} e^{-\delta^2/(8t)} \sum_{i=1}^\infty \sum_{\{j\in\mathbb{N}:j\ne i\}} \left(r_i^{2m} + r_j^{2m}\right) e^{-|z_i-z_j|^2\delta^2/(8t)} \\ &\le \omega_m^2 (4\pi t)^{-m/2} e^{-\delta^2/(8t)} \sum_{i=1}^\infty r_i^{2m} \sum_{z\in\mathbb{Z}^m} e^{-|z|^2\delta^2/(8t)} \\ &\le \omega_m^2 (4\pi t)^{-m/2} e^{-\delta^2/(8t)} \sum_{i=1}^\infty r_i^{2m} \left(1 + 2\int_0^\infty dz\ e^{-|z|^2\delta^2/(8t)}\right)^m,\end{aligned}$$

which gives the bound in (3.4). □

Proof of Theorem 3.2. We first consider the case $\frac{1}{2m} < \alpha < \frac{1}{m}$. By (3.4), it suffices to consider the sum in the left-hand side of (3.8). Since $r \mapsto H_{B(0;r)}(t)$ is increasing, $i \mapsto H_{B(0;ai^{-\alpha})}(t)$ is decreasing. Hence

$$\begin{aligned}\sum_{i=1}^\infty H_{B(z_i;r_i)}(t) &= \sum_{i=1}^\infty H_{B(0;ai^{-\alpha})}(t) \\ &\le \int_0^\infty di\ H_{B(0;ai^{-\alpha})}(t) \\ &= \int_0^\infty di (ai^{-\alpha})^{2m} \int_{B(0;1)} dx \int_{B(0;1)} dy\ p_{\mathbb{R}^m}(axi^{-\alpha}, ayi^{-\alpha};t).\end{aligned} \qquad (3.14)$$

A straightforward application of Tonelli's theorem gives the formulae under (3.5) and (3.6). To obtain a lower bound for the left-hand side of (3.14), we use the monotonicity of $i \mapsto H_{B(0;ai^{-\alpha})}(t)$ once more, and obtain that

$$\sum_{i=1}^{\infty} H_{B(z_i;r_i)}(t) \geq \int_1^{\infty} di\, H_{B(0;ai^{-\alpha})}(t)$$
$$= c_{\alpha,m} t^{(m\alpha-1)/(2\alpha)} - \int_0^1 di\, H_{B(0;ai^{-\alpha})}(t). \qquad (3.15)$$

The last term in the right-hand side of (3.15) is bounded in absolute value by

$$\int_0^1 di\, H_{B(0;ai^{-\alpha})}(t) \leq \int_0^1 di\, |B(0;ai^{-\alpha})| = \omega_m a^m (1-\alpha m)^{-1}.$$

This completes the proof of the assertion under (3.5) and (3.6). □

Consider the case $\frac{1}{m} < \alpha < \frac{1}{m-1}$. By (3.4), and scaling we have that

$$F_\Omega(t) = \sum_{i=1}^{\infty} F_{B(0;ai^{-\alpha})}(t) + O(e^{-\delta^2/(16t)})$$
$$= \sum_{i=1}^{\infty} (ai^{-\alpha})^m F_{B(0;1)}(a^{-2} i^{2\alpha} t) + O(e^{-\delta^2/(16t)}). \qquad (3.16)$$

In a similar way to the proof of (3.5),(3.6), we approximate the sum with respect to i by an integral. However, $i \mapsto F_{B(0;1)}(a^{-2} i^{2\alpha} t)$ is increasing, whereas $i \mapsto (ai^{-\alpha})^m$ is decreasing.

Lemma 3.3. *Let $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be increasing, and let $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be decreasing. If fg is summable, then*

$$\left| \sum_{i=1}^{\infty} f(i)g(i) - \int_1^{\infty} dx\, f(x)g(x) \right| \leq \sum_{i=1}^{\infty} f(i+1)\left(g(i) - g(i+1)\right). \qquad (3.17)$$

Proof. We have that

$$\int_i^{i+1} dx\, f(x)g(x) \geq f(i)g(i+1) = f(i)g(i) - f(i)\left(g(i) - g(i+1)\right),$$

and so

$$\sum_{i=1}^{\infty} f(i)g(i) \leq \int_1^{\infty} dx\, f(x)g(x) + \sum_{i=1}^{\infty} f(i)\left(g(i) - g(i+1)\right). \qquad (3.18)$$

Similarly

$$\int_i^{i+1} dx\, f(x)g(x) \leq f(i+1)g(i) = f(i+1)g(i+1) + f(i+1))\left(g(i) - g(i+1)\right),$$

and

$$\sum_{i=1}^{\infty} f(i+1)g(i+1) \geq \int_1^{\infty} dx\, f(x)g(x) - \sum_{i=1}^{\infty} f(i+1)\left(g(i) - g(i+1)\right).$$

So

$$\sum_{i=1}^{\infty} f(i)g(i) \geq \int_1^{\infty} dx\, f(x)g(x) - \sum_{i=1}^{\infty} f(i+1)\left(g(i) - g(i+1)\right). \tag{3.19}$$

Inequality (3.17) follows from (3.18), (3.19) and $f(i) \leq f(i+1)$. □

Let $f(x) = F_{B(0;1)}(a^{-2}x^{2\alpha}t)$, and $g(x) = a^m x^{-m\alpha}$. Using $g(x) - g(x+1) \leq a^m m\alpha x^{-m\alpha-1}$, we obtain that

$$\begin{aligned}
0 \leq \sum_{i=1}^{\infty} f(i+1)\left(g(i) - g(i+1)\right) &\leq a^m m\alpha \sum_{i=1}^{\infty} i^{-m\alpha-1} F_{B(0;1)}(a^{-2}(i+1)^{2\alpha}t) \\
&\leq a^m m\alpha \sum_{i=1}^{\infty} i^{-m\alpha-1} F_{B(0;1)}(a^{-2}(2i)^{2\alpha}t) \\
&\leq a^{m-1} m^2 \omega_m \alpha 2^{\alpha} \pi^{-1/2} \sum_{i=1}^{\infty} i^{-m\alpha+\alpha-1} t^{1/2} \\
&= O(t^{1/2}),
\end{aligned} \tag{3.20}$$

where we have used that (Proposition 8 in [8]) $F_{B(0;1)}(t) \leq m\omega_m \pi^{-1/2} t^{1/2}$. This gives that

$$\int_0^1 dx\, f(x)g(x) \leq m\omega_m \pi^{-1/2} a^{m-1} t^{1/2} \int_0^1 dx\ x^{-m\alpha+\alpha} = O(t^{1/2}). \tag{3.21}$$

By (3.16),(3.17),(3.20), and (3.21) we conclude that

$$\begin{aligned}
F_{\Omega}(t) &= \int_0^{\infty} dx\, a^m x^{-m\alpha} F_{B(0;1)}(a^{-2}x^{2\alpha}t) + O(t^{1/2}) \\
&= d_{\alpha,m} t^{(m\alpha-1)/(2\alpha)} + O(t^{1/2}),
\end{aligned}$$

where $d_{\alpha,m}$ is given by (3.8). This completes the proof of (3.7).

Consider the cases $m > 2$ and $\frac{1}{m-1} < \alpha < \frac{1}{m-2}$ or $m = 2$ and $\frac{1}{m-1} < \alpha$. Then Ω has finite measure, and finite perimeter. Let $I \in \mathbb{N}$, and apply Theorem 2 from [10] to a ball of radius r:

$$|H_{B(0;r)} - |B(0;r)| + \pi^{-1/2}\mathrm{Per}(B(0;r))t^{1/2}| \leq c_m r^{m-2} t,\ t > 0, \tag{3.22}$$

where $c_m = 2^{m+2} m^3 \omega_m$. Then

$$\begin{aligned}
H_{\Omega}(t) &\geq \sum_{i=1}^{I} H_{B(0;ai^{-\alpha})}(t) \\
&\geq |\Omega| - \pi^{-1/2}\mathrm{Per}(\Omega)t^{1/2} - \sum_{i=I+1}^{\infty} \omega_m a^m i^{-\alpha m} - c_m \sum_{i=1}^{I} (ai^{-\alpha})^{m-2} t.
\end{aligned} \tag{3.23}$$

The third term in the right-hand side of (3.23) is $O(I^{1-\alpha m})$. The fourth term is $O(I^{1-\alpha(m-2)})t$. The choice $I = \lfloor t^{-1/(2\alpha)} \rfloor$ gives the $O(t^{(m\alpha-1)/(2\alpha)})$ remainder in the lower bound.

To obtain an upper bound, we let $J \in \mathbb{N}$, and note that by (3.4),

$$
\begin{aligned}
H_\Omega(t) &\le \sum_{i=1}^{\infty} H_{B(0;ai^{-\alpha})}(t) + O(e^{-\delta^2/(16t)})\\
&\le \sum_{i=1}^{J} H_{B(0;ai^{-\alpha})}(t) + \sum_{i=J+1}^{\infty} |B(0;ai^{-\alpha})| + O(e^{-\delta^2/(16t)})\\
&\le \sum_{i=1}^{J}\Big(|B(0;ai^{-\alpha})| - \pi^{-1/2}\mathrm{Per}(B(0;ai^{-\alpha}))t^{1/2} + c_m(ai^{-\alpha})^{m-2}t\Big)\\
&\quad + \sum_{i=J+1}^{\infty} |B(0;ai^{-\alpha})| + O(e^{-\delta^2/(16t)})\\
&\le |\Omega| - \pi^{-1/2}\mathrm{Per}(\Omega)t^{1/2} + \pi^{-1/2}m\omega_m \sum_{i=J+1}^{\infty}(ai^{-\alpha})^{m-1}t^{1/2}\\
&\quad + c_m\sum_{i=1}^{J}(ai^{-\alpha})^{m-2}t + O(e^{-\delta^2/(16t)}),\ t\downarrow 0.
\end{aligned}
\tag{3.24}
$$

The third term in the right-hand side of (3.24) is $O(J^{1-\alpha(m-1)})t^{1/2}$. The fourth term in the right-hand side of (3.24) is $O(J^{1-\alpha(m-2)})t$. The choice $J = \lfloor t^{-1/(2\alpha)}\rfloor$ gives a remainder $O(t^{(m\alpha-1)/(2\alpha)})$ for the upper bound, and completes the proof of (3.9).

Next consider the case $\alpha = \frac{1}{m-2}$. The sum of the third and fourth terms in the right-hand side of (3.23) equals, up to constants, $I^{-2/(m-2)} + t\log I$. We now choose $I = \lfloor t^{-(m-2)/2}\rfloor$, and obtain the remainder in (3.10). Similarly, the sum of the third and fourth terms in the right-hand side of (3.24) is of order $J^{-1/(m-2)}t^{1/2} + t\log J$. We now choose $J = \lfloor t^{-(m-2)/2}\rfloor$ to obtain the same remainder.

Finally, consider the case $m > 2, \alpha > (m-2)^{-1}$. Then the uniform remainder in the right-hand side of (3.22) is summable. Hence by (3.4),

$$
|H_\Omega(t) - |\Omega| + \pi^{-1/2}\mathrm{Per}(\Omega)t^{1/2}| \le c_m\sum_{i\in\mathbb{N}}(ai^{-\alpha})^{m-2}t + O(e^{-\delta^2/(16t)}),\ t\downarrow 0,
$$

and we obtain (3.11). □

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