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Ultracontractivity and Eigenvalues: Weyl's Law for the Dirichlet-to-Neumann Operator

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Abstract. We show an interesting relation between ultracontractivity and Weyl asymptotics. Then both properties are studied for their behaviour with respect to perturbation. The results are used to establish Weyl's law for the Dirichlet-to-Neumann operator associated with $-\Delta + V$, where V is a measurable bounded potential. In particular, we show that its eigenvalues determine the surface area of the domain.

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1. Introduction

Consider an unbounded self-adjoint operator A with compact resolvent which is bounded below on a space $L_2(\Omega)$. Let $\lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues, repeated with multiplicity. For all $\lambda \in \mathbb{R}$ let

$$N(\lambda) = \#\{n \in \mathbb{N} : \lambda_n \leq \lambda\}$$

be the counting function. We say that A admits *Weyl asymptotics* if the limit $\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^\kappa}$ exists in $(0, \infty)$ for some $\kappa > 0$. The prototype is the Laplace operator with Dirichlet boundary conditions on a bounded domain Ω in $\mathbb{R}^d$ and Weyl's famous result says that for $\kappa = \frac{d}{2}$ the limit exists and is proportional to the volume of Ω (cf. [4]). In this paper we show that Weyl asymptotics are strongly related to ultracontractivity. Recall that the semigroup S generated by $-A$ on $L_2(\Omega)$ is called *ultracontractive* if

$$\|S_t\|_{2\to\infty} \leq c\, t^{-\kappa/2}$$

for all $t \in (0, 1]$ and some $c, \kappa > 0$. Using duality it follows that $\|S_t\|_{1\to\infty} \leq 2^\kappa c^2 t^{-\kappa}$ for all $t \in (0, 1]$, which is equivalent to saying that for all $t \in (0, 1]$

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the operator S_t is a kernel operator whose kernel $K_t(\cdot,\cdot)$ on $\Omega \times \Omega$ is bounded by $2^\kappa c^2 t^{-\kappa}$. Ultracontractivity has been studied intensively (see [3,10,25] and references therein). We show in Theorem 2.7 that ultracontractivity is equivalent to an upper Weyl bound and a growth condition on the eigenvalues.

Our main example is the Dirichlet-to-Neumann operator D_V on $L_2(\Gamma)$, where Γ is the boundary of a Lipschitz domain Ω and $V \in L_\infty(\Omega)$ is real valued. Its graph consists of those pairs $(\varphi, \psi) \in L_2(\Gamma) \times L_2(\Gamma)$ such that there exists a $u \in H^1(\Omega)$ satisfying

$$\begin{cases} \operatorname{Tr} u = \varphi, \\ -\Delta u + V\, u = 0 \text{ weakly on } \Omega, \\ \partial_\nu u = \psi. \end{cases} \tag{1}$$

Here ∂_ν is a weakly defined version of the normal derivative, see Sect. 6. It is a lower-bounded self-adjoint operator with compact resolvent and indeed, our perturbation results allow us to prove Weyl's law

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d-1}} = c_d\, \sigma(\Gamma) \tag{2}$$

for this operator, where $c_d > 0$ is a universal constant and $\sigma(\Gamma)$ is the surface area of Γ. We show this if Ω has a C^∞-boundary, but in our perturbation and regularity studies we merely assume Ω to be Lipschitz, since these results have independent value. The eigenvalues of D_V can be described by a sort of Steklov problem (see the remark at the end of the last section). The first one to consider such a Steklov problem was Sandgren [27] in 1955, who proved Weyl's law (2) for $V = 0$. Later such Steklov problems and also the asymptotic behaviour of the eigenvalues have been studied intensively by Koženikov. We mention in particular his article [20] which contains Weyl's law for D_V with V a C^∞-function and which heavily uses pseudo-differential calculus. Our perturbation results in connection with ultracontractivity give a very new transparent proof. It is based on elementary form methods. Deliberately we choose the Laplacian with a bounded potential to avoid technical arguments.

There is a wealth of results on Weyl's formula for the Laplace operator and many other operators as well as many sophisticated properties on the counting function are known. As an example of such results we mention Netrusov–Safarov [23] and [8] for a general survey on Weyl's formula in physics and mathematics.

The outline of this paper is as follows. By a theorem of Karamata the asymptotics of the counting function is equivalent to a limit of the trace of the semigroup generated by $-D_V$. In Sect. 2 we study the relation between various trace estimates and the connection with the notion of ultracontractivity of a semigroup. In Sect. 3 we prove a perturbation result for ultracontractivity and in Sect. 4 for traces of semigroups. If Ω has a C^∞-boundary and $V = 0$, then the Dirichlet-to-Neumann operator D_0 is a pseudo-differential operator of order one and it is equal to the square root of the Laplace–Beltrami operator on Γ, up to a pseudo-differential operator of order zero. In Sect. 5 we use Weyl's law for the Laplace–Beltrami operator on Γ together with the perturbation result of Sect. 4 to prove a Weyl asymptotics for D_0. In Sect. 6 we add the potential V and prove Weyl's law for D_V.

2. Weyl Asymptotics and Ultracontractivity

Let $\Omega \subset \mathbb{R}^d$ be a bounded open non-empty set. Then $L_2(\Omega)$ has an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ consisting of eigenfunctions of the Dirichlet Laplacian. So if $n \in \mathbb{N}$, then $e_n \in H_0^1(\Omega)$ and there exists a $\lambda_n^D \in \mathbb{R}$ such that $-\Delta e_n = \lambda_n^D e_n$. We may assume that $0 < \lambda_1^D \le \lambda_2^D \le \ldots$. Note that $\lim_{n\to\infty} \lambda_n^D = \infty$. *Weyl's law* tells that

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{\omega_d}{(2\pi)^d} |\Omega|,$$

where $N\colon \mathbb{R} \to \mathbb{N}_0$ is the counting function given by

$$N(\lambda) = \#\{n \in \mathbb{N} : \lambda_n \le \lambda\},$$

the volume of the unit ball in $\mathbb{R}^d$ is denoted by ω_d and $|\Omega|$ is the volume of Ω. Weyl's law holds for arbitrary bounded open non-empty sets (see for example [4] Section 6 for a proof). One possible proof uses Karamata's Tauberian theorem.

Proposition 2.1. (Karamata) *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a lower bounded sequence of real numbers such that the series $\sum e^{-\lambda_n t}$ converges for all $t > 0$. Let $\kappa > 0$ and $a \in \mathbb{R}$. Then the following are equivalent.*

(a) $\lim_{t\downarrow 0} t^\kappa \sum_{n=1}^{\infty} e^{-\lambda_n t} = a.$

(b) $\lim_{\lambda\to\infty} \lambda^{-\kappa} N(\lambda) = \frac{a}{\Gamma(\kappa+1)}$, *where* $N(\lambda) = \#\{n \in \mathbb{N} : \lambda_n \le \lambda\}$.

(c) $\lim_{n\to\infty} \frac{\lambda_n}{n^{1/\kappa}} = \left(\frac{\Gamma(\kappa+1)}{a}\right)^{1/\kappa}.$

Proof. For the equivalence of (a) and (b) see Karamata [19] Satz A. The proof of the equivalence of (b) and (c) is elementary. □

Note that $\sum_{n=1}^\infty e^{-t\lambda_n^D} = \operatorname{Tr} e^{t\Delta^D}$, the trace of the operator $e^{t\Delta^D}$, for all $t > 0$, where $(e^{t\Delta^D})_{t>0}$ is the semigroup generated by Δ^D on $L_2(\Omega)$.

Thus Karamata's theorem establishes an equivalence between the asymptotic behaviour of the eigenvalues in the sense of Weyl and the asymptotic behaviour of the trace as $t \downarrow 0$. Next we introduce the notion of Weyl limit and Weyl bounds.

Definition 2.2. Let $-A$ be the generator of a C_0-semigroup in a separable Hilbert space H.

(a) We say that A has a *Weyl limit* if there exist $a, \kappa > 0$ such that e^{-tA} is a trace class operator for all $t > 0$ and $\lim_{t\downarrow 0} t^\kappa \operatorname{Tr} e^{-tA} = a$.

(b) We say that A has an *upper Weyl bound* if there exist $a, \kappa > 0$ such that e^{-tA} is a trace class operator for all $t > 0$ and $t^\kappa \operatorname{Tr} e^{-tA} \le a$ for all $t \in (0, 1]$.

(c) We say that A has a *lower Weyl bound* if there exist $a, \kappa > 0$ such that e^{-tA} is a trace class operator for all $t > 0$ and $a \le t^\kappa \operatorname{Tr} e^{-tA}$ for all $t \in (0, 1]$.

(d) We say that A has is *Weyl bounded* if A has an upper Weyl bound and a lower Weyl bound.

The purpose of this section is to establish a relation between the existence of an upper Weyl bound and ultracontractivity of the semigroup.

Let $(X, \mathcal{A}, \mu)$ be a measure space. For simplicity we write $L_p = L_p(X)$ in this section for all $p \in [1, \infty]$. If $p_1, p_2, q_1, q_2 \in [1, \infty]$ and both $Q_1 \colon L_{p_1} \to L_{q_1}$ and $Q_2 \colon L_{p_2} \to L_{q_2}$ are bounded, then we say that Q_1 and Q_2 are *consistent* if $Q_1 u = Q_2 u$ almost everywhere for all $u \in L_{p_1} \cap L_{p_2}$. Let $Q \colon L_2 \to L_2$ be a bounded operator. Let $p, q \in [1, \infty]$. Then we set

$$\|Q\|_{p \to q} = \sup\{\|Qu\|_q : u \in L_2 \cap L_p,\ \|u\|_p \le 1\} \in [0, \infty].$$

If $\|Q\|_{p \to q} < \infty$ and $p < \infty$, then $Q|_{L_2 \cap L_p}$ extends consistently to a bounded operator from L_p into L_q.

Let $\kappa > 0$. Let S be a C_0-semigroup on L_2. We say that S is κ-*ultracontractive* if there exists a $c > 0$ such that

$$\|S_t\|_{2 \to \infty} \le c\, t^{-\kappa/2} \tag{3}$$

for all $t \in (0, 1]$. In the literature *ultracontractive semigroups*, that are semigroups which are κ-ultracontractive for some $\kappa > 0$, are well studied, starting with Davies–Simon [10].

If $\|S_t\|_{p \to p} \le M$ for all $t \in (0, 1]$ and $p \in [1, \infty]$, then $\|S_t\|_{p \to p} \le M\, e^{\omega t}$ for all $t \in (0, \infty)$, where $\omega = \log M$. In that case we may extend S_t to L_p for all $p \in [1, \infty]$, and if $p \in (1, \infty)$ then we obtain a C_0-semigroup whose generator we denote by $-A_p$. It is an open problem whether S then also extends to a C_0-semigroup on L_1 (see [30], [5] Lemma 2.1 and [12] Theorem 2.5). Under this assumption of uniform bounds on $\|S_t\|_{p \to p}$ there are many characterisations of κ-ultracontractivity, see [3] Subsection 7.3.2, [25] Section 6.1 and references therein. In the next theorem we list six of the characteristic properties that we need here.

Theorem 2.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $\kappa > 0$. Let S be a C_0-semigroup on L_2. Suppose there exists an $M > 0$ such that $\|S_t\|_{p \to p} \le M$ for all $t \in (0, 1]$ and $p \in [1, \infty]$.*

(a) *The following are equivalent.*

(i) *The semigroup S is κ-ultracontractive.*

(ii) *There exists a $c > 0$ such that*

$$\|S_t\|_{p \to q} \le c\, t^{-\kappa(\frac{1}{p} - \frac{1}{q})} \tag{4}$$

for all $t \in (0, 1]$ and $p, q \in [1, \infty]$ with $p \le q$.

(iii) *There exist $c > 0$ and $p, q \in [1, \infty]$ with $p < q$ such that*

$$\|S_t\|_{p \to q} \le c\, t^{-\kappa(\frac{1}{p} - \frac{1}{q})}$$

for all $t \in (0, 1]$.

(b) *Suppose in addition that S is a holomorphic semigroup. Then (i) is equivalent to the following statements.*

(iv) *For all $p, q \in [1, \infty)$ and $\omega > \log M$ with $p < q$ there exists a $c > 0$ such that*

$$\|u\|_q^{1+\kappa(\frac{1}{p}-\frac{1}{q})} \leq c \, \|u\|_p \, \|(\omega I + A_q)u\|_q^{\kappa(\frac{1}{p}-\frac{1}{q})}$$

for all $u \in L_p \cap \operatorname{dom}(A_q)$, where $-A_q$ is the generator of the extension of the semigroup on L_q.

(v) *There exist $p, q \in [1, \infty)$, $\omega > \log M$ and $c > 0$ with $p < q$ such that*

$$\|u\|_q^{1+\kappa(\frac{1}{p}-\frac{1}{q})} \leq c \, \|u\|_p \, \|(\omega I + A_q)u\|_q^{\kappa(\frac{1}{p}-\frac{1}{q})}$$

for all $u \in L_p \cap \operatorname{dom}(A_q)$, where $-A_q$ is the generator of the extension of the semigroup on L_q.

(c) *Suppose in addition that $\kappa > 1$ and the operator A is m-sectorial, where $-A$ is the generator of S. Let V be the form domain of the associated m-sectorial form. Then (i) is equivalent to the following statement.*

(vi) $V \subset L_{\frac{2\kappa}{\kappa-1}}$.

Proof. This is well-known (cf. [3] Subsection 7.3.2). □

Ultracontractivity is a condition on the kernels of the semigroup because of the following well-known Dunford–Pettis criterion.

Theorem 2.4. *Let $Q \in \mathcal{L}(L_2)$ and $p \in [1, \infty]$. Then the following are equivalent.*

(a) $\|Q\|_{p \to \infty} < \infty$.

(b) *There exists a measurable function $k \colon X \times X \to \mathbb{C}$ such that*

$$\operatorname*{ess\,sup}_{x \in X} \int_X |k(x,y)|^q \, dy < \infty$$

and for all $f \in L_p$ one has $(Qf)(x) = \int_X k(x,y) \, f(y) \, dy$ for a.e. $x \in X$, where q is the dual exponent of p.

If the statements are valid, then

$$\|Q\|_{p \to \infty} = \Big(\operatorname*{ess\,sup}_{x \in X} \int_X |k(x,y)|^q \, dy \Big)^{1/q},$$

with obvious modifications if $q = \infty$.

If $(X, \mathcal{A}, \mu)$ is a finite measure space and S is an ultracontractive C_0-semigroup on L_2, then S_t is a Hilbert–Schmidt operator for all $t > 0$. Hence $S_{2t} = S_t \circ S_t$ is trace class for all $t > 0$. If in addition S^* is also ultracontractive, then the next lemma gives a useful more precise estimate of the trace.

Lemma 2.5. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $E_1 \colon L_1 \to L_2$ and $E_2 \colon L_2 \to L_\infty$ be two bounded maps. Set $E = E_2 \, E_1$. Then E is trace class and*

$$|\operatorname{Tr} E| \leq \mu(X) \, \|E_1\|_{1 \to 2} \, \|E_2\|_{2 \to \infty}.$$

Proof. The Dunford–Pettis theorem implies that the operator E_2 has a measurable kernel $K : X \times X \to \mathbb{C}$ and

$$\|E_2\|_{2\to\infty} = \operatorname*{ess\,sup}_{x\in X} \Big(\int |K(x,y)|^2 \, dy \Big)^{1/2} .$$

Then

$$\int_X \int_X |K(x,y)|^2 \, dy \, dx \le \mu(X) \Big(\operatorname*{ess\,sup}_{x\in X} \Big(\int |K(x,y)|^2 \, dy \Big)^{1/2} \Big)^2 = \mu(X) \, \|E_2\|_{2\to\infty}^2 .$$

So E_2 is a Hilbert–Schmidt operator and $\|E_2\|_{\mathrm{HS}} \le \mu(X)^{1/2} \, \|E_2\|_{2\to\infty}$.

Similarly E_1^* is a Hilbert–Schmidt operator and

$$\|E_1\|_{\mathrm{HS}} = \|E_1^*\|_{\mathrm{HS}} \le \mu(X)^{1/2} \, \|E_1^*\|_{2\to\infty} = \mu(X)^{1/2} \, \|E_1\|_{1\to 2} .$$

Then $E = E_2 \, E_1$ is a composition of two Hilbert–Schmidt operators. Therefore it is trace class and $|\mathrm{Tr}\, E| \le \|E_1\|_{\mathrm{HS}} \, \|E_2\|_{\mathrm{HS}} \le \mu(X) \, \|E_1\|_{1\to 2} \, \|E_2\|_{2\to\infty}$ as required. □

As a consequence we show that ultracontractivity implies an upper Weyl bound.

Proposition 2.6. *Suppose $\mu(X) < \infty$. Let $\kappa > 0$. Let S be a C_0-semigroup on L_2. Suppose both the semigroups S and S^* are κ-ultracontractive. Then there exists a constant $b > 0$ such that $t^\kappa \, \mathrm{Tr}\, S_t \le b$ for all $t \in (0, 1]$.*

Proof. There exists a $c > 0$ such that $\|S_t\|_{2\to\infty} \vee \|S_t^*\|_{2\to\infty} \le c \, t^{-\kappa/2}$ for all $t \in (0, 1]$. Let $t \in (0, 1]$. Then Lemma 2.5 gives

$$\mathrm{Tr}\, S_{2t} \le |\mathrm{Tr}\,(S_t \, S_t)| \le \mu(X) \, \|S_t\|_{1\to 2} \|S_t\|_{2\to\infty} \le c^2 \, \mu(X) \, t^{-\kappa}$$

and the proposition follows. □

If a semigroup S consists of Hilbert–Schmidt operators, then the generator of S has a compact resolvent. If the semigroup S is in addition self-adjoint, then we can find an orthonormal basis for $L_2(X)$ consisting of eigenfunctions for the generator.

Next we consider self-adjoint compact semigroups. It turns out that in this case ultracontractivity can be characterized by an upper Weyl bound and a growth condition on the eigenfunctions.

Theorem 2.7. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space with $\dim L_2(X) = \infty$. Let S be a self-adjoint C_0-semigroup on $L_2(X)$. Suppose the generator $-A$ of S has compact resolvent. Let $\{\varphi_n : n \in \mathbb{N}\}$ be an orthonormal basis for $L_2(X)$ consisting of eigenfunctions of A. For all $n \in \mathbb{N}$ let $\lambda_n \in \mathbb{R}$ be such that $A\varphi_n = \lambda_n \, \varphi_n$. Then the following are equivalent.*

(i) *The semigroup S is ultracontractive.*

(ii) *The semigroup S is trace class and there exist $c, \kappa > 0$ such that*
   (a) $\sup_{t\in(0,1]} t^\kappa \, \mathrm{Tr}\, S_t < \infty$ *and*
   (b) $\|\varphi_n\|_\infty \le c \, \lambda_n^{\kappa/2}$ *for all $n \in \mathbb{N}$ with $\lambda_n > 0$.*

If the two statements are valid, then for all $t > 0$ *the operator* S_t *has a kernel* K_t *and*

$$K_t(x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x)\, \overline{\varphi_n(y)}$$

for a.e. $(x,y) \in X \times X$ *and the convergence is in* $L_\infty(X \times X)$.

Proof. '(i)⇒(ii)'. Let $\kappa > 0$ be such that S is κ-ultracontractive. Part (a) follows from Proposition 2.6. It remains to show (b). Note that the operator A is unbounded since it has compact resolvent. Hence $\lim_{n\to\infty} \lambda_n = \infty$. Let $c > 0$ be as in (3). Then

$$e^{-\lambda_n t} \|\varphi_n\|_\infty = \|S_t \varphi_n\|_\infty \le c\, t^{-\kappa/2}\, \|\varphi_n\|_2 = c\, t^{-\kappa/2}$$

for all $t \in (0,1]$. Now choose $t = \lambda_n^{-1}$ if $\lambda_n \ge 1$.

'(ii)⇒(i)'. Without loss of generality we may assume that $A \ge I$. There exist $c_1, c_2 \in (0,\infty)$ such that $t^\kappa \operatorname{Tr} S_t \le c_1$ for all $t \in (0,1]$ and $h^\kappa\, e^{-h/2} \le c_2$ for all $h \in [0,\infty)$. Let $\varphi \in L_2(X)$. Then

$$S_t \varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n)_{L_2(X)}\, S_t \varphi_n = \sum_{n=1}^{\infty} (\varphi, \varphi_n)_{L_2(X)}\, e^{-\lambda_n t}\, \varphi_n$$

and hence

$$\begin{aligned} \|S_t \varphi\|_\infty &\le \sum_{n=1}^{\infty} \|\varphi\|_2\, e^{-\lambda_n t}\, \|\varphi_n\|_\infty \\ &\le \|\varphi\|_2 \sum_{n=1}^{\infty} e^{-\lambda_n t}\, c\, \lambda_n^{\kappa/2} \\ &\le c\, \sqrt{c_2}\, \|\varphi\|_2 \sum_{n=1}^{\infty} e^{-\lambda_n t/2}\, t^{-\kappa/2} \\ &= c\, \sqrt{c_2}\, \|\varphi\|_2\, t^{-\kappa/2} \operatorname{Tr} S_{t/2} \le 2^\kappa c\, c_1\, \sqrt{c_2}\, \|\varphi\|_2\, t^{-3\kappa/2} \end{aligned}$$

for all $t \in (0,1]$. So S is ultracontractive.

Finally, suppose that the two statements are valid. For all $n \in \mathbb{N}$ define the element $K^{(n)} \in L_\infty(X \times X)$ by $K^{(n)}(x,y) = \varphi_n(x)\, \overline{\varphi_n(y)}$. Let $t \in (0,1]$. Then with the above notation one obtains

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-\lambda_n t}\, \|K^{(n)}\|_\infty &\le \sum_{n=1}^{\infty} e^{-\lambda_n t}\, c^2\, \lambda_n^\kappa \\ &\le c^2\, c_2 \sum_{n=1}^{\infty} t^{-\kappa}\, e^{-\lambda_n t/2} \le 2^\kappa\, c_1\, c_2\, c^2\, t^{-2\kappa}. \end{aligned}$$

So the series $\sum e^{-\lambda_n t}\, K^{(n)}$ is convergent in $L_\infty(X \times X)$. Define

$$K_t = \sum_{n=1}^{\infty} e^{-\lambda_n t}\, K^{(n)}.$$

Then $\|K_t\|_\infty \le 2^\kappa\, c_1\, c_2\, c^2\, t^{-2\kappa}$. Let $T_t \colon L_2 \to L_2$ be the Hilbert–Schmidt operator with kernel K_t. If $m \in \mathbb{N}$, then $S_t \varphi_m = e^{-\lambda_m t}\, \varphi_m = T_t \varphi_m$. So

$S_t = T_t$ and hence for all $u \in L_2(X)$ one deduces that $(S_t u)(x) = (T_t u)(x) = \int_X K_t(x,y)\, u(y)\, dy$ for a.e. $x \in X$. □

3. Perturbation of Ultracontractivity

In this section we investigate which perturbations preserve ultracontractivity.

Proposition 3.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $\kappa > 0$. Let S and T be two C_0-semigroups on L_2. Suppose that*

$$\sup_{t \in (0,1]} \|S_t\|_{p \to p} < \infty \quad \textit{and} \quad \sup_{t \in (0,1]} \|T_t\|_{p \to p} < \infty$$

for all $p \in [1, \infty]$. For all $q \in (1, \infty)$ let $-A_q$ and $-B_q$ be the generator of the extension of S and T on L_q, respectively. Let $\kappa > 0$.

(a) *Assume there exist $q \in (1, \infty)$ and a bounded operator Q on L_q such that $A_q = B_q + Q$. Then S is κ-ultracontractive if and only if T is κ-ultracontractive.*

(b) *Assume that both semigroups S and T are holomorphic. Suppose there exists a $q \in (1, \infty)$ such that $\mathrm{dom}(A_q) = \mathrm{dom}(B_q)$. Then S is κ-ultracontractive if and only if T is κ-ultracontractive.*

Proof. '(a)'. There exists a $p \in [1, q)$ such that $\kappa(\frac{1}{p} - \frac{1}{q}) < 1$. Suppose that the semigroup S is κ-ultracontractive. Then Theorem 2.3(i)⇒(ii) implies that there exists a $c > 0$ such that

$$\|S_t u\|_q \le c\, t^{-\kappa(\frac{1}{p} - \frac{1}{q})}\, \|u\|_p$$

for all $t \in (0, 1]$ and $u \in L_2 \cap L_p$. Let $M \ge 1$ be such that $\|T_t\|_{q \to q} \le M$ for all $t \in (0, 1]$. Then

$$\begin{aligned} \|(T_t - S_t)u\|_q &\le \int_0^t \|T_{t-s}\, Q\, S_s u\|_q\, ds \le M\, \|Q\|_{q \to q} \int_0^t \|S_s u\|_q\, ds \\ &\le c\, M\, \|Q\|_{q \to q} \int_0^t s^{-\kappa(\frac{1}{p} - \frac{1}{q})}\, \|u\|_p \\ &= \frac{c\, M\, \|Q\|_{q \to q}}{1 - \kappa(\frac{1}{p} - \frac{1}{q})}\, t^{1 - \kappa(\frac{1}{p} - \frac{1}{q})}\, \|u\|_p \end{aligned}$$

for all $t \in (0, 1]$ and $u \in L_2 \cap L_p$. So

$$\|T_t u\|_q \le \Big(c + \frac{c\, M\, \|Q\|_{q \to q}}{1 - \kappa(\frac{1}{p} - \frac{1}{q})} \Big) t^{-\kappa(\frac{1}{p} - \frac{1}{q})}\, \|u\|_p$$

for all $t \in (0, 1]$ and $u \in L_2 \cap L_p$ and the semigroup T is κ-ultracontractive by Theorem 2.3(iii)⇒(i).

'(b)'. This follows from the equivalence of (ii), (iv) and (v) in Theorem 2.3. □

4. Perturbation for Weyl Limits

Now we can show how Weyl limits are preserved under perturbations.

Theorem 4.1. *Let* $(X, \mathcal{A}, \mu)$ *be a finite measure space. Let* S *and* T *be two* C_0*-semigroups in* L_2 *with generators* $-A$ *and* $-B$*, respectively. Suppose that*

$$\sup_{t \in (0,1]} \|S_t\|_{p \to p} < \infty \quad \text{and} \quad \sup_{t \in (0,1]} \|T_t\|_{p \to p} < \infty$$

for all $p \in [1, \infty]$*. Suppose there exist bounded consistent operators* $Q_p \in \mathcal{L}(L_p)$ *for all* $p \in (1, \infty)$ *such that* $A = B + Q_2$*. Let* $\kappa > 0$*. Assume* S *is* κ*-ultracontractive and* $\lim_{t \downarrow 0} t^\kappa \operatorname{Tr} S_t$ *exists. Then*

$$\lim_{t \downarrow 0} t^\kappa \operatorname{Tr} T_t = \lim_{t \downarrow 0} t^\kappa \operatorname{Tr} S_t.$$

Proof. It follows from Proposition 3.1(a) that T is also κ-ultracontractive. Hence by Theorem 2.3 there exists a $c > 0$ such that

$$\|S_t\|_{p \to q} \vee \|T_t\|_{p \to q} \le c\, t^{-\kappa(\frac{1}{p} - \frac{1}{q})}$$

for all $t \in (0, 1]$ and $p, q \in [1, \infty]$ with $p \le q$.

Let $t \in (0, 1]$. Then

$$\begin{aligned} \|S_t - T_t\|_{2 \to \infty} &= \Big\| \int_0^t S_s\, Q_2\, T_{t-s}\, ds \Big\|_{2 \to \infty} \\ &\le \Big(\int_0^{t/2} + \int_{t/2}^t \Big) \|S_s\, Q_2\, T_{t-s}\|_{2 \to \infty}\, ds. \end{aligned}$$

Now if $p \in (2, \infty)$ is such that $\frac{\kappa}{p} < 1$, then

$$\begin{aligned} \int_0^{t/2} \|S_s\, Q_2\, T_{t-s}\|_{2 \to \infty}\, ds &\le \int_0^{t/2} \|S_s\|_{p \to \infty}\, \|Q_p\|_{p \to p}\, \|T_{t-s}\|_{2 \to p}\, ds \\ &\le \int_0^{t/2} c^2\, \|Q_p\|_{p \to p}\, s^{-\frac{\kappa}{p}}\, (t-s)^{-\kappa(\frac{1}{2} - \frac{1}{p})}\, ds \\ &= c_1\, t^{-\kappa/2} \cdot t, \end{aligned}$$

where $c_1 = c^2\, \|Q_p\|_{p \to p} \int_0^{1/2} s^{-\frac{\kappa}{p}}\, (1-s)^{-\kappa(\frac{1}{2} - \frac{1}{p})}\, ds < \infty$. Similarly the second term can be estimated. Hence there exists a $c_2 > 0$ such that

$$\|S_t - T_t\|_{2 \to \infty} \le c_2\, t^{-\frac{\kappa}{2}} \cdot t$$

for all $t \in (0, 1]$. Similarly there exists a $c_3 > 0$ such that

$$\|S_t - T_t\|_{1 \to 2} \le c_3\, t^{-\frac{\kappa}{2}} \cdot t$$

for all $t \in (0, 1]$.

Therefore by Lemma 2.5 it follows that

$$\begin{aligned} |t^\kappa \operatorname{Tr} S_t - t^\kappa \operatorname{Tr} T_t| &= t^\kappa \left| \operatorname{Tr} \Big((S_{t/2} - T_{t/2}) S_{t/2} + T_{t/2} (S_{t/2} - T_{t/2}) \Big) \right| \\ &\le t^\kappa\, \mu(X) \Big(\|S_{t/2} - T_{t/2}\|_{2 \to \infty} \|S_{t/2}\|_{1 \to 2} \\ &\qquad + \|T_{t/2}\|_{2 \to \infty} \|S_{t/2} - T_{t/2}\|_{1 \to 2} \Big) \\ &\le \mu(X)\, 2^{\kappa - 1}\, c\, (c_2 + c_3)\, t \end{aligned}$$

for all $t \in (0, 1]$ and the theorem follows. □

Theorem 4.1 will be essential to obtain conditions which imply that Weyl limits are preserved under perturbation. We conclude this section with comments on the hypothesis made in Theorem 4.1.

Remark 4.2. Adopt the notation and assumptions as in Theorem 4.1. Let $p \in (1, \infty)$ and let $S^{(p)}$ and $T^{(p)}$ be the C_0-semigroups on L_p which are consistent with S and T. Let $-A_p$ and $-B_p$ be the generators of $S^{(p)}$ and $T^{(p)}$. Then one automatically has $A_p = B_p + Q_p$. The reason is as follows. Let $t > 0$ and $u \in L_2 \cap L_p$. Then

$$\begin{aligned} S_t^{(p)} u - T_t^{(p)} u = S_t u - T_t u &= \int_0^t S_s(-A + B) T_{t-s} u \, ds \\ &= -\int_0^t S_s \, Q_2 \, T_{t-s} u \, ds = -\int_0^t S_s^{(p)} \, Q_p \, T_{t-s}^{(p)} u \, ds. \end{aligned}$$

Hence

$$S_t^{(p)} u - T_t^{(p)} u = -\int_0^t S_s^{(p)} \, Q_p \, T_{t-s}^{(p)} u \, ds$$

for all $t > 0$ and $u \in L_p$. Then the claim follows from the next lemma.

Lemma 4.3. *Let S and T be two C_0-semigroups on a Banach space Y with generators $-A$ and $-B$. Let $Q \in \mathcal{L}(Y)$. Suppose that*

$$S_t x - T_t x = -\int_0^t S_s \, Q \, T_{t-s} x \, ds$$

for all $t > 0$ and $x \in Y$. Then $A = B + Q$.

Proof. There exists an $M \geq 1$ such that $\|S_t\| \leq M$ for all $t \in (0, 1]$. Let $x \in Y$. There exists a $t_0 \in (0, 1]$ such that $\|T_t x - x\| \leq \varepsilon$ and $\|S_t \, Qx - Qx\| \leq \varepsilon$ for all $t \in (0, t_0]$. Let $t \in (0, t_0]$. Then

$$\begin{aligned} \left\| \frac{1}{t} \int_0^t S_s \, Q \, T_{t-s} x \, ds - Qx \right\| &\leq \frac{1}{t} \int_0^t \|S_s \, Q \, T_{t-s} x - Qx\| \, ds \\ &\leq \frac{1}{t} \int_0^t \|S_s\| \, \|Q\| \, \|T_{t-s} x - x\| + \|S_s \, Qx - Qx\| \, ds \\ &\leq (M \, \|Q\| + 1)\varepsilon. \end{aligned}$$

So $\lim_{t \downarrow 0} \frac{1}{t} \int_0^t S_s \, Q \, T_{t-s} x \, ds = Qx$.

If $x \in D(A)$, then

$$\lim_{t \downarrow 0} \tfrac{1}{t}(I - T_t)x = \lim_{t \downarrow 0} \tfrac{1}{t}(I - S_t)x + \lim_{t \downarrow 0} \tfrac{1}{t}(S_t - T_t)x = Ax - Qx.$$

Therefore $A - Q \subset B$. Since $A - Q$ generates a C_0-semigroup, one deduces that $A - Q = B$. □

One may ask whether the conditions in Theorem 4.1 can be relaxed, by requiring merely that A is a bounded perturbation of B. In fact, if one of the two semigroups has a bounded generator, then this suffices (see 7.2.2 in [3]). The following example shows that this weaker hypothesis does not suffice in general. More precisely, suppose that S and T be two C_0-semigroups on L_2 which extend consistently to C_0-semigroups $S^{(p)}$ and $T^{(p)}$ on L_p for all $p \in (1,\infty)$ with generators $-A_p$ and $-B_p$. Suppose that there exists a bounded operator $Q_2 \in \mathcal{L}(L_2)$ such that $A_2 = B_2 + Q_2$. If $p \in (1,\infty) \setminus \{2\}$, then in general there does not exists a bounded $Q_p \in \mathcal{L}(L_p)$ such that $A_p = B_p + Q_p$. A counter example is as follows.

Example 4.4. Let Δ^D be the Laplacian on $L_2(0,1)$ with Dirichlet boundary conditions. The semigroup S generated by Δ^D extends consistently to a contraction semigroup $S^{(p)}$ in $L_p(0,1)$ for all $p \in [1,\infty]$, which is a C_0-semigroup if $p \in [1,\infty)$. For all $p \in [1,\infty)$ let $-A_p$ be the generator of $S^{(p)}$. Then

$$D(A_p) = \{u \in W^{2,p}(0,1) : u(0) = u(1) = 0\} \subset C[0,1]$$

for all $p \in (1,\infty)$.

Choose $g \in L_2(0,1)$ such that $g \notin L_p(0,1)$ for all $p \in (2,\infty)$. Define the operator $B\colon L_2(0,1) \to L_2(0,1)$ by

$$Bf = (f,g)_{L_2(0,1)} \, \mathbb{1}_{(0,1)}.$$

Then B is bounded from $L_2(0,1)$ into $L_2(0,1)$, but for all $p \in [1,2)$ the operator B does not extend to a bounded operator from $L_p(0,1)$ into $L_p(0,1)$. If $p \in [1,\infty)$, then $B|_{D(A_p)}$ is bounded and compact from $D(A_p)$ into $L_p(0,1)$, where $D(A_p)$ is provided with the graph norm. Since $S^{(p)}$ is a holomorphic semigroup, it follows from Desch–Schappacher [11] Theorem 1 that $-(A_p+B)$ is the generator of a holomorphic semigroup on $L_p(0,1)$ for all $p \in [1,\infty)$. Clearly the operator $A_2-(A_2+B)$ is bounded on $L_2(0,1)$. But for all $p \in [1,2)$ the operator $A_p - (A_p + B)$ is not bounded on $L_p(0,1)$.

5. Weyl's Law for the Dirichlet-to-Neumann Operator

In this section we prove Weyl's law for the Dirichlet-to-Neumann operator on a domain with C^∞-boundary. The following result can be found in the book of Rosenberg [26], for example.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^d$ be an open connected bounded set with C^∞-boundary Γ. Denote by Δ_{LB} the Laplace–Beltrami operator on Γ. Let N^{LB} be the counting function for the positive operator $-\Delta_{LB}$. Then*

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/2} \, N^{LB}(\lambda) = \frac{\sigma(\Gamma)}{(4\pi)^{(d-1)/2} \, \Gamma(\frac{d+1}{2})},$$

where $\sigma(\Gamma)$ is the $(d-1)$-dimensional Hausdorff measure on Γ.

Proof. It follows from Rosenberg [26] Theorem 3.24 that

$$\lim_{t\downarrow 0} t^{(d-1)/2} \, \mathrm{Tr}\, S_t^{LB} = \frac{\sigma(\Gamma)}{(4\pi)^{(d-1)/2}}.$$

Then the theorem is a consequence of Karamata's Tauberian theorem, Proposition 2.1. □

Now we are able to prove Weyl's law for the Dirichlet-to-Neumann operator D_0 on $L_2(\Gamma)$ associated to the potential $V = 0$ in (1). This result is due to Sandgren [27].

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^d$ be a non-empty open connected bounded subset with C^∞-boundary Γ. Let N be the counting function of the eigenvalues of the Dirichlet-to-Neumann operator on $L_2(\Gamma)$. Then*

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)} N(\lambda) = \frac{\sigma(\Gamma)}{(4\pi)^{(d-1)/2}\,\Gamma(\frac{d+1}{2})}.$$

Proof. We use the notation as in Theorem 5.1. Let $N^{\sqrt{LB}}$ denote the counting function for $\sqrt{-\Delta_{LB}}$. Then $N^{\sqrt{LB}}(\lambda) = N^{LB}(\lambda^2)$ for all $\lambda > 0$. Hence it follows from Theorem 5.1 that

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)} N^{\sqrt{LB}}(\lambda) = \frac{\sigma(\Gamma)}{(4\pi)^{(d-1)/2}\,\Gamma(\frac{d+1}{2})}.$$

Then Karamata's Proposition 2.1 gives

$$\lim_{t\downarrow 0} t^{d-1}\,\mathrm{Tr}\, S_t^{\sqrt{LB}} = \Gamma(d)\,\frac{\sigma(\Gamma)}{(4\pi)^{(d-1)/2}\,\Gamma(\frac{d+1}{2})}, \tag{5}$$

where $S^{\sqrt{LB}}$ is the semigroup generated by $-\sqrt{-\Delta_{LB}}$.

It follows from (C.4) or Proposition C.1 in Appendix C of Chapter 12 in [29] that there exists a pseudo-differential operator Q of order 0 such that $D_0 = \sqrt{-\Delta_{LB}} + Q$. For all $p \in (1,\infty)$ the operator Q extends to a bounded operator Q_p on $L_p(\Gamma)$ by [28] Proposition VI.4 and a coordinate transformation. Let S be the semigroup generated by $-D_0$. Then S satisfies the bounds (4) with $\kappa = d-1$ by [13] Theorem 2.6. Then by perturbation, Theorem 4.1, one deduces from (5) that

$$\lim_{t\downarrow 0} t^{d-1}\,\mathrm{Tr}\, S_t = \Gamma(d)\,\frac{\sigma(\Gamma)}{(4\pi)^{(d-1)/2}\,\Gamma(\frac{d+1}{2})}$$

and the theorem follows by using Karamata's theorem again. □

If Ω has merely a Lipschitz boundary, then Weyl upper bounds are valid.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded connected set with Lipschitz boundary Γ. Consider the semigroup S generated by $-D_0$, where D_0 is the Dirichlet-to-Neumann operator.

Proposition 5.3. *There exists a constant $b > 0$ such that $t^{d-1}\,\mathrm{Tr}\, S_t \le b$ for all $t \in (0, 1]$.*

Proof. By [13] Theorem 2.6 the semigroup S is $(d-1)$-ultracontractive. Then the upper Weyl bounds are a consequence of Proposition 2.6. □

In the next proposition we prove a lower bound, under the assumption that the kernel of S satisfies Poisson upper bounds. It is an open problem whether the kernel of S has Poisson upper bounds if the domain Ω merely has a Lipschitz boundary. If Ω has a C^∞-boundary, then these Poisson upper bounds have been proved in [13] Theorem 1.1 and independently in [16] Theorem 1. In work in progress, [14], these upper bounds are also proved if Ω

has a $C^{1,\varepsilon}$-boundary, where $\varepsilon \in (0,1]$. Since S is a positive semigroup (see [6] page 67), the kernel of S is positive.

Proposition 5.4. *Suppose the kernel K of the semigroup S satisfies Poisson upper bounds, that is, there exists a $c > 0$ such that*

$$K_t(x,y) \leq c\,(t \wedge 1)^{-(d-1)} \frac{1}{\left(1 + \dfrac{|x-y|}{t}\right)^d}$$

for all $x, y \in \Gamma$ and $t \in (0,\infty)$. Then there exists an $a > 0$ such that $a \leq t^{d-1} \operatorname{Tr} S_t$ for all $t \in (0,1]$.

Proof. Set

$$c' = \sup_{t\in(0,1]} \sup_{x\in\Gamma} t^{-(d-1)} \int_\Gamma \frac{1}{\left(1 + \dfrac{|x-y|}{t}\right)^{d-\frac{1}{2}}} \, d\sigma(y).$$

Then $c' < \infty$ by a quadrature estimate. Let $r \geq 1$ be such that $\frac{c\,c'}{\sqrt{1+r}} \leq \frac{1}{2}$. There exists a $c'' > 0$ such that $\sigma(B_\Gamma(x,rt)) \leq c''\, t^{d-1}$ for all $x \in \Gamma$ and $t \in (0,1]$.

Now let $t \in (0,1]$. Then

$$\begin{aligned}
\int_{\Gamma\setminus B_\Gamma(x,rt)} K_t(x,y)\, d\sigma(y) &\leq \int_{\Gamma\setminus B_\Gamma(x,rt)} c\, t^{-(d-1)} \frac{1}{\left(1 + \dfrac{|x-y|}{t}\right)^d} \, d\sigma(y) \\
&\leq c\, \frac{1}{\sqrt{1+r}} \int_\Gamma t^{-(d-1)} \frac{1}{\left(1 + \dfrac{|x-y|}{t}\right)^{d-\frac{1}{2}}} \, d\sigma(y) \\
&\leq c\, c' \frac{1}{\sqrt{1+r}} \leq \frac{1}{2}
\end{aligned}$$

for all $x \in \Gamma$. Hence

$$\int_{B_\Gamma(x,rt)} K_t(x,y)\, d\sigma(y) = 1 - \int_{\Gamma\setminus B_\Gamma(x,rt)} K_t(x,y)\, d\sigma(y) \geq \frac{1}{2}$$

for almost every $x \in \Gamma$, since $\int_\Gamma K_t(x,y)\, d\sigma(y) = (S_t \mathbb{1})(x) = \mathbb{1}(x) = 1$ for almost every $x \in \Gamma$. Therefore

$$\begin{aligned}
\operatorname{Tr} S_{2t} = \|S_t\|_{\mathrm{HS}}^2 &= \int_\Gamma \int_\Gamma |K_t(x,y)|^2 \, d\sigma(y)\, d\sigma(x) \\
&\geq \int_\Gamma \int_{B_\Gamma(x,rt)} |K_t(x,y)|^2 \, d\sigma(y)\, d\sigma(x) \\
&\geq \int_\Gamma \frac{1}{\sigma(B_\Gamma(x,rt))} \left| \int_{B_\Gamma(x,rt)} K_t(x,y)\, d\sigma(y) \right|^2 d\sigma(x)
\end{aligned}$$

$$\geq \frac{1}{4} \int_{\Gamma} \frac{1}{\sigma(B_{\Gamma}(x, rt))} \, d\sigma(x)$$
$$\geq \frac{\sigma(\Gamma)}{4c''} t^{-(d-1)}$$

and the proof is complete. □

So if the kernel of S satisfies Poisson upper bounds, then it follows from Propositions 5.3 and 5.4 that there are $a, b > 0$ such that

$$a \leq t^{d-1} \operatorname{Tr} S_t \leq b$$

for all $t \in (0, 1]$.

One might hope that in general for a positive self-adjoint operator A for which there are constants $a, b, \kappa > 0$ such that $S_t = e^{-tA}$ is trace class for all $t > 0$ and

$$a \leq t^{\kappa} \operatorname{Tr} S_t \leq b$$

for all $t \in (0, 1]$, it would follow that $\lim_{t \downarrow 0} t^{\kappa} \operatorname{Tr} S_t$ exists. Unfortunately, this is false by the following counter example.

Example 5.5. We first show that there are $N_0, N_1, \ldots \in \mathbb{N}_0$ and $t_0, t_1, \ldots \in (0, 1]$ such that $1 = t_0$, $t_{n+1} < t_n \leq 2^{-n}$ for all $n \in \mathbb{N}_0$, $0 = N_0 < N_1 < N_2 < \ldots$ and if $\lambda_1, \lambda_2, \ldots \in [1, \infty)$ are defined by

$$\lambda_n = \begin{cases} n & \text{if } N_{2k} < n \leq N_{2k+1}, \\ 2n & \text{if } N_{2k+1} < n \leq N_{2k+2}, \end{cases} \tag{6}$$

for all $k \in \mathbb{N}_0$, then

$$\Big| t_{2k+1} \sum_{n=1}^{N_{2k+1}} e^{-\lambda_n t_{2k+1}} - 1 \Big| \leq 2^{-k}, \tag{7}$$

$$\sum_{n=N_{2k+1}+1}^{\infty} e^{-n t_{2k+1}} \leq 2^{-k},$$

$$\Big| t_{2k+2} \sum_{n=1}^{N_{2k+2}} e^{-\lambda_n t_{2k+2}} - \frac{1}{2} \Big| \leq 2^{-k}, \text{ and} \tag{8}$$

$$\sum_{n=N_{2k+2}+1}^{\infty} e^{-n t_{2k+2}} \leq 2^{-k}$$

for all $k \in \mathbb{N}_0$. The proof is by induction.

Set $N_0 = 0$ and $t_0 = 1$. Let $k \in \mathbb{N}_0$ and suppose that N_{2k} and t_{2k} are defined. For all $n \in \mathbb{N}$ define $\lambda'_n \in [1, \infty)$ by

$$\lambda'_n = \begin{cases} n & \text{if there exists an } l \in \{0, \ldots, k-1\} \text{ such that } N_{2l} < n \leq N_{2l+1}, \\ 2n & \text{if there exists an } l \in \{0, \ldots, k-1\} \text{ such that } N_{2l+1} < n \leq N_{2l+2}, \\ n & \text{if } n > N_{2k}. \end{cases}$$

Define $A\colon (0,\infty) \to \mathbb{R}$ by

$$A(t) = \sum_{n=1}^{\infty} e^{-\lambda'_n t}.$$

Since $\lim_{t\downarrow 0} t \sum_{n=N_{2k}+1}^{\infty} e^{-nt} = 1$, it follows that $\lim_{t\downarrow 0} |t\, A(t) - 1| = 0$. Hence there exists a $t_{2k+1} \in (0, t_{2k} \wedge 2^{-(2k+1)})$ such that

$$|t_{2k+1}\, A(t_{2k+1}) - 1| \le 2^{-(k+1)}. \tag{9}$$

Next, there exists an $N_{2k+1} \in \mathbb{N}$ such that $N_{2k} < N_{2k+1}$ and

$$\sum_{n=N_{2k+1}+1}^{\infty} e^{-nt_{2k+1}} \le 2^{-(k+1)}. \tag{10}$$

Then (7) follows from the triangle inequality from (9) and (10).

For all $n \in \mathbb{N}$ define $\lambda''_n \in [1,\infty)$ by

$$\lambda''_n = \begin{cases} n & \text{if there exists an } l \in \{0,\ldots,k\} \text{ such that } N_{2l} < n \le N_{2l+1}, \\ 2n & \text{if there exists an } l \in \{0,\ldots,k-1\} \text{ such that } N_{2l+1} < n \le N_{2l+2}, \\ 2n & \text{if } n > N_{2k+1}. \end{cases}$$

Further, define $B\colon (0,\infty) \to \mathbb{R}$ by

$$B(t) = \sum_{n=1}^{\infty} e^{-\lambda''_n t}.$$

Since $\lim_{t\downarrow 0} t \sum_{n=N_{2k+1}+1}^{\infty} e^{-2nt} = \frac{1}{2}$, it follows that $\lim_{t\downarrow 0} |t\, B(t) - \frac{1}{2}| = 0$. Hence there exists a $t_{2k+2} \in (0, t_{2k+1} \wedge 2^{-(2k+2)})$ such that

$$\left| t_{2k+2}\, B(t_{2k+2}) - \frac{1}{2} \right| \le 2^{-(k+1)}.$$

Next, there exists an $N_{2k+2} \in \mathbb{N}$ such that $N_{2k+1} < N_{2k+2}$ and

$$\sum_{n=N_{2k+2}+1}^{\infty} e^{-nt_{2k+2}} \le 2^{-(k+1)}.$$

(Note that the term is $e^{-nt_{2k+2}}$, not $e^{-2nt_{2k+2}}$.) Then

$$\begin{aligned} \left| t_{2k+2} \sum_{n=1}^{N_{2k+2}} e^{-\lambda_n t_{2k+2}} - \frac{1}{2} \right| &\le 2^{-(k+1)} + t_{2k+2} \sum_{n=N_{2k+2}+1}^{\infty} e^{-2nt_{2k+2}} \\ &\le 2^{-(k+1)} + \sum_{n=N_{2k+2}+1}^{\infty} e^{-nt_{2k+2}} \le 2^{-k} \end{aligned}$$

and (8) is valid.

By induction $t_0, t_1, \ldots$ and $N_0, N_1, \ldots$ are defined. Define $\lambda_1, \lambda_2, \ldots$ by (6). Let A be the self-adjoint multiplication operator in ℓ_2 such that $Ae_n = \lambda_n\, e_n$ for all $n \in \mathbb{N}$. Let S be the semigroup generated by $-A$.

If $k \in \mathbb{N}$, then

$$\left| t_{2k+2} \sum_{n=1}^{\infty} e^{-\lambda_n t_{2k+2}} - \frac{1}{2} \right| \leq \left| t_{2k+2} \sum_{n=1}^{N_{2k+2}} e^{-\lambda_n t_{2k+2}} - \frac{1}{2} \right| + t_{2k+2} \sum_{n=N_{2k+2}+1}^{\infty} e^{-\lambda_n t_{2k+2}}$$
$$\leq 2^{-k} + \sum_{n=N_{2k+2}+1}^{\infty} e^{-n t_{2k+2}} \leq 2^{-(k-1)}.$$

Similarly,

$$\left| t_{2k+1} \sum_{n=1}^{\infty} e^{-\lambda_n t_{2k+1}} - 1 \right| \leq 2^{-(k-1)}.$$

So

$$\lim_{k \to \infty} t_{2k+1} \operatorname{Tr} S_{t_{2k+1}} = 1$$

and

$$\lim_{k \to \infty} t_{2k+2} \operatorname{Tr} S_{t_{2k+2}} = \frac{1}{2}.$$

Therefore $\lim_{t \downarrow} t \operatorname{Tr} S_t$ does not exists.

It is elementary to show that

$$\tfrac{1}{2} e^{-2} \leq t \operatorname{Tr} S_t \leq 1$$

for all $t \in (0, 1]$.

6. Weyl's Law for the Dirichlet-to-Neumann Operator with Potential

Let $\Omega \subset \mathbb{R}^d$ be open bounded connected with Lipschitz boundary. Since many intermediate results in this section are valid on domains which do not have a C^∞-boundary, we state the lemmas with appropriate conditions on the boundary. Let $\Gamma = \partial\Omega$ and we provide Γ with the $(d-1)$-dimensional surface measure. We recall that for all $p \in [1, \infty)$ there exists a unique bounded operator $\operatorname{Tr} : W^{1,p}(\Omega) \to L_p(\Gamma)$ such that $\operatorname{Tr} u = u|_\Gamma$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. The operator Tr is called the *trace operator*.

Let $p \in [1, \infty)$ and $u \in W^{1,p}(\Omega)$ with $\Delta u \in L_p(\Omega)$. If $\psi \in L_p(\Gamma)$, then we write $\partial_\nu u = \psi$ if

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (\Delta u)\, \overline{v} = \int_\Gamma \psi\, \overline{v} \qquad (11)$$

for all $v \in C_c^\infty(\mathbb{R}^d)$. Note that ψ is unique, if it exists. We say that $\partial_\nu u \in L_p(\Gamma)$ if there exists a $\psi \in L_p(\Gamma)$ such that $\partial_\nu u = \psi$. Then $\partial_\nu u$ is called the *normal derivative* of u. Let p' be the dual exponent of p. If $p \neq 1$, then it follows that (11) is valid for all $v \in W^{1,p'}(\Omega)$ since $\{v|_\Omega : v \in C_c^\infty(\mathbb{R}^d)\}$ is dense in $W^{1,p'}(\Omega)$.

Remark 6.1. Let $p \in (1, \infty)$ and $u \in W^{2,p}(\Omega)$. Then $\partial_\nu u \in L_p(\Gamma)$. In fact, denote by $\nu \colon \Gamma \to \mathbb{R}^d$ the exterior normal. Thus $\nu = (\nu_1, \ldots, \nu_d) \in (L_\infty(\Gamma))^d$. Let $\psi = \sum_{j=1}^d \nu_j \operatorname{Tr}(\partial_j u)$. Then it follows from the divergence theorem, see for example [1] A6.8(1), that $\partial_\nu u = \psi$.

Denote by Δ^D the *Dirichlet Laplacian* in $L_2(\Omega)$, i.e., Δ^D is the operator in $L_2(\Omega)$ defined by

$$\operatorname{dom}(\Delta^D) = \big\{u \in H_0^1(\Omega) : \Delta u \in L_2(\Omega)\big\}$$

and $\Delta^D u = \Delta u$, the distributional derivative. If $\mathfrak{a}^D \colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ is the sesquilinear form defined by

$$\mathfrak{a}^D(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v},$$

then $-\Delta^D$ is the operator associated with $\mathfrak{a}^D$. Hence Δ^D is self-adjoint and $-\Delta^D$ is positive.

There is a remarkable result on the normal derivative of functions in the domain $\operatorname{dom}(\Delta^D)$ of the Dirichlet Laplacian, which is a consequence of results of Jerison–Kenig and Gesztesy–Mitrea.

Proposition 6.2. *If* $u \in \operatorname{dom}(\Delta^D)$*, then* $\partial_\nu u \in L_2(\Gamma)$*.*

Proof. If $u \in \operatorname{dom}(\Delta^D)$, then $u \in H^{3/2}(\Omega)$ by [18] Theorem B.2. This implies that $\partial_\nu u \in L_2(\Gamma)$ by [15] Lemma 2.4. □

Let $V \in L_\infty(\Omega, \mathbb{R})$ be a real-valued potential. We emphasize that we do not assume that V is positive. Clearly $\Delta^D - V$ generates a holomorphic C_0-semigroup on $L_2(\Omega)$. We will assume throughout that

$$0 \notin \sigma(-\Delta^D + V). \tag{12}$$

Since $-\Delta^D + V$ has compact resolvent, this is equivalent with 0 not being an eigenvalue of $-\Delta^D + V$. Then it follows that $(-\Delta^D + V)^{-1} \in \mathcal{L}(L_2(\Omega))$ with range $\operatorname{dom}(\Delta^D)$. Hence Proposition 6.2 implies that $\partial_\nu(-\Delta^D + V)^{-1}$ is a linear map from $L_2(\Omega)$ into $L_2(\Gamma)$. Actually, this map is bounded.

Lemma 6.3. $\partial_\nu(-\Delta^D + V)^{-1} \in \mathcal{L}(L_2(\Omega), L_2(\Gamma))$.

Proof. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $L_2(\Omega)$ and $\psi \in L_2(\Gamma)$. Suppose that $\lim w_n = 0$ in $L_2(\Omega)$ and $\lim \partial_\nu(-\Delta^D + V)^{-1} w_n = \psi$ in $L_2(\Gamma)$. Set $u_n = (-\Delta^D + V)^{-1} w_n$ for all $n \in \mathbb{N}$. Then $u_n \in D(\Delta^D) \subset H_0^1(\Omega)$ and $-\Delta^D u_n = w_n - V u_n$ for all $n \in \mathbb{N}$. Since $0 \notin \sigma(-\Delta^D + V)$ it follows that $\lim u_n = 0$ in $L_2(\Omega)$. Because

$$\int_\Omega |\nabla u_n|^2 = (-\Delta^D u_n, u_n)_{L_2(\Omega)} = (w_n - V u_n, u_n)_{L_2(\Omega)}$$

for all $n \in \mathbb{N}$, one deduces that $\lim \int_\Omega |\nabla u_n|^2 = 0$. Let $v \in H^1(\Omega)$. Then

$$\begin{aligned}(\psi, \operatorname{Tr} v)_{L_2(\Gamma)} &= \lim_{n\to\infty} (\partial_\nu u_n, \operatorname{Tr} v)_{L_2(\Gamma)} \\ &= \lim_{n\to\infty} \int_\Omega \nabla u_n \cdot \overline{\nabla v} + \int_\Omega (\Delta u_n)\,\overline{v} \\ &= \lim_{n\to\infty} \int_\Omega \nabla u_n \cdot \overline{\nabla v} - \int_\Omega (w_n - V\,u_n)\,\overline{v} = 0.\end{aligned}$$

Hence $\psi = 0$. By the closed graph theorem the lemma follows. □

We next consider the Dirichlet problem. Let $\varphi \in L_2(\Gamma)$ and $u \in H^1(\Omega)$. We say that u is a *solution of the Dirichlet problem (13)* if

$$\begin{bmatrix} -\Delta u + V\,u = 0 \text{ as distribution on } \Omega \quad \text{and} \\ \operatorname{Tr} u = \varphi. \end{bmatrix} \tag{13}$$

Clearly a necessary condition is that

$$\varphi \in \operatorname{Tr} H^1(\Omega) = \{\operatorname{Tr} v : v \in H^1(\Omega)\},$$

the *trace space* of Ω. Obviously, if $\varphi \in \operatorname{Tr} H^1(\Omega)$ and $u \in H^1(\Omega)$, then u is a solution of the Dirichlet problem (13) if and only if $\operatorname{Tr} u = \varphi$ and

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega V\,u\,\overline{v} = 0$$

for all $v \in H_0^1(\Omega)$.

The following consequence of Proposition 6.2 is proved in [9] Corollary 5.4. We include a proof for completeness.

Proposition 6.4. *For all $\varphi \in \operatorname{Tr} H^1(\Omega)$ there exists a unique solution $u \in H^1(\Omega)$ of the Dirichlet problem* (13)*. Moreover, there exists a bounded operator*

$$\gamma_V : L_2(\Gamma) \to L_2(\Omega)$$

such that $\gamma_V \varphi$ is the solution of (13) *for all $\varphi \in \operatorname{Tr} H^1(\Omega)$. Finally,*

$$\gamma_V = -\Big(\partial_\nu(-\Delta^D + V)^{-1}\Big)^*.$$

Note that $\partial_\nu(-\Delta^D + V)^{-1} \in \mathcal{L}(L_2(\Omega), L_2(\Gamma))$ by Lemma 6.3.

Proof. Let $\varphi \in \operatorname{Tr} H^1(\Omega)$. Then there exists a $u_0 \in H^1(\Omega)$ such that $\varphi = \operatorname{Tr} u_0$. Define the form $\mathfrak{a}_V^D : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}_V^D(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega V\,u\,\overline{v}.$$

Then $\mathfrak{a}_V^D$ is $L_2(\Omega)$-elliptic and continuous. We show that $\mathfrak{a}_V^D$ is not degenerate. Let $u \in H_0^1(\Omega)$ and suppose that $\mathfrak{a}_V^D(u, v) = 0$ for all $v \in H_0^1(\Omega)$. Then $(-\Delta^D + V)u = 0$. So $0 \in \sigma(-\Delta^D + V)$, which contradicts the assumption (12). Define $\alpha : H_0^1(\Omega) \to \mathbb{C}$ by

$$\alpha(v) = \int_\Omega \nabla u_0 \cdot \overline{\nabla v} + \int_\Omega V\,u_0\,\overline{v}.$$

Then α is continuous and anti-linear. Hence by the Fredholm–Lax–Milgram lemma, [7] Lemma 4.1, there exists a unique $w \in H_0^1(\Omega)$ such that

$$\mathfrak{a}_V^D(w, v) = \alpha(v)$$

for all $v \in H_0^1(\Omega)$. Define $u = u_0 - w$. Then

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega V\, u\, \overline{v} = 0$$

for all $v \in H_0^1(\Omega)$ and $\operatorname{Tr} u = \varphi$. So u is a solution of the Dirichlet problem (13). Since $-\Delta^D + V$ is injective, it is the only solution.

Let $\varphi \in \operatorname{Tr} H^1(\Omega)$. Let $u \in H^1(\Omega)$ be the solution of the Dirichlet problem (13). Then $(-\Delta + V)u = 0$ weakly on Ω and $\operatorname{Tr} u = \varphi$. Let $w \in L_2(\Omega)$ and write $v = (-\Delta^D + V)^{-1}w$. Then $v \in \operatorname{dom}(\Delta^D) \subset H_0^1(\Omega)$ and $\partial_\nu v \in L_2(\Gamma)$ by Proposition 6.2. Moreover, $-\Delta v + V\, v = w$ as distribution. Note that $\int_\Omega (Vu)\, \overline{\tau} = \int_\Omega (\Delta u)\, \overline{\tau} = \int_\Omega u\, \overline{\Delta \tau} = -\int_\Omega \nabla u \cdot \overline{\nabla \tau}$ for all $\tau \in C_c^\infty(\Omega)$. Approximating v by C_c^∞-functions gives $\int_\Omega (Vu)\, \overline{v} = -\int_\Omega \nabla u \cdot \overline{\nabla v}$. Then

$$\begin{aligned}
\int_\Gamma \varphi\, \overline{\partial_\nu (-\Delta^D + V)^{-1} w} &= \int_\Gamma (\operatorname{Tr} u)\, \overline{\partial_\nu v} \\
&= \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega u\, \overline{\Delta v} \\
&= \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega u\, \overline{V\, v - w} \\
&= -\int_\Omega u\, \overline{w}.
\end{aligned}$$

Therefore $(\partial_\nu(-\Delta^D + V)^{-1})^* \varphi = -u$. Since $(\partial_\nu(-\Delta^D + V)^{-1})^*$ is continuous by Lemma 6.3, the proposition follows. □

We now consider the Dirichlet-to-Neumann operator D_V in $L_2(\Gamma)$ which is the main object of our study in this section. It was defined in (1) and it has a characterisation via forms. Define the form $\mathfrak{a}_V \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}_V(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega V\, u\, \overline{v}.$$

Then D_V is the operator in $L_2(\Gamma)$ associated with the pair $(\mathfrak{a}_V, \operatorname{Tr})$, see [7] Sections 2 and 7. In particular, the operator D_V is self-adjoint and lower bounded by [7] Theorems 4.5 and 4.15. Of special interest is D_0, the Dirichlet-to-Neumann operator with $V = 0$ which we considered in the previous section, since $-D_0$ generates a submarkovian semigroup on $L_2(\Gamma)$ and therefore has a canonical extension to $L_p(\Gamma)$ for all $p \in [1, \infty]$, see [6] Section 4.4. If $V \geq 0$ then the semigroup generated by $-D_V$ is ultracontractive by [13] Theorem 2.6. For general $V \in L_\infty(\Omega, \mathbb{R})$ the kernel expansion of Theorem 2.7 has an immediate consequence for the kernel of the semigroup generated by $-D_V$ in case this semigroup is ultracontractive.

Theorem 6.5. *Let $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Let $V \in L_\infty(\Omega, \mathbb{R})$. Suppose that $0 \notin \sigma(-\Delta^D + V)$ and that the semigroup S^V generated by $-D_V$ is ultracontractive. Then the kernel of S^V is continuous.*

Proof. Let $\varphi \in \mathrm{dom}(D_V)$ be an eigenfunction with eigenvalue μ. Then there exists a $u \in H^1(\Omega)$ such that $\mathrm{Tr}\, u = \varphi$, $(-\Delta + V)u = 0$ weakly on Ω and $\partial_\nu u = \mu\, \varphi$. Hence

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega V\, u\, \overline{v} = \int_\Gamma \mu\, \mathrm{Tr}\, u\, \overline{\mathrm{Tr}\, v}$$

for all $v \in H^1(\Omega)$. By [24] Theorem 3.14(ii) we conclude that $u \in C(\overline{\Omega})$. Then $\varphi = \mathrm{Tr}\, u \in C(\Gamma)$. Now the theorem follows from Theorem 2.7. □

Corollary 6.6. *Let $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Let $V \in L_\infty(\Omega, \mathbb{R})$. Suppose that $0 \notin \sigma(-\Delta^D + V)$ and that the semigroup S^V generated by $-D_V$ is ultracontractive. Then $S^V_t L_1(\Gamma) \subset C(\Gamma)$ for all $t > 0$.*

We shall now prove that the operator D_V is a bounded perturbation of D_0. Recall that $\gamma_0, \gamma_V : L_2(\Gamma) \to L_2(\Omega)$ are the bounded operators introduced in Proposition 6.4. We denote by M_V the multiplication operator defined by the function V on $L_p(\Omega)$, where $p \in [1, \infty]$ will be clear from the context. Then Proposition 6.4 implies that the operator $(\gamma_0)^*\, M_V\, \gamma_V$ is bounded on $L_2(\Gamma)$.

Proposition 6.7. $D_V = D_0 + (\gamma_0)^*\, M_V\, \gamma_V$.

Proof. Let $\varphi \in \mathrm{dom}(D_V)$ and $\psi \in \mathrm{dom}(D_0)$. Set $u = \gamma_V \varphi$ and $v = \gamma_0 \psi$. Then

$$\begin{aligned}(D_V \varphi, \psi)_{L_2(\Gamma)} - (\varphi, D_0 \psi)_{L_2(\Gamma)} &= \mathfrak{a}_V(u, v) - \mathfrak{a}_0(u, v) \\ &= \int_\Gamma V\, u\, \overline{v} = (M_V\, \gamma_V \varphi, \gamma_0 \psi)_{L_2(\Gamma)} \\ &= ((\gamma_0)^*\, M_V\, \gamma_V \varphi, \psi)_{L_2(\Gamma)}.\end{aligned}$$

The operator $(\gamma_0)^*\, M_V\, \gamma_V$ is bounded on $L_2(\Gamma)$. Hence $\varphi \in \mathrm{dom}(D_0^*) = \mathrm{dom}(D_0)$ and similarly $\psi \in \mathrm{dom}(D_V)$. Therefore

$$\begin{aligned}((D_V - D_0)\varphi, \psi)_{L_2(\Gamma)} &= (D_V \varphi, \psi)_{L_2(\Gamma)} - (\varphi, D_0 \psi)_{L_2(\Gamma)} \\ &= ((\gamma_0)^*\, M_V\, \gamma_V \varphi, \psi)_{L_2(\Gamma)}\end{aligned}$$

and the proposition follows by density of $\mathrm{dom}(D_0)$ in $L_2(\Gamma)$. □

Remark 6.8. In [9] Theorem 5.2 the equality $\mathrm{dom}(D_V) = \mathrm{dom}(D_0) = H^1(\Gamma)$ is proved.

Since the semigroup S generated by $-D_0$ is submarkovian, it follows that for all $p \in [1, \infty]$ the semigroup S extends consistently to a contraction semigroup on $L_p(\Gamma)$ and this semigroup is a C_0-semigroup for all $p \in [1, \infty)$. We denote the generator by $-D_{0,p}$. We wish to extend Proposition 6.7 to $L_p(\Gamma)$ and also to prove that the semigroup generated by $-D_V$ extends consistently to a C_0-semigroup for all $p \in [1, \infty)$.

If $p \in [1, \infty)$, then the Beurling–Deny criteria imply that the semigroup generated by the Dirichlet Laplacian Δ^D also extends consistently to a C_0-semigroup on $L_p(\Omega)$, whose generator we denote by Δ_p^D. Since $V \in L_\infty(\Omega, \mathbb{R})$, the semigroup generated by $-(-\Delta^D + V)$ extends consistently to a quasi-contractive semigroup on $L_p(\Omega)$ for all $p \in [1, \infty]$, which is a C_0-semigroup for all $p \in [1, \infty)$. The generator is $-(-\Delta_p^D + V)$. By [5] Theorem 3.1 the semigroup generated by $-(-\Delta^D + V)$ has Gaussian kernel bounds. Hence it follows from [2] Corollary 4.3 (or Kunstmann–Vogt [21] Proposition 4) that $\sigma(-\Delta_p^D + V) = \sigma(-\Delta_p^D + V)$ for all $p \in [1, \infty)$. In particular, the operator $-\Delta_p^D + V$ is invertible because $0 \notin \sigma(-\Delta^D + V)$ by assumption (12).

We need regularity properties of the operators $-\Delta_p^D + V$. If Ω has a $C^{1,1}$-boundary, then

$$\operatorname{dom}\left(\Delta_p^D\right) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

for all $p \in (1, \infty)$ by [17] Theorem 2.4.2.5. Hence if Ω has a $C^{1,1}$-boundary and $p \in (1, \infty)$, then

$$\operatorname{dom}(-\Delta_p^D + V) = \operatorname{dom}(\Delta_p^D) \subset W^{2,p}(\Omega)$$

and the operator $(-\Delta_p^D + V)^{-1}$ is continuous from $L_p(\Omega)$ into $W^{2,p}(\Omega)$.

We next show that the operator $(\gamma_0)^* \, M_V \, \gamma_V$ extends consistently to a bounded operator on $L_p(\Gamma)$ for all $p \in [1, \infty]$, thus including the two end-points.

Lemma 6.9. *Suppose Ω has a $C^{1,1}$-boundary. Then one has the following.*

(a) *Let $p \in (1, \infty)$ and $q \in [p, \infty]$ be such that $\frac{1}{q} > \frac{1}{p} \frac{d-p}{d-1}$. Then $(\gamma_V)^*$ extends consistently to a bounded operator from $L_p(\Omega)$ into $L_q(\Gamma)$.*

(b) *Let $p \in (1, \infty)$. Then γ_V extends consistently to a bounded operator from $L_p(\Omega)$ into $L_p(\Gamma)$.*

(c) *Let $p \in [1, \infty]$. Then the operator $(\gamma_0)^* \, M_V \, \gamma_V$ extends consistently to a bounded operator on $L_p(\Gamma)$.*

Proof. '(a)'. The map $(-\Delta_p^D + V)^{-1}$ is continuous from $L_p(\Omega)$ into $W^{2,p}(\Omega)$ and ∂_ν is continuous from $W^{2,p}(\Omega)$ into $W^{1-\frac{1}{p},p}(\Gamma)$ by [22] Theorem 2.5.5, where we used that Ω has a $C^{1,1}$-boundary. Now use the Sobolev embedding theorem [22] Theorems 2.4.2 and 2.4.6.

'(b)'. This follows from the previous statement and duality.

'(c)'. Let $p_1 \in (d, \infty)$. The embedding $L_\infty(\Gamma) \to L_{p_1}(\Gamma)$ is continuous and γ_V extends continuously from $L_{p_1}(\Gamma)$ into $L_{p_1}(\Omega)$ by Statement (b). Clearly M_V is continuous from $L_{p_1}(\Omega)$ into $L_{p_1}(\Omega)$. Finally, the operator $(\gamma_V)^* = -\partial_\nu \, (-\Delta_p^D + V)^{-1}$ extends consistently to a continuous operator from $L_{p_1}(\Omega)$ into $L_\infty(\Gamma)$ by Statement (a). So $(\gamma_0)^* \, M_V \, \gamma_V$ extends consistently to a bounded operator from $L_\infty(\Gamma)$ into $L_\infty(\Gamma)$. Then the statement follows by duality and interpolation. □

Proposition 6.10. *Suppose Ω has a $C^{1,1}$-boundary. Then for all $p \in [1, \infty]$ the semigroup T generated by $-D_V$ extends consistently to a quasi-contractive* *semigroup on $L_p(\Gamma)$, which is a C_0-semigroup if $p \in [1,\infty)$. Moreover, there exists a $c > 0$ such that*

$$\|T_t\|_{p\to q} \le c\, t^{-(d-1)(\frac{1}{p}-\frac{1}{q})} \tag{14}$$

for all $t \in (0,1]$ and $p, q \in [1,\infty]$ with $p \le q$.

Proof. Recall that the semigroup S generated by $-D_0$ is submarkovian, the semigroup S extends consistently to a contraction semigroup $S^{(p)}$ on $L_p(\Gamma)$ for all $p \in [1,\infty]$ and this semigroup is a C_0-semigroup for all $p \in [1,\infty)$. We denoted the generator by $-D_{0,p}$.

It follows from Lemma 6.9(c) that for all $p \in [1,\infty]$ there exists a bounded operator Q_p on $L_p(\Gamma)$ such that Q_p is consistent with the operator $(\gamma_0)^* M_V \gamma_V$. Also $D_V = D_0 + Q_2$ by Proposition 6.7. Let $p \in [1,\infty)$. Let $T^{(p)}$ be the C_0-semigroup generated by $-(D_{0,p} + Q_p)$. Then $T^{(p)}$ is a quasi-contractive semigroup. Moreover,

$$T_t^{(p)} \varphi = \lim_{n\to\infty} \Big(S_{t/n}^{(p)} \, e^{-\frac{t}{n} Q_p} \Big)^n \varphi$$

for all $t > 0$ and $\varphi \in L_p(\Gamma)$. Since $D_V = D_0 + Q_2$ it follows that $T^{(p)}$ is consistent with the semigroup T. If $p = \infty$ then we define $T_t^{(\infty)} = (T_t^{(1)})^*$ for all $t > 0$. Then $T^{(\infty)}$ is a semigroup on $L_\infty(\Gamma)$ which is consistent with T.

Since S satisfies the ultracontractivity bounds (4) with $\kappa = d - 1$ by [13] Theorem 2.6 and Q_2 is bounded, it follows from Proposition 3.1(a) that the ultracontractivity bounds (14) for T are valid. □

Remark 6.11. If $d \ge 3$, then as in the proof of [13] Theorem 2.6 we can deduce $(d-1)$-ultracontractivity of the semigroup generated by $-D_0$ by the criterion formulated in Theorem 2.3(c). In fact, the form domain of D_0 is $\mathrm{dom}(D_0^{1/2}) = \mathrm{Tr}\, H^1(\Omega) \subset L_q(\Gamma)$, where $\frac{1}{q} = \frac{1}{2} - \frac{1}{2}\,\frac{1}{d-1}$ and the inclusion follows from [22] Theorem 2.4.2. So Condition (vi) in Theorem 2.3(c) is satisfied with $\kappa = d-1$. For $d = 2$ the proof is more involved and we refer to [13] Theorem 2.6.

Now we are able to prove the main theorem of this section.

Theorem 6.12. *Suppose Ω has a C^∞-boundary. Let $V \in L_\infty(\Omega, \mathbb{R})$ and suppose $0 \notin \sigma(-\Delta^D + V)$. Let N_V be the counting function associated with the Dirichlet-to-Neumann operator D_V on $L_2(\Gamma)$. Then*

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)} N_V(\lambda) = \frac{\sigma(\Gamma)}{(4\pi)^{(d-1)/2}\, \Gamma(\frac{d+1}{2})}.$$

Proof. The theorem follows immediately from Theorems 5.2, 4.1 and Propositions 6.7 and 6.10. □

We conclude with comments on the eigenvalues of D_V. Let $\lambda \in \mathbb{R}$. Because $0 \notin \sigma(-\Delta^D + V)$ it follows that $\lambda \in \sigma(D_V)$ if and only if there exists a $u \in H^1(\Omega)$ with $u \ne 0$, such that

$$\begin{cases} -\Delta u + V\, u = 0 \text{ weakly on } \Omega, \\ \partial_\nu u = \lambda\, \mathrm{Tr}\, u. \end{cases}$$

Such kind of problem is sometimes called a *Steklov eigenvalue problem*. If $V = 0$, then Theorem 6.12 is contained in the paper [27] of Sandgren. If V is of class C^∞ and also for more general elliptic operators than the Laplacian, the Steklov eigenvalue problem is studied on manifolds by Koženikov [20], who also proved Weyl's law for those operators.

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