**Integral Equations
and Operator Theory**

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Sharp Spectral Estimates for the Perturbed Landau Hamiltonian with L^p Potentials

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Abstract. We establish a sharp uniform estimate on the size of the spectral clusters of the Landau Hamiltonian with (possibly complex-valued) L^p potentials as the cluster index tends to infinity.

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1. Introduction

Consider the Hamiltonian $H_{0,\perp}$ of a particle confined to a two-dimensional surface in a constant magnetic field $B = (0, 0, 1)$, perpendicular to the surface. In the symmetric gauge, the so-called Landau Hamiltonian is given by

$$H_{0,\perp} = \left(-\mathrm{i}\frac{\partial}{\partial x} + \frac{y}{2}\right)^2 + \left(-\mathrm{i}\frac{\partial}{\partial y} - \frac{x}{2}\right)^2, \quad (x, y) \in \mathbb{R}^2. \tag{1.1}$$

The spectrum of $H_{0,\perp}$ is pure point, consisting of eigenvalues of infinite multiplicity, given by

$$\sigma(H_{0,\perp}) = \{\lambda_k = 2k + 1 : k \in \mathbb{N}_0\},$$

and each eigenvalue λ_k (called the kth Landau level) is of infinite multiplicity. We are interested in the spectrum of the perturbed Landau Hamiltonian

$$H = H_{0,\perp} + V \tag{1.2}$$

where $V \in L^p(\mathbb{R}^2)$ is a (possibly complex-valued) potential. In particular, we derive sharp bounds (depending only on the L^p-norm of V) on the location of the spectrum of H lying in the kth spectral cluster

$$\Lambda_k = \{z \in \mathbb{C} : |\lambda_k - \mathrm{Re}z| \leq 1\} \tag{1.3}$$

as $k \to \infty$. In some cases, sharp spectral bounds are known. For example, if $V \in L^\infty(\mathbb{R}^2)$, then by standard perturbation theory

$$\sigma(H) \cap \Lambda_k \subset \{z \in \mathbb{C} : |\lambda_k - z| \leq \|V\|_\infty\}. \tag{1.4}$$

On the other hand, if V is a real-valued smooth function of compact support, then it was shown in [20] that there exists a constant C, depending on V, such that

$$\sigma(H) \cap \Lambda_k \subset [\lambda_k - C\lambda_k^{-1/2}, \lambda_k + C\lambda_k^{-1/2}] \qquad (1.5)$$

for k sufficiently large. We emphasize here that even though (1.5) is much more precise than (1.4), the latter gives a *uniform* bound for a whole *class of potentials*, not just for a single potential.

The estimate (1.5) was later proven in [23] by different techniques and for more general potentials, namely for continuous V such that $\sup_{x \in \mathbb{R}^2}(1 + x^2)^{\rho/2}|V(x)| < \infty$ for some $\rho > 1$. The $O(\lambda_k^{-1/2})$—decay for the size of the kth cluster is sharp, i.e. the eigenvalue clusters have size $\geq c\lambda_k^{-1/2}$ for some $c > 0$ unless $V \equiv 0$. For the long-range case $\rho < 1$, the optimal decay rate of the spectral cluster size is $\mathcal{O}(\lambda_k^{-\rho/2})$ [22]. In both cases, a trace formula for the asymptotic distribution of eigenvalues in the kth cluster, as k tends to infinity, is obtained. These results admit a semiclassical interpretation in terms of the so-called 'averaging principle' according to which a good approximation is obtained by replacing the perturbation by its average along the orbits of the free dynamics.

In this article, we will prove *upper bounds* on the size of spectral clusters under perturbations by *rough potentials*, i.e. V merely in L^p for some $p < \infty$. It is cetrainly beyond the scope of our techniques to establish asymptotics of the eigenvalue distribution; in fact, our results indicate that the L^p scale is not the appropriate one for this problem since the size of spectral clusters tends to zero slower than in (1.5). On the other hand, our bounds depend only on the L^p norm of the potential, i.e. they are uniform for the whole class of L^p potentials, and *in this sense they are sharp*.

Our motivation for considering spectral cluster bounds originally came from its potential application to Lieb-Thirring type estimates for complex-valued perturbations of the Landau Hamiltonian. Non-selfadjoint differential operators have received considerable attention in recent years and play an important role in contemparary mathematical physics, e.g. in optical model of nuclear scattering, the analysis of resonances using complex scaling or the scattering of atoms by periodic electric fields. We refer to [3, 4, 28] for many more applications and motivations.

From a mathematical point of view, it is an interesting and challenging problem to find suitable tools that are robust and powerful enough to be useful in the non-selfadjoint case. For instance, Sobolev inequalities can be used to control eigenvalues of the Schrödinger operator $-\Delta + V$ lying outside some fixed conic neighborhood of the essential spectrum, see e.g. [5, 6, 10, 25]. A fundamental insight of Safronov [21] was that methods from stationary scattering theory could be used to bound eigenvalues of $-\Delta + V$ close to the essential spectrum. The paper [9] of Frank was the first where much deeper, uniform Sobolev inequalities (due to Kenig, Ruiz and Sogge [14]) were applied in connection with eigenvalue inequalities for complex-valued potentials. Similar uniform estimates were subsequently established in [1, 8].

   The principal aim of this note is to prove sharp estimates on the size of the spectral clusters that depend only on an L^p-norm of V. Our main result (Theorem 2.1) holds more generally in all even dimensions. As in the case of [9], our result depends on a uniform resolvent estimate; here we use the sharp spectral projection estimates of Koch and Ricci [16], which are in turn based on dispersive estimates of Koch and Tataru [18]. We also prove a weaker version of our main result in the odd-dimensional case (Theorem 3.1), that holds under the extra assumption of weak coupling. As an extension of the method used there, we give a new proof of a unique continuation theorem (Theorem 4.1), based on Carleman estimates with linear weights.

   We close this introduction with some remarks about the Schrödinger operator with constant magnetic field in higher dimensions. It is well known that in this case the magnetic-field 2—form can be identified with an anti-symmetric $d \times d$ matrix B. Assuming that $B \neq 0$, set $2l := \dim \operatorname{Ran} B \in 2\mathbb{N}$, $m := \dim \operatorname{Ker} B \in \mathbb{N}_0$, so that $d = 2l + m$. Then there exist real numbers $b_1 \geq \cdots \geq b_l > 0$ such that the nonzero eigenvalues of B, counted with multiplicities, coincide with $\{\pm b_j\}_{j=1}^l$. In the present article we will only consider the special cases $m = 0, 1$ and $b_1 = \cdots = b_l = 1$ for any $l \in \mathbb{N}$. The physically relevant cases $d = 2$ or $d = 3$ are included.

   Notation We will use the standard notation $a \lesssim b$ if there exists a non-negative constant C such that $a \leq Cb$. If we want to emphasize the dependence of the constant on some parameter s, we write $a \lesssim_s b$. If $a \lesssim b \lesssim a$, we write $a \approx b$. The natural numbers are denoted by $\mathbb{N} = \{1, 2, \ldots\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We write the norm of a bounded linear operator $T : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ as $\|T\|_{L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)}$. If $p = q = 2$, we omit the subscript and simply write $\|T\|$. The space of all bounded linear operators between two normed spaces X, Y is denoted by $\mathcal{B}(X, Y)$. The spectrum of a closed operator T is denoted by $\sigma(T)$. The essential spectrum is defined by $\sigma_e(T) = \{z \in \mathbb{C} : T - z \text{ is not Fredholm}\}$ and $\sigma_d(T)$ denotes the discrete spectrum, i.e. the set of isolated eigenvalues of finite multiplicity. We occasionally use the Japanese bracket, defined as $\langle x \rangle := (1 + |x|^2)^{1/2}$.

2. Spectral Cluster Estimates in Even Dimensions

In $d = 2n$ dimensions, we consider the generalization of (2.1), i.e.

$$H_{0,\perp} = \sum_{j=1}^n \left(-i\frac{\partial}{\partial x_j} + \frac{y_j}{2} \right)^2 + \left(-i\frac{\partial}{\partial y_j} - \frac{x_j}{2} \right)^2, \quad (x, y) \in \mathbb{R}^{2n}. \quad (2.1)$$

Its eigenvalues are given by $\lambda_k := 2k + n$, $k \in \mathbb{N}_0$.

   Suppose $V \in L^r(\mathbb{R}^d)$ is a (possibly complex-valued) potential with $r \in [d/2, \infty]$ if $d > 2$ and $r \in (1, \infty]$ if $d = 2$. The perturbed operator (defined in the sense of sectorial forms) then satisfies the spectral estimate [2, Theorem 5.1]

$$\sigma(H) \subset \left\{ z \in \mathbb{C} : |\operatorname{Im} z|^{1 - \frac{d}{2r}} \lesssim 1 + \|V\|_{L^r(\mathbb{R}^d)} \right\}.$$

In particular, if $r > d/2$, this implies that $|\mathrm{Im}\,z| \lesssim 1$ for all $z \in \sigma(H)$. Our main result is the following refinement.

Theorem 2.1. *Let $d \in 2\mathbb{N}$, and let $V \in L^r(\mathbb{R}^d)$ with $r \in [d/2, \infty]$ if $d > 2$ and $r \in (1, \infty]$ if $d = 2$. Then for all $k \in \mathbb{N}$ with $C\|V\|_r \lambda_k^\rho < 1/2$ (C being the constant in (2.5)) we have*

$$\sigma(H) \cap \Lambda_k \subset \{z \in \mathbb{C} : \delta(z) \lesssim \|V\|_{L^r(\mathbb{R}^d)} \lambda_k^{\nu(r)}\}, \qquad (2.2)$$

where $\delta(z) := \mathrm{dist}(z, \sigma(H_{0,\perp}))$ and

$$\nu(r) := \begin{cases} \frac{d}{2r} - 1 & \text{if } \frac{d}{2} \leq r \leq \frac{d+1}{2}, \\ -\frac{1}{2r} & \text{if } \frac{d+1}{2} \leq r \leq \infty. \end{cases} \qquad (2.3)$$

Moreover, the estimate is sharp in the following sense: For every k as above there exists $V \in L^r(\mathbb{R}^d)$, real-valued and $V \leq 0$, such that

$$\sigma(H) \cap \{z \in \mathbb{R} : |z - \lambda_k| \approx \|V\|_{L^r(\mathbb{R}^d)} \lambda_k^{\nu(r)}\} \neq \emptyset. \qquad (2.4)$$

The proof of Theorem 2.1 will be based on the following proposition, which is the main technical result. In the following, $R_{0,\perp}$ denotes the resolvent of $H_{0,\perp}$.

Proposition 2.2. *Let $d \in 2\mathbb{N}$, $q \in [2, 2d/(d-2)]$ if $d > 2$ and $q \in [2, \infty)$ if $d = 2$. Assume $z \in \Lambda_{k_0} \cap \rho(H_{0,\perp})$. Then there is a constant $C > 0$ such that for all $z \in \rho(H_{0,\perp})$, we have*

$$\|R_{0,\perp}(z)\|_{L^{q'}(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \leq C(1 + |\mathrm{Re}\,z|)^{\rho(q)}(1 + \delta(z)^{-1}), \qquad (2.5)$$

where

$$\rho(q) := \begin{cases} \frac{1}{q} - \frac{1}{2} & \text{if } 2 \leq q \leq \frac{2(d+1)}{d-1}, \\ \frac{d-2}{2} - \frac{d}{q} & \text{if } \frac{2(d+1)}{d-1} \leq q \leq \infty. \end{cases} \qquad (2.6)$$

Proof. 1. We first consider the easy spectral region: $\mathrm{Re}\,z \leq 2$ or $|\mathrm{Im}\,z| \geq \gamma \mathrm{Re}\,z$, $\mathrm{Re}\,z > 2$. Here, $\gamma > 0$ is arbitrarily small, but fixed. We first notice that a straightforward application of the diamagnetic inequality, together with the Sobolev embedding $H^1 \hookrightarrow L^q$ (and its dual) and a scaling argument yields

$$\|R_{0,\perp}(-1 - |z|)\|_{L^{q'}(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \leq \|(-\Delta + 1 + |z|)^{-1}\|_{L^{q'}(\mathbb{R}^d) \to L^q(\mathbb{R}^d)}$$
$$\lesssim (1 + |z|)^{d\left(\frac{1}{2} - \frac{1}{q}\right) - 1}.$$

This is better than (2.5). Now assume that $\mathrm{Re}\,z \leq 2$ and consider the resolvent difference $D(z) := R_{0,\perp}(z) - R_{0,\perp}(-1 - |z|)$. By what we already proved, it is sufficient to show that (2.5) holds for $D(z)$ instead of $R_{0,\perp}(z)$. We write

$$D(z) = (1 + z + |z|)(H_{0,\perp} + 1 + |z|)^{-1/2} R_{0,\perp}(z)(H_{0,\perp} + 1 + |z|)^{-1/2}. \quad (2.7)$$

Using the diamagnetic inequality for the kernel of the heat semigroup,

$$\left| e^{-tH_{0,\perp}}(x,y) \right| \leq e^{t\Delta}(x,y), \qquad (2.8)$$

and the representation

$$(H_{0,\perp} + 1 + |z|)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t(H_{0,\perp} + 1 + |z|)} t^{-1/2} \, \mathrm{d}t,$$

we obtain the following pointwise estimate for the kernel,

$$\left|(H_{0,\perp}+1+|z|)^{-1/2}(x,y)\right| \le (-\Delta+1+|z|)^{-1/2}(x,y). \qquad (2.9)$$

Then (2.7) and (2.9) yield

$$\|D(z)\|_{L^{q'}(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \le (2|z|+1)\|(-\Delta+1+|z|)^{-1/2}\|^2_{L^2\to L^q}\delta(z)^{-1}$$

$$\lesssim (1+|z|)^{d\left(\frac{1}{2}-\frac{1}{q}\right)-1}(1+\delta(z)^{-1}).$$

Here, we used Sobolev embedding (and its dual) again, as well as the identity

$$\|R_{0,\perp}(z)\|_{L^2\to L^2} = \delta(z)^{-1}.$$

The case $|\mathrm{Im}z| \ge \gamma\mathrm{Re}z$, $\mathrm{Re}z > 2$, is similar; here we estimate

$$\|D(z)\|_{L^{q'}(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \lesssim \frac{|z|^{d\left(\frac{1}{2}-\frac{1}{q}\right)-1}}{\sin\gamma},$$

which is again better than (2.5).

2. We now consider the nontrivial spectral region: $\mathrm{Re}z > 2$ and $|\mathrm{Im}z| < \gamma\mathrm{Re}z$. We first prove the bound

$$\|R_{0,\perp}(z)\|_{L^{q'}(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim (\mathrm{Re}z)^{\rho(q)/2}(1+\delta(z)^{-1}). \qquad (2.10)$$

Let P_k be the spectral projection onto the eigenspace corresponding to λ_k. The dual version of the spectral projection estimates in [16] reads

$$\|P_k\|_{L^{q'}(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim \lambda_k^{\rho(q)/2}, \qquad (2.11)$$

By orthogonality of the spectral projections, we have

$$\|R_{0,\perp}(z)f\|^2_{L^2(\mathbb{R}^d)} = \sum_{k=0}^{\infty}\frac{1}{|z-\lambda_k|^2}\|P_kf\|^2_{L^2(\mathbb{R}^d)} \lesssim \|f\|^2_{L^{q'}(\mathbb{R}^d)}\sum_{k=0}^{\infty}\frac{\lambda_k^{\rho(q)}}{|z-\lambda_k|^2}$$

$$\lesssim (\mathrm{Re}z)^{\rho(q)}\left(1+\delta(z)^{-1}\right)^2\|f\|^2_{L^{q'}(\mathbb{R}^d)}.$$

This proves (2.10). In the last step we used the following estimates for the sum over k:

$$\sum_{\lambda_k\le\mathrm{Re}z/2}\frac{\lambda_k^{\rho(q)}}{|z-\lambda_k|^2} + \sum_{\lambda_k\ge 2\mathrm{Re}z}\frac{\lambda_k^{\rho(q)}}{|z-\lambda_k|^2} \lesssim (\mathrm{Re}z)^{\rho(q)-1},$$

$$\sum_{\mathrm{Re}z/2<\lambda_k<\mathrm{Re}z-1}\frac{\lambda_k^{\rho(q)}}{|z-\lambda_k|^2} + \sum_{\mathrm{Re}z+1<\lambda_k<2\mathrm{Re}z}\frac{\lambda_k^{\rho(q)}}{|z-\lambda_k|^2} \lesssim (\mathrm{Re}z)^{\rho(q)}.$$

These can easily be proved by estimating the sums by integrals and changing variables appropriately.

3. Let $u \in C_0^\infty(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$. Set $f := (H_{0,\perp}-z)u$ and $\mu := \mathrm{Re}z$. Without loss of generality we may assume that $\mu > 1$. We fix $z_0 = (x_0,y_0) \in \mathbb{R}^d$ and denote the ball with center z_0 and radius r by $B_r(z_0)$. We claim that

$$\mu^{\frac{1}{2(d+1)}}\|u\|_{L^{\frac{2(d+1)}{d-1}}(B_{\sqrt{\mu}}(z_0))} \lesssim \|u\|_{L^2(B_{2\sqrt{\mu}}(z_0))} + \mu^{-\frac{1}{2(d+1)}}\|f\|_{L^{\frac{2(d+1)}{d+3}}(B_{2\sqrt{\mu}}(z_0))}$$

$$(2.12)$$

where the implicit constant is independent of u, μ and z_0. Proceeding as in [16] we set

$$\overline{x} := \frac{x - x_0}{\sqrt{\mu}}, \quad \overline{y} := \frac{y - y_0}{\sqrt{\mu}} \tag{2.13}$$

and

$$\overline{u}(\overline{x}, \overline{y}) = \mathrm{e}^{-\frac{\mathrm{i}}{2}(x_0 y - y_0 x)} u(x, y), \quad \overline{f}(\overline{x}, \overline{y}) = \mathrm{e}^{-\frac{\mathrm{i}}{2}(x_0 y - y_0 x)} f(x, y). \tag{2.14}$$

We then have

$$L_\mu \overline{u} = \mu \overline{f}, \tag{2.15}$$

where

$$L_\mu := \sum_{j=1}^n \left(-\mathrm{i}\frac{\partial}{\partial x_j} + \frac{\mu}{2} y_j \right)^2 + \left(-\mathrm{i}\frac{\partial}{\partial y_j} - \frac{\mu}{2} x_j \right)^2 - \mu^2 - \mathrm{i}(\mathrm{Im}z)\mu.$$

The operator L_μ is normal and satisfies the assumptions of [19, Theorem 7] with $\delta \approx \mu^{-1}$ there. From (2.15) and [19, Theorem 7 B)] it then follows that

$$\|\overline{u}\|_{W_\mu^{\frac{1}{d+1}, \frac{2(d+1)}{d-1}}(B_1(0))} \lesssim \mu^{1/2}\|\overline{u}\|_{L^2(B_2(0))} + \mu\|\overline{f}\|_{W_\mu^{-\frac{1}{d+1}, \frac{2(d+1)}{d+3}}(B_2(0))}. \tag{2.16}$$

Here, $W_\mu^{s,p}$ is the semiclassical Sobolev space with norm

$$\|u\|_{W_\mu^{s,p}} := \|(\mu^2 - \Delta)^{s/2} u\|_{L^p}.$$

Note that in the region $\{|\xi| \lesssim \mu\}$, we have $\|u\|_{W_\mu^{s,p}} \approx \mu^s \|u\|_{L^p}$, while in the (elliptic) region $\{|\xi| \gg \mu\}$, we have $\|u\|_{W_\mu^{s,p}} \gtrsim \mu^s \|u\|_{L^p}$ for $s \geq 0$ and $\|u\|_{W_\mu^{s,p}} \lesssim \mu^s \|u\|_{L^p}$ for $s \leq 0$. These estimates follow from standard Bernstein inequalities, see e.g. [27, Appendix A]. Therefore, (2.16) implies that

$$\mu^{\frac{1}{d+1}}\|\overline{u}\|_{L^{\frac{2(d+1)}{d-1}}(B_1(0))} \lesssim \mu^{1/2}\|\overline{u}\|_{L^2(B_2(0))} + \mu\mu^{-\frac{1}{d+1}}\|\overline{f}\|_{L^{\frac{2(d+1)}{d+3}}(B_2(0))}.$$

By the change of variables (2.13)–(2.14), this is equivalent to (2.12).

4. By a covering argument (2.12) implies that

$$\mu^{\frac{1}{2(d+1)}}\|u\|_{L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d)} \lesssim \|u\|_{L^2(\mathbb{R}^d)} + \mu^{-\frac{1}{2(d+1)}}\|f\|_{L^{\frac{2(d+1)}{d+3}}(\mathbb{R}^d)}. \tag{2.17}$$

Recalling that $\mu = \lambda_{k_0} + \mathcal{O}(1)$, $f = (H_{0,\perp} - z)u$ and $\rho(q) = 1/(d+1)$ where $q = 2(d+1)/(d-1)$, and combining (2.10) with (2.17), we arrive at

$$\|u\|_{L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d)} \lesssim (\mathrm{Re}z)^{-\frac{1}{d+1}}(1 + \delta(z)^{-1})\|(H_{0,\perp} - z)u\|_{L^{\frac{2(d+1)}{d+3}}(\mathbb{R}^d)}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Since the latter is a core for $H_{0,\perp}$ ([30]), the above inequality is equivalent to (2.5) with $q = 2(d+1)/(d-1)$; see e.g. the proof of [2, Theorem C.3] for details of this argument. The general case follows by interpolation between this case and the cases $q = 2$ and $q = 2d/(d-2)$. In the former case, (2.5) is trivial. In the latter case, it follows from [2, Theorem C.3]. $\qquad\square$

Proof of Theorem 2.1. 1. Let $q = 2r'$. We then have $\nu(r) = \rho(q)$ (see (2.3) and (2.6)), and $\nu(r) \leq 0$ for $r \in [d/2, \infty]$. Moreover, in this range of r, we have $q = 2r' \in [2, 2d/(d-2)]$. Assume $z \in \sigma(H) \cap \Lambda_k$. By the Birman-Schwinger principle, Hölder's inequality and (2.5) we have

$$1 \leq \|V^{1/2} R_{0,\perp}(z)|V|^{1/2}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C\|V\|_{L^r(\mathbb{R}^d)}[1 + \delta(z)^{-1}]\lambda_k^{\rho(q)}. \tag{2.18}$$

Here we have set $V^{1/2} := |V|^{1/2} \operatorname{sgn}(V)$ with

$$\operatorname{sgn}(z) := \begin{cases} \frac{\bar{z}}{|z|} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

By assumption that $C\|V\|_{L^r(\mathbb{R}^d)}\lambda_k^{\rho(q)} \leq 1/2$, we infer from (2.18) that

$$\frac{1}{2} \leq C\|V\|_{L^r(\mathbb{R}^d)}\delta(z)^{-1}\lambda_k^{\rho(q)}.$$

This proves (2.2).

2. In order to show that the result is sharp we claim that it is sufficient to prove that for any fixed $k_0 \in \mathbb{N}$ there exists $V \in L^r(\mathbb{R}^d)$ (depending on k_0) such that $\|V\|_{L^r} = 1$, $V \leq 0$, and

$$\||V|^{1/2} P_{k_0} |V|^{1/2}\| \geq c_0 \lambda_{k_0}^{\rho(q)}. \tag{2.19}$$

Here, c_0 is some k_0-independent constant. To prove the claim, we define the Birman-Schwinger operators

$$Q(z; V) := |V|^{1/2} R_{0,\perp}(z)|V|^{1/2}$$

where $z \in \rho(H_{0,\perp})$ and $V \in L^r(\mathbb{R}^d)$ satisfies (2.19). Since

$$Q'(z; V) = |V|^{1/2} R_{0,\perp}(z)^2 |V|^{1/2} \geq 0, \quad z \in \rho(H_{0,\perp}) \cap \mathbb{R},$$

and $\|Q(z; V)\| \lesssim \lambda_{k_0}^{\rho(q)} < 1$ for $z \in [\lambda_{k_0} - 1/2, \lambda_{k_0} + 1/2] \setminus \{\lambda_{k_0}\}$ and k_0 sufficiently large, the claim will follow by a standard application of the Birman-Schwinger principle once we prove that there exists $V \in L^r(\mathbb{R}^d)$ such that the operator $Q(a; V)$, with $a := \lambda_{k_0} - \frac{1}{2}c_0\lambda_{k_0}^{\rho(q)}$, has an eigenvalue $\mu \geq 1$. We write

$$Q(a; V) = Q_0(a; V) + Q_1(a; V),$$

where

$$Q_0(z; V) := \frac{1}{\lambda_{k_0} - z}|V|^{1/2} P_{k_0} |V|^{1/2}, \quad Q_1(z; V) := \sum_{k \neq k_0} \frac{1}{\lambda_k - z}|V|^{1/2} P_k |V|^{1/2}.$$

By (2.5), we have $\|Q_1(a; V)\| = \mathcal{O}(\lambda_{k_0}^{\rho(q)})$. Moreover, since $Q_0(a; V)$ is non-negative and compact [24, Lemma 5.1], it follows that $\mu_0(V) := \|Q_0(a; V)\|$ is its largest eigenvalue. Let $\psi \in L^2(\mathbb{R}^d)$ be the corresponding normalized eigenfunction. Then

$$\|(Q(a; V) - \mu_0(V))\psi\|_2 = \|Q_1(a; V)\psi\|_2 = \mathcal{O}(\lambda_{k_0}^{\rho(q)}).$$

Since $Q(a; V)$ is selfadjoint, this implies that

$$\sigma(Q(a; V)) \cap [\mu_0(V) - \mathcal{O}(\lambda_{k_0}^{\rho(q)}), \mu_0(V) + \mathcal{O}(\lambda_{k_0}^{\rho(q)})] \neq \emptyset. \qquad (2.20)$$

Choosing $c = c_0/2$, we have by (2.19)

$$\mu_0(V) = \|Q_0(a; V)\| = \frac{1}{c}\lambda_{k_0}^{-\rho(q)}\|V^{1/2}P_{k_0}V^{1/2}\| \geq \frac{c_0}{c} = 2. \qquad (2.21)$$

It follows from (2.20) that $Q(a : V)$ has an eigenvalue $\mu \geq 1$ for k_0 sufficiently large.

3. It remains to prove the claim (2.19). We use the fact that the spectral projection estimates (2.11) are sharp. In the TT^* version, this means that

$$\|P_k\|_{L^{q'} \to L^q} \geq 2c_0\lambda_k^{\rho(q)}, \quad k \in \mathbb{N}. \qquad (2.22)$$

By Hölder's inequality and a duality argument, we have

$$\|P_k\|_{L^{q'} \to L^q} = \sup_{\|W_1\|_{L^{2r}}=\|W_2\|_{L^{2r}}=1} \|W_1P_kW_2\|. \qquad (2.23)$$

Moreover, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
|\langle W_1P_kW_2f, g\rangle| = |\langle P_kW_2f, P_kW_1g\rangle| &\leq \|P_kW_2f\|\|P_kW_1g\| \\
&= \langle W_2P_kW_2f, f\rangle^{1/2}\langle W_1P_kW_1g, g\rangle^{1/2} \\
&\leq \|W_1P_kW_1\|^{1/2}\|W_2P_kW_2\|^{1/2}\|f\|\|g\|,
\end{aligned}$$

and hence

$$\|W_1P_kW_2\| \leq \|W_1P_kW_1\|^{1/2}\|W_2P_kW_2\|^{1/2}. \qquad (2.24)$$

Combining (2.22)–(2.24), we get

$$\sup_{\|W\|_{L^{2r}}=1} \|WP_kW\| \geq 2c_0\lambda_k^{\rho(q)}.$$

Therefore, we can choose a normalized $W \in L^{2r}(\mathbb{R}^d)$ such that

$$\|WP_{k_0}W\| \geq c_0\lambda_{k_0}^{\rho(q)}.$$

The claim (2.19) follows with $V = W^2$. $\qquad\qquad\qquad\qquad\qquad \square$

3. Spectral Cluster Estimates in Odd Dimensions

Consider the Hamiltonian with constant magnetic field in $d = 2n + 1$ dimensions,

$$H_0 = \sum_{j=1}^n \left(-i\frac{\partial}{\partial x_j} + \frac{y_j}{2}\right)^2 + \left(-i\frac{\partial}{\partial y_j} - \frac{x_j}{2}\right)^2 - \frac{\partial^2}{\partial z^2}, \quad (x, y, z) \in \mathbb{R}^{2n+1}. \tag{3.1}$$

The spectrum of H_0 is purely absolutely continuous and $\{\lambda_k\}_{k\in\mathbb{N}_0}$ play the role of thresholds. In the following we will use the notation $x_\perp = (x, y) \in \mathbb{R}^{2n}$. To distinguish the spectral parameter from the coordinate z, we will call the latter x_d instead. The notation $\rho(q, 2n)$ and $\nu(q, 2n)$ will be used to denote the exponents in (2.6) and (2.3), respectively, but with d substituted by $2n$. The resolvents of H_0 and $H_0 + V$ will be denoted by R_0 and R, respectively.

Our main result in this section is a partial analogue of Theorem 2.1 in odd dimensions. This result is weaker than in the even-dimensional case since it only holds in the weak coupling regime and requires an additional weight. Moreover, in contrast to Theorem 2.1, it is relevant only for non-selfadjoint perturbations since the spectrum of H_0 equals $[(d-1)/2, \infty)$, i.e. there are no gaps. We suspect that the conclusion remains true for embedded eigenvalues, but we do not pursue this question here.

Before stating the theorem we recall that the essential spectrum is stable under the perturbations we consider, i.e. $\sigma_{\mathrm{e}} = [(d-1)/2, \infty)$; the proof is a standard application of Weyl's theorem and is omitted. Moreover, we always have that $\sigma_{\mathrm{e}}(H) \cup \sigma_{\mathrm{d}}(H) = \sigma(H)$, see e.g. [11, Theorem XII.2.1].

Theorem 3.1. *Let $d \in 2\mathbb{N} + 1$, and let $\|\langle x_d \rangle^s V\|_{L^r(\mathbb{R}^d)} = 1$ with $r \in ((d-1)/2, \infty)$ and $s > (2r)^{-1}$. Fix $k \in \mathbb{N}$, and let $0 < \epsilon \leq (2C_s \lambda_k^{1/2 + \rho(q,2n)})^{-1}$ (C_s being the constant in (3.2)). Then*

$$\sigma_{\mathrm{d}}(H_0 + \epsilon V) \cap \Lambda_k \subset \{z \in \mathbb{C} : |z - \lambda_k|^{1/2} \lesssim \epsilon \lambda_k^{\nu(r,2n)}\}.$$

Theorem 3.1 follows from the following proposition in the same way as in the even-dimensional case.

Proposition 3.2. *Let $d \in 2\mathbb{N} + 1$ and $2 < q < 2(d-1)/(d-3)$. If $s > 1/2 - 1/q$, then we have*

$$\|\langle x_d \rangle^{-s} R_0(z) \langle x_d \rangle^{-s}\|_{L^{q'}(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \leq C_s \langle \mathrm{Re} z \rangle^{\rho(q,2n)} ((\mathrm{Re} z)_+^{\frac{1}{2}} + \delta(z)^{-\frac{1}{2}}). \tag{3.2}$$

Here, $\delta(z)$ is the distance of z to the threshold set $\{\lambda_k\}_{k \in \mathbb{N}_0}$.

Remark 3.3. The reason that we have to restrict to the weak coupling regime in Theorem 3.1 is the presence of the factor $(\mathrm{Re} z)_+^{1/2}$ in the estimate above.

Lemma 3.4. *Fix $x_d \in \mathbb{R}$ and assume that $\mathrm{Re} z \geq 2$. For $2 \leq q \leq \infty$, we have the estimate, for every $f \in C_0^\infty(\mathbb{R}^d)$,*

$$\|R_0(z)f(\cdot, x_d)\|_{L^q(\mathbb{R}^{2n})} \lesssim \int_{-\infty}^\infty \Big\{ |x_d - y_d|^{-1-2\rho(q,2n)}$$
$$+ (\mathrm{Re} z)^{\rho(q,2n)}((\mathrm{Re} z)^{\frac{1}{2}}$$
$$+ \delta(z)^{-\frac{1}{2}}) \Big\} \|f(\cdot, y_d)\|_{L^{q'}(\mathbb{R}^{2n})} \, \mathrm{d} y_d.$$

Proof. We have

$$R_0(z)f = \sum_{k=0}^\infty (P_k \otimes (-\partial_{x_d}^2 - (z - \lambda_k))^{-1})f.$$

The resolvent kernel of $(-\partial_{x_d}^2 - \mu)^{-1}$ is given by

$$(-\partial_z^2 - \mu)^{-1}(x_d, y_d) = \frac{\mathrm{e}^{\mathrm{i}\sqrt{\mu}|x_d - y_d|}}{2\mathrm{i}\sqrt{\mu}}$$

where $\sqrt{\cdot} : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}^+$ is the principal branch of the square root. Therefore,

$$R_0(z)f(\cdot, x_d) = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\sqrt{z-\lambda_k}|x_d-y_d|}}{2i\sqrt{z-\lambda_k}}(P_k \otimes \mathbf{1})f(\cdot, y_d) \, dy_d.$$

By Minkowski's inequality, it follows that

$$\|R_0(z)f(\cdot, x_d)\|_{L^q(\mathbb{R}^{2n})}$$
$$\leq \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\text{Im}\sqrt{z-\lambda_k}|x_d-y_d|}}{|z-\lambda_k|^{1/2}} \|(P_k \otimes \mathbf{1})f(\cdot, y_d)\|_{L^q(\mathbb{R}^{2n})} \, dy_d.$$

Thus, by [16],

$$\|R_0(z)f(\cdot, x_d)\|_{L^q(\mathbb{R}^{2n})}$$
$$\lesssim \sum_{k=0}^{\infty} \lambda_k^{\rho(q,2n)} \int_{-\infty}^{\infty} \frac{e^{-\text{Im}\sqrt{z-\lambda_k}|x_d-y_d|}}{|z-\lambda_k|^{1/2}} \|f(\cdot, y_d)\|_{L^{q'}(\mathbb{R}^{2n})} \, dy_d.$$

By Fubini's theorem it remains to prove that

$$\sum_{k=0}^{\infty} \lambda_k^{\rho(q,2n)} \frac{e^{-\text{Im}\sqrt{z-\lambda_k}|t|}}{|z-\lambda_k|^{1/2}} \lesssim |t|^{-1-2\rho(q,2n)} + (\text{Re}z)^{\rho(q,2n)}((\text{Re}z)^{\frac{1}{2}} + \delta(z)^{-\frac{1}{2}}).$$

$$(3.3)$$

We write $z = \lambda_{k_0} + \beta + i\tau$, where $|\beta| \leq 1$, and $w = z - \lambda_k$. Then for $|k-k_0| \geq 1$ we have

$$|w|^2 = (2(k_0 - k) + \beta)^2 + \tau^2 \geq 2|k - k_0| - |\beta| \geq |k - k_0|.$$

Moreover, if $k > 2k_0$, then $\text{Re}w < 0$, which implies that

$$\text{Im}\sqrt{w} \geq \frac{\sqrt{|w|}}{\sqrt{2}} \geq \frac{\sqrt{k}}{2}, \quad k > 2k_0.$$

We can thus estimate the sum in (3.3) by

$$\int_0^{k_0-1} \frac{r^{\rho(q,2n)}}{(k_0 - r)^{1/2}} \, dr + 3\frac{k_0^{\rho(q,2n)}}{\delta(z)^{1/2}} + \int_{k_0+1}^{2k_0} \frac{r^{\rho(q,2n)}}{(r - k_0)^{1/2}} \, dr$$
$$+ \int_{2k_0}^{\infty} \frac{r^{\rho(q,2n)}e^{-\frac{1}{2}\sqrt{r}|t|}}{\sqrt{r}} \, dr + \mathcal{O}(1).$$

Note that the errors made by replacing sums by integrals have been absorbed in the $\mathcal{O}(1)$ term. Splitting the first integral into a contribution from the region $r > k_0/2$ and its complement and changing variables $r \to t^2 r$ in the last integral, we obtain the estimate (3.3). $\qquad\square$

We now prove a generalization of Proposition 3.2. To state it, we introduce the following mixed Lebesgue space:

$$\mathscr{X}_q := \left(L_{x_d}^{\frac{2}{1-2\rho(q,2n)}}(\mathbb{R}) \cap L_{x_d}^1(\mathbb{R}) \right) \otimes L_{x^\perp}^{q'}(\mathbb{R}^{2n}).$$

Lemma 3.5. *Let $d \in 2\mathbb{N}+1$ and $2 < q < 2(d-1)/(d-3)$. Then we have*

$$\|R_0(z)\|_{\mathcal{X}_q \to \mathcal{X}_q^*} \lesssim (1+|\mathrm{Re}z|)^{\rho(q,2n)}((\mathrm{Re}z)_+^{\frac{1}{2}} + \delta(z)^{-\frac{1}{2}}).$$

Proof. By duality and by density of $C_0^\infty(\mathbb{R}^d)$ in $\mathcal{X}_q$, it suffices to prove that, for $\mathrm{Re}z \geq 2$,

$$\sup_{\substack{f,g \in C_0^\infty(\mathbb{R}^d) \\ \|f\|_{\mathcal{X}_q}=\|g\|_{\mathcal{X}_q}=1}} |\langle R_0(z)f,g\rangle| \lesssim (\mathrm{Re}z)^{\rho(q,2n)}((\mathrm{Re}z)^{\frac{1}{2}} + \delta(z)^{-\frac{1}{2}}). \tag{3.4}$$

Hence, let $f,g \in C_0^\infty(\mathbb{R}^d)$. By Hölder's inequality and Lemma 3.4, we have the estimate

$$
\begin{aligned}
|\langle R_0(z)f,g\rangle| &= \left| \int_{-\infty}^\infty \langle R_0(z)f(\cdot,x_d), g(\cdot,x_d)\rangle_{L_{x_\perp}^2} \, dx_d \right| \\
&\leq \int_{-\infty}^\infty \|R_0(z)f(\cdot,x_d)\|_{L_{x_\perp}^q} \|g(\cdot,x_d)\|_{L_{x_\perp}^{q'}} \, dx_d \\
&\lesssim \int_{-\infty}^\infty \int_{-\infty}^\infty \Big\{ |x_d - y_d|^{-1-2\rho(q,2n)} \\
&\quad + A(z) \Big\} \|f(\cdot,y_d)\|_{L_{x_\perp}^{q'}} \|g(\cdot,x_d)\|_{L_{x_\perp}^{q'}} \, dy_d \, dx_d,
\end{aligned}
$$

where $A(z)$ is the constant on the right hand side of (3.4). The claim follows from the one-dimensional Hardy-Littlewood-Sobolev inequality (note that the condition $2 < q < 2(d-1)/(d-1)$ implies that $\rho(q,2n) < 0$). $\qquad\square$

Proof that Lemma 3.5 $\Longrightarrow$ Proposition 3.2. By Hölder's inequality (in one dimension) and duality, we have that, for $s > 1/2 - 1/q$.

$$\mathcal{B}(\mathcal{X}_q, \mathcal{X}_q^*) \subset \mathcal{B}(L_{\langle x_d \rangle^s}^{q'}(\mathbb{R}^d), L_{\langle x_d \rangle^{-s}}^q(\mathbb{R}^d)) \qquad\qquad\square$$

4. Unique Continuation

Recall the definition of the weak unique continuation property (w.u.c.p.): A partial differential operator $P(x,D)$ is said to have the w.u.c.p. if the following holds. Let $\Omega \subset \mathbb{R}^d$ be open and connected, and assume that $P(x,D)u = 0$ in Ω where u is compactly supported in Ω. Then $u \equiv 0$ in Ω.

Theorem 4.1. *Let $d \in 2\mathbb{N}+1$. Assume that $V \in L^{d/2}(\mathbb{R}^d)$. Then $H_0 + V$ has the w.u.c.p.*

The proof is a standard application (see e.g. [15]) of the following Carleman estimate.

Theorem 4.2. *Let $d \in 2\mathbb{N}+1$, and let $I \subset \mathbb{R}$ be a compact interval. There exists a constant $C_I > 0$ such that for any $u \in C_0^\infty(\mathbb{R}^{2n} \times I)$ and $\tau \in \mathbb{R}$, with $\mathrm{dist}(\tau^2, 2\mathbb{N}+n) \geq 1/2$, we have the estimate*

$$\|e^{\tau x_d} u\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq C_I \|e^{\tau x_d} H_0 u\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}. \tag{4.1}$$

Remark 4.3. Theorem 4.1 is a special case of [29, Theorem 1]. It also follows from more general results of [17,18]. However, we choose to include it here since the proof is based on a simpler Carleman estimate with linear weight, in the spirit of [14].

Proof of Theorem 4.2. We follow the procedure of Jerison's [12] proof of the unique continuation theorem of Jerison and Kenig [13]. The proof is similar to [7, Theorem 1.2], except that we use the spectral projection estimates of Koch and Ricci [16] for the twisted Laplacian (2.1) instead of the spectral cluster estimates of Sogge [26]. We recall (a special case of) the main result in [16]:

$$\|P_k u\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^{2n})} \lesssim \lambda_k^{-\frac{1}{d}} \|u\|_{L^2(\mathbb{R}^{2n})}, \tag{4.2}$$

$$\|P_k u\|_{L^2(\mathbb{R}^{2n})} \lesssim \lambda_k^{-\frac{1}{d}} \|u\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^{2n})}. \tag{4.3}$$

Adopting the notation of [7], we denote by G_τ the inverse of the conjugated operator

$$e^{\tau x_d} H_0 e^{-\tau x_d} = D_{x_d}^2 + 2i\tau D_{x_d} - \tau^2 + H_{0,\perp}.$$

It will be sufficient to prove that

$$\|G_\tau f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^{2n})} \lesssim_I \|f\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^{2n})}$$

for all $f \in C_0^\infty(\mathbb{R}^{2n} \times I)$. Using the eigenfunction expansion of $H_{0,\perp}$, we obtain

$$G_\tau f(x_\perp, x_d) = \sum_{k=0}^\infty \int_{-\infty}^\infty m_\tau(x_d - y_d, \lambda_k)(P_k \otimes \mathbf{1}) f(x_\perp, y_d) \, dy_d$$

where

$$m_\tau(x_d - y_d, \lambda_k) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{i(x_d-y_d)\eta}}{\eta^2 + 2i\tau\eta - \tau^2 + \lambda_k} \, d\eta.$$

Using the spectral projection estimates (4.2) and proceeding as in the proof of [7], we arrive at

$$\|G_\tau f(\cdot, x_d)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^{2n})} \lesssim \sum_{k=0}^\infty \int_{-\infty}^\infty \frac{|m_\tau(x_d - y_d, \lambda_k)|}{(1 + 2k)^{\frac{1}{d}}} \|f(\cdot, y_d)\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^{2n})} \, dy_d.$$

By the straightforward estimate

$$|m_\tau(x_d - y_d, \lambda_k)| \lesssim \frac{e^{-|\tau-\lambda_k||x_d-y_d|}}{\sqrt{\lambda_k}},$$

see Lemma 2.3 in [7], one can sum up the previous estimates (estimate the sum by an integral and change variables $k \to \lambda = \sqrt{2k+n}$):

$$\sum_{k=0}^\infty (1 + 2k)^{-\frac{1}{d}} |m_\tau(x_d - y_d, \lambda_k)|$$

$$\lesssim \int_0^\infty \lambda^{-\frac{2}{d}} e^{-|\tau-\lambda||x_d-y_d|} \, d\lambda \lesssim 1 + |x_d - y_d|^{\frac{2}{d}-1}.$$

Thus

$$\|G_\tau f(\cdot, x_d)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^{2n})} \lesssim |I|^{\frac{1}{2} - \frac{1}{d}} \|f\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$
$$+ \int_{-\infty}^{\infty} |x_d - y_d|^{\frac{2}{d} - 1} \|f(\cdot, y_d)\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^{2n})} \, dy_d.$$

An application of the one-dimensional Hardy-Littlewood-Sobolev inequality yields

$$\|G_\tau f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \lesssim \|f\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}. \qquad \square$$

Acknowledgements

References

[1] Cuenin, J.-C.: Eigenvalue bounds for Dirac and fractional Schrödinger operators with complex potentials. J. Funct. Anal. **272**(7), 2987–3018 (2017)

[2] Cuenin, J.-C., Kenig, C.E.: L^p resolvent estimates for magnetic Schrödinger operators with unbounded background fields. Commun. Partial Differ. Equ. **42**(2), 235–260 (2017)

[3] Davies, E.B.: Non-self-adjoint differential operators. Bull. Lond. Math. Soc. **34**(5), 513–532 (2002)

[4] Davies, E.B.: Linear Operators and Their Spectra, Volume 106 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2007)

[5] Demuth, M., Hansmann, M., Katriel, G.: On the discrete spectrum of non-selfadjoint operators. J. Funct. Anal. **257**(9), 2742–2759 (2009)

[6] Demuth, M., Hansmann, M., Katriel, G.: Eigenvalues of non-selfadjoint operators: a comparison of two approaches. In: Mathematical Physics, Spectral Theory and Stochastic Analysis, Volume 232 of Oper. Theory Adv. Appl., pp. 107–163. Birkhäuser/Springer Basel AG, Basel (2013)

[7] Dos Santos Ferreira, D., Kenig, C.E., Salo, M.: Determining an unbounded potential from Cauchy data in admissible geometries. Commun. Partial Differ. Equ. **38**(1), 50–68 (2013)

[8] Frank, R.L., Sabin, J.: Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform Sobolev estimates. ArXiv e-prints, April (2014)

[9] Frank, R.L.: Eigenvalue bounds for Schrödinger operators with complex potentials. Bull. Lond. Math. Soc. **43**(4), 745–750 (2011)

[10] Frank, R.L., Laptev, A., Lieb, E.H., Seiringer, R.: Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials. Lett. Math. Phys. **77**(3), 309–316 (2006)

[11] Gohberg, I., Goldberg, S., Kaashoek, M. A.: Classes of linear operators. Vol. I, volume 49 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel (1990)

[12] Jerison, D.: Carleman inequalities for the Dirac and Laplace operators and unique continuation. Adv. Math. **62**(2), 118–134 (1986)

[13] Jerison, D., Kenig, C.E.: Unique continuation and absence of positive eigenvalues for Schrödinger operators. Ann. Math. (2) **121**(3), 463–494 (1985). With an appendix by E. M. Stein

[14] Kenig, C.E., Ruiz, A., Sogge, C.D.: Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. Duke Math. J. **55**(2), 329–347 (1987)

[15] Kenig, C.E.: Restriction theorems, Carleman estimates, uniform Sobolev inequalities and unique continuation. In: Harmonic analysis and partial differential equations (El Escorial, 1987), volume 1384 of Lecture Notes in Math., pp. 69–90. Springer, Berlin (1989)

[16] Koch, H., Ricci, F.: Spectral projections for the twisted Laplacian. Stud. Math. **180**(2), 103–110 (2007)

[17] Koch, H., Tataru, D.: Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients. Commun. Pure Appl. Math. **54**(3), 339–360 (2001)

[18] Koch, H., Tataru, D.: Dispersive estimates for principally normal pseudodifferential operators. Commun. Pure Appl. Math. **58**(2), 217–284 (2005)

[19] Koch, H., Tataru, D.: Carleman estimates and absence of embedded eigenvalues. Commun. Math. Phys. **267**(2), 419–449 (2006)

[20] Korotyaev, E., Pushnitski, A.: A trace formula and high-energy spectral asymptotics for the perturbed Landau Hamiltonian. J. Funct. Anal. **217**(1), 221–248 (2004)

[21] Laptev, A., Safronov, O.: Eigenvalue estimates for Schrödinger operators with complex potentials. Commun. Math. Phys. **292**(1), 29–54 (2009)

[22] Lungenstrass, T., Raikov, G.: A trace formula for long-range perturbations of the Landau Hamiltonian. Ann. Henri Poincaré **15**(8), 1523–1548 (2014)

[23] Pushnitski, A., Raikov, G., Villegas-Blas, C.: Asymptotic density of eigenvalue clusters for the perturbed Landau Hamiltonian. Commun. Math. Phys. **320**(2), 425–453 (2013)

[24] Raĭkov, G.D.: Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential. I. Behaviour near the essential spectrum tips. Commun. Partial Differ. Equ. **15**(3), 407–434 (1990)

[25] Sambou, D.: Lieb-Thirring type inequalities for non-self-adjoint perturbations of magnetic Schrödinger operators. J. Funct. Anal. **266**(8), 5016–5044 (2014)

[26] Sogge, C.D.: Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds. J. Funct. Anal. **77**(1), 123–138 (1988)

[27] Tao, T.: Nonlinear dispersive equations, volume 106 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis

[28] Trefethen, L.N., Embree, M.: Spectra and pseudospectra. In: The behavior of nonnormal matrices and operators. Princeton University Press, Princeton (2005)

[29] Wolff, T.H.: A property of measures in R^N and an application to unique continuation. Geom. Funct. Anal. **2**(2), 225–284 (1992)

[30] Yajima, K.: Schrödinger evolution equations with magnetic fields. J. Anal. Math. **56**, 29–76 (1991)

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