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Factorizations of Kernels and Reproducing Kernel Hilbert Spaces

Rani Kumari, Jaydeb Sarkar, Srijan Sarkar and Dan Timotin

Abstract. The paper discusses a series of results concerning reproducing kernel Hilbert spaces, related to the factorization of their kernels. In particular, it is proved that for a large class of spaces isometric multipliers are trivial. One also gives for certain spaces conditions for obtaining a particular type of dilation, as well as a classification of Brehmer type submodules.

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1. Introduction

Reproducing kernel Hilbert space theory is an interdisciplinary subject that arises from the interaction between function theory, system theory and operator theory. The main aim of this paper is to investigate the structure of factors of a kernel function and to relate it with reproducing kernel Hilbert spaces and operators acting on them.

The precise definition of reproducing kernels is given in Sect. 2; they may be either scalar or operator valued (the latter type being less familiar to operator theorists). Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{B}(\mathcal{E})$ be the set of all bounded linear operators on $\mathcal{E}$. If k_1 is a scalar-valued kernel and K_2 is a $\mathcal{B}(\mathcal{E})$-valued kernel on Λ, then $K = K_1 K_2$, where $K_1 = k_1 I_{\mathcal{E}}$, is also a $\mathcal{B}(\mathcal{E})$-valued kernel on Λ. We intend to study in the sequel factorizations of reproducing kernels of the above type and relate function and operator theoretic results on $\mathcal{H}_K$ with those of on $\mathcal{H}_{k_1}$ and $\mathcal{H}_{K_2}$.

The paper is organized as follows. In Sects. 2 and 3 we recall basic facts concerning reproducing kernel Hilbert spaces, multipliers, and modules over the polynomials. Section 4 is devoted to a presentation of tensor products of reproducing kernel spaces, which are intrinsically related to products of kernels. This part is generally known, but we did not find a suitable reference that would gather all the results we needed.

New results start with Sect. 5, in which we prove that for a large class of reproducing kernel Hilbert spaces $\mathcal{H}_K$ isometric multipliers are trivial. This in particular implies that the reproducing kernel Hilbert spaces with proper isometrically isomorphic shift invariant subspaces are rare.

In Sect. 6, we prove that a reproducing kernel Hilbert module $\mathcal{H}_K$ (see the definition in Sect. 6) defined over a domain Ω in $\mathbb{C}^n$ dilates to $\mathcal{H}_{k_1} \otimes \mathcal{E}$, for some Hilbert space $\mathcal{E}$, if and only if $K = k_1 L$ for some $\mathcal{B}(\mathcal{E})$-valued kernel L on Ω. Finally, in Sect. 7 we obtain a complete classification of Brehmer type submodules of a large class of reproducing kernel Hilbert modules; in particular, we prove that the Brehmer submodules and doubly commuting submodules of the Hardy module $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ are the same.

2. Preliminaries

In this section we briefly recall some basic facts concerning kernels and reproducing kernel Hilbert spaces. As a general reference for reproducing kernel Hilbert spaces, see [1,3]. For vector-valued reproducing kernel Hilbert spaces, see [12, Chapter 10].

Let Λ be a set and $\mathcal{E}$ be a Hilbert space. An operator-valued function $K : \Lambda \times \Lambda \to \mathcal{B}(\mathcal{E})$ is called a *kernel* (cf. [1,12]) and is denoted by $K(\lambda, \mu) \succ 0$, if

$$\sum_{p,q=1}^{m} \langle K(x_p, x_q)\eta_q, \eta_p\rangle_{\mathcal{E}} \geq 0, \tag{2.1}$$

for all $\{x_j\}_{j=1}^m \subseteq \Lambda$ and $\{\eta_j\}_{j=1}^m \subseteq \mathcal{E}$ and $m \in \mathbb{N}$. In this case there exists a Hilbert space $\mathcal{H}_K$ of $\mathcal{E}$-valued functions on Λ such that $\{K(\cdot, \lambda)\eta : \lambda \in \Lambda, \eta \in \mathcal{E}\}$ is a total set in $\mathcal{H}_K$ and

$$\langle f(\lambda), \eta\rangle_{\mathcal{E}} = \langle f, K(\cdot, \lambda)\eta\rangle_{\mathcal{H}_K} \quad (\eta \in \mathcal{E}, \lambda \in \Lambda). \tag{2.2}$$

In particular, we have

$$\|K(\cdot, \lambda)\eta\|^2_{\mathcal{H}_K} = \langle K(\lambda, \lambda)\eta, \eta\rangle_{\mathcal{E}} = \|K(\lambda, \lambda)^{1/2}\eta\|_{\mathcal{E}}. \tag{2.3}$$

Remark 2.1. If $\Phi : \Lambda \to \mathcal{B}(\mathcal{E}_*, \mathcal{E})$ for some Hilbert spaces $\mathcal{E}, \mathcal{E}_*$, then it is easy to see that $K(\lambda, \mu) := \Phi(\lambda)\Phi(\mu)^*$ is a kernel with values in $\mathcal{B}(\mathcal{E})$. Conversely, if $K : \Lambda \times \Lambda \to \mathcal{B}(\mathcal{E})$ is a kernel, then we may write $K(\lambda, \mu) = \Phi(\mu)\Phi(\lambda)^*$, with $\mathcal{E}_* = \mathcal{H}_K$ and $\Phi(\lambda) = K(\cdot, \lambda)$.

Let $\mathcal{E}$ be a Hilbert space and K_1 and K_2 be two $\mathcal{B}(\mathcal{E})$-valued kernel on Λ. We will write this sometimes as $K(\lambda, \mu) \succ 0$; then $K_1 \prec K_2$ will mean that $(K_2 - K_1)(\lambda, \mu) \succ 0$.

The following lemma is known, but for lack of an appropriate reference we supply a proof for completeness.

Lemma 2.2. *If $K_1(\lambda, \mu) \prec K_2(\lambda, \mu)$ and $L_1(\lambda, \mu) \prec L_2(\lambda, \mu)$, then*

$$K_1(\lambda, \mu) \otimes L_1(\lambda, \mu) \prec K_2(\lambda, \mu) \otimes L_2(\lambda, \mu).$$

Proof. Using (2.1), we have to prove that for nonnegative matrices A_1, A_2, B_1, B_2, if $A_1 \le A_2$ and $B_1 \le B_2$, then $A_1 \otimes B_1 \le A_2 \otimes B_2$. One can suppose that B_1, B_2 are invertible (otherwise one adds a small multiple of the identity and pass to the limit). Therefore

$$B_1^{-1/2} A_1 B_1^{-1/2} \le I, \quad B_2^{-1/2} A_2 B_2^{-1/2} \le I,$$

whence (since the tensor product of two contractions is a contraction)

$$(B_1^{-1/2} \otimes B_2^{-1/2})(A_1 \otimes A_2)\left(B_1^{-1/2} \otimes B_2^{-1/2}\right) \le I \otimes I$$

(the identities acting on the corresponding spaces). It remains to multiply on the right and on the left with $B_1^{1/2} \otimes B_2^{1/2}$. □

The proof of the following simple lemma is left to the reader.

Lemma 2.3. *Let K be a $\mathcal{B}(\mathcal{E})$-valued kernel on Λ and $\mathcal{H}_K$ the corresponding reproducing kernel Hilbert space. Suppose $\rho : \Lambda' \to \Lambda$ is a bijection. Then $\mathcal{H}' := \{f \circ \rho : f \in \mathcal{H}\}$ endowed with the scalar product*

$$\langle f \circ \rho, g \circ \rho \rangle_{\mathcal{H}'} := \langle f, g \rangle_{\mathcal{H}},$$

is a reproducing kernel Hilbert space of functions on Λ', with the $\mathcal{B}(\mathcal{E})$-valued kernel

$$K'(\lambda', \mu') = K(\rho(\lambda'), \rho(\mu')).$$

Moreover, the map $f \mapsto f \circ \rho$ is unitary from $\mathcal{H}$ to $\mathcal{H}'$.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Hilbert spaces and $K_j : \Lambda \times \Lambda \to \mathcal{B}(\mathcal{E}_j)$, $j = 1, 2$, be two kernels. A function $\varphi : \Lambda \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ is said to be a *multiplier* if

$$\varphi f \in \mathcal{H}_{K_2} \quad \text{for every } f \in \mathcal{H}_{K_1}.$$

We will denote by $\mathcal{M}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$ the space of all multipliers from $\mathcal{H}_{K_1}$ into $\mathcal{H}_{K_2}$. When $K_1 = K_2$, we will simply denote it by $\mathcal{M}(\mathcal{H}_{K_1})$. From the closed graph theorem it follows that each multiplier $\varphi \in \mathcal{M}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$ induces a bounded multiplication operator M_φ from $\mathcal{H}_{K_1}$ to $\mathcal{H}_{K_2}$, where

$$(M_\varphi f)(\lambda) = (\varphi f)(\lambda) = \varphi(\lambda) f(\lambda) \quad (f \in \mathcal{H}_{K_1}, \lambda \in \Lambda).$$

For each $\varphi \in \mathcal{M}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$, $\lambda \in \Lambda$ and $\eta \in \mathcal{E}_2$ we have

$$M_\varphi^*(K_2(\cdot, \lambda)\eta) = K_1(\cdot, \lambda)\varphi(\lambda)^*\eta. \tag{2.4}$$

We shall call a multiplier $\varphi \in \mathcal{M}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$ *partially isometric* or *isometric* if the induced multiplication operator M_φ has the corresponding property.

It will be the case sometimes below that the same function $\phi : \Lambda \to \mathcal{B}(\mathcal{E})$ will be a multiplier for different reproducing kernel Hilbert spaces with $\mathcal{B}(\mathcal{E})$-valued kernels. In this case we will write the multiplier M_ϕ^K to make the kernel (and the space) explicit.

A criterion for multipliers is given in [12, Theorem 10.22]: $\phi : \Lambda \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ is a multiplier if and only if there exists $c > 0$ such that

$$\phi(\lambda) K_1(\lambda, \mu) \phi(\mu)^* \prec c^2 K_2(\lambda, \mu), \tag{2.5}$$

and the smallest such c is precisely the norm of M_ϕ.

An important particular case are the *quasiscalar kernels*. These are $\mathcal{B}(\mathcal{E})$-valued kernels of the form

$$K(\lambda,\mu) = k(\lambda,\mu) I_{\mathcal{E}} \quad (\lambda,\mu \in \Lambda),$$

where k is a scalar-valued kernel on Λ and $\mathcal{E}$ is a Hilbert space. It follows then from (2.3) that

$$\|K(\cdot,\lambda)\eta\|_{\mathcal{H}_K} = k(\lambda,\lambda)\|\eta\|_{\mathcal{E}}. \tag{2.6}$$

We also note that as Hilbert spaces, one has

$$\mathcal{H}_K = \mathcal{H}_k \otimes \mathcal{E}.$$

Therefore, for a fixed orthonormal basis $\{e_j\}$ in $\mathcal{E}$, the general form of $F \in \mathcal{H}_K$ is given by

$$F = \sum_j f_j \otimes e_j,$$

with $f_j \in \mathcal{H}_k$ and $\sum_j \|f_j\|^2_{\mathcal{H}_k} < \infty$.

Now let k be a scalar kernel and $\lambda \in \Lambda$. By virtue of (2.2), it follows that the functions in $\mathcal{H}_k$ vanishing at λ are given by

$$\mathcal{H}_k \ominus \{k(\cdot,\lambda)\} = \{f \in \mathcal{H}_k : f(\lambda) = 0\}.$$

For quasiscalar kernels, we have the following:

Lemma 2.4. *Let k be a scalar kernel, $\mathcal{E}$ a Hilbert space, $\{e_j\}$ an orthonormal basis in $\mathcal{E}$, and $K = kI_{\mathcal{E}}$ the corresponding quasiscalar kernel. If $\lambda \in \Lambda$, then $\mathcal{H}_K \ominus \{k(\cdot,\lambda)x : x \in \mathcal{E}\}$ is given by*

$$\left\{ F = \sum_j f_j \otimes e_j : f_j \in \mathcal{H}_k,\ f_j(\lambda) = 0,\ \sum_j \|f_j\|^2_{\mathcal{H}_k} < \infty \right\}.$$

Proof. Let us denote by X the space in the right hand side of the equality. If $F \in X$, then it is immediate that F is orthogonal to any function $k(\cdot,\lambda)x$.

Conversely, suppose $g = \sum_j g_j \otimes e_j$ is orthogonal to X, that is,

$$0 = \langle g, F\rangle = \sum_j \langle g_j, f_j\rangle,$$

for all $F = \sum_j f_j \otimes e_j \in X$. In particular, each g_j is orthogonal to the space $\{f \in \mathcal{H}_k : f(\lambda) = 0\}$, and is thus a scalar multiple of $k(\cdot,\lambda)$. Therefore $g = k(\cdot,\lambda)x$ for some $x \in \mathcal{E}$. □

3. Kernels and Modules

We now consider a bounded domain Ω in $\mathbb{C}^n$ and a $\mathcal{B}(\mathcal{E})$-valued kernel K on Ω. In what follows, $\boldsymbol{z}$ will denote the element $(z_1,\ldots,z_n)$ in $\mathbb{C}^n$.

Let $K(\boldsymbol{z},\boldsymbol{w})$ be holomorphic in $\{z_1,\ldots,z_n\}$ and anti-holomorphic in $\{w_1 \ldots, w_n\}$ and $\mathcal{H}_K$ be the corresponding reproducing kernel Hilbert space. Then $\mathcal{H}_K$ is a set of $\mathcal{E}$-valued holomorphic functions on Ω and

$$\{K(\cdot,\boldsymbol{w})\eta : \boldsymbol{w} \in \Omega, \eta \in \mathcal{E}\},$$

is a total set in $\mathcal{H}_K$, that is,

$$\mathcal{H}_K = \overline{\text{span}}\{K(\cdot, \boldsymbol{w})\eta : \boldsymbol{w} \in \Omega, \eta \in \mathcal{E}\} \subseteq \mathcal{O}(\Omega, \mathcal{E}).$$

In what follows, we always assume that for any $\lambda \in \Lambda$ the function $K(\cdot, \lambda) : \Omega \to \mathcal{B}(\mathcal{E})$ is not identically zero.

We say that $\mathcal{H}_K$ is a *reproducing kernel Hilbert module* if

$$z_j \mathcal{H}_K \subseteq \mathcal{H}_K \quad (j = 1, \ldots, n).$$

In this case the multiplication operator tuple $(M_{z_1}, \ldots, M_{z_n})$, defined by

$$(M_{z_j} f)(\boldsymbol{w}) = w_j f(\boldsymbol{w}) \quad (\boldsymbol{w} \in \Omega, f \in \mathcal{H}_K),$$

induces a $\mathbb{C}[\boldsymbol{z}]$-module action on $\mathcal{H}_K$ as follows (cf. [6]):

$$p \cdot h = p(M_{z_1}, \ldots, M_{z_n})h \quad (p \in \mathbb{C}[z_1, \ldots, z_n], h \in \mathcal{H}_K).$$

A closed subspace $\mathcal{S}$ of $\mathcal{H}_K$ is said to be a *submodule* if $\mathcal{S}$ is M_{z_j}-invariant, $j = 1, \ldots, n$. Here the $\mathbb{C}[\boldsymbol{z}]$-module action on $\mathcal{S}$ is induced by the restriction of the multiplication operator tuple $\boldsymbol{M}_z|_{\mathcal{S}} = (M_{z_1}|_{\mathcal{S}}, \ldots, M_{z_n}|_{\mathcal{S}})$.

Note also that a submodule of a reproducing kernel Hilbert module is also a reproducing kernel Hilbert module.

If $\mathcal{H}_K$ is a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$, and the constant functions $\eta \in \mathcal{E}$ belong to $\mathcal{H}_K$, then of course $\mathbb{C}[\boldsymbol{z}]\mathcal{E} \subset \mathcal{H}_K$. The following lemma is often used in concrete cases.

Lemma 3.1. *Suppose $\mathcal{H}_{K_i} \subseteq \mathcal{O}(\Omega, \mathcal{E}_i)$, $i = 1, 2$ are reproducing kernel Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$, and $T : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2}$ satisfies*

$$T M_{z_j} = M_{z_j} T \quad (j = 1, \ldots, n).$$

If $\mathbb{C}[\boldsymbol{z}]\mathcal{E} \subset \mathcal{H}_{K_1}$ and is dense therein, then T is a multiplier.

Proof. Define $\Phi : \Omega \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ by $\Phi(z)\eta = T(\eta)$, where $T(\eta)$ is the action of T on the constant function $z \mapsto \eta \in \mathcal{E}_1$. The intertwining assumption in the statement implies that $T(p(z)\eta) = M_\Phi P(z)\eta$ for any polynomial p and $\eta \in \mathcal{E}$. If $\mathbb{C}[\boldsymbol{z}]\mathcal{E}$ is dense in $\mathcal{H}_{K_1}$, it follows that $T = M_\Phi$. $\square$

Let $\mathcal{H}_{K_i} \subseteq \mathcal{O}(\Omega, \mathcal{E}_i)$, $i = 1, 2$, be reproducing kernel Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$. We say that they are unitarily equivalent if there exists a unitary $U : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2}$ that satisfies

$$U M_{z_j} = M_{z_j} U \quad (j = 1, \ldots, n).$$

Corollary 3.2. *Suppose $\mathcal{H}_{K_i} \subseteq \mathcal{O}(\Omega, \mathcal{E}_i)$, $i = 1, 2$ are reproducing kernel Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$, and $\mathbb{C}[\boldsymbol{z}]\mathcal{E} \subset \mathcal{H}_{K_1}$ and is dense therein. Then $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ are unitarily equivalent if and only if there exists a unitary multiplier M_Φ such that $U = M_\Phi$.*

4. Tensor Products of Kernels

Our purpose in this section is to explore the relationship between kernels and functions defined on a set Λ and associated objects defined on the diagonal of $\Lambda \times \Lambda$. In the scalar case one can use as references [3,6,8].

Let $\mathcal{E}_i$ are Hilbert spaces and K_i are $\mathcal{B}(\mathcal{E}_i)$-valued kernels on Λ, $i = 1, 2$. Then the Hilbert tensor product $\mathcal{H}_{\mathcal{K}_1} \otimes \mathcal{H}_{\mathcal{K}_2}$ is a reproducing kernel Hilbert space on $\Lambda \times \Lambda$, with the $\mathcal{B}(\mathcal{E}_1 \otimes \mathcal{E}_2)$-valued kernel

$$(K_1 \otimes K_2)((\lambda_1, \lambda_2), (\mu_1, \mu_2)) = K_1(\lambda_1, \mu_1) \otimes K_2(\lambda_2, \mu_2).$$

More precisely, the map defined on simple tensors by $f \otimes g \mapsto f(\lambda_1)g(\lambda_2)$ extends to a unitary operator from $\mathcal{H}_{\mathcal{K}_1} \otimes \mathcal{H}_{\mathcal{K}_2}$ onto $\mathcal{H}_{K_1 \otimes K_2}$, which allows the identification of these two spaces.

For clarity, it is useful to make apparent the argument of functions, typically $\lambda \in \Lambda$ and $(\lambda_1, \lambda_2) \in \Lambda \times \Lambda$. So, for instance, we will write $K(\lambda, \mu)$ rather than $K(\cdot, \mu)$ in order to denote the function $\lambda \mapsto K(\lambda, \mu)$.

Now let $\Delta = \{(\lambda, \lambda) : \lambda \in \Lambda\}$ be the diagonal of $\Lambda \times \Lambda$ and let $\mathcal{N}$ be the set of all functions in $\mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}$ vanishing on Δ, that is,

$$\mathcal{N} = \{g \in \mathcal{H}_{k_1} \otimes \mathcal{H}_{k_2} : g(\lambda, \lambda) = 0, \ \lambda \in \Lambda\}.$$

Define also $\delta : \Lambda \to \Delta$ to be the bijection

$$\delta(\lambda) = (\lambda, \lambda) \quad (\lambda \in \Lambda).$$

The scalar case of the next lemma appears in [3, I.8]; we include the proof of the vector case for completion.

Lemma 4.1. *With the above notations,*

$$(K_1 * K_2)(\lambda, \mu) := K_1(\lambda, \mu) \otimes K_2(\lambda, \mu).$$

is a $\mathcal{B}(\mathcal{E}_1 \otimes \mathcal{E}_2)$-valued reproducing kernel for the Hilbert space of functions on Λ defined by $\{f \circ \delta : f \in \mathcal{N}^\perp\}$, endowed with the scalar product

$$\langle f \circ \delta, g \circ \delta \rangle_{\mathcal{H}} := \langle f, g \rangle_{\mathcal{N}^\perp}.$$

The map $f \mapsto f \circ \delta$ is unitary from $\mathcal{N}^\perp$ to $\mathcal{H}_{K_1 * K_2}$.

Proof. Note first that $\mathcal{N}^\perp$ is spanned by the set

$$S := \{K_1(\lambda_1, \mu)x_1 \otimes K_2(\lambda_2, \mu)x_2 : \mu \in \Lambda, x_1 \in \mathcal{E}_1, x_2 \in \mathcal{E}_2\}.$$

Indeed, for any $F \in \mathcal{N}$ we have

$$\langle F, K_1(\lambda_1, \mu)x_1 \otimes K_2(\lambda_2, \mu)x_2 \rangle = \langle F(\mu, \mu), x_1 \otimes x_2 \rangle = 0, \tag{4.1}$$

whence $S \subset \mathcal{N}^\perp$. On the other hand, if $F \in S^\perp$, then (4.1) is true for all $\mu \in \Lambda$ and $x_1 \in \mathcal{E}_1$, $x_2 \in \mathcal{E}_2$. By linearity we may deduce that $\langle F(\mu, \mu), \xi \rangle = 0$ for all $\xi \in \mathcal{E}_1 \otimes \mathcal{E}_2$, whence $F \in \mathcal{N}$.

It follows then easily that the restrictions of the functions in $\mathcal{N}^\perp$ to Δ form a reproducing kernel Hilbert space, with kernel given by $K_1(\lambda, \mu) \otimes K_2(\lambda, \mu)$. The proof is finished by applying Lemma 2.3, with $\rho = \delta$. □

The proof of the above lemma yields the following useful result:

Corollary 4.2. *The formula* $(\pi F)(\lambda) := F(\lambda, \lambda)$ *defines a coisometry from* $\mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}$ *to* $\mathcal{H}_{K_1 * K_2}$*, with* $\ker \pi = \mathcal{N}$*. Also,*

$$\pi^*((K_1 * K_2)(\lambda, \mu)(x_1 \otimes x_2) = K_1(\lambda_1, \mu)x_1 \otimes K_2(\lambda_2, \mu)x_2.$$

Proof. We observe that, for any $x_j, y_j \in \mathcal{E}_j$, $j = 1, 2$ and $\mu, \nu \in \Lambda$,

$$\begin{aligned}\langle K_1(\lambda_1, \mu)x_1 \otimes K_2(\lambda_2, \mu)x_2, K_1(\lambda_1, \nu)y_1 \otimes K_2(\lambda_2, \nu)y_2\rangle \\ = \langle K_1(\nu, \mu)x_1, y_1\rangle\langle K_2(\nu, \mu)x_2, y_2\rangle \\ = \langle K_1(\lambda, \mu)x_1 \otimes K_2(\lambda, \mu)x_2, K_1(\lambda, \nu)y_1 \otimes K_2(\lambda, \nu)y_2\rangle.\end{aligned}$$

Then $X : \mathcal{H}_{K_1 * K_2} \to \mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}$ defined by

$$\pi^*((K_1 * K_2)(\lambda, \mu)(x_1 \otimes x_2) = K_1(\lambda_1, \mu)x_1 \otimes K_2(\lambda_2, \mu)x_2,$$

for $x_j \in \mathcal{E}_j$, $j = 1, 2$, and $\mu \in \Lambda$, is an isometry. By the proof of the previous lemma we have $\ker X^* = (\operatorname{ran} L)^\perp = \mathcal{N}$ and the result now follows by defining $\pi = X^*$. □

In the scalar case, the previous result is [8, Theorem 1.1].

Suppose now that $F_1 : \Lambda \to \mathcal{B}(\mathcal{E}_1)$ is a multiplier on $\mathcal{H}_{K_1}$ and $F_2 : \Lambda \to \mathcal{B}(\mathcal{E}_2)$ is a multiplier on $\mathcal{H}_{K_2}$. Then the $F_1 \otimes F_2 : \Lambda \otimes \Lambda \to \mathcal{B}(\mathcal{E}_1 \otimes \mathcal{E}_2)$ is a multiplier on $\mathcal{H}_{K_1 \otimes K_2}$, and $M_{F_1 \otimes F_2} = M_{F_1} \otimes M_{F_2}$. The space $\mathcal{N}$ is invariant to multipliers on $\mathcal{H}_{K_1 \otimes K_2}$, and therefore $\mathcal{N}^\perp$ is invariant to adjoints of multipliers.

Lemma 4.3. *If* F_1 *is a multiplier on* $\mathcal{H}_{K_1}$ *and* F_2 *is a multiplier on* $\mathcal{H}_{K_2}$*, then the function* $F_1 * F_2 : \Lambda \to \mathcal{B}(\mathcal{E}_1 \otimes \mathcal{E}_2)$*, defined by* $(F_1 * F_2)(\lambda) = F_1(\lambda) \otimes F_2(\lambda)$*, is a multiplier on* $\mathcal{H}_{K_1 * K_2}$*. Moreover*

$$M_{F_1 * F_2}^{K_1 * K_2} = \pi \left(M_{F_1}^{K_1} \otimes M_{F_2}^{K_2}\right) \pi^*. \tag{4.2}$$

Proof. The assumption implies that (2.5) is satisfied for the two multipliers, so

$$F_1(\lambda)K_1(\lambda, \mu)F_1(\mu)^* \prec c_1^2 K_1(\lambda, \mu), \quad F_2(\lambda)K_2(\lambda, \mu)F_2(\mu)^* \prec c_2^2 K_2(\lambda, \mu).$$

By Lemma 2.2, we have

$$\begin{aligned}(F_1(\lambda) \otimes F_2(\lambda))\big(K_1(\lambda, \mu) \otimes K_2(\lambda, \mu)\big)(F_1(\mu) \otimes F_2(\mu))^* \\ \prec c_1^2 c_2^2 \big(K_1(\lambda, \mu) \otimes K_2(\lambda, \mu)\big),\end{aligned}$$

which means precisely that

$$(F_1 * F_2)(\lambda)(K_1 * K_2)(\lambda, \mu)(F_1 * F_2)(\mu) \prec c_1^2 c_2^2 (K_1 * K_2)(\lambda, \mu).$$

Again using (2.5) it follows that $F_1 * F_2$ is a multiplier on $\mathcal{H}_{K_1 * K_2}$ (of norm at most $c_1 c_2$).

To obtain Formula (4.2), we will check its adjoint on the reproducing kernels $(K_1 * K_2)(\lambda, \mu)(x_1 \otimes x_2)$, where $\mu \in \Lambda, x_1 \in \mathcal{E}_1, x_2 \in \mathcal{E}_2$ are fixed, while $\lambda \in \Lambda$ is the variable. According to (2.4), we have

$$\begin{aligned}\left(M_{F_1 * F_2}^{K_1 * K_2}\right)^* (K_1 * K_2)(\lambda, \mu)(x_1 \otimes x_2)K_1(\lambda, \mu) \\ = K_1(\lambda, \mu)F_1(\mu)^* x_1 \otimes K_2(\lambda, \mu)F_2(\mu)^* x_1.\end{aligned}$$

On the other hand, by Corollary 4.2

$$\pi^*(K_1 * K_2)(\lambda,\mu)(x_1\otimes x_2) = K_1(\lambda_1,\mu)x_1 \otimes K_2(\lambda_2,\mu)x_2.$$

Then, applying again (2.4),

$$\begin{aligned}\left(\left(M_{F_1}^{K_1}\right)^* \otimes \left(M_{F_2}^{K_2}\right)^*\right) \pi^*((K_1 * K_2)(\lambda,\mu)(x_1\otimes x_2) \\ = \big(K_1(\lambda_1,\mu)F_1(\mu)^*x_1\big) \otimes \big(K_2(\lambda_2.\mu)F_2(\mu)^*x_2\big),\end{aligned}$$

Therefore

$$\begin{aligned}\pi\left(\left(M_{F_1}^{K_1}\right)^* \otimes \left(M_{F_2}^{K_2}\right)^*\right) \pi^*((K_1 * K_2)(\lambda,\mu)(x_1\otimes x_2) \\ = \big(K_1(\lambda,\mu)F_1(\mu)^*x_1\big) \otimes \big(K_2(\lambda,\mu)F_2(\mu)^*x_2\big),\end{aligned}$$

and (4.2) is thus proved. □

If one of the kernels is scalar-valued, say $\dim \mathcal{E}_2 = 1$, the kernel $K_1 * k_2$ becomes simply the product k_2K_1. Then Lemma 4.3 says that f_2F_1 is a multiplier on $\mathcal{H}_{k_2K_1}$.

5. Isometric Multipliers

In this section, we study the isometric multipliers of reproducing kernel Hilbert spaces corresponding to quasiscalar kernels, generalizing results known about certain concrete kernels. It is an open problem to extend them to non-quasiscalar kernels, for which the methods used here do not seem to be appropriate.

Let k be a scalar-valued kernel on a set Λ and let $\mathcal{H}_k$ be the corresponding reproducing kernel Hilbert space. For each λ and μ in Λ, define a relation $\sim_k$ as follows: $\lambda \sim_k \mu$ if there exist $m \in \mathbb{N}$ and $\{\lambda_1,\dots,\lambda_m\} \subseteq \Lambda$ such that

$$\lambda_1 = \lambda, \lambda_m = \mu, \quad \text{and } k(\lambda_j,\lambda_{j+1}) \neq 0 \ \text{ for } \ 1 \le j \le m-1.$$

Then $\sim_k$ is an equivalence relation on Λ. In particular, if λ,μ are in two different equivalence classes, then $k(\lambda,\mu) = 0$.

Suppose k_1, k_2 are two scalar-valued reproducing kernels on Λ, $K_1 = k_1 I_{\mathcal{E}_1}$, $K_2 = k_2 I_{\mathcal{E}_2}$. If ϕ is a multiplier on $\mathcal{H}_{K_1}$, it follows from Lemma 4.3, applied for $F_1 = \phi$ and $F_2 = I$, that $\phi * I$ is also a multiplier of $\mathcal{H}_{K_1 * K_2}$, and

$$\left\| M_{\phi * I}^{K_1 * K_2} \right\| \le \left\| M_\phi^{K_1} \right\|. \tag{5.1}$$

Theorem 5.1. *Let k_1, k_2 be two scalar-valued reproducing kernels on Λ, and $K_1 = k_1 I_{\mathcal{E}_1}$, $K_2 = k_2 I_{\mathcal{E}_2}$. Denote $k = k_1k_2$, $K = K_1 * K_2$, and suppose the following conditions are satisfied:*

1. *the map $M_\phi^{K_1} \mapsto M_{\phi * I}^K$ from $\mathcal{M}(\mathcal{H}_{K_1})$ to $\mathcal{M}(\mathcal{H}_K)$ preserves the norm;*
2. *$\mathcal{H}_{k_1} \cap \mathcal{H}_{k_2}$ is dense in $\mathcal{H}_{k_1}$.*

If $M_{\phi * I}^K$ is an isometric multiplier in $\mathcal{M}(\mathcal{H}_K)$, then it is a constant isometry on each of the equivalence classes of $\sim_{k_1}$.

In particular, if $k_1(\lambda,\mu) \neq 0$ for any λ, μ, then $\sim_{k_1}$ has a single equivalence class, and the conclusion becomes that φ is a constant isometry.

Proof. We use the notation of the previous section; so $\Delta = \{(\lambda,\lambda) : \lambda \in \Lambda\}$, $\mathcal{N} = \{F \in \mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2} : F(\lambda,\lambda) = 0,\ \lambda \in \Lambda\}$, and $(\pi F)(\lambda) = F(\lambda,\lambda)$ defines a coisometry from $\mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}$ to $\mathcal{H}_K$ with $\ker \pi = \mathcal{N}$. Then $M_\phi^{K_1}$ is a contraction by assumption (1), and we have by Lemma 4.3

$$M^K_{\varphi\otimes I} = \pi\left(M_\varphi^{K_1} \otimes I_{\mathcal{H}_{K_2}}\right)\pi^*, \tag{5.2}$$

whence

$$M^K_{\varphi\otimes I}\pi = \pi\left(M_\varphi^{K_1} \otimes I_{\mathcal{H}_{K_2}}\right)\pi^*\pi = \pi\left(M_\varphi^{K_1} \otimes I_{\mathcal{H}_{K_2}}\right)P_{\mathcal{N}^\perp}.$$

Now for $F \in \mathcal{N}^\perp$ and using the fact that $M^K_{\varphi\otimes I}$ is an isometry, we have

$$\left\|\pi\left(M_\varphi^{K_1} \otimes I_{\mathcal{H}_{K_2}}\right)F\right\| = \left\|M_\varphi^K \pi F\right\| = \|\pi F\| = \|F\|,$$

where the last equality follows from the fact that π is an isometry on $(\ker\pi)^\perp$. Hence, since $M_\varphi^{K_1} \otimes I_{k_2}$ is a contraction, we have

$$\|F\| \geq \left\|\left(M_\varphi^{K_1} \otimes I_{\mathcal{H}_{K_2}}\right)F\right\| \geq \left\|\pi\left(M_\varphi^{K_1} \otimes I_{\mathcal{H}_{K_2}}\right)F\right\| = \|F\|,$$

and hence

$$\left\|\pi\left(M_\varphi^{K_1} \otimes I_{\mathcal{H}_{K_2}}\right)F\right\| = \left\|\left(M_\varphi^{K_1} \otimes I_{\mathcal{H}_{K_2}}\right)F\right\|.$$

Consequently, $(M_\varphi^{K_1} \otimes I_{\mathcal{H}_{K_2}})F \in (\ker\pi)^\perp = \mathcal{N}^\perp$, that is,

$$\left(M_\varphi^{K_1} \otimes I_{\mathcal{H}_{K_2}}\right)\mathcal{N}^\perp \subseteq \mathcal{N}^\perp.$$

In particular, since $k_1(\lambda_1,\mu)x_1 \otimes k_2(\lambda_2,\mu)x_2 \in \mathcal{N}^\perp$ for $\mu \in \Lambda$, $x_1 \in \mathcal{E}_1$, $x_2 \in \mathcal{E}_2$ (here λ_1, λ_2 are the argument variables), we have

$$M_\varphi^{K_1} k_1(\lambda_1,\mu)x_1 \otimes k_2(\lambda_2,\mu)x_2 \in \mathcal{N}^\perp \quad (\mu \in \Lambda,\ x_1 \in \mathcal{E}_1,\ x_2 \in \mathcal{E}_2).$$

Now, if $f, g \in \mathcal{H}_{k_1} \cap \mathcal{H}_{k_2}$, $y_1 \in \mathcal{E}_1$, $y_2 \in \mathcal{E}_2$, then

$$f(\lambda_1)y_1 \otimes g(\lambda_2)y_2 - g(\lambda_1)y_1 \otimes f(\lambda_2)y_2 \in \mathcal{N},$$

and therefore

$$\begin{aligned}
0 &= \left\langle \left(M_\varphi^{K_1} k_1(\lambda_1,\mu)x_1 \otimes k_2(\lambda_2,\mu)\right)x_2, f(\lambda_1)y_1 \otimes g(\lambda_2)y_2 \right.\\
&\quad \left. - g(\lambda_1)y_1 \otimes f(\lambda_2)y_2\right\rangle_{\mathcal{H}_{k_1}\otimes\mathcal{H}_{k_2}}\\
&= \langle \varphi(\lambda_1)k_1(\lambda_1,\mu)x_1, f(\lambda_1)y_1\rangle_{\mathcal{H}_{K_1}} \langle k_2(\lambda_2,\mu)x_2, g(\lambda_2)y_2\rangle_{\mathcal{H}_{K_2}}\\
&\quad - \langle \varphi(\lambda_1)k_1(\lambda_1,\mu), g(\lambda_1)y_1\rangle_{\mathcal{H}_{K_1}} \langle k_2(\lambda_2,\mu)x_2, f(\lambda_2)y_2\rangle_{\mathcal{H}_{K_2}}\\
&= \langle \varphi(\lambda_1)k_1(\lambda_1,\mu)x_1, f(\lambda_1)y_1\rangle_{\mathcal{H}_{K_1}} \overline{g(\mu)}\langle x_2, y_2\rangle -\\
&\quad - \langle \varphi(\lambda_1)k_1(\lambda_1,\mu), g(\lambda_1)y_1\rangle_{\mathcal{H}_{K_1}} \overline{f(\mu)}\langle x_2, y_2\rangle.
\end{aligned}$$

Applying assumption (2), the above formula is valid by continuity for any $f, g \in \mathcal{H}_{k_1}$.

Fix $\mu \in \Lambda$. Take $f \perp k_1(\lambda,\mu)$ (so $f(\mu) = 0$) and $g = k_1(\cdot,\mu)$ (so $g(\mu) \neq 0$); also, assume $\langle x_2, y_2\rangle \neq 0$. It follows from the preceding equation that

$$\langle \varphi(\lambda_1)k_1(\lambda_1,\mu)x_1, f(\lambda_1)y_1\rangle_{\mathcal{H}_{K_1}} = 0$$

for all $x_1, y_1 \in \mathcal{E}_1$. Therefore the function $\varphi(\lambda_1)k_1(\lambda_1,\mu)x_1 = M_\varphi^{k_1}k_1(\lambda_1,\mu)x_1$ is orthogonal to the space spanned by the functions $f(\lambda_1)y_1 \in \mathcal{H}_{K_1}$ with

$f \in \mathcal{H}_{k_1}$, $f(\mu) = 0$, and $y_1 \in \mathcal{E}_1$. If we identify $\mathcal{H}_{K_1}$ with $\mathcal{H}_{k_1} \otimes \mathcal{E}_1$, this space becomes the space spanned by $f \otimes y_1$, $f(\mu) = 0$. We may then apply Lemma 2.4 to conclude that

$$\varphi(\lambda_1)k_1(\lambda_1, \mu)x_1 = k_1(\lambda_1, \mu)x_1'$$

for some $x_1' \in \mathcal{E}_1$. But we have, for all $y \in \mathcal{E}_1$,

$$\begin{aligned} \langle k_1(\lambda_1, \mu)x_1', k_1(\lambda_1, \mu)y\rangle &= \langle M_\varphi^{k_1} k_1(\lambda_1, \mu)x_1, k_1(\lambda_1, \mu)y\rangle \\ &= \left\langle k_1(\lambda_1, \mu)x_1, (M_\varphi^{k_1})^* k_1(\lambda_1, \mu)y\right\rangle \\ &= \langle k_1(\lambda_1, \mu)x_1, \varphi(\mu)^* k_1(\lambda_1, \mu)y\rangle \\ &= \langle \varphi(\mu)k_1(\lambda_1, \mu)x_1, k_1(\lambda_1, \mu)y\rangle. \end{aligned}$$

Therefore

$$k_1(\lambda_1, \mu)x_1' = \varphi(\lambda_1)k_1(\lambda_1, \mu)x_1 = \varphi(\mu)k_1(\lambda_1, \mu)x_1$$

for all $x_1 \in \mathcal{E}_1$. In this relation λ is still the argument of the functions in the two sides of the equality, but we may deduce from here the pointwise equality

$$\varphi(\lambda_1)k_1(\lambda_1, \mu) = \varphi(\mu)k_1(\lambda_1, \mu).$$

for all $\lambda_1, \mu \in \Lambda$. So, if $k_1(\lambda_1, \mu) \neq 0$, then $\varphi(\lambda_1) = \varphi(\mu)$. From the definition of $\sim_{k_1}$ it follows that on each of its equivalence classes the multiplier ϕ on $\mathcal{H}_{K_1}$ is a constant operator. Regarding again $\mathcal{H}_{K_1}$ as $\mathcal{H}_{k_1} \otimes \mathcal{E}_1$, it follows that $M_\phi^{K_1} = I_{\mathcal{H}_{k_1}} \otimes \Phi$ for some $\Phi \in \mathcal{B}(\mathcal{E}_1)$. Therefore, in order for $M_\phi^{K_1}$ to be an isometry, Φ must be an isometry; this finishes the proof of the theorem. $\square$

Corollary 5.2. *Let k_1, k_2 be two scalar-valued reproducing kernels on Λ, and $K_1 = k_1 I_{\mathcal{E}_1}$, $K = k_1 k_2 I_{\mathcal{E}_1}$. Suppose the following conditions are satisfied:*

1. *the map $M_\phi^{K_1} \mapsto M_\phi^K$ from $\mathcal{M}(\mathcal{H}_{K_1})$ to $\mathcal{M}(\mathcal{H}_K)$ is surjective and preserves the norm;*
2. *$\mathcal{H}_{k_1} \cap \mathcal{H}_{k_2}$ is dense in $\mathcal{H}_{k_1}$.*

Then any isometric multiplier in $\mathcal{M}(\mathcal{H}_K)$ is a constant isometry on each of the equivalence classes of $\sim_{k_1}$.

There is an important case in which condition (1) in the above corollary is satisfied, which we will present as a separate statement.

Corollary 5.3. *Let $\Lambda = \Omega$ be a domain in $\mathbb{C}^n$ and k_1, k_2 are analytic in the first variable, $K_1 = k_1 I_{\mathcal{E}_1}$, $K = k_1 k_2 I_{\mathcal{E}_1}$. Suppose the following conditions are satisfied:*

1. *$\mathcal{M}(\mathcal{H}_{K_1})$ coincides with the uniformly bounded $\mathcal{B}(\mathcal{E}_1)$-valued analytic functions and for any $\phi \in \mathcal{M}(\mathcal{H}_{K_1})$ we have*
$$\|M_\phi\|_{\mathcal{H}_{K_1}} = \sup_\lambda \|\phi(\lambda)\|; \tag{5.3}$$
2. *$\mathcal{H}_{k_1} \cap \mathcal{H}_{k_2}$ is dense in $\mathcal{H}_{k_1}$.*

Then any isometric multiplier in $\mathcal{M}(\mathcal{H}_K)$ is a constant isometry on each of the equivalence classes of $\sim_{k_1}$.

Proof. Let $\phi \in \mathcal{M}(\mathcal{H}_K)$. Then by (2.4), we have

$$\sup_{\lambda} \|\phi(\lambda)\| \leq \|M_\phi^K\|_{\mathcal{B}(\mathcal{H}_K)}.$$

Hence condition (1) imply that $M_\phi^{K_1} \in \mathcal{M}(\mathcal{H}_{K_1})$. From (5.1) it follows that $\|M_\phi^{K_1}\| = \|M_\phi^K\|$. We may then apply Corollary 5.2 to conclude the proof. □

Remark 5.4. Under the same assumptions and notations as in Corollary 5.3, suppose also that polynomials are in $\mathcal{H}_{k_1}$ as well as in $\mathcal{H}_{k_2}$. Then a sufficient condition for (2) is that they are dense in $\mathcal{H}_{k_1}$.

Using now Corollary 3.2, it is easy to derive the following result.

Theorem 5.5. *Let $\mathcal{E}$ be a Hilbert space, Ω be a domain in $\mathbb{C}^n$ and $\mathcal{H}_{k_1}, \mathcal{H}_{k_2} \subseteq \mathcal{O}(\Omega)$ are reproducing kernel Hilbert spaces. Let $K_1 = k_1 I_{\mathcal{E}}$ and $K = k_1 k_2 I_{\mathcal{E}}$. Suppose the following conditions are satisfied:*

1. *$\mathcal{H}_{k_1}$ is a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$.*
2. *$\mathbb{C}[\boldsymbol{z}] \subseteq \mathcal{H}_{k_1} \cap \mathcal{H}_{k_2}$ and $\mathbb{C}[\boldsymbol{z}]$ is dense in $\mathcal{H}_{k_1}$.*
3. *$\mathcal{M}(\mathcal{H}_{K_1}) = H^\infty_{\mathcal{B}(\mathcal{E})}(\Omega)$ and for each $\varphi \in \mathcal{M}(\mathcal{H}_{K_1})$ we have*
$$\|M_\phi\|_{\mathcal{H}_{K_1}} = \sup_{\lambda \in \Omega} \|\phi(\lambda)\|.$$
4. *$z_1 \sim_{k_1} z_2$ for any $z_1, z_2 \in \Omega$. In particular, this is true if $k_1(z_1, z_2) \neq 0$ for any $z_1, z_2 \in \Omega$.*

Let $\varphi \in \mathcal{M}(\mathcal{H}_K)$ be a multiplier and $\mathcal{S}$ be a submodule of $\mathcal{H}_K$. Then

(i) *M_φ is an isometric multiplier if and only if there exists an isometry $V \in \mathcal{B}(\mathcal{E})$ such that $M_\varphi = I_{\mathcal{H}_{k_1 k_2}} \otimes V$.*

(ii) *$\mathcal{S} \subseteq \mathcal{H}_K$ is unitarily equivalent to $\mathcal{H}_K$ if and only if there exists a closed subspace $\tilde{\mathcal{E}}$ of $\mathcal{E}$ such that $\mathcal{S} = \mathcal{H}_{k_1 k_2} \otimes \tilde{\mathcal{E}}$.*

Example. Let $n \geq 1$, $\alpha > -1$, and consider the space

$$A^2_\alpha(\mathbb{B}^n) := \left\{ f \text{ holomorphic in } \mathbb{B}^n : \|f\|_\alpha = \int_{\mathbb{B}^n} |f(\boldsymbol{z})|^2 (1 - |\boldsymbol{z}|^2)^\alpha \, dV(\boldsymbol{z}) < \infty \right\} \tag{5.4}$$

where dV is the volume measure on $\mathbb{B}^n$. It is usually called the weighted Bergman space. Moreover, $A^2_\alpha(\mathbb{B}^n) = \mathcal{H}_g$, where the reproducing kernel g is given by the formula

$$g_\alpha(\boldsymbol{z}, \boldsymbol{w}) = \left(1 - \sum_{i=1}^n z_i \bar{w}_i\right)^{-\alpha-n-1}, \quad \boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n \tag{5.5}$$

(see, for instance, [17, Ch. 2]).

We may apply Theorem 5.5 to the reproducing kernel module $A^2_\alpha(\mathbb{B}^n)$. We take

$$k_1(\boldsymbol{z}, \boldsymbol{w}) = \left(1 - \sum_{i=1}^n z_i \bar{w}_i\right)^{-n}, \quad k_1(\boldsymbol{z}, \boldsymbol{w}) = \left(1 - \sum_{i=1}^n z_i \bar{w}_i\right)^{-\alpha-1}.$$

Then k_1 is the reproducing kernel of the Hardy space on the unit ball, while k_2 is a kernel by the binomial formula. The hypotheses of the theorem are easily checked; the only nonobvious one, (3), is a consequence of the usual formula for the Hardy space norm (see [17, Ch. 4]). Therefore:

(i) M_φ is an isometry if and only if $M_\varphi = I_{A^2_\alpha(\mathbb{B}^n)} \otimes V$ for some isometry $V \in \mathcal{B}(\mathcal{E})$.

(ii) $\mathcal{S}$ is unitarily equivalent to $A^2_\alpha(\mathbb{B}^n) \otimes \mathcal{E}$ if and only if $\mathcal{S} = A^2_\alpha(\mathbb{B}^n) \otimes \tilde{\mathcal{E}}$ for some closed subspace $\tilde{\mathcal{E}}$ of $\mathcal{E}$.

In particular, (i) generalizes Proposition 4.2 in [11], which proves the result for the case $\alpha = 0$ (the usual Bergman space). Part (ii) is related to the rigidity of submodules of weighted Bergman modules (see [7,10,13,14]).

6. Factorizations of Kernels and Dilations

Suppose Ω is a domain in $\mathbb{C}^n$. A scalar-valued reproducing kernel $g : \Omega \times \Omega \to \mathbb{C}$ is said to be a *good kernel* if:

1. g is analytic in the first variable;
2. $\mathcal{H}_g$ is a reproducing kernel Hilbert module;
3. $$\bigcap_{j=1}^{n} \ker(M^*_{z_j} - \bar{w}_j I_{\mathcal{H}_g}) = \mathbb{C} g(\cdot, \boldsymbol{w}) \quad (\boldsymbol{w} \in \Omega);$$
4. there exists a $\boldsymbol{w}_0 \in \Omega$ such that
$$g(\cdot, \boldsymbol{w}_0) \equiv 1.$$

We say then that $\mathcal{H}_g \subseteq \mathcal{O}(\Omega, \mathbb{C})$ is a *good reproducing kernel Hilbert module.*

The definition of a good kernel is motivated by the study of sharp kernels introduced by Agrawal and Salinas [2] (see also Cowen and Douglas [4]).

Note that if g is a scalar valued kernel on a set Λ and the function $g(\cdot, \lambda_0)$ is non-vanishing for some $\lambda_0 \in \Lambda$ then one can assume, after renormalizing, that $g(\cdot, \lambda_0) \equiv 1$.

Let $\mathcal{H}_g$ be a good reproducing kernel Hilbert module over Ω and $\mathcal{H}_K \subseteq \mathcal{O}(\Omega, \mathcal{E})$ be a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. We say that $\boldsymbol{M}_z = (M_{z_1}, \ldots, M_{z_n})$ on $\mathcal{H}_K$ *dilates* (cf. [5,15]) to $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$ on $\mathcal{H}_g \otimes \mathcal{E}$, or $\mathcal{H}_K$ dilates to $\mathcal{H}_g \otimes \mathcal{E}$, for some Hilbert space $\mathcal{E}$, if there exists an isometry $\Pi : \mathcal{H} \to \mathcal{H}_g \otimes \mathcal{E}$ such that
$$(M^*_{z_i} \otimes I_{\mathcal{E}})\Pi = \Pi M^*_{z_i} \quad (i = 1, \ldots, n).$$

Our main result in this section is the following theorem which relates dilation of a reproducing kernel Hilbert module to a good reproducing Hilbert module with the factorization of the kernel.

Theorem 6.1. *Let $\mathcal{E}$ and $\mathcal{E}_*$ be two Hilbert spaces and $\mathcal{H}_g$ be a good reproducing kernel Hilbert module on Ω and $\mathcal{H}_K \subseteq \mathcal{O}(\Omega, \mathcal{E})$ be a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Then the following conditions are equivalent:*

1. *$\mathcal{H}_K$ dilates to $\mathcal{H}_g \otimes \mathcal{E}_*$.*
2. *There exists a holomorphic function $\Phi : \Omega \to \mathcal{B}(\mathcal{E}_*, \mathcal{E})$ such that*
$$K(\boldsymbol{z}, \boldsymbol{w}) = g(\boldsymbol{z}, \boldsymbol{w})\Phi(z)\Phi(\boldsymbol{w})^* \quad (\boldsymbol{z}, \boldsymbol{w} \in \Omega).$$

Proof. Assume (2) holds. Then for each $\boldsymbol{z}, \boldsymbol{w} \in \Omega$ and $\eta, \zeta \in \mathcal{E}_*$, we have

$$\begin{aligned}\langle K(\cdot, \boldsymbol{w})\eta, K(\cdot, \boldsymbol{z})\zeta\rangle_{\mathcal{H}_K} &= \langle K(\boldsymbol{z}, \boldsymbol{w})\eta, \zeta\rangle_{\mathcal{E}_*}\\ &= \langle g(\boldsymbol{z}, \boldsymbol{w})\Phi(\boldsymbol{z})\Phi(\boldsymbol{w})^*\eta, \zeta\rangle_{\mathcal{E}_*}\\ &= g(\boldsymbol{z}, \boldsymbol{w})\langle \Phi(\boldsymbol{z})\Phi(\boldsymbol{w})^*\eta, \zeta\rangle_{\mathcal{E}_*}\\ &= \langle g(\cdot, \boldsymbol{w}), g(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_g}\langle \Phi(\boldsymbol{w})^*\eta, \Phi(\boldsymbol{z})^*\zeta\rangle_{\mathcal{E}_*}\\ &= \langle g(\cdot, \boldsymbol{w}) \otimes \Phi(\boldsymbol{w})^*\eta, g(\cdot, \boldsymbol{z}) \otimes \Phi(\boldsymbol{z})^*\zeta\rangle_{\mathcal{H}_g\otimes\mathcal{E}_*}.\end{aligned}$$

This allows us to define an isometry $\Pi : \mathcal{H}_K \to \mathcal{H}_g \otimes \mathcal{E}_*$ by

$$\Pi(K(\cdot, \boldsymbol{w})\eta) = g(\cdot, \boldsymbol{w}) \otimes \Phi(\boldsymbol{w})^*\eta \quad (\boldsymbol{w} \in \Omega, \eta \in \mathcal{E}_*).$$

Using this, on one hand, we have

$$\begin{aligned}(\Pi M_{z_j}^*)(K(\cdot, \boldsymbol{w})\eta) &= \Pi(\bar{w}_j K(\cdot, \boldsymbol{w})\eta)\\ &= \bar{w}_j \Pi(K(\cdot, \boldsymbol{w})\eta)\\ &= \bar{w}_j (g(\cdot, \boldsymbol{w}) \otimes \Phi(\boldsymbol{w})^*\eta),\end{aligned}$$

and on the other hand, by (2.4), we have

$$\begin{aligned}(M_{z_j} \otimes I_{\mathcal{E}_*})^*\Pi(K(\cdot, \boldsymbol{w})\eta) &= (M_{z_j} \otimes I_{\mathcal{E}_*})^*(g(\cdot, \boldsymbol{w}) \otimes \Phi(\boldsymbol{w})^*\eta)\\ &= \bar{w}_j (g(\cdot, \boldsymbol{w}) \otimes \Phi(\boldsymbol{w})^*\eta),\end{aligned}$$

where $\eta \in \mathcal{E}$ and $\boldsymbol{w} \in \Omega$. Therefore

$$(M_{z_j} \otimes I_{\mathcal{E}_*})^*\Pi = \Pi M_j^* \quad (j = 1, \ldots, n), \tag{6.1}$$

and hence $\mathcal{H}_K$ dilates to $\mathcal{H}_g \otimes \mathcal{E}_*$. This proves (1).

Assume now (1) holds. Then there exists an isometry $\Pi : \mathcal{H}_K \to \mathcal{H}_g \otimes \mathcal{E}_*$ such that (6.1) holds. Then for $\boldsymbol{w} \in \Omega$ and $\eta \in \mathcal{E}$ and $j = 1, \ldots, n$, we have

$$\begin{aligned}(M_{z_j} \otimes I_{\mathcal{E}_*})^*(\Pi K(\cdot, \boldsymbol{w})\eta) &= ((M_{z_j} \otimes I_{\mathcal{E}_*})^*\Pi)(K(\cdot, \boldsymbol{w})\eta)\\ &= \Pi M_{z_j}^*(K(\cdot, \boldsymbol{w})\eta)\\ &= \bar{w}_j(\Pi K(\cdot, \boldsymbol{w})\eta).\end{aligned}$$

In particular,

$$\Pi(K(\cdot, \boldsymbol{w})\eta) \in \bigcap_{j=1}^{n} \ker\left((M_{z_j} \otimes I_{\mathcal{E}_*})^* - \bar{w}_j I_{\mathcal{H}_g\otimes\mathcal{E}_*}\right) = g(\cdot, \boldsymbol{w}) \otimes \mathcal{E}_*.$$

Then for each $\boldsymbol{w} \in \Omega$ there exists a linear map $\Phi(\boldsymbol{w}) : \mathcal{E}_* \to \mathcal{E}$ such that

$$\Pi(K(\cdot, \boldsymbol{w})\eta) = g(\cdot, \boldsymbol{w}) \otimes \Phi(\boldsymbol{w})^*\eta \quad (\eta \in \mathcal{E}).$$

Observe that if $\boldsymbol{w} \in \Omega$ and $\eta \in \mathcal{E}$ we have

$$\begin{aligned}\|\Phi(\boldsymbol{w})^*\eta\|_{\mathcal{E}_*} &= \frac{1}{\|g(\cdot, \boldsymbol{w})\|_{\mathcal{H}_g}}\|\Pi(K(\cdot, \boldsymbol{w})\eta)\|_{\mathcal{H}_g\otimes\mathcal{E}_*}\\ &\leq \frac{1}{\|g(\cdot, \boldsymbol{w})\|_{\mathcal{H}_g}}\|(K(\cdot, \boldsymbol{w})\eta)\|_{\mathcal{H}_K}\\ &\leq \frac{1}{\|g(\cdot, \boldsymbol{w})\|_{\mathcal{H}_g}}\|K(\boldsymbol{w}, \boldsymbol{w})^{\frac{1}{2}}\|_{\mathcal{B}(\mathcal{E})}\|\eta\|_{\mathcal{E}},\end{aligned}$$

where the last inequality follows from the fact that

$$\begin{aligned}\|(K(\cdot,\boldsymbol{w})\eta)\|^2_{\mathcal{H}_K} &= \langle K(\cdot,\boldsymbol{w})\eta, K(\cdot,\boldsymbol{w})\eta\rangle_{\mathcal{H}_K} \qquad (\text{by}(2.2))\\ &= \langle K(\boldsymbol{w},\boldsymbol{w})\eta,\eta\rangle_{\mathcal{E}}\\ &= \|K(\boldsymbol{w},\boldsymbol{w})^{\frac{1}{2}}\eta\|^2_{\mathcal{E}}.\end{aligned}$$

Therefore $\Phi(\boldsymbol{w})^*$, $\boldsymbol{w}\in\Omega$, is a bounded linear operator. For $\eta,\zeta\in\mathcal{E}$ we now have

$$\begin{aligned}\langle K(\boldsymbol{z},\boldsymbol{w})\eta,\zeta\rangle_{\mathcal{E}} &= \langle K(\cdot,\boldsymbol{w})\eta, K(\cdot,\boldsymbol{z})\zeta\rangle_{\mathcal{H}_K}\\ &= \langle \Pi(K(\cdot,\boldsymbol{w})\eta), \Pi(K(\cdot,\boldsymbol{z})\zeta)\rangle_{\mathcal{H}_g\otimes\mathcal{E}_*}\\ &= \langle g(\cdot,\boldsymbol{w})\otimes\Phi(\boldsymbol{w})^*\eta, g(\cdot,\boldsymbol{z})\otimes\Phi(\boldsymbol{z})^*\zeta\rangle_{\mathcal{H}_g\otimes\mathcal{E}_*}\\ &= g(\boldsymbol{z},\boldsymbol{w})\langle\Phi(\boldsymbol{w})^*\eta,\Phi(\boldsymbol{z})^*\zeta\rangle_{\mathcal{E}_*}\\ &= \langle g(\boldsymbol{z},\boldsymbol{w})\Phi(\boldsymbol{z})\Phi(\boldsymbol{w})^*\eta,\zeta\rangle_{\mathcal{E}_*},\end{aligned}$$

and hence

$$K(\boldsymbol{z},\boldsymbol{w}) = g(\boldsymbol{z},\boldsymbol{w})\Phi(\boldsymbol{z})\Phi(\boldsymbol{w})^* \quad (\boldsymbol{z},\boldsymbol{w}\in\Omega).$$

Finally, since

$$\begin{aligned}\langle\Phi(\boldsymbol{w})\zeta,\eta\rangle_{\mathcal{E}} &= \langle\zeta,\Phi(\boldsymbol{w})^*\eta\rangle_{\mathcal{E}_*}\\ &= \langle g(\cdot,\boldsymbol{w}_0)\otimes\zeta, g(\cdot,\boldsymbol{w})\otimes\Phi(\boldsymbol{w})^*\eta\rangle_{\mathcal{H}_g\otimes\mathcal{E}_*}\\ &= \langle g(\cdot,\boldsymbol{w}_0)\otimes\zeta, \Pi(K(\cdot,\boldsymbol{w})\eta)\rangle_{\mathcal{H}_g\otimes\mathcal{E}_*}\\ &= \langle\Pi^*(g(\cdot,\boldsymbol{w}_0)\otimes\zeta), K(\cdot,\boldsymbol{w})\eta\rangle_{\mathcal{H}_g\otimes\mathcal{E}_*},\end{aligned}$$

for each $\eta\in\mathcal{E}$ and $\zeta\in\mathcal{E}_*$, and since $\boldsymbol{w}\mapsto K(\cdot,\boldsymbol{w})$ is anti-holomorphic, we conclude that $\boldsymbol{w}\mapsto\Phi(\boldsymbol{w})$ is holomorphic. This shows that (3) holds and completes the proof of the theorem. □

The next corollary follows by taking into account Remark 2.1.

Corollary 6.2. *Let $\mathcal{E}$ be a Hilbert spaces and $\mathcal{H}_g$ be a good reproducing kernel Hilbert module on Ω and $\mathcal{H}_K\subseteq\mathcal{O}(\Omega,\mathcal{E})$ be a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Then the following conditions are equivalent:*

1. *There exists a Hilbert space $\mathcal{E}_*$ such that the equivalent conditions in the statement of Theorem 6.1 hold.*
2. *There exists a $\mathcal{B}(\mathcal{E})$-valued kernel L on Ω, holomorphic in the first and anti-holomorphic in the second variable, such that $K = gL$.*

Theorem 6.1 and Corollary 6.2 represent a generalization of the dilation results of quasi-free Hilbert modules (see Theorems 1 and 2 in [9]) to reproducing kernel Hilbert modules. Let us also note that, moreover, our argument does not rely on localizations of Hilbert modules.

7. Submodules of Reproducing Kernel Hilbert Modules

Suppose $p(\boldsymbol{z},\boldsymbol{w}) = \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{N}^n} a_{\boldsymbol{kl}}\boldsymbol{z}^{\boldsymbol{k}}\bar{\boldsymbol{w}}^{\boldsymbol{l}}$ is a polynomial in $(z_1,\ldots,z_n)$ and $(\bar{w}_1,\ldots,\bar{w}_n)$. For a commuting tuple $\boldsymbol{T}=(T_1,\ldots,T_n)$ on a Hilbert space $\mathcal{H}$, we define $p(\boldsymbol{T},\boldsymbol{T}^*)$ by

$$p(\boldsymbol{T}, \boldsymbol{T}^*) = \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \boldsymbol{T}^{\boldsymbol{k}} \boldsymbol{T}^{*\boldsymbol{l}}.$$

Note the order of the factors, which is important since we have not assumed that the tuple $\boldsymbol{T}$ is doubly commuting.

We will often consider in this section a good kernel g with the property that g^{-1} is a polynomial. We will then write

$$g^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{l}},$$

having always in mind that the sum is finite.

The following standard relationship between factorized kernels and operator positivity of multiplication tuples on reproducing kernel Hilbert modules is well known (cf. Theorem 4 in [9]).

Proposition 7.1. *Let $\mathcal{H}_K \subseteq \mathcal{O}(\Omega, \mathcal{E})$ be a reproducing kernel Hilbert module and g be a good kernel on Ω with g^{-1} a polynomial. Then $g^{-1}(\boldsymbol{M}_z, \boldsymbol{M}_z) \geq 0$ on $\mathcal{H}_K$ if and only if there exists a kernel L on Ω such that $K = gL$.*

Proof. It is enough to prove that $g^{-1}(\boldsymbol{M}_z, \boldsymbol{M}_z^*) \geq 0$ if and only if $g^{-1}K$ is positive definite. Indeed, for $\{\boldsymbol{w}_j\}_{j=1}^m \subseteq \Omega$, $\{\eta_j\}_{j=1}^m \subseteq \mathcal{E}$ and $m \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{i,j=1}^m \langle (g^{-1}K)(\boldsymbol{w}_i, \boldsymbol{w}_j)\eta_j, \eta_i\rangle \\
&\quad = \sum_{i,j=1}^m \langle g^{-1}(\boldsymbol{w}_i, \boldsymbol{w}_j) K(\boldsymbol{w}_i, \boldsymbol{w}_j)\eta_j, \eta_i\rangle \\
&\quad = \sum_{i,j=1}^m \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \boldsymbol{w}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{l}} \langle K(\cdot, \boldsymbol{w}_j)\eta_j, K(\cdot, \boldsymbol{w}_i)\eta_i\rangle \\
&\quad = \sum_{i,j=1}^m \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \left\langle \boldsymbol{M}_z^{*\boldsymbol{l}} K(\cdot, \boldsymbol{w}_j)\eta_j, \boldsymbol{M}_z^{*\boldsymbol{k}} K(\cdot, \boldsymbol{w}_i)\eta_i\right\rangle \\
&\quad = \sum_{i,j=1}^m \left\langle \left(\sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \boldsymbol{M}_z^{\boldsymbol{k}} \boldsymbol{M}_z^{*\boldsymbol{l}} \right) K(\cdot, \boldsymbol{w}_j)\eta_j, K(\cdot, \boldsymbol{w}_i)\eta_i \right\rangle \\
&\quad = \sum_{i,j=1}^m \langle g^{-1}(\boldsymbol{M}_z, \boldsymbol{M}_z^*) K(\cdot, \boldsymbol{w}_j)\eta_j, K(\cdot, \boldsymbol{w}_i)\eta_i\rangle.
\end{aligned}$$

This completes the proof. □

This and Theorem 6.1 immediately yield the following generalization of Theorem 6 in [9].

Theorem 7.2. *Let $\mathcal{H}_K \subseteq \mathcal{O}(\Omega, \mathcal{E})$ be a reproducing kernel Hilbert module and g be a good kernel on Ω with g^{-1} a polynomial. Then the following assertions are equivalent:*

1. $g^{-1}(\boldsymbol{M}_z, \boldsymbol{M}_z^*) \geq 0$ *on* $\mathcal{H}_K$.
2. *There exists a kernel L on Ω such that $K = gL$.*

3. *There exists a Hilbert space* $\mathcal{E}_*$ *such that* $\mathcal{H}_K$ *dilates to* $\mathcal{H}_g \otimes \mathcal{E}_*$.

We now turn to the study of submodules of good reproducing kernel Hilbert modules. To this end, we first need the following simple lemma.

Lemma 7.3. *Let* $\mathcal{H}_g$ *on* Ω *be a good reproducing kernel Hilbert module over* $\mathbb{C}[\boldsymbol{z}]$, *with* $g^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{l}}$ *a polynomial. Let* $P_{g(\cdot, \boldsymbol{w}_0)}$ *be the orthogonal projection of* $\mathcal{H}_g$ *onto the one dimensional subspace generated by* $g(\cdot, \boldsymbol{w}_0) \equiv 1$. *Then*

$$\sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \boldsymbol{M}_z^{\boldsymbol{k}} \boldsymbol{M}_z^{*\boldsymbol{l}} = P_{g(\cdot, \boldsymbol{w}_0)}.$$

Proof. For each $\boldsymbol{z}, \boldsymbol{w} \in \Omega$ we compute

$$\begin{aligned}
\left\langle \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \boldsymbol{M}_z^{\boldsymbol{k}} \boldsymbol{M}_z^{*\boldsymbol{l}} g(\cdot, \boldsymbol{w}), g(\cdot, \boldsymbol{z}) \right\rangle &= \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \left\langle \boldsymbol{M}_z^{\boldsymbol{k}} \boldsymbol{M}_z^{*\boldsymbol{l}} G(\cdot, \boldsymbol{w}), g(\cdot, \boldsymbol{z}) \right\rangle \\
&= \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \left\langle \boldsymbol{M}_z^{*\boldsymbol{l}} g(\cdot, \boldsymbol{w}), \boldsymbol{M}_z^{*\boldsymbol{k}} g(\cdot, \boldsymbol{z}) \right\rangle \\
&= \left(\sum_{i,j=0}^{k} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{l}} a_{\boldsymbol{kl}} \right) \langle g(\cdot, \boldsymbol{w}), g(\cdot, \boldsymbol{z}) \rangle \\
&= g^{-1}(\boldsymbol{z}, \boldsymbol{w}) g(\boldsymbol{z}, \boldsymbol{w}) = 1 \\
&= \langle P_{g(\cdot, \boldsymbol{w}_0)} g(\cdot, \boldsymbol{w}), g(\cdot, \boldsymbol{z}) \rangle.
\end{aligned}$$

This completes the proof of the lemma. □

Let $\mathcal{H}_g$ be as in the previous lemma and $\mathcal{E}$ be a Hilbert space. Let $\mathcal{S}$ be a submodule of $\mathcal{H}_g \otimes \mathcal{E}$; that is, $\mathcal{S}$ is a joint $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$ invariant subspace of $\mathcal{H}_g \otimes \mathcal{E}$. Then $\mathcal{S}$ is a module over $\mathbb{C}[\boldsymbol{z}]$ with module multiplication operators $\boldsymbol{R}_z = (R_{z_1}, \ldots, R_{z_n})$, where

$$R_{z_i} = M_{z_i}|_{\mathcal{S}} \quad (i = 1, \ldots, n).$$

We say that $\mathcal{S}$ is a *Brehmer submodule* if

$$g^{-1}(\boldsymbol{R}_z, \boldsymbol{R}_z^*) = \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{kl}} \boldsymbol{R}_z^{\boldsymbol{k}} \boldsymbol{R}_{\boldsymbol{z}}^{*\boldsymbol{l}} \geq 0.$$

In the following we characterize Brehmer submodules in terms of partial isometric multipliers. The idea of the proof is to invoke the dilation result, Theorem 7.2, to submodules of good reproducing kernel Hilbert modules (cf. [15]).

Theorem 7.4. *Let* $\mathcal{E}$ *be a Hilbert space and* g *be a good kernel with* g^{-1} *a polynomial. Let* $\mathcal{S}$ *be a submodule of* $\mathcal{H}_g \otimes \mathcal{E}$. *Then* $\mathcal{S}$ *is a Brehmer submodule of* $\mathcal{H}_g \otimes \mathcal{E}$ *if and only if there exists a Hilbert space* $\mathcal{E}_*$ *and a partial isometric multiplier* $\Theta \in \mathcal{M}(\mathcal{H}_g \otimes \mathcal{E}_*, \mathcal{H}_g \otimes \mathcal{E})$ *such that*

$$\mathcal{S} = \Theta(\mathcal{H}_g \otimes \mathcal{E}_*).$$

Proof. Let $\mathcal{S}$ be a Brehmer submodule, that is,

$$g^{-1}(\boldsymbol{R}_z, \boldsymbol{R}_z^*) \geq 0.$$

By Theorem 7.2, there exists a Hilbert space $\mathcal{E}_*$ such that $\mathcal{S}$ dilates to $\mathcal{H}_g \otimes \mathcal{E}_*$. Therefore there exists an isometry $\pi : \mathcal{S} \to \mathcal{H}_g \otimes \mathcal{E}_*$ such that

$$\pi R_{z_i}^* = (M_{z_i} \otimes I_{\mathcal{E}_*})^* \pi \quad (i = 1, \ldots, n).$$

Let $i : \mathcal{S} \to \mathcal{H}_g \otimes \mathcal{E}$ be the inclusion map and $\Pi = i \circ \pi^*$. Then $\Pi : \mathcal{H}_g \otimes \mathcal{E}_* \to \mathcal{H}_g \otimes \mathcal{E}$ is a partial isometry with

$$\operatorname{ran} \Pi = \mathcal{S},$$

and

$$\Pi(M_{z_i} \otimes I_{\mathcal{E}_*}) = (M_{z_i} \otimes I_{\mathcal{E}})\Pi \quad (i = 1, \ldots, n).$$

This yields that $\Pi = M_\Theta$ for some partial isometric multiplier $\Theta \mathcal{M}(\mathcal{H}_g \otimes \mathcal{E}_*, \mathcal{H}_g \otimes \mathcal{E})$ and $\mathcal{S} = \Theta(\mathcal{H}_g \otimes \mathcal{E}_*)$.

Conversely, let $\mathcal{S} = \Theta(\mathcal{H}_g \otimes \mathcal{E}_*)$ for some partial isometric multiplier $\Theta \in \mathcal{M}(\mathcal{H}_g \otimes \mathcal{E}_*, \mathcal{H}_g \otimes \mathcal{E})$. Then

$$P_{\mathcal{S}} = M_\Theta M_\Theta^*.$$

Hence

$$\begin{aligned} g^{-1}(\boldsymbol{R}_z, \boldsymbol{R}_z^*) &= \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{k}\boldsymbol{l}} \boldsymbol{R}_z^{\boldsymbol{k}} \boldsymbol{R}_{\boldsymbol{z}}^{*\boldsymbol{l}} \\ &= \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{k}\boldsymbol{l}} \boldsymbol{M}_z^{\boldsymbol{k}} P_{\mathcal{S}} \boldsymbol{M}_z^{*\boldsymbol{l}} \\ &= \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{k}\boldsymbol{l}} \boldsymbol{M}_z^{\boldsymbol{k}} M_\Theta M_\Theta^* \boldsymbol{M}_z^{*\boldsymbol{l}} \\ &= M_\Theta \left(\sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}^n} a_{\boldsymbol{k}\boldsymbol{l}} \boldsymbol{M}_z^{\boldsymbol{k}} \boldsymbol{M}_z^{*\boldsymbol{l}} \right) M_\Theta^* \\ &= M_\Theta P_{g(\cdot, \boldsymbol{w}_0)} M_\Theta^* \qquad \text{(by Lemma 7.3)} \\ &\geq 0. \end{aligned}$$

This completes the proof of the theorem. □

Example. The weighted Bergman spaces $A_\alpha^2(\mathbb{B}^n)$ defined by (5.4) are reproducing kernel Hilbert spaces defined on $\Omega = \mathbb{D}^n$. The corresponding kernel g_α is given by (5.5) and is easily seen to be a good kernel. Moreover,

$$g_\alpha^{-1}(\boldsymbol{z}, \boldsymbol{w})^{-1} = \left(1 - \sum_{i=1}^n z_i \bar{w}_i\right)^{\alpha+n+1}, \quad \boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n$$

is a polynomial for $\alpha \in \mathbb{N}$. Therefore Theorem 7.4 yields a characterization of Brehmer submodules of $A_\alpha^2(\mathbb{B}^n) \otimes \mathcal{E}$ for nonegative integers α. In particular, for $\alpha = 0$ one obtains the case of the classical Bergman space.

Now we consider the important case when $\Omega = \mathbb{D}^n$ and $\mathcal{H}_g = H^2(\mathbb{D}^n)$ and $n \geq 2$. A submodule $\mathcal{S}$ of $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ is said to be *doubly commuting* (cf. [16]) if

$$[R_{z_i}^*, R_{z_j}] := R_{z_i}^* R_{z_j} - R_{z_j} R_{z_i}^* = 0,$$

for all $1 \leq i \neq j \leq n$.

The next theorem is proved in [16].

Theorem 7.5. *A submodule $\mathcal{S}$ of $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ is doubly commuting if and only if there exists a Hilbert space $\mathcal{E}_*$ and an inner multiplier $\Theta \in \mathcal{M}(H^2(\mathbb{D}^n) \otimes \mathcal{E}_*, H^2(\mathbb{D}^n) \otimes \mathcal{E}) = H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ such that*

$$\mathcal{S} = \Theta(H^2(\mathbb{D}^n) \otimes \mathcal{E}_*).$$

In the following, we prove that the class of doubly commuting submodules and the class of Brehmer submodules of $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ are the same.

Theorem 7.6. *Let $\mathcal{E}$ be a Hilbert space. Then $\mathcal{S}$ is a Brehmer submodule of $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ if and only if $\mathcal{S}$ is a doubly commuting submodule.*

Proof. If $\mathcal{S}$ is a doubly commuting submodule, it follows from Theorems 7.4 and 7.5 that it is a Brehmer submodule.

Conversely, suppose $\mathcal{S}$ is a Brehmer submodule. By Theorem 7.4, there exists a Hilbert space $\mathcal{E}_*$ and a partial isometry $M_\Theta : H^2(D^n) \otimes \mathcal{E}_* \to H^2(\mathbb{D}^n) \otimes \mathcal{E}$, for some multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$, such that

$$\mathcal{S} = \Theta(H^2(\mathbb{D}^n) \otimes \mathcal{E}_*).$$

It is easy to see that the closed subspace $\ker M_\Theta$ is a submodule of $H^2(\mathbb{D}^n) \otimes \mathcal{E}_*$. We claim that the orthogonal of $\ker M_\Theta$ is also a submodule of $H^2(\mathbb{D}^n) \otimes \mathcal{E}_*$. Indeed, if $f \in (\ker M_\Theta)^\perp$, then

$$\|f\| = \|M_{z_i} M_\Theta f\| = \|M_\Theta M_{z_i} f\| \leq \|M_{z_i} f\| = \|f\|,$$

and hence the inequality becomes an equality. But then

$$\|M_\Theta M_{z_i} f\| = \|M_{z_i} f\|,$$

yields $z_i f \in (\ker M_\Theta)^\perp$ for all $i = 1, \ldots, n$, and hence $(\ker M_\Theta)^\perp$ is a submodule of $H^2(\mathbb{D}^n) \otimes \mathcal{E}_*$, or equivalently that $(\ker M_\Theta)^\perp$ is a joint $(M_{z_1}, \ldots, M_{z_n})$-reducing subspace of $H^2(\mathbb{D}^n) \otimes \mathcal{E}_*$. Now the reducing subspaces of $H^2(\mathbb{D}^n) \otimes \mathcal{E}_*$ are known to be of the form $H^2(\mathbb{D}^n) \otimes \tilde{\mathcal{E}}_*$ for some $\tilde{\mathcal{E}}_* \subseteq \mathcal{E}_*$. Then $\mathcal{S}$ is the image of the isometric multiplier $M_\Theta|_{H^2(\mathbb{D}^n) \otimes \tilde{\mathcal{E}}_*}$, so $\mathcal{S}$ is doubly commuting by the result quoted above. This completes the proof of the theorem. □

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Rani Kumari, Jaydeb Sarkar (✉) and Srijan Sarkar
Statistics and Mathematics Unit
Indian Statistical Institute
8th Mile, Mysore Road
Bangalore 560059
India
e-mail: jay@isibang.ac.in;
jaydeb@gmail.com

Rani Kumari
e-mail: rani_vs@isibang.ac.in

Srijan Sarkar
e-mail: srijan_rs@isibang.ac.in

Dan Timotin
Institute of Mathematics of the Romanian Academy
P.O. Box 1-764
014700 Bucharest
Romania
e-mail: Dan.Timotin@imar.ro

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