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Norm and Essential Norm of a Weighted Composition Operator on the Bloch Space

Xiaosong Liu and Songxiao Li

Abstract. Some new estimates for the norm and essential norm of a weighted composition operator on the Bloch space are given in this paper.

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1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For $a \in \mathbb{D}$, let σ_a be the automorphism of $\mathbb{D}$ exchanging 0 for a, namely $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$. For $0 < p < \infty$, the Bergman space A^p consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dxdy$ denote the normalized area Lebesgue measure. The Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$, is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\beta = \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

Under the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_\beta$, the Bloch space is a Banach space. From Theorem 1 of [1], we see that

$$\|f\|_\beta \approx \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^2}.$$

See [23] for more information of the Bloch space.

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For $0 < p < \infty$, let H^p denote the Hardy space of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

We say that an $f \in H(\mathbb{D})$ belongs to the $BMOA$ space, if

$$\|f\|_*^2 = \sup_{I \subseteq \partial\mathbb{D}} \frac{1}{|I|} \int_I |f(\zeta) - f_I|^2 \frac{d\zeta}{2\pi} < \infty,$$

where $f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{d\zeta}{2\pi}$. It is well known that $BMOA$ is a Banach space under the norm $\|f\|_{BMOA} = |f(0)| + \|f\|_*$. From [6], we have

$$\|f\|_* \approx \sup_{w \in \mathbb{D}} \|f \circ \sigma_w - f(w)\|_{H^2}.$$

Throughout the paper, $S(\mathbb{D})$ denotes the set of all analytic self-maps of $\mathbb{D}$. Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. The composition operator C_φ and the multiplication operator M_u are defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \ (M_u f)(z) = u(z) f(z), \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

The weighted composition operator uC_φ, induced by u and φ, is defined as follows.

$$(uC_\varphi f)(z) = u(z) f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

It is clear that the weighted composition operator uC_φ is the composition of C_φ and M_u.

It is well known that C_φ is bounded on $BMOA$ for any $\varphi \in S(\mathbb{D})$ by Littlewood's subordination theorem. The compactness of the operator $C_\varphi : BMOA \to BMOA$ was studied in [2,5,17,19,20]. Based on results in [2] and [17], Wulan in [19] showed that $C_\varphi : BMOA \to BMOA$ is compact if and only if

$$\lim_{n\to\infty} \|\varphi^n\|_* = 0 \quad and \quad \lim_{|\varphi(a)|\to 1} \|\sigma_a \circ \varphi\|_* = 0.$$

In [20], Wulan, Zheng and Zhu further showed that $C_\varphi : BMOA \to BMOA$ is compact if and only if $\lim_{n\to\infty} \|\varphi^n\|_* = 0$. In [8], Laitila gave some function theoretic characterizations for the boundedness and compactness of the operator $uC_\varphi : BMOA \to BMOA$. In [4], Colonna used the idea of [20] and showed that $uC_\varphi : BMOA \to BMOA$ is compact if and only if

$$\lim_{n\to\infty} \|u\varphi^n\|_* = 0 \ \ and \ \ \lim_{|\varphi(a)|\to 1} \Big(\log \frac{2}{1 - |\varphi(a)|^2}\Big) \|u \circ \sigma_a - u(a)\|_{H^2} = 0.$$

Motivated by results in [4], Laitila and Lindström gave estimates for norm and essential norm of the weighted composition operator $uC_\varphi : BMOA \to BMOA$ in [9]. Among others, they showed that, under the assumption of the boundedness of uC_φ on $BMOA$,

$$\|uC_\varphi\|_{e,BMOA\to BMOA} \approx$$
$$\limsup_{n\to\infty} \|u\varphi^n\|_* + \limsup_{|\varphi(a)|\to 1} \Big(\log \frac{2}{1 - |\varphi(a)|^2}\Big) \|u \circ \sigma_a - u(a)\|_{H^2}.$$

Recall that the essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators K mapping X into Y, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y\}:K} \text{ is compact}$$

where X, Y are Banach spaces and $\|\cdot\|_{X\to Y}$ is the operator norm.

By Schwarz–Pick Lemma, it is easy to see that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded for any $\varphi \in S(\mathbb{D})$. The compactness of C_φ on $\mathcal{B}$ was studied in [10,12,18,20,22]. In [20], Wulan, Zheng and Zhu proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{n\to\infty} \|\varphi^n\|_{\mathcal{B}} = 0$. In [22], Zhao obtained the exact value for the essential norm of $C_\varphi : \mathcal{B} \to \mathcal{B}$ as follows.

$$\|C_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} = \left(\frac{e}{2}\right) \limsup_{n\to\infty} \|\varphi^n\|_{\mathcal{B}}.$$

In [14], Ohno, Stroethoff and Zhao studied the boundedness and compactness of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$. In [3], Colonna provided a new characterization of the boundedness and compactness of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ by using $\|u\varphi^n\|_{\mathcal{B}}$. The essential norm of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ was studied in [7,11,13]. In [11], the authors proved that

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \approx \max\left(\limsup_{|\varphi(z)|\to 1} \frac{|u(z)\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2},\ \limsup_{|\varphi(z)|\to 1} \log\frac{e}{1-|\varphi(z)|^2}|u'(z)|(1-|z|^2)\right)$$

In [7], the authors obtained a new estimate for the essential norm of $uC_\varphi : \mathcal{B} \to \mathcal{B}$, i.e., they showed that

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \approx \max\left(\limsup_{j\to\infty} \|I_u(\varphi^j)\|_{\mathcal{B}},\ \limsup_{j\to\infty} (\log j)\|J_u(\varphi^j)\|_{\mathcal{B}}\right),$$

where $I_u f(z) = \int_0^z f'(\zeta)u(\zeta)d\zeta$, $J_u f(z) = \int_0^z f(\zeta)u'(\zeta)d\zeta$.

Motivated by the work of [3,4,9,20], the aim of this article is to give some new estimates for the norm and essential norm of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$. The techniques we use are strongly inspired by the work on $BMOA$ done by Laitila and his collaborators (see [9]).

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \lesssim b$ means that there is a positive constant C such that $a \le Cb$. Moreover, if both $a \lesssim b$ and $b \lesssim a$ hold, then one says that $a \approx b$.

2. Norm of uC_φ on the Bloch Space

In this section we give some estimates for the norm of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$. For this purpose, we need some lemmas which we state as follows. The following lemma can be found in [23].

Lemma 2.1. *Let $f \in \mathcal{B}$. Then*

$$|f(z)| \lesssim \log\frac{2}{1-|z|^2}\|f\|_{\mathcal{B}},\ z \in \mathbb{D}.$$

Lemma 2.2. *For* $2 \leq p < \infty$ *and* $f \in \mathcal{B}$,

$$\sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^2} \approx \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^p}.$$

Proof. Using the Hölder inequality, we get

$$\sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^2} \leq \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^p}, \tag{2.1}$$

for $2 \leq p < \infty$.

On the other hand, there exists a constant $C > 0$ such that (see [21, p.38])

$$\sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^p} \leq C\|f\|_{\mathcal{B}} \lesssim \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^2}, \tag{2.2}$$

which, combined with (2.1), implies the desired result. □

Lemma 2.3. [16] *For* $f \in A^2$,

$$\|f\|_{A^2}^2 \approx |f(0)|^2 + \int_{\mathbb{D}} |f'(w)|^2(1-|w|^2)^2 dA(w).$$

The classical Nevanlinna counting function N_φ and the generalized Nevanlinna counting functions $N_{\varphi,\gamma}$ for φ are defined by (see [15])

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}\{w\}} \log\frac{1}{|z|} \quad and \quad N_{\varphi,\gamma}(w) = \sum_{z\in\varphi^{-1}\{w\}} \Big(\log\frac{1}{|z|}\Big)^\gamma,$$

respectively, where $\gamma > 0$ and $w \in \mathbb{D}\backslash\{\varphi(0)\}$.

Lemma 2.4. [16] *Let* $\varphi \in S(\mathbb{D})$ *and* $f \in A^2$. *Then*

$$\|f\circ\varphi\|_{A^2}^2 \approx |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(w)|^2 N_{\varphi,2}(w) dA(w).$$

Lemma 2.5. [15] *Let* $\varphi \in S(\mathbb{D})$ *and* $\gamma > 0$. *If* $\varphi(0) \neq 0$ *and* $0 < r < |\varphi(0)|$, *then*

$$N_{\varphi,\gamma}(0) \leq \frac{1}{r^2}\int_{r\mathbb{D}} N_{\varphi,\gamma} dA.$$

Lemma 2.6. *Let* $\varphi \in S(\mathbb{D})$ *such that* $\varphi(0) = 0$. *If* $\sup_{0<|w|<1} |w|^2 N_{\varphi,2}(w) < \delta$, *then*

$$N_{\varphi,2}(w) \leq \frac{4\delta}{(\log 2)^2}\Big(\log\frac{1}{|w|}\Big)^2 \tag{2.3}$$

when $\frac{1}{2} \leq |w| < 1$.

Proof. See the proof of Lemma 2.1 in [17]. □

Lemma 2.7. *For all* $g \in A^2$ *and* $\phi \in S(\mathbb{D})$ *such* $g(0) = \phi(0) = 0$, *we have*

$$\|g\circ\phi\|_{A^2} \lesssim \|\phi\|_{A^2}\|g\|_{A^2}. \tag{2.4}$$

In particular, for all $f \in \mathcal{B}$, $a \in \mathbb{D}$ *and* $\varphi \in S(\mathbb{D})$,

$$\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^2} \lesssim \|\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a\|_{A^2}\|f\circ\sigma_a - f(a)\|_{A^2}.$$

Proof. Let $\phi \in S(\mathbb{D})$ such that $\phi(0) = 0$. Then,

$$\|\sigma_z \circ \phi - \sigma_z(\phi(0))\|_{A^2}^2 = \int_{\mathbb{D}} \frac{(1-|z|^2)^2 |\phi(w)|^2}{|1-\bar{z}\phi(w)|^2} dA(w) \le 4\|\phi\|_{A^2}^2. \quad (2.5)$$

From Lemma 2.3 and (2.5) we obtain

$$\begin{aligned} \|\sigma_z \circ \phi - \sigma_z(\phi(0))\|_{A^2}^2 &= \int_{\mathbb{D}} |(\sigma_z \circ \phi)'|^2 (\log \frac{1}{|w|})^2 dA(w) \\ &= \int_{\mathbb{D}} N_{\sigma_z \circ \phi, 2} dA(w) \le 4\|\phi\|_{A^2}^2. \end{aligned} \quad (2.6)$$

For $z \in \mathbb{D} \setminus \{0\}$, from Lemma 4.2 in [16] and Lemma 2.5, we have

$$|z|^2 N_{\phi,2}(z) = |z|^2 N_{\sigma_z \circ \phi, 2}(0) \le \int_{|z|\mathbb{D}} N_{\sigma_z \circ \phi, 2}(w) dA(w) \le 4\|\phi\|_{A^2}^2. \quad (2.7)$$

So, by Lemma 2.6 we get

$$N_{\phi,2}(z) \le \frac{16}{(\log 2)^2} \|\phi\|_{A^2}^2 (\log \frac{1}{|z|})^2, \quad (2.8)$$

for $z \in \mathbb{D} \setminus \frac{1}{2}\mathbb{D}$. Thus,

$$\int_{\mathbb{D}\setminus \frac{1}{2}\mathbb{D}} |g'(z)|^2 N_{\phi,2}(z) dA(z) \le \frac{16}{(\log 2)^2} \|\phi\|_{A^2}^2 \|g\|_{A^2}^2. \quad (2.9)$$

In addition, for $z \in \mathbb{D}$ and $g \in A^2$, from Theorems 4.14 and 4.28 of [23], we have $|g'(z)| \le (1-|z|^2)^{-2} \|g\|_{A^2}$. Then,

$$\begin{aligned} \int_{\frac{1}{2}\mathbb{D}} |g'(z)|^2 N_{\phi,2}(z) dA(z) &\le 16\|g\|_{A^2}^2 \int_{\frac{1}{2}\mathbb{D}} N_{\phi,2}(z) dA(z) \\ &\le 16\|\phi\|_{A^2}^2 \|g\|_{A^2}^2. \end{aligned} \quad (2.10)$$

Since $g(0) = 0$, by Lemma 2.4 we have

$$\|g \circ \phi\|_{A^2}^2 \approx \int_{\mathbb{D}} |g'(z)|^2 N_{\phi,2}(z) dA(z). \quad (2.11)$$

Combine with (2.9), (2.10) and (2.11), we obtain

$$\|g \circ \phi\|_{A^2} \lesssim \|\phi\|_{A^2} \|g\|_{A^2},$$

as desired. In particular, for all $f \in \mathcal{B}$, $a \in \mathbb{D}$ and $\varphi \in S(\mathbb{D})$, if we set

$$g = f \circ \sigma_{\varphi(a)} - f(\varphi(a)), \ \phi = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a,$$

we get

$$\|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{A^2} \lesssim \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_{A^2} \|f \circ \sigma_a - f(a)\|_{A^2}. \qquad \square$$

For the simplicity of the rest of this paper, we introduce the following abbreviation. Set

$$\begin{aligned} \alpha(u, \varphi, a) &= |u(a)| \cdot \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_{A^2}, \\ \beta(u, \varphi, a) &= \log \frac{2}{1-|\varphi(a)|^2} \|u \circ \sigma_a - u(a)\|_{A^2}, \end{aligned}$$

where $a \in \mathbb{D}$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$.

Theorem 2.8. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then*

$$\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}} \approx |u(0)| \log \frac{2}{1-|\varphi(0)|^2} + \sup_{a\in\mathbb{D}} \alpha(u,\varphi,a) + \sup_{a\in\mathbb{D}} \beta(u,\varphi,a).$$

Proof. First we give the upper estimate for $\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}$. For all $f \in \mathcal{B}$, using the triangle inequality, we get

$$\begin{aligned} &\|(uC_\varphi f)\circ\sigma_a - (uC_\varphi f)(a)\|_{A^2} \\ &= \|(u\circ\sigma_a - u(a))\cdot(f\circ\varphi\circ\sigma_a - f(\varphi(a))) \\ &\quad + u(a)(f\circ\varphi\circ\sigma_a - f(\varphi(a))) + (u\circ\sigma_a - u(a))f(\varphi(a))\|_{A^2} \\ &\le \|(u\circ\sigma_a - u(a))\cdot(f\circ\varphi\circ\sigma_a - f(\varphi(a)))\|_{A^2} \\ &\quad + |u(a)|\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^2} + |f(\varphi(a))|\|u\circ\sigma_a - u(a)\|_{A^2}. \end{aligned} \tag{2.12}$$

By Lemmas 2.1 and 2.7, we have

$$\begin{aligned} &|u(a)|\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^2} + |f(\varphi(a))|\|u\circ\sigma_a - u(a)\|_{A^2} \\ &\lesssim \alpha(u,\varphi,a)|\|f\circ\sigma_a - f(a)\|_{A^2} + \log\frac{2}{1-|\varphi(a)|^2}\|u\circ\sigma_a - u(a)\|_{A^2}\|f\|_{\mathcal{B}} \\ &\lesssim \big(\alpha(u,\varphi,a) + \beta(u,\varphi,a)\big)\|f\|_{\mathcal{B}}. \end{aligned} \tag{2.13}$$

From Lemmas 2.1 and 2.2, we get

$$\begin{aligned} &\sup_{a\in\mathbb{D}} \|(u\circ\sigma_a - u(a))\cdot(f\circ\varphi\circ\sigma_a - f(\varphi(a)))\|_{A^2} \\ &\lesssim \sup_{a\in\mathbb{D}} \log 2\|u\circ\sigma_a - u(a)\|_{A^2}\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^2} \\ &\lesssim \sup_{a\in\mathbb{D}} \log\frac{2}{1-|\varphi(a)|^2}\|u\circ\sigma_a - \psi(a)\|_{A^2}\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^2} \\ &\lesssim \sup_{a\in\mathbb{D}} \beta(u,\varphi,a)\|f\circ\varphi\|_{\mathcal{B}} \lesssim \sup_{a\in\mathbb{D}} \beta(u,\varphi,a)\|f\|_{\mathcal{B}}. \end{aligned} \tag{2.14}$$

Then, by (2.12), (2.13) and (2.14), we have

$$\sup_{a\in\mathbb{D}} \|(uC_\varphi f)\circ\sigma_a - (uC_\varphi f)(a)\|_{A^2} \lesssim \big(\sup_{a\in\mathbb{D}} \alpha(u,\varphi,a) + \sup_{a\in\mathbb{D}} \beta(u,\varphi,a)\big)\|f\|_{\mathcal{B}}.$$

In addition, by Lemma 2.1, $|(uC_\varphi f)(0)| \lesssim |u(0)| \log \frac{2}{1-|\varphi(0)|^2}\|f\|_{\mathcal{B}}$, we get

$$\begin{aligned} &\|uC_\varphi f\|_{\mathcal{B}} \\ &\approx |(uC_\varphi f)(0)| + \sup_{a\in\mathbb{D}} \|(uC_\varphi f)\circ\sigma_a - (uC_\varphi f)(a)\|_{A^2} \\ &\lesssim |u(0)| \log\frac{2}{1-|\varphi(0)|^2}\|f\|_{\mathcal{B}} + \sup_{a\in\mathbb{D}} \alpha(u,\varphi,a)\|f\|_{\mathcal{B}} + \sup_{a\in\mathbb{D}} \beta(u,\varphi,a)\|f\|_{\mathcal{B}}, \end{aligned}$$

which implies

$$\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}} \lesssim |u(0)| \log\frac{2}{1-|\varphi(0)|^2} + \sup_{a\in\mathbb{D}} \alpha(u,\varphi,a) + \sup_{a\in\mathbb{D}} \beta(u,\varphi,a). \tag{2.15}$$

Next we find the lower estimate for $\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}$. Let $f = 1$. It is easy to see that $\|u\|_{\mathcal{B}} \le \|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}$. For any $a \in \mathbb{D}$, set

$$f_a(z) = \sigma_{\varphi(a)}(z) - \varphi(a), \quad z \in \mathbb{D}. \tag{2.16}$$

Then, $f_a(0) = 0$, $f_a(\varphi(a)) = -\varphi(a)$, $\|f_a\|_{\mathcal{B}} \le 4$ and $\|f_a\|_\infty \le 2$. Using the triangle inequality, we get

$$\begin{aligned}\alpha(u,\varphi,a) &= |u(a)|\cdot\|\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a - \varphi(a) + \varphi(a)\|_{A^2}\\ &= \|u(a)\cdot(f_a\circ\varphi\circ\sigma_a - f_a(\varphi(a)))\|_{A^2}\\ &\le \|(u\circ\sigma_a - u(a))\cdot f_a\circ\varphi\circ\sigma_a\|_{A^2}\\ &\quad + \|(u\circ\sigma_a)\cdot f_a\circ\varphi\circ\sigma_a - u(a)f_a(\varphi(a))\|_{A^2}\\ &\le 2\|u\circ\sigma_a - u(a)\|_{A^2} + \|(uC_\varphi f_a)\circ\sigma_a - (uC_\varphi f_a)(a)\|_{A^2}\\ &\le 2\|u\|_{\mathcal{B}} + 4\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}} \le 6\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}. \end{aligned} \tag{2.17}$$

Set

$$h_a(z) = \log\frac{2}{1-\overline{\varphi(a)}z}, \quad z\in\mathbb{D}. \tag{2.18}$$

Then, $h_a \in \mathcal{B}$, $h_a(\varphi(a)) = \log\frac{2}{1-|\varphi(a)|^2}$ and $\sup_{a\in\mathbb{D}}\|h_a\|_{\mathcal{B}} \le 2+\log 2$. Using the triangle inequality and Lemma 2.7, we obtain

$$\begin{aligned}&\beta(u,\varphi,a)\\ &= \|\log\frac{2}{1-|\varphi(a)|^2}\cdot(u\circ\sigma_a - u(a))\|_{A^2}\\ &= \|h_a(\varphi(a))(u\circ\sigma_a - u(a))\|_{A^2}\\ &\le \|(h_a\circ\varphi\circ\sigma_a - h_a(\varphi(a)))\cdot(u\circ\sigma_a - u(a))\|_{A^2}\\ &\quad + \|(u\circ\sigma_a)\cdot h_a\circ\varphi\circ\sigma_a - u(a)h_a(\varphi(a))\|_{A^2}\\ &\quad + \|u(a)(h_a\circ\varphi\circ\sigma_a - h_a(\varphi(a)))\|_{A^2}\\ &\lesssim \|(h_a\circ\varphi\circ\sigma_a - h_a(\varphi(a)))\cdot(u\circ\sigma_a - u(a))\|_{A^2}\\ &\quad + \|(uC_\varphi h_a)\circ\sigma_a - (uC_\varphi h_a)(a)\|_{A^2} + \alpha(u,\varphi,a)\|h_a\circ\sigma_a - h_a(a)\|_{A^2}\\ &\lesssim \|(h_a\circ\varphi\circ\sigma_a - h_a(\varphi(a)))\cdot(u\circ\sigma_a - u(a))\|_{A^2}\\ &\quad + (2+\log 2)\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}} + (2+\log 2)\alpha(u,\varphi,a). \end{aligned} \tag{2.19}$$

By Lemmas 2.2 and 2.7, we have

$$\begin{aligned}&\|(h_a\circ\varphi\circ\sigma_a - h_a(\varphi(a)))\cdot(u\circ\sigma_a - u(a))\|_{A^2}\\ &\lesssim \|(h_a\circ\varphi\circ\sigma_a - h_a(\varphi(a)))\|_{A^2}\|u\circ\sigma_a - u(a)\|_{A^2}\\ &\le \|h_a\circ\varphi\|_{\mathcal{B}}\|u\|_{\mathcal{B}} \lesssim \|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}. \end{aligned} \tag{2.20}$$

Combining (2.17), (2.19) and (2.20), we have

$$\sup_{a\in\mathbb{D}}\alpha(u,\varphi,a) + \sup_{a\in\mathbb{D}}\beta(u,\varphi,a) \lesssim \|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}.$$

Moreover,

$$|u(0)|\log\frac{2}{1-|\varphi(0)|^2} = |(uC_\varphi h_0)(0)| \le (2+\log 2)\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}} \lesssim \|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}.$$

Therefore,

$$|u(0)|\log\frac{2}{1-|\varphi(0)|^2} + \sup_{a\in\mathbb{D}}\alpha(u,\varphi,a) + \sup_{a\in\mathbb{D}}\beta(u,\varphi,a) \lesssim \|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}. \quad \square$$

Lemma 2.9. *Suppose that $uC_\varphi:\mathcal{B}\to\mathcal{B}$ is bounded. Then*

$$\sup_{a\in\mathbb{D}}\|uC_\varphi(\sigma_{\varphi(a)}-\varphi(a))\|_{\mathcal{B}}\approx\sup_{n\geq 0}\|u\varphi^n\|_{\mathcal{B}} \tag{2.21}$$

and

$$\limsup_{|\varphi(a)|\to 1}\|uC_\varphi(\sigma_{\varphi(a)}-\varphi(a))\|_{\mathcal{B}}\lesssim\limsup_{n\to\infty}\|u\varphi^n\|_{\mathcal{B}}. \tag{2.22}$$

Proof. From Corollary 2.1 of [3], we see that

$$\sup_{a\in\mathbb{D}}\|uC_\varphi\sigma_{\varphi(a)}\|_{\mathcal{B}}\approx\sup_{n\geq 0}\|u\varphi^n\|_{\mathcal{B}}.$$

Then (2.21) follows immediately.

The Taylor expansion of $\sigma_{\varphi(a)}-\varphi(a)$ is

$$\sigma_{\varphi(a)}-\varphi(a)=-\sum_{n=0}^{\infty}\left(\overline{\varphi(a)}\right)^n(1-|\varphi(a)|^2)z^{n+1}.$$

Then, by the boundedness of $uC_\varphi:\mathcal{B}\to\mathcal{B}$ we have

$$\|uC_\varphi(\sigma_{\varphi(a)}-\varphi(a))\|_{\mathcal{B}}\leq(1-|\varphi(a)|^2)\sum_{n=0}^{\infty}|\varphi(a)|^n\|u\varphi^{n+1}\|_{\mathcal{B}}.$$

For each N, set

$$M_1=:\sum_{n=0}^{N}|\varphi(a)|^n\|u\varphi^{n+1}\|_{\mathcal{B}}.$$

Then we get

$$\begin{aligned}&\|uC_\varphi(\sigma_{\varphi(a)}-\varphi(a))\|_{\mathcal{B}}\\ &\leq(1-|\varphi(a)|^2)\sum_{n=0}^{N}|\varphi(a)|^n\|u\varphi^{n+1}\|_{\mathcal{B}}\\ &\quad+(1-|\varphi(a)|^2)\sum_{n=N+1}^{\infty}|\varphi(a)|^n\|u\varphi^{n+1}\|_{\mathcal{B}}\\ &\leq M_1(1-|\varphi(a)|^2)+\big((1-|\varphi(a)|^2)\sum_{n=N+1}^{\infty}|\varphi(a)|^n\sup_{n\geq N+1}\|u\varphi^{n+1}\|_{\mathcal{B}}\\ &\leq M_1(1-|\varphi(a)|^2)+2\sup_{n\geq N+1}\|u\varphi^{n+1}\|_{\mathcal{B}}.\end{aligned}$$

Taking $\limsup_{|\varphi(a)|\to 1}$ in the last inequality and then letting $N\to\infty$, we get the desired result. □

Proposition 2.10. *Let $\varphi\in S(\mathbb{D})$ and $u\in H(\mathbb{D})$. Then the following claims hold.*

(i) *For $a\in\mathbb{D}$, let $f_a(z)=\sigma_{\varphi(a)}-\varphi(a)$. Then*

$$\alpha(u,\varphi,a)\lesssim\frac{\beta(u,\varphi,a)}{\log\frac{2}{1-|\varphi(a)|^2}}+\|(uC_\varphi f_a)\circ\sigma_a-(uC_\varphi f_a)(a)\|_{A^2}.$$

(ii) *For* $a \in \mathbb{D}$, *let* $g_a = \frac{h_a^2}{h_a(\varphi(a))}$, *where* $h_a(z) = \log \frac{2}{1-\overline{\varphi(a)}z}$. *Then*

$$\beta(u,\varphi,a) \lesssim \alpha(u,\varphi,a) + \|(g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a))\|_{A^2} + \|(uC_\varphi g_a) \circ \sigma_a - (uC_\varphi g_a)(a)\|_{A^2}.$$

(iii) *For all* $f \in \mathcal{B}$ *and* $a \in \mathbb{D}$,

$$\begin{aligned}&\|(uC_\varphi f) \circ \sigma_a - (uC_\varphi f)(a)\|_{A^2}\\ &\lesssim \|(u \circ \sigma_a - u(a)) \cdot (f \circ \varphi \circ \sigma_a - f(\varphi(a)))\|_{A^2}\\ &\quad + (\alpha(u,\varphi,a) + \beta(u,\varphi,a))\|f\|_{\mathcal{B}}.\end{aligned}$$

(iv) *For all* $f \in \mathcal{B}$ *and* $a \in \mathbb{D}$,

$$\begin{aligned}&\|(u \circ \sigma_a - u(a)) \cdot (f \circ \varphi \circ \sigma_a - f(\varphi(a)))\|_{A^2}\\ &\lesssim \|f\|_{\mathcal{B}} \min\left\{\sup_{w\in\mathbb{D}} \beta(u,\varphi,w), \frac{\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}}{\sqrt{\log \frac{2}{1-|\varphi(a)|^2}}}\right\}.\end{aligned}$$

Proof. (i) It is easy to see that $\|f_a \circ \varphi \circ \sigma_a\|_\infty \le 2$. For any $a \in \mathbb{D}$, we get

$$\begin{aligned}\alpha(u,\varphi,a) &= |u(a)|\|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{A^2}\\ &= \|(u \circ \sigma_a - u(a)) \cdot f_a \circ \varphi \circ \sigma_a - (uC_\varphi f_a) \circ \sigma_a - (uC_\varphi f_a)(a)\|_{A^2}\\ &\lesssim \|u \circ \sigma_a - u(a)\|_{A^2} + \|(uC_\varphi f_a) \circ \sigma_a - (uC_\varphi f_a)(a)\|_{A^2}\\ &\le \frac{\beta(u,\varphi,a)}{\log \frac{2}{1-|\varphi(a)|^2}} + \|(uC_\varphi f_a) \circ \sigma_a - (uC_\varphi f_a)(a)\|_{A^2}.\end{aligned}$$

(ii) It is obvious that $g_a(\varphi(a)) = \log \frac{2}{1-|\varphi(a)|^2}$. Since $(g_a \circ \sigma_{\varphi(a)} - g_a(\varphi(a)))(0) = 0$,

$$g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a)) = g_a \circ \sigma_{\varphi(a)} \circ (\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a) - g_a(\varphi(a)),$$

by Lemma 2.7 and the fact that $\sup_{a\in\mathbb{D}} \|g_a\|_{\mathcal{B}} < \infty$ we obtain

$$|u(a)|\|g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))\|_{A^2} \lesssim \alpha(u,\varphi,a) \sup_{a\in\mathbb{D}} \|g_a\|_{\mathcal{B}} \lesssim \alpha(u,\varphi,a).$$

By the triangle inequality we get

$$\begin{aligned}&\beta(u,\varphi,a)\\ &= \|g_a(\varphi(a)) \cdot (u \circ \sigma_a - u(a))\|_{A^2}\\ &= \|(g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a))\\ &\quad + u(a)(g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))) - (u(a) g_a \circ \varphi \circ \sigma_a - u(a) g_a(\varphi(a)))\|_{A^2}\\ &\le \|(g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a))\|_{A^2}\\ &\quad + |u(a)|\|g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))\|_{A^2} + \|(uC_\varphi g_a) \circ \sigma_a - (uC_\varphi g_a)(a)\|_{A^2}\\ &\lesssim \alpha(u,\varphi,a) + \|(g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a))\|_{A^2}\\ &\quad + \|(uC_\varphi g_a) \circ \sigma_a - (uC_\varphi g_a)(a)\|_{A^2},\end{aligned}$$

as desired.

(iii) See the proof of Theorem 2.8.

(iv) Using the fact that $\log 2 \le \log \frac{2}{1-|\varphi(a)|^2}$ and Theorem 2.8, we have

$$\sup_{a\in\mathbb{D}} \|u\circ\sigma_a - u(a)\|_{A^2} \le \sup_{a\in\mathbb{D}} \beta(u,\varphi,a) \lesssim \|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}. \tag{2.23}$$

By Lemma 2.2 and the Hölder inequality, we obtain

$$\begin{aligned}
&\|(u\circ\sigma_a - u(a))\cdot(f\circ\varphi\circ\sigma_a - f(\varphi(a)))\|_{A^2}^2\\
&= \|(u\circ\sigma_a - u(a))^2(f\circ\varphi\circ\sigma_a - f(\varphi(a)))^2\|_{A^1}\\
&\le \|u\circ\sigma_a - u(a)\|_{A^2}\|u\circ\sigma_a - u(a)\|_{A^4}\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^8}^2\\
&\le \|u\circ\sigma_a - u(a)\|_{A^2}\sup_{a\in\mathbb{D}}\|u\circ\sigma_a - u(a)\|_{A^4}\sup_{a\in\mathbb{D}}\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^8}^2\\
&\lesssim \beta(u,\varphi,a)\sup_{a\in\mathbb{D}}\|u\circ\sigma_a - u(a)\|_{A^2}\frac{\sup_{a\in\mathbb{D}}\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^2}^2}{\log\frac{2}{1-|\varphi(a)|^2}}.
\end{aligned}$$

Then, by the boundedness of C_φ on $\mathcal{B}$ and (2.23), we obtain

$$\begin{aligned}
&\beta(u,\varphi,a)\sup_{a\in\mathbb{D}}\|u\circ\sigma_a - u(a)\|_{A^2}\frac{\sup_{a\in\mathbb{D}}\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^2}^2}{\log\frac{2}{1-|\varphi(a)|^2}}\\
&\lesssim (\sup_{a\in\mathbb{D}}\beta(u,\varphi,a))^2\frac{\sup_{a\in\mathbb{D}}\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^2}^2}{\log\frac{2}{1-|\varphi(a)|^2}}\\
&\lesssim \sup_{a\in\mathbb{D}}\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{A^2}^2\min\left\{\sup_{a\in\mathbb{D}}\beta(u,\varphi,a), \frac{\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\right\}^2\\
&\lesssim \|f\circ\varphi\|_{\mathcal{B}}^2\min\left\{\sup_{a\in\mathbb{D}}\beta(u,\varphi,a), \frac{\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\right\}^2\\
&\lesssim \|f\|_{\mathcal{B}}^2\min\left\{\sup_{a\in\mathbb{D}}\beta(u,\varphi,a), \frac{\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\right\}^2,
\end{aligned}$$

as desired. □

Theorem 2.11. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Suppose that uC_φ is bounded on $\mathcal{B}$. Then*

$$\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}} \approx |u(0)|\log\frac{2}{1-|\varphi(0)|^2} + \sup_{n\ge 0}\|u\varphi^n\|_{\mathcal{B}} + \sup_{a\in\mathbb{D}}\beta(u,\varphi,a).$$

Proof. For any $f \in \mathcal{B}$, by (iii) and (iv) of Proposition 2.10, we get

$$\|uC_\varphi f\|_\beta \lesssim \sup_{a\in\mathbb{D}}\big(\alpha(u,\varphi,a) + \beta(u,\varphi,a)\big)\|f\|_{\mathcal{B}}.$$

By Lemma 2.9 and (i) of Proposition 2.10, we have

$$\begin{aligned}
\alpha(u,\varphi,a) &\lesssim \beta(u,\varphi,a)/\log\frac{2}{1-|\varphi(a)|^2} + \sup_{a\in\mathbb{D}}\|uC_\varphi f_a\|_{\mathcal{B}}\\
&\lesssim \beta(u,\varphi,a) + \sup_{n\ge 0}\|u\varphi^n\|_{\mathcal{B}}.
\end{aligned}$$

Thus,

$$\|uC_\varphi f\|_\beta \lesssim \big(\sup_{a\in\mathbb{D}} \beta(u,\varphi,a) + \sup_{n\geq 0} \|u\varphi^n\|_{\mathcal{B}}\big)\|f\|_{\mathcal{B}}.$$

In addition, $(uC_\varphi f)(0)| = |u(0)||f(\varphi(0))| \lesssim |u(0)||\log \frac{2}{1-|\varphi(0)|^2}\|f\|_{\mathcal{B}}$. Thus,

$$\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}} \lesssim |u(0)|\log\frac{2}{1-|\varphi(0)|^2} + \sup_{n\geq 0}\|u\varphi^n\|_{\mathcal{B}} + \sup_{a\in\mathbb{D}}\beta(u,\varphi,a).$$

On the other hand, let $p_n(z) = z^n$. Then $p_n \in \mathcal{B}$ for all $n \geq 0$. Thus

$$\sup_{n\geq 0}\|u\varphi^n\|_{\mathcal{B}} = \sup_{n\geq 0}\|(uC_\varphi)p_n\|_{\mathcal{B}} \leq \|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}} < \infty,$$

which, together with Theorem 2.8, implies

$$|u(0)|\log\frac{2}{1-|\varphi(0)|^2} + \sup_n \|u\varphi^n\|_{\mathcal{B}} + \sup_{a\in\mathbb{D}}\beta(u,\varphi,a) \lesssim \|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}. \qquad \square$$

Corollary 2.12. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded if and only if*

$$\sup_{n\geq 0}\|u\varphi^n\|_{\mathcal{B}} < \infty \text{ and } \sup_{a\in\mathbb{D}}\log\frac{2}{1-|\varphi(a)|^2}\|u\circ\sigma_a - u(a)\|_{A^2} < \infty.$$

3. Essential Norm of uC_φ on the Bloch Space

In this section we characterize the essential norm of the weighted composition operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ in several forms, specially we will give characterizations in terms of the Bloch norm of $u\varphi^n$. For $t \in (0,1)$, we define

$$E(\varphi,a,t) = \{z \in \mathbb{D} : |(\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a)(z)| > t\}.$$

Similarly to the proof of Lemma 9 of [9], we get the following result. Since the proof is similar, we omit the details.

Lemma 3.1. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then*

$$\widetilde{\gamma} := \limsup_{r\to 1}\limsup_{t\to 1}\sup_{|\varphi(a)|\leq r}\left(\int_{E(\varphi,a,t)}|u(\sigma_a(z))|^4 dA(z)\right)^{1/4} \lesssim \limsup_{n\to\infty}\|u\varphi^n\|_{\mathcal{B}}.$$

Theorem 3.2. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded. Then*

$$\begin{aligned}\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} &\approx \limsup_{n\to\infty}\|u\varphi^n\|_{\mathcal{B}} + \limsup_{|\varphi(a)|\to 1}\|uC_\varphi g_a\|_{\mathcal{B}}\\ &\approx \widetilde{\alpha} + \widetilde{\beta} + \widetilde{\gamma}\\ &\approx \widetilde{\alpha} + \limsup_{|\varphi(a)|\to 1}\|uC_\varphi g_a\|_{\mathcal{B}} + \widetilde{\gamma}\\ &\approx \limsup_{n\to\infty}\|u\varphi^n\|_{\mathcal{B}} + \widetilde{\beta},\end{aligned}$$

where $\widetilde{\alpha} = \limsup_{|\varphi(a)|\to 1} \alpha(u,\varphi,a)$, $\widetilde{\beta} = \limsup_{|\varphi(a)|\to 1} \beta(u,\varphi,a)$ *and*

$$g_a(z) = \Big(\log \frac{2}{1-\overline{\varphi(a)}z}\Big)^2 \Big(\log \frac{2}{1-|\varphi(a)|^2}\Big)^{-1}.$$

Proof. Set $f_n(z) = z^n$. It is well known that $f_n \in \mathcal{B}$ and $f_n \to 0$ weakly in $\mathcal{B}$ as $n \to \infty$. Then

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \gtrsim \limsup_{n\to\infty} \|uC_\varphi f_n\|_{\mathcal{B}} = \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}}. \tag{3.1}$$

Choose $a_n \in \mathbb{D}$ such that $|\varphi(a_n)| \to 1$ as $n \to \infty$. It is easy to check that g_{a_n} are uniformly bounded in $\mathcal{B}$ and converges weakly to zero in $\mathcal{B}$ (see [14]). By these facts we obtain

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \gtrsim \limsup_{n\to\infty} \|uC_\varphi g_{a_n}\|_{\mathcal{B}} = \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}}. \tag{3.2}$$

By (3.1) and (3.2), we obtain

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \gtrsim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}} + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}}. \tag{3.3}$$

From (i) of Proposition 2.10, we see that

$$\alpha(u,\varphi,a) \lesssim \frac{\beta(u,\varphi,a)}{\log \frac{2}{1-|\varphi(a)|^2}} + \|uC_\varphi f_a\|_{\mathcal{B}},$$

which together with Lemma 2.9 implies that

$$\widetilde{\alpha} = \limsup_{|\varphi(a)|\to 1} \alpha(u,\varphi,a) \lesssim \limsup_{|\varphi(a)|\to 1} \|uC_\varphi f_a\|_{\mathcal{B}} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}}. \tag{3.4}$$

From (ii) and (iv) of Proposition 2.10, we see that

$$\begin{aligned}\beta(u,\varphi,a) &\lesssim \alpha(u,\varphi,a) + \|(g_a\circ\varphi\circ\sigma_a - g_a(\varphi(a)))\cdot(u\circ\sigma_a - u(a))\|_{A^2} \\ &\quad + \|(uC_\varphi g_a)\circ\sigma_a - (uC_\varphi g_a)(a)\|_{A^2} \\ &\lesssim \alpha(u,\varphi,a) + \|g_a\|_{\mathcal{B}} \frac{\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}}{\sqrt{\log \frac{2}{1-|\varphi(a)|^2}}} + \|uC_\varphi g_a\|_{\mathcal{B}},\end{aligned}$$

which implies that

$$\widetilde{\beta} = \limsup_{|\varphi(a)|\to 1} \beta(u,\varphi,a) \lesssim \widetilde{\alpha} + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}}. \tag{3.5}$$

By Lemma 3.1, (3.3), (3.4) and (3.5), we have

$$\begin{aligned}\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} &\gtrsim \widetilde{\alpha} + \widetilde{\gamma} + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}} \\ &\gtrsim \widetilde{\alpha} + \widetilde{\gamma} + \widetilde{\beta},\end{aligned}$$

and

$$\begin{aligned}\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} &\gtrsim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}} + \widetilde{\alpha} + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}} \\ &\gtrsim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}} + \widetilde{\beta}.\end{aligned}$$

Next we give the upper estimate for $\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}}$. For $n \geq 0$, we define the linear operator on $\mathcal{B}$ by $(K_n f)(z) = f(\frac{n}{n+1}z)$. It is easy to check that K_n is a compact operator on $\mathcal{B}$. Thus

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \leq \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}}\leq 1} \|uC_\varphi(I - K_n)f\|_{\mathcal{B}},$$

where I is the identity operator. Let $S_n = I - K_n$. Then,

$$\begin{aligned}\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} &\leq \liminf_{n\to\infty} \|uC_\varphi S_n\|_{\mathcal{B}} \\ &= \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{B}}\leq 1} \big(|u(0)(S_n f)(\varphi(0))| + \|(uC_\varphi S_n f\|_\beta\big) \\ &= \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{B}}\leq 1} \|uC_\varphi S_n f\|_\beta. \end{aligned} \tag{3.6}$$

Let $f \in \mathcal{B}$ such that $\|f\|_{\mathcal{B}} \leq 1$. Fix $n \geq 0$, $r \in (0,1)$ and $t \in (\frac{1}{2}, 1)$. Then

$$\begin{aligned}\|uC_\varphi S_n f\|_\beta &\approx \sup_{a\in\mathbb{D}} \|(uC_\varphi S_n f)\circ\sigma_a - (uC_\varphi S_n f)(a)\|_{A^2} \\ &\leq \sup_{|\varphi(a)|\leq r} \|(uC_\varphi S_n f)\circ\sigma_a - (uC_\varphi S_n f)(a)\|_{A^2} \\ &\quad + \sup_{|\varphi(a)|> r} \|(uC_\varphi S_n f)\circ\sigma_a - (uC_\varphi S_n f)(a)\|_{A^2}. \end{aligned} \tag{3.7}$$

By (iii) and (iv) of Proposition 2.10, we have

$$\begin{aligned}&\sup_{|\varphi(a)|> r} \|(uC_\varphi S_n f)\circ\sigma_a - (uC_\varphi S_n f)(a)\|_{A^2} \\ &\lesssim \|S_n f\|_{\mathcal{B}} \sup_{|\varphi(a)|> r} \left(\alpha(u,\varphi,a) + \beta(u,\varphi,a) + \frac{\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\right). \end{aligned} \tag{3.8}$$

In addition,

$$\begin{aligned}&\sup_{|\varphi(a)|\leq r} \|(uC_\varphi S_n f)\circ\sigma_a - (uC_\varphi S_n f)(a)\|_{A^2} \\ &\leq \sup_{|\varphi(a)|\leq r} \big(|(S_n f)(\varphi(a))|\|u\circ\sigma_a - u(a)\|_{A^2} \\ &\quad + \|u\circ\sigma_a\cdot((C_\varphi S_n f)\circ\sigma_a - (C_\varphi S_n f)(a))\|_{A^2}\big) \\ &\leq \|u\|_{\mathcal{B}} \max_{|w|\leq r} |(S_n f)(w)| + I_1^{1/2} + I_2^{1/2}, \end{aligned} \tag{3.9}$$

where

$$I_1 = \sup_{|\varphi(a)|\leq r} \int_{\mathbb{D}\setminus E(\varphi,a,t)} |(u\circ\sigma_a)(z)\cdot((S_n f)\circ\varphi\circ\sigma_a(z) - (S_n f)(\varphi(a)))|^2 dA(z),$$

$$I_2 = \sup_{|\varphi(a)|\leq r} \int_{E(\varphi,a,t)} |(u\circ\sigma_a)(z)\cdot((S_n f)\circ\varphi\circ\sigma_a(z) - (S_n f)(\varphi(a)))|^2 dA(z).$$

Let $\varphi_a = \sigma_{\varphi(a)}\circ\varphi\circ\sigma_a$. Then by (3.19) in [8, p. 37], we have

$$\begin{aligned}&|(S_n f)\circ\sigma_{\varphi(a)}\circ\varphi_a(z) - (S_n f\circ\varphi)(a)| \\ &\lesssim \sup_{|w|\leq t} |\big((S_n f)\circ\sigma_{\varphi(a)}\big)(w) - (S_n f)(\varphi(a))| \end{aligned}$$

for $z \in \mathbb{D} \backslash E(\varphi, a, t)$. Since

$$\begin{aligned}\|u \circ \sigma_a \cdot \varphi_a\|_{A^2} &\leq \|u \circ \sigma_a - u(a)\|_{A^2} \|\varphi_a\|_\infty + |u(a)| \|\varphi_a\|_2 \\ &\lesssim \sup_{a \in \mathbb{D}} \|u \circ \sigma_a - u(a)\|_{A^2} + \alpha(u, \varphi, a_n) \\ &\lesssim \|uC_\varphi\|_{\mathcal{B} \to \mathcal{B}},\end{aligned}$$

we have

$$\begin{aligned}I_1 &\lesssim \sup_{|\varphi(a)| \leq r} \sup_{|w| \leq t} |((S_n f) \circ \sigma_{\varphi(a)})(w) - (S_n f)(\varphi(a))|^2 \|u \circ \sigma_a \cdot \varphi_a\|_{A^2}^2 \\ &\lesssim \|uC_\varphi\|_{\mathcal{B} \to \mathcal{B}}^2 \sup_{|z| \leq \frac{t+r}{1+tr}} |(S_n f)(z)|^2.\end{aligned}$$

By Lemma 2.2, we get

$$\begin{aligned}&\|(S_n f) \circ \varphi \circ \sigma_a - (S_n f)(\varphi(a))\|_{A^4}^2 \\ &\leq \sup_{a \in \mathbb{D}} \|(S_n f) \circ \varphi \circ \sigma_a - (S_n f)(\varphi(a))\|_{A^4}^2 \\ &\lesssim \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^2}^2 \leq 1,\end{aligned}$$

which implies that

$$\begin{aligned}I_2 &\leq \sup_{|\varphi(a)| \leq r} \left(\int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/2} \\ &\quad \times \|(S_n f) \circ \varphi \circ \sigma_a - (S_n f)(\varphi(a))\|_{A^4}^2 \\ &\leq \sup_{|\varphi(a)| \leq r} \left(\int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/2}.\end{aligned}$$

By combining the above estimates, for $r \in (0, 1)$ and $t \in (\frac{1}{2}, 1)$, we obtain

$$\begin{aligned}&\|uC_\varphi S_n f\|_\beta \\ &\lesssim \sup_{|\varphi(a)| > r} \left(\alpha(u, \varphi, a) + \beta(u, \varphi, a) + \frac{\|uC_\varphi\|_{\mathcal{B} \to \mathcal{B}}}{\sqrt{\log \frac{2}{1 - |\varphi(a)|^2}}} \right) \\ &\quad + \sup_{|\varphi(a)| \leq r} \left(\int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/4} \\ &\quad + \sup_{|z| \leq \frac{t+r}{1+tr}} |(S_n f)(z)| \|uC_\varphi\|_{\mathcal{B} \to \mathcal{B}}.\end{aligned}$$

Taking the supremum over $\|f\|_{\mathcal{B}} \leq 1$ and letting $n \to \infty$, we obtain

$$\begin{aligned}\|uC_\varphi\|_{e, \mathcal{B} \to \mathcal{B}} &\lesssim \sup_{|\varphi(a)| > r} \left(\alpha(u, \varphi, a) + \beta(u, \varphi, a) + \frac{\|uC_\varphi\|_{\mathcal{B} \to \mathcal{B}}}{\sqrt{\log \frac{2}{1 - |\varphi(a)|^2}}} \right) \\ &\quad + \sup_{|\varphi(a)| \leq r} \left(\int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/4},\end{aligned}$$

which implies that

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \lesssim \widetilde{\alpha} + \widetilde{\beta} + \widetilde{\gamma}.$$

By (3.4), (3.5) and Lemma 3.1, we get

$$\begin{aligned}\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} &\lesssim \widetilde{\beta} + \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}} \\ &\lesssim \widetilde{\alpha} + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}} + \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}} \\ &\lesssim \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}} + \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}}.\end{aligned}$$

By (ii), (iv) of Proposition 2.10, we have

$$\begin{aligned}&\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \\ &\lesssim \sup_{|\varphi(a)|>r} \left(\alpha(u,\varphi,a) + \beta(u,\varphi,a) + \frac{\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\right) \\ &\quad + \sup_{|\varphi(a)|\le r} \left(\int_{E(\varphi,a,t)} |u(\sigma_a(z))|^4 dA(z)\right)^{1/4} \\ &\lesssim \sup_{|\varphi(a)|>r} \Big(\alpha(u,\varphi,a) + \|(uC_\varphi g_a)\circ\sigma_a - (uC_\varphi g_a)(a)\|_{A^2} \\ &\quad + \|(u\circ\sigma_a - u(a))\cdot(g_a\circ\varphi\circ\sigma_a - g_a(\varphi(a)))\|_{A^2} + \frac{\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\Big) \\ &\quad + \left(\int_{E(\varphi,a,t)} |u(\sigma_a(z))|^4 dA(z)\right)^{1/4} \\ &\lesssim \sup_{|\varphi(a)|>r} \Bigg(\alpha(u,\varphi,a) + \|uC_\varphi g_a\|_{\mathcal{B}} + \|g_a\|_{\mathcal{B}} \frac{\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}} \\ &\quad + \frac{\|uC_\varphi\|_{\mathcal{B}\to\mathcal{B}}}{\sqrt{\log\frac{2}{1-|\varphi(a)|^2}}}\Bigg) + \left(\int_{E(\varphi,a,t)} |u(\sigma_a(z))|^4 dA(z)\right)^{1/4},\end{aligned}$$

which implies that

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \lesssim \widetilde{\alpha} + \limsup_{|\varphi(a)|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}} + \widetilde{\gamma}. \qquad \square$$

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Xiaosong Liu
Department of Mathematics
Jiaying University
Meizhou 514015
China
e-mail: gdxsliu@163.com

Songxiao Li (✉)
Institute of System Engineering
Macau University of Science and Technology
Avenida Wai Long
Taipa
Macau
e-mail: jyulsx@163.com

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