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Integral Equations and Operator Theory

Remarks on the Operator-Norm Convergence of the Trotter Product Formula

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Abstract. We revise the operator-norm convergence of the Trotter product formula for a pair $\{A, B\}$ of generators of semigroups on a Banach space. Operator-norm convergence holds true if the dominating operator A generates a holomorphic contraction semigroup and B is a A-infinitesimally small generator of a contraction semigroup, in particular, if B is a bounded operator. Inspired by studies of evolution semigroups it is shown in the present paper that the operator-norm convergence generally fails even for bounded operators B if A is not a holomorphic generator. Moreover, it is shown that operator norm convergence of the Trotter product formula can be arbitrary slow.

Keywords. Semigroups, Bounded perturbations, Trotter product formula, Darboux–Riemann sums, Operator-norm convergence.

1. Introduction and Main Results

Recall that the product formula

$$e^{-\tau C} = \lim_{n\to\infty} \left(e^{-\tau A/n} e^{-\tau B/n}\right)^n, \quad \tau \geq 0,$$

was established by S. Lie (in 1875) for matrices where $C := A + B$. The proof is based on the telescopic representation

$$\begin{aligned}&\left(e^{-\tau A/n} e^{-\tau B/n}\right)^n - e^{-\tau C}\\ &= \sum_{k=0}^{n-1} \left(e^{-\tau A/n} e^{-\tau B/n}\right)^{n-1-k} \left(e^{-\tau A/n} e^{-\tau B/n} - e^{-\tau C/n}\right) e^{-k\tau C/n},\end{aligned} \tag{1.1}$$

$n \in \mathbb{N}$, and expansion

$$e^{-\tau X} = I - \tau X + O(\tau^2), \qquad \tau \longrightarrow 0,$$

for a matrix X in the operator-norm topology $\|\cdot\|$. Indeed, using this expansion one obtains the estimate:

$$\|e^{-\tau A/n} e^{-\tau B/n} - e^{-\tau C/n}\| = O((\tau/n)^2).$$

Then from (1.1) we get the existence of a constant $c_0 > 0$ such that the following estimate holds

$$\begin{aligned}&\left\|\left(e^{-\tau A/n}e^{-\tau B/n}\right)^n - e^{-\tau C}\right\|\\ &\quad\le c_0\frac{\tau^2}{n^2}\sum_{k=0}^{n-1} e^{\frac{n-1-k}{n}\tau\|A\|}e^{\frac{n-1-k}{n}\tau\|B\|}e^{\tau\frac{k}{n}\|C\|}.\end{aligned}$$

Since $\|C\| \le \|A\| + \|B\|$, one obtains inequality

$$\begin{aligned}&\left\|\left(e^{-\tau A/n}e^{-\tau B/n}\right)^n - e^{-\tau C}\right\|\\ &\quad\le c_0\frac{\tau^2}{n^2}\sum_{k=0}^{n-1} e^{\tau\frac{n-1}{n}(\|A\|+\|B\|)} \le c_0\frac{\tau^2}{n}e^{\tau(\|A\|+\|B\|)},\end{aligned}$$

which yields that

$$\sup_{\tau\in[0,T]}\left\|\left(e^{-\tau A/n}e^{-\tau B/n}\right)^n - e^{-\tau C}\right\| = O(1/n), \tag{1.2}$$

as $n \to \infty$ for any $T > 0$. Note that this proof carries through verbatim for bounded operators A and B on Banach spaces.

Trotter [7] has extended this result to unbounded operators A and B on Banach spaces, but in the strong operator topology. He proved that if A and B are generators of contraction semigroups on a separable Banach space such that the algebraic sum $A + B$ is a densely defined closable operator and the closure $C = \overline{A + B}$ is a generator of a contraction semigroup, then

$$e^{-\tau C} = \text{s-}\lim_{n\to\infty}\left(e^{-\tau A/n}e^{-\tau B/n}\right)^n, \tag{1.3}$$

uniformly in $\tau \in [0, T]$ for any $T > 0$. It is obvious that this result holds if B is a bounded operator.

Considering the Trotter product formula on a Hilbert space Kato has shown in [4] that for non-negative operators A and B the Trotter formula (1.3) holds in the *strong* operator topology if $\mathrm{dom}(\sqrt{A}) \cap \mathrm{dom}(\sqrt{B})$ is dense in the Hilbert space and $C = A\dot{+}B$ is the form-sum of operators A and B. Later on it was shown in [3] that the relation (1.2) holds if the algebraic sum $C = A + B$ is already a self-adjoint operator. Therefore, (1.2) is valid in particular, if B is a bounded self-adjoint operator.

The historically first result concerning the operator-norm convergence of the Trotter formula in a Banach space is due to [1]. Since the concept of self-adjointness is missing for Banach spaces it was assumed that the *dominating* operator A is a generator of a *contraction holomorphic* semigroup and B is a generator of a contraction semigroup. In Theorem 3.6 of [1] it was shown that if $0 \in \rho(A)$ and if there is a $\alpha \in [0, 1)$ such that $\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B)$ and $\mathrm{dom}(A^*) \subseteq \mathrm{dom}(B^*)$, then for any $T > 0$ one has

$$\sup_{\tau\in[0,T]}\left\|\left(e^{-\tau A/n}e^{-\tau B/n}\right)^n - e^{-\tau C}\right\| = O(\ln(n)/n^{1-\alpha}). \tag{1.4}$$

Note that the assumption $0 \in \rho(A)$ was made for simplicity and that the assumption $\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B)$ yields that the operator B is infinitesimally small with respect to A. Taking into account [5, Corollary IX.2.5] one gets that the well-defined algebraic sum $C = A + B$ is a generator of a contraction holomorphic semigroup. By Theorem 3.6 of [1] the convergence rate (1.4) improves if B is a bounded operator, i.e. $\alpha = 0$. Then for any $T > 0$ one gets

$$\sup_{\tau\in[0,T]} \left\| \left(e^{-\tau A/n} e^{-\tau B/n}\right)^n - e^{-\tau C} \right\| = O((\ln(n))^2/n).$$

Summarizing, the question arises whether the Trotter product formula converges in the operator-norm if A is a generator of a contraction (but not holomorphic) semigroup and B is a bounded operator? The aim of the present paper is to give an answer to this question for a certain class of generators.

It turns out that an appropriate class for that is the class of generators of *evolution* semigroups. To proceed further we need the notion of a *propagator*, or a *solution operator* [6].

A strongly continuous map $U(\cdot,\cdot) : \Delta \longrightarrow \mathcal{B}(X)$, where $\Delta := \{(t,s) : 0 < s \le t \le T\}$ and $\mathcal{B}(X)$ is the set of bounded operators on the separable Banach space X, is called a *propagator* if the conditions

$$\text{(i)}\ \sup_{(t,s)\in\Delta} \|U(t,s)\|_{\mathcal{B}(X)} < \infty,$$

$$\text{(ii)}\ U(t,s) = U(t,r)U(r,s),\ \ 0 < s \le r \le t \le T,$$

are satisfied. Let us consider the Banach space $L^p(\mathcal{I}, X)$, $\mathcal{I} := [0,T]$, $p \in [1,\infty)$. The operator $\mathcal{K}$ is an evolution generator of the evolution semigroup $\{e^{-\tau\mathcal{K}}\}_{\tau\ge0}$ if there is a propagator such that the representation

$$\left(e^{-\tau\mathcal{K}} f\right)(t) = U(t, t-\tau)\chi_{\mathcal{I}}(t-\tau) f(t-\tau), \quad f \in L^p(\mathcal{I}, X), \tag{1.5}$$

holds for a.e. $t \in \mathcal{I}$ and $\tau \ge 0$ [6]. Since $e^{-\tau\mathcal{K}} f = 0$ for $\tau \ge T$, the evolution generator $\mathcal{K}$ can never be a generator of a holomorphic semigroup.

A simple example of an evolution generator is the differentiation operator:

$$\begin{gathered}(D_0 f)(t) := \partial_t f(t),\\ f \in \mathrm{dom}(D_0) := \{f \in H^{1,p}(\mathcal{I}, X) : f(0) = 0\}.\end{gathered} \tag{1.6}$$

Then by (1.6) one obviously gets the contraction shift semigroup:

$$(e^{-\tau D_0} f)(t) = \chi_{\mathcal{I}}(t-\tau) f(t-\tau), \quad f \in L^p(\mathcal{I}, X), \tag{1.7}$$

for a.e. $t \in \mathcal{I}$ and $\tau \ge 0$. Hence, (1.5) implies that the corresponding propagator of the non-holomorphic evolution semigroup $\{e^{-\tau D_0}\}_{\tau\ge0}$ is given by $U_{D_0}(t,s) = I$, $(t,s) \in \Delta$.

Note that in [6] we considered the operator $\mathcal{K}_0 := \overline{D_0 + \mathcal{A}}$, where $\mathcal{A}$ is the multiplication operator induced by a generator A of a holomorphic contraction semigroup on X. More precisely

$$\begin{gathered}(\mathcal{A} f)(t) := A f(t),\ \text{and}\ (e^{-\tau\mathcal{A}} f)(t) = e^{-\tau A} f(t),\\ f \in \mathrm{dom}(\mathcal{A}) := \{f \in L^p(\mathcal{I}, X) : A f(\cdot) \in L^p(\mathcal{I}, X)\}.\end{gathered}$$

Then the perturbation of the shift semigroup (1.7) by $\mathcal{A}$ corresponds to the semigroup with generator $\mathcal{K}_0$. One easily checks that $\mathcal{K}_0$ is an evolution generator of a contraction semigroup on $L^p(\mathcal{I}, X)$ that is never holomorphic.

Indeed, since the generators D_0 and $\mathcal{A}$ commute, the representation (1.5) for evolution semigroup $\{e^{-\tau\mathcal{K}_0}\}_{\tau\geq 0}$ takes the form:

$$\left(e^{-\tau\mathcal{K}_0}f\right)(t) = e^{-\tau A}\chi_{\mathcal{I}}(t-\tau)f(t-\tau), \quad f \in L^p(\mathcal{I}, X),$$

for a.e. $t \in \mathcal{I}$ and $\tau \geq 0$ with propagator $U_0(t,s) = e^{-(t-s)A}$. Therefore, again $e^{-\tau\mathcal{K}_0}f = 0$ for $\tau \geq T$.

Furthermore, if $B(\cdot)$ is a *strongly measurable* family of generators of contraction semigroups on X, i.e. $B(\cdot) : \mathcal{I} \longrightarrow \mathcal{G}(1,0)$ (see [5], Ch.IX, §1.4), then the induced multiplication operator $\mathcal{B}$:

$$\begin{aligned} &(\mathcal{B}f)(t) := B(t)f(t) \\ &f \in \mathrm{dom}(\mathcal{B}) := \left\{ f \in L^p(\mathcal{I}, X) : \begin{matrix} f(t) \in \mathrm{dom}(B(t)) \text{ for a.e. } t \in \mathcal{I} \\ B(t)f(t) \in L^p(\mathcal{I}, X) \end{matrix} \right\}, \end{aligned} \tag{1.8}$$

is a generator of a contraction semigroup on $L^p(\mathcal{I}, X)$.

In [6] it was assumed that $\{B(t)\}_{t\in\mathcal{I}}$ is a strongly measurable family of generators of contraction semigroups and that A is a generator of a bounded holomorphic semigroup with $0 \in \rho(A)$ for simplicity. Moreover, we supposed that the following conditions are satisfied:

(i) $\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B(t))$ for a.e. $t \in \mathcal{I}$ and some $\alpha \in (0,1)$ such that

$$\operatorname*{ess\,sup}_{t\in\mathcal{I}} \|B(t)A^{-\alpha}\|_{\mathcal{B}(X)} < \infty\,;$$

(ii) $\mathrm{dom}(A^*) \subseteq \mathrm{dom}(B(t)^*)$ for a.e. $t \in \mathcal{I}$ such that

$$\operatorname*{ess\,sup}_{t\in\mathcal{I}} \|B(t)^*(A^{-1})^*\|_{\mathcal{B}(X)} < \infty\,;$$

(iii) there is a $\beta \in (\alpha, 1)$ and $L_\beta > 0$ such that

$$\|A^{-1}(B(t)-B(s))A^{-\alpha}\|_{\mathcal{B}(X)} \leq L_\beta|t-s|^\beta, \quad t,s \in \mathcal{I}. \tag{1.9}$$

Under these assumptions it turns out that $\mathcal{K} := \mathcal{K}_0 + \mathcal{B}$ is a generator of a contraction evolution semigroup, i.e there is a propagator $\{U(t,s)\}_{(t,s)\in\Delta}$ such that the representation (1.5) is valid. Moreover, we prove in [6] the Trotter product formula converges in the operator norm with convergence rate $O(1/n^{\beta-\alpha})$:

$$\sup_{\tau\geq 0}\left\|\left(e^{-\tau\mathcal{K}_0/n}e^{-\tau\mathcal{B}/n}\right)^n - e^{-\tau\mathcal{K}}\right\|_{\mathcal{B}(L^p(\mathcal{I},X))} = O(1/n^{\beta-\alpha})\ .$$

We comment that if $B(\cdot) : \mathcal{I} \longrightarrow \mathcal{B}(X)$ is a Hölder continuous function with Hölder exponent $\beta \in (0,1)$, then the assumptions (i)–(iii) are satisfied for any $\alpha \in (0,\beta)$. Then our results [6] yield that

$$\sup_{\tau\geq 0}\left\|\left(e^{-\tau\mathcal{K}_0/n}e^{-\tau\mathcal{B}/n}\right)^n - e^{-\tau\mathcal{K}}\right\|_{\mathcal{B}(L^p(\mathcal{I},X))} = O(1/n^{\gamma}), \tag{1.10}$$

holds for any $\gamma \in (0,\beta)$. In particular, if $A := I$ and $B(\cdot) : \mathcal{I} \longrightarrow \mathcal{B}(X)$ is Hölder continuous, then assumptions (i)–(iii) are satisfied. Hence the convergence rate estimate (1.10) is valid for $\mathcal{K}_0 = D_0 + I$ and $\mathcal{K} = \mathcal{K}_0 + \mathcal{B} =$

$D_0 + I + \mathcal{B}$. Using that we immediately get from (1.10) the convergence rate estimate

$$\sup_{\tau\geq 0}\left\|\left(e^{-\tau D_0/n}e^{-\tau\mathcal{B}/n}\right)^n - e^{-\tau(D_0+\mathcal{B})}\right\|_{\mathcal{B}(L^p(\mathcal{I},X))} = O(1/n^\gamma) \qquad (1.11)$$

from (1.10).

In the following we will present a couple of results which are obtained independently of [6]. The first result concerns the problem of sharpness of convergence rate $O(1/n^\gamma)$, $\gamma \in (0,\beta)$. It turns out that $O(1/n^\gamma)$ is almost sharp. To show this we consider the simple case, when $X = \mathbb{C}$ and $\mathcal{I} := [0,1]$. If $\mathcal{A} = I$ and $\mathcal{B}$ is equal to the multiplication operator Q induced by a bounded measurable function $q(\cdot) : \mathcal{I} \longrightarrow \mathbb{C}$ in $L^p(\mathcal{I})$, then one can verify that the condition (1.9) is equivalent to $q(\cdot) \in C^{0,\beta}(\mathcal{I})$, for definition of $C^{0,\beta}(\mathcal{I})$ see below. In this case we get below the convergence rate

$$\sup_{\tau\geq 0}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathcal{B}(L^p(\mathcal{I},X))} = O(1/n^\beta) \qquad (1.12)$$

which is slightly better than (1.11) obtained from [6]. This result remains true if $q(\cdot)$ is Lipschitz continuous, i.e. for $\beta = 1$.

For the same example we can show some further results which go beyond [6]: If $q(\cdot)$ is *only* continuous, then

$$\sup_{\tau\geq 0}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathcal{B}(L^p(\mathcal{I},X))} = o(1). \qquad (1.13)$$

Moreover, for any convergent to zero sequence $\delta_n > 0$, $n \in \mathbb{N}$, there exists a continuous function $q(\cdot)$ such that

$$\sup_{\tau\geq 0}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathcal{B}(L^p(\mathcal{I},X))} = \omega(\delta_n), \qquad (1.14)$$

where the Landau *symbol* $\omega(\cdot)$ is defined below.

Finally, we answer the question posed above. We give an example of a bounded measurable function $q(\cdot)$, which induces a bounded multiplication operator, such that

$$\limsup_{n\to\infty}\sup_{\tau\geq 0}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathcal{B}(L^p(\mathcal{I},X))} > 0. \qquad (1.15)$$

Hence, in contrast to the holomorphic case, when the *dominating* operator is a generator of a holomorphic semigroup (1.4), the Trotter product formula (1.15) with dominating generator D_0, may *not* converge in the operator-norm.

The paper is organized as follows. In Sect. 2 we reformulate the convergence of the Trotter product formula in terms of the corresponding evolutions semigroups. In Sect. 3 we prove the results (1.12)-(1.15).

We conclude this section by few remarks concerning **notation** used in this paper.

1. We use a definition of the *generator* C of a semigroup (1.3), which differs from the standard one by a *minus* [5].
2. Furthermore, we widely use the so-called *Landau symbols*:

$$g(n) = O(f(n)) \Longleftrightarrow \limsup_{n\to\infty}\left|\frac{g(n)}{f(n)}\right| < \infty,$$

$$g(n) = o(f(n)) \Longleftrightarrow \limsup_{n\to\infty} \left|\frac{g(n)}{f(n)}\right| = 0,$$
$$g(n) = \Theta(f(n)) \Longleftrightarrow 0 < \liminf_{n\to\infty} \left|\frac{g(n)}{f(n)}\right| \leq \limsup_{n\to\infty} \left|\frac{g(n)}{f(n)}\right| < \infty,$$
$$g(n) = \omega(f(n)) \Longleftrightarrow \limsup_{n\to\infty} \left|\frac{g(n)}{f(n)}\right| = \infty.$$

3. We use the notation $C^{0,\beta}(\mathcal{I}) = \{f : \mathcal{I} \to \mathbb{C}$: there is some $K > 0$ such that $|f(x) - f(y)| \leq K|x-y|^\beta\}$ for $\beta \in (0,1]$.

2. Trotter Product Formula and Evolution Semigroups

Below we consider the Banach space $L^p(\mathcal{I}, X)$ for $\mathcal{I} := [0,T]$, $p \in [1,\infty)$. Recall that semigroup $\{\mathcal{U}(\tau)\}_{\tau\geq 0}$, on the Banach space $L^p(\mathcal{I}, X)$ is called an *evolution* semigroup if there is a propagator $\{U(t,s)\}_{(t,s)\in\Delta}$ such that the representation (1.5) holds.

Let $\mathcal{K}_0$ be the generator of an evolution semigroup $\{\mathcal{U}_0(\tau)\}_{\tau\geq 0}$ and let $\mathcal{B}$ be a multiplication operator induced by a measurable family $\{B(t)\}_{t\in\mathcal{I}}$ of generators of contraction semigroups. Note that in this case the multiplication operator $\mathcal{B}$ (1.8) is a generator of a contraction semigroup $(e^{-\tau\,\mathcal{B}} f)(t) = e^{-\tau\,B(t)} f(t)$, on the Banach space $L^p(\mathcal{I}, X)$. Since $\{\mathcal{U}_0(\tau)\}_{\tau\geq 0}$ is an evolution semigroup, then by definition (1.5) there is a propagator $\{U_0(t,s)\}_{(t,s)\in\Delta}$ such that the representation

$$(\mathcal{U}_0(\tau)f)(t) = U_0(t, t-\tau)\chi_{\mathcal{I}}(t-\tau)f(t-\tau), \quad f \in L^p(\mathcal{I}, X),$$

is valid for a.e. $t \in \mathcal{I}$ and $\tau \geq 0$. Then we define

$$G_j(t,s;n) := U_0(s + j\tfrac{(t-s)}{n}, s + (j-1)\tfrac{(t-s)}{n})e^{-\frac{(t-s)}{n}B\left(s+(j-1)\frac{(t-s)}{n}\right)}$$

where $j \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, $(t,s) \in \Delta$, and we set

$$V_n(t,s) := \prod_{j=1}^{n\,\leftarrow} G_j(t,s;n), \quad n \in \mathbb{N}, \quad (t,s) \in \Delta,$$

where the product is increasingly ordered in j from the right to the left. Then a straightforward computation shows that the representation

$$\left(\left(e^{-\tau\mathcal{K}_0/n}e^{-\tau\mathcal{B}/n}\right)^n f\right)(t) = V_n(t, t-\tau)\chi_{\mathcal{I}}(t-\tau)f(t-\tau), \tag{2.1}$$

$f \in L^p(\mathcal{I}, X)$, holds for each $\tau \geq 0$ and a.e. $t \in \mathcal{I}$.

Proposition 2.1. *Let $\mathcal{K}$ and $\mathcal{K}_0$ be generators of evolution semigroups on the Banach space $L^p(\mathcal{I}, X)$ for some $p \in [1,\infty)$. Further, let $\{B(t) \in \mathcal{G}(1,0)\}_{t\in\mathcal{I}}$ be a strongly measurable family of generators of contraction semigroups on X. Then*

$$\begin{aligned}\sup_{\tau\geq 0}\left\|e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n}e^{-\tau\mathcal{B}/n}\right)^n\right\|_{\mathcal{B}(L^p(\mathcal{I},X))} \\ = \operatorname*{ess\,sup}_{(t,s)\in\Delta}\|U(t,s) - V_n(t,s)\|_{\mathcal{B}(X)}, \quad n \in \mathbb{N}.\end{aligned} \tag{2.2}$$

Proof. Let $\{L(\tau)\}_{\tau\geq 0}$ be the left-shift semigroup on the Banach space $\mathfrak{X} = L^p(\mathcal{I}, X)$:

$$(L(\tau)f)(t) = \chi_{\mathcal{I}}(t+\tau)f(t+\tau), \quad f \in L^p(\mathcal{I}, X).$$

Using that we get

$$\begin{aligned}&\left(L(\tau)\left(e^{-\tau\mathcal{K}} - \left(e^{-\tau/n\mathcal{K}_0}e^{-\tau\mathcal{B}/n}\right)^n\right)f\right)(t)\\ &\quad= \{U(t+\tau,t) - V_n(t+\tau,t)\}\,\chi_{\mathcal{I}}(t+\tau)f(t),\end{aligned}$$

for $\tau \geq 0$ and a.e. $t \in \mathcal{I}$. It turns out that for each $n \in \mathbb{N}$ the operator $L(\tau)\left(e^{-\tau\mathcal{K}} - \left(e^{-\tau/n\mathcal{K}_0}e^{-\tau\mathcal{B}/n}\right)^n\right)$ is a multiplication operator induced by $\{(U(t+\tau,t) - V_n(t+\tau,t))\chi_{\mathcal{I}}(t+\tau)\}_{t\in\mathcal{I}}$. Therefore,

$$\begin{aligned}&\left\|L(\tau)\left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n}e^{-\tau\mathcal{B}/n}\right)^n\right)\right\|_{\mathcal{B}(\mathfrak{X})}\\ &\quad= \operatorname*{ess\,sup}_{t\in\mathcal{I}}\|U(t+\tau,t) - V_n(t+\tau,t)\|_{\mathcal{B}(X)}\chi_{\mathcal{I}}(t+\tau),\end{aligned}$$

for each $\tau \geq 0$. Note that one has

$$\begin{aligned}&\sup_{\tau\geq 0}\left\|L(\tau)\left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n}e^{-\tau\mathcal{B}/n}\right)^n\right)\right\|_{\mathcal{B}(\mathfrak{X})}\\ &\quad= \operatorname*{ess\,sup}_{\tau\geq 0}\left\|L(\tau)\left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n}e^{-\tau\mathcal{B}/n}\right)^n\right)\right\|_{\mathcal{B}(\mathfrak{X})}.\end{aligned}$$

This is based on the fact that if $F(\cdot) : \mathbb{R}_+ \longrightarrow \mathcal{B}(\mathfrak{X})$ is strongly continuous, then $\sup_{\tau\geq 0}\|F(\tau)\|_{\mathcal{B}(\mathfrak{X})} = \operatorname*{ess\,sup}_{\tau\geq 0}\|F(\tau)\|_{\mathcal{B}(\mathfrak{X})}$. Hence, we find

$$\begin{aligned}&\sup_{\tau\geq 0}\left\|L(\tau)\left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n}e^{-\tau\mathcal{B}/n}\right)^n\right)\right\|_{\mathcal{B}(\mathfrak{X})}\\ &\quad= \operatorname*{ess\,sup}_{\tau\geq 0}\operatorname*{ess\,sup}_{t\in\mathcal{I}}\|U(t+\tau,t) - V_n(t+\tau,t))\|_{\mathcal{B}(X)}\chi_{\mathcal{I}}(t+\tau).\end{aligned}$$

Further, if $\Phi(\cdot,\cdot) : \mathbb{R}_+ \times \mathcal{I} \longrightarrow \mathcal{B}(X)$ is a strongly measurable function, then

$$\operatorname*{ess\,sup}_{(\tau,t)\in\mathbb{R}_+\times\mathcal{I}}\|\Phi(\tau,t)\|_{\mathcal{B}(X)} = \operatorname*{ess\,sup}_{\tau\geq 0}\operatorname*{ess\,sup}_{t\in\mathcal{I}}\|\Phi(\tau,t)\|_{\mathcal{B}(X)}.$$

Then, taking into account two last equalities, one obtains

$$\begin{aligned}&\sup_{\tau\geq 0}\left\|L(\tau)\left(e^{-\tau\mathcal{K}} - \left(e^{-\tau\mathcal{K}_0/n}e^{-\tau\mathcal{B}/n}\right)^n\right)\right\|_{\mathcal{B}(\mathfrak{X})}\\ &\quad= \operatorname*{ess\,sup}_{(\tau,t)\in\mathbb{R}_+\times\mathcal{I}}\|U(t+\tau,t) - V_n(t+\tau,t)\|_{\mathcal{B}(X)}\chi_{\mathcal{I}}(t+\tau) =\\ &\quad= \operatorname*{ess\,sup}_{(t,s)\in\Delta}\|U(t,s) - V_n(t,s)\|_{\mathcal{B}(X)},\end{aligned}$$

that proves (2.2) $\square$

3. Bounded Perturbations of the Shift Semigroup Generator

3.1. Basic Facts

We study bounded perturbations of the evolution generator D_0 (1.6). To do this aim we consider $\mathcal{I} = [0,1]$, $X = \mathbb{C}$ and we denote by $L^p(\mathcal{I})$ the Banach space $L^p(\mathcal{I}, \mathbb{C})$.

For $t \in \mathcal{I}$, let $q : t \mapsto q(t) \in L^\infty(\mathcal{I})$. Then, q induces a bounded multiplication operator Q on the Banach space $L^p(\mathcal{I})$:

$$(Qf)(t) = q(t)f(t), \;\; f \in L^p(\mathcal{I}).$$

For simplicity we assume that $q \geq 0$. Then Q generates on $L^p(\mathcal{I})$ a contraction semigroup $\{e^{-\tau Q}\}_{\tau \geq 0}$. Since generator Q is bounded, the closed operator $\mathcal{A} := D_0 + Q$, with domain $\mathrm{dom}(\mathcal{A}) = \mathrm{dom}(D_0)$, is generator of a semigroup on $L^p(\mathcal{I})$. By [7], the Trotter product formula in the strong topology follows immediately

$$\left(e^{-\tau D_0/n} e^{-\tau Q/n}\right)^n f \to e^{-\tau(D_0+Q)} f, \quad f \in L^p(\mathcal{I}), \tag{3.1}$$

uniformly in $\tau \in [0, T]$ on bounded time intervals.

Following [2, §5], we define on $X = \mathbb{C}$ a family of bounded operators $\{V(t)\}_{t \in \mathcal{I}}$ by

$$V(t) := e^{-\int_0^t ds q(s)}.$$

Note that for almost every $t \in \mathcal{I}$ these operators are positive. Then $V^{-1}(t)$ exists and it has the form

$$V^{-1}(t) = e^{\int_0^t ds q(s)}.$$

The operator families $\{V(t)\}_{t \in \mathcal{I}}$ and $\{V^{-1}(t)\}_{t \in \mathcal{I}}$ induce two bounded multiplication operators $\mathcal{V}$ and $\mathcal{V}^{-1}$ on $L^p(\mathcal{I})$, respectively. Then invertibility implies that $\mathcal{V}\,\mathcal{V}^{-1} = \mathcal{V}^{-1}\,\mathcal{V} = I|_{L^p}$. Using the operator $\mathcal{V}$ one easily verifies that $D_0 + Q$ is similar to D_0, i.e. one has

$$\mathcal{V}^{-1}(D_0 + Q)\mathcal{V} = D_0, \text{ or } D_0 + Q = \mathcal{V} D_0 \mathcal{V}^{-1}.$$

Hence, the semigroup generated on $L^p(\mathcal{I})$ by $D_0 + Q$ gets the explicit form:

$$\begin{aligned} \left(e^{-\tau(D_0+Q)} f\right)(t) &= \left(\mathcal{V} e^{-\tau D_0} \mathcal{V}^{-1} f\right)(t) \\ &= e^{-\int_{t-\tau}^t q(y)dy} f(t-\tau)\chi_{\mathcal{I}}(t-\tau). \end{aligned} \tag{3.2}$$

Since by (1.5) the propagator $U(t,s)$ that corresponds to evolution semigroup (3.2) is defined by

$$\left(e^{-\tau(D_0+Q)}\right) f(t) = U(t, t-\tau) f(t-\tau)\chi_{\mathcal{I}}(t-\tau),$$

we deduce that it is equal to $U(t,s) = e^{-\int_s^t dy\, q(y)}$.

Now we study the corresponding Trotter product formula. For a fixed $\tau \geq 0$ and $n \in \mathbb{N}$, we define approximation V_n by

$$\left(\left(e^{-\tau D_0/n} e^{-\tau Q/n}\right)^n f\right)(t) =: V_n(t, t-\tau)\chi_{\mathcal{I}}(t-\tau) f(t-\tau).$$

Then by straightforward calculations, similar to (2.1), one finds that

$$V_n(t,s) = e^{-\frac{t-s}{n}\sum_{k=0}^{n-1} q(s+k\frac{t-s}{n})}, \quad (t,s) \in \Delta.$$

Proposition 3.1. *Let $q \in L^\infty(\mathcal{I})$ be non-negative. Then*

$$\sup_{\tau\geq 0}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathcal{B}(L^p(\mathcal{I}))}$$
$$= \Theta\left(\operatorname*{ess\,sup}_{(t,s)\in\Delta}\left|\int_s^t q(y)dy - \frac{t-s}{n}\sum_{k=0}^{n-1} q(s+k\tfrac{t-s}{n})\right|\right)$$

as $n \to \infty$, where Θ is the Landau symbol defined in Sect. 1.

Proof. First, by Proposition 2.1 and by $U(t,s) = e^{-\int_s^t dy\, q(y)}$ we obtain

$$\sup_{\tau\geq 0}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathcal{B}(L^p(\mathcal{I}))}$$
$$= \operatorname*{ess\,sup}_{(t,s)\in\Delta}\left|e^{-\int_s^t dy\, q(y)} - e^{-\frac{t-s}{n}\sum_{k=0}^{n-1} q(s+k\frac{t-s}{n})}\right|. \tag{3.3}$$

Then, using the inequality

$$e^{-\max\{x,y\}}|x-y| \leq |e^{-x} - e^{-y}| \leq |x-y|, \quad 0 \leq x, y,$$

for $0 \leq s < t \leq 1$ one finds the estimates

$$e^{-\|q\|_{L^\infty}} R_n(t,s;q) \leq$$
$$\left|e^{-\int_s^t dy\, q(y)} - e^{-\frac{t-s}{n}\sum_{k=0}^{n-1} q(s+k\frac{t-s}{n})}\right| \leq R_n(t,s;q),$$

where

$$R_n(t,s,q) := \left|\int_s^t dy\, q(y) - \frac{t-s}{n}\sum_{k=0}^{n-1} q(s+k\tfrac{t-s}{n})\right|, \quad (t,s) \in \Delta. \tag{3.4}$$

Hence, for the left-hand side of (3.3) we get the estimate

$$e^{-\|q\|_{L^\infty}} R_n(q) \leq \sup_{\tau\geq 0}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathcal{B}(L^p)} \leq R_n(q),$$

where $R_n(q) := \operatorname*{ess\,sup}_{(t,s)\in\Delta} R_n(t,s;q)$, $n \in \mathbb{N}$. These estimates together with definition of Θ prove the assertion. □

Note that by virtue of (3.4) and Proposition 3.1 the operator-norm convergence rate of the Trotter product formula for the pair $\{D_0, Q\}$ coincides with the convergence rate of the integral Darboux–Riemann sum approximation of the Lebesgue integral.

3.2. Examples

First we consider the case of a real Hölder-continuous function $q \in C^{0,\beta}(\mathcal{I})$.

Theorem 3.2. *If $q \in C^{0,\beta}(\mathcal{I})$ is non-negative, then*

$$\sup_{\tau\geq 0}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\| = O(1/n^\beta),$$

as $n \to \infty$.

Proof. One has

$$\int_s^t dy\, q(y) - \frac{t-s}{n}\sum_k^{n-1} q(s+\tfrac{k}{n}(t-s))$$
$$= \sum_{k=0}^{n-1} \int_{\frac{k}{n}(t-s)}^{\frac{k+1}{n}(t-s)} dy\,\left(q(s+y) - q(s+\tfrac{k}{n}(t-s))\right),$$

which yields the estimate

$$\Big|\int_s^t dy\, q(y) - \frac{t-s}{n}\sum_k^{n-1} q(s+\tfrac{k}{n}(t-s))\Big|$$
$$\le \sum_{k=0}^{n-1} \int_{\frac{k}{n}(t-s)}^{\frac{k+1}{n}(t-s)} dy\,\left|q(s+y) - q(s+\tfrac{k}{n}(t-s))\right|.$$

Since $q \in C^{0,\beta}(\mathcal{I})$, there is a constant $L_\beta > 0$ such that for $y \in [\frac{k}{n}(t-s), \frac{k+1}{n}(t-s)]$ one has

$$\left|q(s+y) - q(s+\tfrac{k}{n}(t-s)\right| \le L_\beta|y - \tfrac{k}{n}(t-s)|^\beta \le L_\beta \frac{(t-s)^\beta}{n^\beta}.$$

Hence, we find

$$\Big|\int_s^t q(y)dy - \frac{t-s}{n}\sum_k^{n-1} q(s+\tfrac{k}{n}(t-s))\Big| \le L_\beta \frac{(t-s)^{1+\beta}}{n^\beta} \le L_\beta \frac{1}{n^\beta},$$

which proves

$$\operatorname*{ess\,sup}_{(t,s)\in\Delta} \Big|\int_s^t q(y)dy - \frac{t-s}{n}\sum_k^{n-1} q(s+\tfrac{k}{n}(t-s))\Big| = O\left(\frac{1}{n^\beta}\right).$$

Applying now Proposition 3.1 one completes the proof. □

It is a natural question: what happens, when q is only continuous?

Theorem 3.3. *If $q : \mathcal{I} \to \mathbb{C}$ is continuous and non-negative, then*

$$\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\| = o(1), \tag{3.5}$$

as $n \to \infty$.

Proof. Since $q(\cdot)$ is continuous, then for any $\varepsilon > 0$ there is $\delta > 0$ such that for $|y-x| < \delta$ we have $|q(y)-q(x)| < \varepsilon$, $y, x \in \mathcal{I}$. Therefore, if $1/n < \delta$, then for $y \in (\frac{k}{n}(t-s), \frac{k+1}{n}(t-s))$ we have

$$|q(s+y) - q(s+\tfrac{k}{n}(t-s))| < \varepsilon, \quad (t,s) \in \Delta.$$

Hence,

$$\Big|\int_s^t q(y)dy - \frac{t-s}{n}\sum_k^{n-1} q(s+\tfrac{k}{n}(t-s))\Big| \le \varepsilon(t-s) \le \varepsilon,$$

which yields

$$\operatorname*{ess\,sup}_{(t,s)\in\Delta}\Big|\int_s^t q(y)dy - \frac{t-s}{n}\sum_k^{n-1} q(s+\tfrac{k}{n}(t-s))\Big| = o(1).$$

Now it remains only to apply Proposition 3.1. □

We comment that for a general continuous q one can say nothing about the convergence rate. Indeed, it can be shown that in (3.5) the convergence to zero can be arbitrary slow.

Theorem 3.4. *Let $\delta_n > 0$ be a sequence with $\delta_n \to 0$ as $n \to \infty$. Then there exists a continuous function $q : \mathcal{I} = [0,1] \to \mathbb{R}$ such that*

$$\sup_{\tau\geq 0}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathcal{B}(L^p(\mathcal{I}))} = \omega(\delta_n) \tag{3.6}$$

as $n \to \infty$, where ω is the Landau symbol defined in Sect. 1.

Proof. Taking into account Theorem 6 of [8], we find that for any sequence $\{\delta_n\}_{n\in\mathbb{N}}$, $\delta_n > 0$ satisfying $\lim_{n\to\infty}\delta_n = 0$ there exists a continuous function $f(\cdot) : [0, 2\pi] \longrightarrow \mathbb{R}$ such that

$$\left|\int_0^{2\pi} f(x)\,dx - \frac{2\pi}{n}\sum_{k=1}^{n} f(2k\pi/n)\right| = \omega(\delta_n),$$

as $n \to \infty$. Setting $q(y) := f(2\pi(1-y))$, $y \in [0,1]$, we get a continuous function $q(\cdot) : [0,1] \longrightarrow \mathbb{R}$, such that

$$\left|\int_0^1 q(y)dy - \frac{1}{n}\sum_{k=0}^{n-1} q(k/n)\right| = \omega(\delta_n).$$

Because $q(\cdot)$ is continuous we find

$$\begin{aligned}\operatorname*{ess\,sup}_{(t,s)\in\Delta}\Big|\int_s^t q(y)\,dy - \frac{t-s}{n}\sum_{n=0}^{n-1} q(s+k\tfrac{t-s}{n})\Big| \\ \geq \Big|\int_0^1 q(y)\,dy - \frac{1}{n}\sum_{k=0}^{n-1} q(k/n)\Big|,\end{aligned}$$

which yields

$$\operatorname*{ess\,sup}_{(t,s)\in\Delta}\Big|\int_s^t q(y)\,dy - \frac{t-s}{n}\sum_{n=0}^{n-1} q(s+k\tfrac{t-s}{n})\Big| = \omega(\delta_n).$$

Applying now Proposition 3.1 we prove (3.6). □

Our final comment concerns the case when q is only *measurable*. Then it can happen that the Trotter product formula for that pair $\{D_0, Q\}$ does not converge in the operator-norm topology.

Theorem 3.5. *There is a non-negative function $q \in L^\infty([0,1])$ such that*

$$\limsup_{n\to\infty} \sup_{\tau\geq 0} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n} e^{-\tau Q/n}\right)^n \right\|_{\mathcal{B}(L^p(\mathcal{I}))} > 0. \tag{3.7}$$

Proof. Let us introduce the open intervals

$$\begin{aligned} \Delta_{0,n} &:= (0, \tfrac{1}{2^{2n+2}}), \\ \Delta_{k,n} &:= (t_{k,n} - \tfrac{1}{2^{2n+2}}, t_{k,n} + \tfrac{1}{2^{2n+2}}), \quad k = 1,2,\ldots,2^n-1, \\ \Delta_{2^n,n} &:= (1 - \tfrac{1}{2^{2n+2}}, 1), \end{aligned}$$

$n \in \mathbb{N}$, where

$$t_{k,n} = \frac{k}{2^n}, \quad k = 0,\ldots,n, \quad n \in \mathbb{N}.$$

Notice that $t_{0,n} = 0$ and $t_{2^n,n} = 1$. One easily checks that the intervals $\Delta_{k,n}$, $k = 0,\ldots,2^n$, are mutually disjoint. We introduce the open sets

$$\mathcal{O}_n = \bigcup_{k=0}^{2^n} \Delta_{k,n} \subseteq \mathcal{I}, \quad n \in \mathbb{N}.$$

and

$$\mathcal{O} = \bigcup_{n\in\mathbb{N}} \mathcal{O}_n \subseteq \mathcal{I}.$$

Then it is clear that

$$|\mathcal{O}_n| = \frac{1}{2^{n+1}}, \quad n \in \mathbb{N}, \quad \text{and} \quad |\mathcal{O}| \leq \frac{1}{2}.$$

Therefore, the Lebesgue measure of the closed set $\mathcal{C} := \mathcal{I} \setminus \mathcal{O} \subseteq \mathcal{I}$ can be estimated by

$$|\mathcal{C}| \geq \frac{1}{2}.$$

Using the characteristic function $\chi_{\mathcal{C}}(\cdot)$ of the set $\mathcal{C}$ we define

$$q(t) := \chi_{\mathcal{C}}(t), \quad t \in \mathcal{I}.$$

The function $q(\cdot)$ is measurable and it satisfies $0 \leq q(t) \leq 1$, $t \in \mathcal{I}$.

Let $\varepsilon \in (0,1)$. We choose $s \in (0,\varepsilon)$ and $t \in (1-\varepsilon, 1)$ and we set

$$\xi_{k,n}(t,s) := s + k\,\frac{t-s}{2^n}, \quad k = 0,\ldots,2^n-1, \quad n \in \mathbb{N}, \quad (t,s) \in \Delta.$$

Note that $\xi_{k,n}(t,s) \in (0,1)$, $k = 0,\ldots,2^n-1$, $n \in \mathbb{N}$. Moreover, we have

$$t_{k,n} - \xi_{k,n}(t,s) = k\frac{1}{2^n} - s - k\,\frac{t-s}{2^n} = k\frac{1-t+s}{2^n} - s,$$

which leads to the estimate

$$|t_{k,n} - \xi_{k,n}(t,s)| \leq \varepsilon(\frac{k}{2^{n-1}} + 1), \quad k = 0,\ldots,2^n-1, \quad n \in \mathbb{N}.$$

Hence

$$|t_{k,n} - \xi_{k,n}(t,s)| \leq 3\varepsilon, \quad k = 0,\ldots,2^n-1, \quad n \in \mathbb{N}.$$

Let $\varepsilon_n := 1/(3 \cdot 2^{2n+2})$ for $n \in \mathbb{N}$. Then we get that $\xi_{k,n}(t,s) \in \Delta_{k,n}$ for $k = 0,\ldots,2^n-1$, $n \in \mathbb{N}$, $s \in (0,\varepsilon_n)$ and for $t \in (1-\varepsilon_n, 1)$.

Now let

$$S_n(t,s;q) := \frac{t-s}{n} \sum_{k=0}^{n-1} q(s + k\tfrac{t-s}{n}), \quad n \in \mathbb{N}, \quad (t,s) \in \Delta.$$

We consider

$$S_{2^n}(t,s;q) = \frac{t-s}{n} \sum_{k=0}^{2^n-1} q(s + k\tfrac{t-s}{2^n}) = \frac{t-s}{n} \sum_{k=0}^{2^n-1} q(\xi_{k,n}(t,s)),$$

$n \in \mathbb{N}$, $(t,s) \in \Delta$. If $s \in (0, \varepsilon_n)$ and $t \in (1-\varepsilon_n, 1)$, then $S_{2^n}(t,s;q) = 0$, $n \in \mathbb{N}$ and

$$\left| \int_s^t q(y)\,dy - S_{2^n}(t,s;q) \right| = \int_s^t q(y)dy, \quad n \in \mathbb{N},$$

for $s \in (0, \varepsilon_n)$ and $t \in (1-\varepsilon_n, 1)$. In particular, this yields

$$\operatorname*{ess\,sup}_{(t,s)\in\Delta} \left| \int_s^t q(y)dy - S_{2^n}(t,s;q) \right| \geq \operatorname*{ess\,sup}_{(t,s)\in\Delta} \int_s^t q(y)dy \geq \int_{\mathcal{I}} \chi_{\mathcal{C}}(y)dy \geq \frac{1}{2}.$$

Hence, we obtain

$$\limsup_{n\to\infty} \operatorname*{ess\,sup}_{(t,s)\in\Delta} \left| \int_s^t q(y)dy - S_{2^n}(t,s;q) \right| \geq \frac{1}{2},$$

and applying Proposition 3.1 we finish the prove of (3.7). □

We note that Theorem 3.5 does not exclude the convergence of the Trotter product formula for the pair $\{D_0, Q\}$ in the *strong* operator topology. Examples of this dichotomy are known for the Trotter-Kato product formula in Hilbert spaces [3]. By virtue of (3.1) and (3.7), Theorem 3.5 yields an example of this dichotomy in Banach spaces.

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