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Interpolation Problems for Certain Classes of Slice Hyperholomorphic Functions

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Abstract. A general interpolation problem (which includes as particular cases the Nevanlinna–Pick and Carathéodory–Fejér interpolation problems) is considered in two classes of slice hyperholomorphic functions of the unit ball of the quaternions. In the Hardy space of the unit ball we present a Beurling–Lax type parametrization of all solutions, and the formula for the minimal norm solution. In the class of functions slice hyperholomorphic in the unit ball and bounded by one in modulus there (that is, in the class of Schur functions in the present framework) we present a necessary and sufficient condition for the problem to have a solution, and describe the set of all solutions in the indeterminate case.

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1. Introduction

There are several approaches to the theory of functions over non commutative algebras and over the quaternions; see e.g., [11,15,18,19,24]. In particular, the notion of slice regularity in [17] comprises quaternionic polynomials and power series with quaternionic coefficients on one side. Quite a number of results have been obtained in this setting, and we refer to the books [6,14,16] and the references therein for more information. First interpolation results in this setting were obtained in [3], and the objective of this paper is to obtain a more unified approach to interpolation theory. To set the framework, we need first to recall some notation and definitions.

Let $\mathbb{H}$ be the algebra of real quaternions $p = x_0 + ix_1 + jx_2 + kx_3$ where $x_\ell \in \mathbb{R}$ and i, j, k are imaginary units such that $ij = k$, $ki = j$, $jk = i$ and $i^2 = j^2 = k^2 = -1$. The conjugate, the absolute value, the real part and the imaginary part of a quaternion p are defined as $\bar{p} = x_0 - ix_1 - jx_2 - kx_3$, $|p| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$, $\operatorname{Re} p = x_0$ and $\operatorname{Im} p = ix_1 + jx_2 + kx_3$, respectively.

By $\mathbb{S}$ we denote the unit sphere of purely imaginary quaternions. Any $I \in \mathbb{S}$ satisfies $I^2 = -1$ so that the set $\mathbb{C}_I = \{x + Iy : x, y \in \mathbb{R}\}$ can be identified with the complex plane. We say that two quaternions p and q are *equivalent* if $p = h^{-1}qh$ for some nonzero $h \in \mathbb{H}$. It turns out (see e.g., [12]) that p and q are equivalent if and only if $\operatorname{Re} p = \operatorname{Re} q$ and $|\operatorname{Im} p| = |\operatorname{Im} q|$, so the set of all quaternions equivalent to a given $p \in \mathbb{H}$ form a 2-sphere which will be denoted by $[p]$. We refer to [25] for systematic treatment of geometry over quaternions.

Definition 1.1. Given an open set $\Omega \subset \mathbb{H}$, a real differentiable function $f : \Omega \to \mathbb{H}$ is called left slice regular or *left slice hyperholomorphic* (or just *slice hyperholomorphic*, in what follows) on Ω if for every $I \in \mathbb{S}$,

$$\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) \equiv 0, \tag{1.1}$$

where f_I stands for the restriction of f to $\Omega \cap \mathbb{C}_I$.

The (slice) derivative of a (left) slice hyperholomorphic function is defined pointwise by the formula

$$f'(x + Iy) = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f_I(x + Iy). \tag{1.2}$$

We will denote by $\mathcal{R}(\Omega, \widetilde{\Omega})$ the set of all functions $f : \Omega \mapsto \widetilde{\Omega} \subset \mathbb{H}$ which are (left) slice hyperholomorphic on Ω and we will write $\mathcal{R}(\Omega)$ in case $\widetilde{\Omega} = \mathbb{H}$. In this paper, we will focus on functions defined and slice hyperholomorphic on the unit ball $\mathbb{B} = \{p \in \mathbb{H} : |p| < 1\}$. Similarly to the complex case, the functions $f \in \mathcal{R}(\mathbb{B})$ admit power series expansion

$$f(p) = \sum_{k=0}^{\infty} p^k f_k \quad (f_k \in \mathbb{H}) \tag{1.3}$$

where the series on the right converges to f uniformly on compact subsets of $\mathbb{B}$; on the other hand, if $\limsup_k |f_k|^{\frac{1}{k}} \le 1$, the power series as in (1.3) converges absolutely on $\mathbb{B}$ and represents a slice hyperholomorphic function. We thus may identify the function from $\mathcal{R}(\mathbb{B})$ with power series of the form (1.3) with radius of convergence at least one. A simple computation shows that the slice derivative of the monomial $p^k\alpha$ equals $p^{k-1}\alpha k$; therefore, for f of the form (1.3) we have

$$f'(p) = \sum_{k=0}^{\infty} p^k f_{k+1}(k + 1), \tag{1.4}$$

and it is readily seen that the latter power series has the same radius of convergence as that in (1.3). We will deal with two classes of slice hyperholomorphic functions. The first is the quaternionic Hardy space $\mathrm{H}^2(\mathbb{B})$ consisting of square summable power series:

$$\mathrm{H}^2(\mathbb{B}) = \left\{ f(p) = \sum_{k=0}^{\infty} p^k f_k : \|f\|^2_{\mathrm{H}^2} := \sum_{k=0}^{\infty} |f_k|^2 < \infty \right\}. \tag{1.5}$$

The space $\mathrm{H}^2(\mathbb{B})$ will be denoted, from now on, simply as H^2. It is a right quaternionic Hilbert space with inner product

$$\langle f,\, g\rangle = \sum_{k=0}^{\infty} \bar{g}_k f_k \quad \text{if} \quad f(p) = \sum_{k=0}^{\infty} p^k f_k,\;\; g(p) = \sum_{k=0}^{\infty} p^k g_k. \tag{1.6}$$

Another class of particular interest is the Schur class $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ of slice hyperholomorphic functions mapping the quaternionic unit ball $\mathbb{B}$ into its closure. Several characterizations of the latter class will be recalled in Theorem 2.2 below.

In the complex case, interpolation theory for Hardy and Schur class functions has been thoroughly studied in the last century. The study of interpolation theory for slice hyperholomorphic Schur functions was initiated in [3] by handling a simple Nevanlinna–Pick problem; see also [1] for the boundary version. In this paper we will consider a more general interpolation problem which we introduce right away. Let us say that the matrix $A \in \mathbb{H}^{n\times n}$ is stable if its right spectrum (see [12,14,23]) is contained in $\mathbb{B}$. For such a matrix, the series

$$\mathcal{G}_{E,A} := \sum_{k=0}^{\infty} A^{*k} E^* E A^k \tag{1.7}$$

converges for every $E \in \mathbb{H}^{1\times n}$ and defines the positive semidefinite matrix $\mathcal{G}_{E,A}$ which is the unique solution to the Stein equation

$$\mathcal{G}_{E,A} - A^* \mathcal{G}_{E,A} A = E^* E. \tag{1.8}$$

The pair (E, A) is called *observable* if $\mathcal{G}_{E,A}$ is positive definite. Given a stable matrix $A \in \mathbb{H}^{n\times n}$ and given $E \in \mathbb{H}^{1\times n}$, one can define a left functional calculus $f \mapsto (E^* f)^{\wedge L}(A^*)$ on $\mathcal{R}(\mathbb{B})$ by

$$(E^* f)^{\wedge L}(A^*) = \sum_{k=0}^{\infty} A^{*k} E^* f_k \quad \text{if} \quad f(p) = \sum_{k=0}^{\infty} p^k f_k. \tag{1.9}$$

Observe that the convergence of the series defining $(E^* f)^{\wedge L}(A^*)$ is guaranteed by the assumption that A is stable. With evaluation (1.9) in hands, we formulate interpolation problems whose data set consists of the triple (A, E, N) with $A \in \mathbb{H}^{n\times n}$, $E, N \in \mathbb{H}^{1\times n}$ such that A is stable and the pair (E, A) is observable.

Problem 1: Find all functions $f \in \mathrm{H}^2$ such that

$$(E^* f)^{\wedge L}(A^*) = N^*. \tag{1.10}$$

Problem 2: Find all functions $f \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ satisfying condition (1.10).

In Section 2.4, we will show that for certain particular choices of A, E and N, the interpolation condition (1.10) amounts to Nevanlinna–Pick or Carathéodory–Fejér type conditions. As we will also see in Section 2.4, interpolation problems that do not appear in the complex case are particular cases of our general scheme.

The main results concerning Problem 1 are presented in Theorem 1.2 below. Here and in what follows, the symbol $\mathbf{I}_n$ denotes the $n \times n$ identity matrix (or shortened to **I**, the size being clear from the context). The notions of adjoint matrices, of Hermitian matrices, of positive semidefinite and positive definite matrices over $\mathbb{H}$ are similar to those over $\mathbb{C}$.

Theorem 1.2. *Given A, E, N as in (1.10), let $\mathcal{G}_{E,A} > 0$ be defined as in (1.7) and let*

$$F(p) = \sum_{k=0}^{\infty} p^k F_k = \sum_{k=0}^{\infty} p^k E A^k \mathcal{G}_{E,A}^{-1} N^*, \tag{1.11}$$

$$B(p) = \sum_{k=0}^{\infty} p^k B_k = 1 - E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* + \sum_{k=1}^{\infty} p^k E A^{k-1}(\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^*. \tag{1.12}$$

Then all solutions f to Problem 1 are parametrized by the formula

$$f = F + B \star h, \quad h \in \mathrm{H}^2, \tag{1.13}$$

where h is a free parameter. Moreover, representation (1.13) is orthogonal in H^2 and

$$\|f\|_{\mathrm{H}^2}^2 = \|F\|_{\mathrm{H}^2}^2 + \|B \star h\|_{\mathrm{H}^2}^2 = \|F\|_{\mathrm{H}^2}^2 + \|h\|_{\mathrm{H}^2}^2 = N\mathcal{G}_{E,A}^{-1}N^* + \|h\|_{\mathrm{H}^2}^2. \tag{1.14}$$

In particular, F is the unique minimal norm solution to Problem 1.

The prof will be given in Sect. 3. We observe that the functions F and B defined in (1.11), (1.12) belong to $\mathcal{R}(\mathbb{B})$ since A is stable. The $\star$-multiplication used in formula (1.13) is the usual convolution multiplication (see (2.1) below). In Sect. 4, we study Problem 2. We show that the problem has a solution if and only if the Pick matrix

$$P = \sum_{k=0}^{\infty} A^{*k}(E^*E - N^*N)A^k$$

is positive semidefinite. If P is singular, the problem has a unique solution. In case P is positive definite, the problem has infinitely many solutions which can be described in terms of a linear fractional formula presented in Theorem 4.5. The statement of the theorem requires more preliminaries, and is postponed to Sect. 4.

2. Slice Hyperholomorphic Functions and Kernels

In this section we collect a number of basic facts needed in the sequel. Interpreting the set $\mathcal{R}(\mathbb{B})$ as the right quaternionic vector space of power series (1.3) converging in $\mathbb{B}$, one can introduce the ring structure on $\mathcal{R}(\mathbb{B})$ using the convolution multiplication

$$(g \star f)(p) = \sum_{k=0}^{\infty} p^k \cdot \left(\sum_{r=0}^{k} g_r f_{k-r}\right) \quad \text{if} \quad f(p) = \sum_{k=0}^{\infty} p^k f_k, \;\; g(p) = \sum_{k=0}^{\infty} p^k g_k, \tag{2.1}$$

which is called (left) *slice hyperholomorphic multiplication* in the present context. As a convolution multiplication of the power series over a noncommutative ring, the $\star$-multiplication is associative and noncommutative. Point evaluation is not multiplicative with respect to the $\star$-multiplication. However we have

$$(g\star f)(p)=\begin{cases} g(p)f(g(p)^{-1}pg(p)) & \text{if } g(p)\neq 0,\\ 0 & \text{if } g(p)=0.\end{cases} \tag{2.2}$$

We also observe that $(g\star f)(x)=g(x)f(x)$ for every $x\in\mathbb{R}$.

If the function $f\in\mathcal{R}(\mathbb{B})$ is as in (2.1), then we can construct its slice hyperholomorphic inverse $f^{-\star}$ as $f^{-\star}(p)=(f^c\star f)^{-1}f^c(p)$ where the *slice hyperholomorphic conjugate* f^c of f is defined by

$$f^c(p)=\sum_{k=0}^{\infty}p^k\overline{f}_k \quad \text{if} \quad f(p)=\sum_{k=0}^{\infty}p^kf_k \tag{2.3}$$

and $f^{-\star}$ is defined in $\mathbb{B}$ outside the zeros of $f^c\star f$. If f satisfies $f(0)=f_0\neq 0$, one can define its $\star$-inverse $f^{-\star}$ using the power series

$$f^{-\star}(p)=\sum_{k=0}p^ka_k$$

with the coefficients a_k defined recursively by

$$a_0=f_0^{-1} \quad \text{and} \quad a_k=-f_0^{-1}\sum_{j=1}^{k}f_ja_{k-j} \ \ (k\geq 1).$$

If $f(p)\neq 0$ for all $p\in\mathbb{B}$, the latter power series converges on $\mathbb{B}$. Equalities $f^{-\star}\star f=f\star f^{-\star}\equiv 1$ and $(g\star f)^{-\star}=f^{-\star}\star g^{-\star}$ are immediate.

2.1. Right Slice Hyperholomorphic Functions

A real differentiable function $f:\Omega\to\mathbb{H}$ is called *right slice hyperholomorphic* on Ω (in notation, $f\in\mathcal{R}^r(\Omega)$) if for every $I\in\mathbb{S}$ its restriction f_I to $\Omega\cap\mathbb{C}_I$ is subject to

$$\frac{\partial}{\partial x}f_I(x+Iy)+\frac{\partial}{\partial y}f_I(x+Iy)I\equiv 0.$$

The results for right slice hyperholomorphic functions are parallel to those for (left) hyperholomorphic ones. A function $f\in\mathcal{R}^r(\mathbb{B})$ can be identified with power series $f(p)=\sum_{k=0}^{\infty}f_kp^k$ converging on $\mathbb{B}$. The set $\mathcal{R}^r(\mathbb{B})$ itself is a left quaternionic vector space and it becomes a ring once we introduce the *right slice hyperholomorphic multiplication*

$$(g\star_r f)(p)=\sum_{k=0}^{\infty}\left(\sum_{r=0}^{k}g_rf_{k-r}\right)p^k \quad \text{if} \quad f(p)=\sum_{k=0}^{\infty}f_kp^k,\ \ g(p)=\sum_{k=0}^{\infty}g_kp^k,$$

which can be written alternatively as

$$(g\star_r f)(p)=\begin{cases} g(f(p)pf(p)^{-1})f(p) & \text{if } f(p)=0,\\ 0 & \text{if } f(p)\neq 0.\end{cases} \tag{2.4}$$

2.2. The Space H^2 and its Contractive Multipliers

Let us recall that a matrix-valued function $K(p,q): \Omega\times\Omega \to \mathbb{H}^{m\times m}$ is called a positive kernel (in notation, $K \succeq 0$) if the block matrix $[K(q_i,q_j)]_{i,j=1}^r$ is positive semidefinite for any choice of finitely many points $q_1,\dots,q_r$. Equivalently,

$$\sum_{i,j=1}^{r} c_i^* K(q_i,q_j)c_j \geq 0 \quad \text{for all} \quad r\in\mathbb{N},\ c_1,\dots,c_r\in\mathbb{H}^m,\ q_1,\dots,q_r\in\Omega.$$

Definition 2.1. We say that the kernel $K(p,q) : \Omega\times\Omega \to \mathbb{H}^{m\times m}$ is *slice sesquihyperholomorphic* on an open set $\Omega\subset\mathbb{H}$ if it is (left) slice hyperholomorphic in p and right slice hyperholomorphic in $\bar{q}$.

The space H^2 introduced in (1.5) as the set of square summable power series, can be alternatively characterized as the reproducing kernel Hilbert space with reproducing kernel

$$k_{\mathrm{H}^2}(p,q)=\sum_{n=0}^{\infty} p^n\bar{q}^n. \tag{2.5}$$

The latter means that the function $k_{\mathrm{H}^2}(\cdot,q)$ belongs to H^2 for every $q\in\mathbb{B}$, and for any function $f\in\mathrm{H}^2$ as in (1.3),

$$\langle f,\, k_{\mathrm{H}^2}(\cdot,q)\rangle_{\mathrm{H}^2}=\sum_{k=0}^{\infty} q^k f_k = f(q). \tag{2.6}$$

Quaternionic Hardy spaces have been introduced in [5]. The next result identifying the class $\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$ with the class of contractive multipliers of the quaternionic Hardy space H^2 can be found in [3,4].

Theorem 2.2. *Let $S:\mathbb{B}\to\mathbb{H}$. The following are equivalent:*

1. *S is slice hyperholomorphic on $\mathbb{B}$ and $|S(p)|\leq 1$ for all $p\in\mathbb{B}$.*
2. *S^c is slice hyperholomorphic on $\mathbb{B}$ and $|S^c(p)|\leq 1$ for all $p\in\mathbb{B}$.*
3. *The operator M_S of left $\star$–multiplication by S*

$$M_S:\ f\mapsto S\star f \tag{2.7}$$

is a contraction on H^2, that is, $\|S\star f\|_{\mathrm{H}^2}\leq\|f\|_{\mathrm{H}^2}$ for all $f\in\mathrm{H}^2$.

4. *The kernel*

$$K_S(p,q)=\sum_{k=0}^{\infty} p^k(1-S(p)\overline{S(q)})\bar{q}^k \tag{2.8}$$

is positive on $\mathbb{B}\times\mathbb{B}$.

Remark 2.3. If $S\in\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$, then the contractive operator $M_S:\mathrm{H}^2\to\mathrm{H}^2$ relates the kernels (2.5) and (2.8) via the reproducing kernel property (2.6) as follows:

$$K_S(p,q)=\langle (I-M_SM_S^*)k_{\mathrm{H}^2}(\cdot,q),\, k_{\mathrm{H}^2}(\cdot,p)\rangle_{\mathrm{H}^2}\quad (p,q\in\mathbb{H}). \tag{2.9}$$

2.3. Interpolation Condition (1.10)

We now present several examples of the interpolation condition (1.10) corresponding to specific choices of A, E and N. Observe that if A is block diagonal and if E and N are decomposed accordingly as

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 \end{bmatrix},$$

then

$$(E^* f)^{\wedge L}(A^*) = \begin{bmatrix} (E_1^* f)^{\wedge L}(A_1^*) \\ (E_2^* f)^{\wedge L}(A_2^*) \end{bmatrix}, \tag{2.10}$$

and condition (1.8) is equivalent to the system $(E_j^* f)^{\wedge L}(A_j^*) = N_j^*$; $j = 1, 2$. The matrix A is stable if and only if A_1 and A_2 are both stable, and the observability of (E, A) implies observability of (E_1, A_1) and (E_2, A_2) (but not vice versa). The significance of the observability assumption will be explained below; now we present some particular examples of A and E.

Example 2.4. Given $a_1, \ldots, a_n \in \mathbb{H}$, let

$$A = \begin{bmatrix} \overline{a}_1 & & 0 \\ & \ddots & \\ 0 & & \overline{a}_n \end{bmatrix} \in \mathbb{H}^{n\times n}, \quad E = \begin{bmatrix} 1 \ldots 1 \end{bmatrix} \in \mathbb{H}^{1\times n}. \tag{2.11}$$

It is not hard to show that $\mathcal{G}_{E,A}$ is singular if at least three of diagonal entries in A belong to the same conjugacy class. On the other hand, if none three of a_i's belong to the same conjugacy class, then the Vandermonde matrix

$$V = \begin{bmatrix} E^* \ A^*E^* \ \cdots \ A^{*n-1}E^* \end{bmatrix} = \begin{bmatrix} a_i^{j-1} \end{bmatrix}_{i,j=1}^{n}$$

is invertible by a result from [22]. Since $0 \le VV^* \le \mathcal{G}_{E,A}$, we may conclude that the pair (2.11) is observable if and only if the diagonal entries of A are distinct and none three of them belong to the same conjugacy class. Furthermore, for (A, E) as in (2.11) and f as in (1.3),

$$(E^* f)^{\wedge L}(A^*) = \sum_{k=0}^{\infty} \begin{bmatrix} a_1^k \\ \vdots \\ a_n^k \end{bmatrix} f_k = \begin{bmatrix} f(a_1) \\ \vdots \\ f(a_n) \end{bmatrix}, \tag{2.12}$$

and letting $N = \begin{bmatrix} \overline{c}_1 \cdots \overline{c}_n \end{bmatrix}$, we conclude that the general interpolation condition (1.10) amounts to $f(a_i) = c_i$ for $i = 1, \ldots, n$.

Example 2.5. Given $a \in \mathbb{H}$, let

$$A = \begin{bmatrix} \overline{a} & 1 & \ldots & & 0 \\ 0 & \overline{a} & 1 & & \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & 0 & \overline{a} \end{bmatrix}, \quad E = \begin{bmatrix} 1 \ 0 \ldots 1 \end{bmatrix}. \tag{2.13}$$

The pair (E, A) is observable since $VV^* \le \mathcal{G}_{E,A}$ and the matrix

$$V = \begin{bmatrix} E^* \ A^*E^* \ \cdots \ A^{*n-1}E^* \end{bmatrix}$$

is upper triangular and with all diagonal entries equal one (hence, V is invertible). For f as in (1.3), we have

$$(E^*f)^{\wedge L}(A^*) = \begin{bmatrix} \sum_{k=0}^{\infty} a^k f_k \\ \sum_{k=0}^{\infty} a^k f_{k+1}\cdot(k+1) \\ \vdots \\ \sum_{k=0}^{\infty} a^k f_{k+n-1}\cdot\binom{k+n-1}{k} \end{bmatrix} = \begin{bmatrix} f(a) \\ f'(a) \\ \vdots \\ \frac{f^{(n-1)}(a)}{(n-1)!}, \end{bmatrix} \tag{2.14}$$

and letting $N = \begin{bmatrix}\bar{c}_0 \ \bar{c}_1 \ \ldots \ \bar{c}_{n-1}\end{bmatrix}$, we conclude that the general interpolation condition (1.10) amounts to conditions $f^{(j)}(a) = j!c_j$ for $j = 0, \ldots, n-1$.

The two latter examples are well known from the classical complex theory. The quaternionic peculiarity is reflected in the first example by the assumption that none three of interpolation nodes belong to the same conjugacy class. This assumption has been discussed in detail in [3] in the context of Problem 2. Note that combining Examples 2.4 and 2.5 with (2.10) produces the multipoint version of the Carathéodory–Fejér problem which prescribes the values of the unknown interpolant along with several of its derivatives at finitely many points. It can be shown that the pair (E, A) corresponding to this problem is observable if and only if none three of right eigenvalues of A belong to the same conjugacy class. The next numerical example is purely quaternionic.

Example 2.6. Let $A = \begin{bmatrix} i/2 & 1 \\ 0 & j/2 \end{bmatrix}$ and $E = \begin{bmatrix}1 \ 0\end{bmatrix}$. Then a tedious calculation shows that for $f \in \mathcal{R}(\mathbb{B})$

$$(E^*f)^{\wedge L}(A^*) = \begin{bmatrix} f(-i/2) \\ \frac{i-j}{2}f(-i/2) - \frac{i-j}{2}f(i/2) + \frac{1+k}{2}f'(-i/2) \end{bmatrix},$$

and letting $N = \begin{bmatrix}\bar{c}_1 \ \bar{c}_2\end{bmatrix}$ we come up with interpolation conditions

$$f(-i/2) = c_1, \quad f(i/2) + if'(-i/2) = (i-j)c_2 + c_1.$$

It is commonly known that in the complex case, the problems with interpolation conditions mixing the values of the interpolant and its derivatives at distinct points do not admit nice (descriptions of) solution sets; in the quaternionic setting, some problems of this type appear as particular cases of the general scheme (1.10), i.e., are of the same type as Nevanlinna–Pick or Carathéodory–Fejér conditions.

3. Interpolation by Hardy Space Functions

In the complex case, Nevanlinna–Pick interpolation problem in the Hardy space $\mathrm{H}^2(\mathbb{D})$ of the unit disk was first considered in [27]. Using the state-space approach and realization formulas, a more general problem of the form

(1.10) was considered in [9] for matrix-valued functions on $\mathbb{D}$ and adapted in [2] for matrix-valued Hardy functions. The main result related to this section is Theorem 1.2. Its proof will follow from several simple lemmas.

Lemma 3.1. *The function F given in (1.11) solves Problem 1 and* $\|F\|^2_{\mathrm{H}^2} = N\mathcal{G}_{E,A}^{-1}N^*$.

Proof. By (1.7), we have

$$\sum_{k=0}^{\infty}|F_k|^2 = \sum_{k=0}^{\infty} N\mathcal{G}_{E,A}^{-1}A^{*k}E^*EA^k\mathcal{G}_{E,A}^{-1}N^*$$
$$= N\mathcal{G}_{E,A}^{-1}\left(\sum_{k=0}^{\infty}A^{*k}E^*EA^k\right)\mathcal{G}_{E,A}^{-1}N^* = N\mathcal{G}_{E,A}^{-1}N^*,$$

so that $F \in \mathrm{H}^2$ and $\|F\|^2_{\mathrm{H}^2} = N\mathcal{G}_{E,A}^{-1}N^*$. Furthermore, by (1.9),

$$(E^*F)^{\wedge L}(A^*) = \sum_{k=0}^{\infty}A^{*k}E^*F_k = \sum_{k=0}^{\infty}A^{*k}E^*EA^k\mathcal{G}_{E,A}^{-1}N^* = N^*$$

which means that F is a solution of the Problem 1. □

Lemma 3.2. *Let F and B be defined in (1.11), (1.12). Then*

$$\langle F,\, B\star h\rangle_{\mathrm{H}^2} = 0 \quad \text{and} \quad \|B\star h\|_{\mathrm{H}^2} = \|h\|_{\mathrm{H}^2} \quad \text{for all} \quad h\in\mathrm{H}^2. \tag{3.1}$$

Furthermore, for every $p,q\in\mathbb{B}$,

$$K_B(p,q) := \sum_{k=0}^{\infty}p^k(1-B(p)B(q)^*)\overline{q}^k$$
$$= \left(\sum_{k=0}^{\infty}p^kEA^k\right)\mathcal{G}_{E,A}^{-1}\left(\sum_{k=0}^{\infty}A^{*k}E^*\overline{q}^k\right). \tag{3.2}$$

Proof. We verify the first equality in (3.1) for the monomial $h(p) = p^m$ $(m\geq 0)$. By (1.11) and (1.12),

$$\begin{aligned}
\langle F,\, B\star h\rangle_{\mathrm{H}^2} &= \sum_{k=0}^{\infty}\overline{B}_kF_{k+m}\\
&= (1-E(\mathbf{I}-A)^{-1}\mathcal{G}_{E,A}^{-1}E^*)EA^m\mathcal{G}_{E,A}^{-1}N^*\\
&\quad+\sum_{k=1}^{\infty}E(\mathbf{I}-A)^{-1}\mathcal{G}_{E,A}^{-1}(\mathbf{I}-A^*)A^{*k-1}E^*EA^{k+m}\mathcal{G}_{E,A}^{-1}N^*\\
&= (1-E(\mathbf{I}-A)^{-1}\mathcal{G}_{E,A}^{-1}E^*)EA^m\mathcal{G}_{E,A}^{-1}N^*\\
&\quad+E(\mathbf{I}-A)^{-1}\mathcal{G}_{E,A}^{-1}(\mathbf{I}-A^*)\mathcal{G}_{E,A}A^{m+1}\mathcal{G}_{E,A}^{-1}N^*\\
&= E(\mathbf{I}-A)^{-1}\mathcal{G}_{E,A}^{-1}\left(\mathcal{G}_{E,A}(\mathbf{I}-A)-E^*E+(\mathbf{I}-A^*)\mathcal{G}_{E,A}A\right)\\
&\quad\times A^m\mathcal{G}_{E,A}^{-1}N^*\\
&= E(\mathbf{I}-A)^{-1}\mathcal{G}_{E,A}^{-1}\left(\mathcal{G}_{E,A}-E^*E+A^*\mathcal{G}_{E,A}A\right)A^m\mathcal{G}_{E,A}^{-1}N^* = 0,
\end{aligned}$$

where for the last equality we made use of (1.8). By the right linearity we get the latter equality for all polynomials and subsequently, for all $h \in \mathrm{H}^2$, since polynomials are dense in H^2. Similarly,

$$\begin{aligned}
\|B\|_{\mathrm{H}^2}^2 &= (1 - E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}E^*)(1 - E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&\quad + E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)\left(\sum_{k=1}^{\infty} A^{*k-1}E^*EA^{k-1}\right) \\
&\quad \times (\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&= 1 - E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}E^* - E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&\quad + E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}E^*E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&\quad + E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)\mathcal{G}_{E,A}(\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&= 1 - E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}\left[(\mathbf{I} - A^*)\mathcal{G}_{E,A} + \mathcal{G}_{E,A}(\mathbf{I} - A) - E^*E\right. \\
&\quad \left. - (\mathbf{I} - A^*)\mathcal{G}_{E,A}(\mathbf{I} - A)\right]\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&= 1 - E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}\left[\mathcal{G}_{E,A} - E^*E - A^*\mathcal{G}_{E,A}A\right]\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&= 1,
\end{aligned}$$

and since the operator M_p of left $\star$-multiplication by the independent variable p is isometric on H^2, we have $\|B \star h\|_{\mathrm{H}^2} = \|h\|_{\mathrm{H}^2} = 1$ for the monomial $h(p) = p^m$. On the other hand, if $m \geq 1$, then

$$\begin{aligned}
\langle B,\, B \star h\rangle_{\mathrm{H}^2} &= \sum_{k=0}^{\infty} \overline{B}_k B_{k+m} \\
&= \overline{B}_0 B_m + \sum_{k=1}^{\infty} \overline{B}_k B_{k+m} \\
&= (1 - E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}E^*)EA^{m-1}(\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&\quad + E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)\left(\sum_{k=1}^{\infty} A^{*k-1}E^*EA^{m+k-1}\right) \\
&\quad \times (\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&= (1 - E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}E^*)EA^{m-1}(\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&\quad + E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)\mathcal{G}_{E,A}A^m(\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* \\
&= E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}\left(E^*E + A^*\mathcal{G}_{E,A}A - \mathcal{G}_{E,A}\right) \\
&\quad \times A^{m-1}(\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* = 0.
\end{aligned}$$

Since M_p is isometric, we have $\left\langle B \star p^m,\, B \star p^k\right\rangle_{\mathrm{H}^2} = 0$ for all $m \neq k$. Therefore, for a quaternion polynomial $h(p) = h_0 + \cdots + p^\ell h_\ell$, we have

$$\|B \star h\|_{\mathrm{H}^2}^2 = \sum_{i,j=0}^{\ell} \left\langle B \star p^i h_i,\, B \star p^j h_j\right\rangle_{\mathrm{H}^2}$$

$$= \sum_{j=0}^{\ell} \left\langle B \star p^j h_j,\, B \star p^j h_j \right\rangle_{\mathrm{H}^2}$$
$$= \sum_{j=0}^{\ell} \langle Bh_j,\, Bh_j \rangle_{\mathrm{H}^2} = \sum_{j=0}^{\ell} |h_j|^2 = \|h\|^2_{\mathrm{H}^2},$$

which proves the second equality in (3.1) for all quaternion polynomials, which in turn extends to the whole H^2 by an approximation argument.

To verify (3.2), we write (1.12) in the form $B(p) = D + p\mathbf{e}(p)M$, where

$$B(p) = D + p\mathbf{e}(p)M, \quad \text{where} \quad \mathbf{e}(p) = \sum_{k=0}^{\infty} p^k E A^k,$$
$$D = 1 - E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^*, \quad M = (\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^*. \tag{3.3}$$

Let us observe the following three equalities:

$$DD^* = 1 - E\mathcal{G}_{E,A}^{-1}E^*, \quad DM^* = -E\mathcal{G}_{E,A}^{-1}A^*, \quad MM^* = \mathcal{G}_{E,A}^{-1} - A\mathcal{G}_{E,A}^{-1}A^*. \tag{3.4}$$

Indeed, making use of (1.8), we have

$$\begin{aligned} DD^* &= (1 - E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^*)(1 - E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}E^* \\ &= 1 - E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^* - E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}E^* \\ &\quad + E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}(\mathcal{G}_{E,A} - A^*\mathcal{G}_{E,A}A)(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}E^* \\ &= 1 - E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}(\mathcal{G}_{E,A}(\mathbf{I} - A) \\ &+ (\mathbf{I} - A^*)\mathcal{G}_{E,A} - \mathcal{G}_{E,A} + A^*\mathcal{G}_{E,A}A) \\ &\quad \times (\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}E^* \\ &= 1 - E\mathcal{G}_{E,A}^{-1}E^* \end{aligned}$$

which confirms the first equality in (3.4). Furthermore,

$$\begin{aligned} DM^* &= (1 - E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^*)E(\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*) \\ &= E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}((\mathbf{I} - A^*)\mathcal{G}_{E,A}\mathcal{G}_{E,A} + A^*\mathcal{G}_{E,A}A) \\ &\quad \times (\mathbf{I} - A)^{-1}\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*) \\ &= -E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}A^*(\mathbf{I} - A^*) = -E\mathcal{G}_{E,A}^{-1}A^*, \end{aligned}$$

which verifies the second equality in (3.4). The verification of the third equality in (3.4) is straightforward and will be omitted. Let us also observe that

$$E + p\mathbf{e}(p)A = E + \sum_{k=0}^{\infty} p^{k+1}EA^{k+1} = \sum_{k=0}^{\infty} p^k EA^k = \mathbf{e}(p).$$

Making use of the latter equality along with (3.3), we have

$$\begin{aligned}
B(p)\overline{B(q)} &= (D + p\mathbf{e}(p)M)(D^* + M^*\mathbf{e}(q)^*\overline{q})\\
&= DD^* + p\mathbf{e}(p)MD^* + DM^*\mathbf{e}(q)^*\overline{q} + p\mathbf{e}(p)MM^*\mathbf{e}(q)^*\overline{q}\\
&= 1 - E\mathcal{G}_{E,A}^{-1}E^* - p\mathbf{e}(p)A\mathcal{G}_{E,A}^{-1}E^* - E\mathcal{G}_{E,A}^{-1}A^*\mathbf{e}(q)^*\overline{q}\\
&\quad + p\mathbf{e}(p)(\mathcal{G}_{E,A}^{-1} - A\mathcal{G}_{E,A}^{-1}A^*)\mathbf{e}(q)^*\overline{q}\\
&= 1 - E\mathcal{G}_{E,A}^{-1}E^* - (\mathbf{e}(p) - E)\mathcal{G}_{E,A}^{-1}E^* - E\mathcal{G}_{E,A}^{-1}(\mathbf{e}(q)^* - E^*)\\
&\quad + p\mathbf{e}(p)\mathcal{G}_{E,A}^{-1}\mathbf{e}(q)^*\overline{q} - (\mathbf{e}(p) - E)\mathcal{G}_{E,A}^{-1}(\mathbf{e}(q)^* - E^*)\\
&= 1 + p\mathbf{e}(p)\mathcal{G}_{E,A}^{-1}\mathbf{e}(q)^*\overline{q} - \mathbf{e}(p)\mathcal{G}_{E,A}^{-1}\mathbf{e}(q)^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{k=0}^{\infty} p^k(1 - B(p)\overline{B(q)})\overline{q}^k &= \sum_{k=0}^{\infty} p^k(\mathbf{e}(p)\mathcal{G}_{E,A}^{-1}\mathbf{e}(q)^* - p\mathbf{e}(p)\mathcal{G}_{E,A}^{-1}\mathbf{e}(q)^*\overline{q})\overline{q}^k\\
&= \mathbf{e}(p)\mathcal{G}_{E,A}^{-1}\mathbf{e}(q)^*,
\end{aligned}$$

which proves (3.2). □

Lemma 3.3. *A function g belongs to H^2 and satisfies the homogeneous interpolation condition*

$$(E^*g)^{\wedge L}(A^*) = 0 \tag{3.5}$$

if and only if it is of the form $g = B \star h$ for some $h \in \mathrm{H}^2$, where B is given in (1.12).

Proof. Observe that again due to (1.8),

$$\begin{aligned}
(E^*B)^{\wedge L}(A^*) &= \sum_{k=0}^{\infty} A^{*k}E^*B_k\\
&= E^* - E^*E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^*\\
&\quad + A^*\sum_{k=1}^{\infty} A^{*k-1}E^*EA^{k-1}(\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^*\\
&= E^* - E^*E\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^*\\
&\quad + A^*\mathcal{G}_{E,A}(\mathbf{I} - A)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^*\\
&= \left((\mathbf{I} - A^*)\mathcal{G}_{E,A} - E^*E + A^*\mathcal{G}_{E,A}(\mathbf{I} - A)\right)\mathcal{G}_{E,A}^{-1}(\mathbf{I} - A^*)^{-1}E^*\\
&= 0.
\end{aligned}$$

Since for all $m \geq 0$ and $a \in \mathbb{H}$, we have

$$(E^*B \star p^m a)^{\wedge L}(A^*) = A^*(E^*B)^{\wedge L}(A^*)a,$$

we conclude by linearity and continuity that for any $h \in \mathrm{H}^2$, the function $g = B \star h$ satisfies condition (3.5). By the second equality in (3.1), $B \star h$ belongs to H^2.

Conversely, let (3.5) hold for a function $g \in \mathrm{H}^2$. By (1.6), this means that g is orthogonal (in the metric of H^2) to the function $\mathbf{e}x$ for any $x \in \mathbb{H}^n$.

In other words, g belongs to the orthogonal complement $\mathcal{M}^\perp$ of the right linear span

$$\mathcal{M} = \mathrm{span}_{\mathbf{r}}\,\{\mathbf{e}(p)x : \ x \in \mathbb{H}^n\}\,.$$

Using the formula for the reproducing kernel of a finite dimensional reproducing quaternionic kernel Hilbert space (see for instance [7, Theorem 9.6, p. 461]), we see that $\mathcal{M}$ is the (right) reproducing kernel Hilbert space with reproducing kernel

$$\mathbf{k}_{\mathcal{M}}(p,q) = \mathbf{e}(p)\mathcal{G}_{E,A}^{-1}\mathbf{e}(q)^*.$$

Hence, $\mathcal{M}^\perp$ admits the reproducing kernel

$$\mathbf{k}_{\mathcal{M}^\perp}(p,q) = \sum_{k=0}^{\infty} p^k\overline{q}^k - \mathbf{e}(p)\mathcal{G}_{E,A}^{-1}\mathbf{e}(q)^* = \sum_{k=0}^{\infty} p^k B(p)\overline{B(q)}\overline{q}^k, \tag{3.6}$$

where the last equality follows from (3.2). Since g belongs to $\mathcal{M}^\perp$, it follows that $g = B \star h$ for some $h \in \mathrm{H}^2$. □

Proof of Theorem 1.2: It follows from Lemmas 3.2 and 3.3 that any function of the form (1.13) satisfies (1.14) and is a solution of Problem 1. Conversely, let f be any solution of this problem, and let F be defined by (1.13). The function $f - F$ satisfies the homogeneous interpolation problem. By Lemma 3.3 $f - F = B \star h$, where B is given by (1.14) and $h \in \mathrm{H}^2$. Formula (1.14) follows from the orthogonality in (1.13), from the formula for the norm of F in Lemma 3.1 and from the fact that M_B is an isometry from H^2 into itself. This concludes the proof. □

4. Interpolation by Schur-class Functions

We start with several preliminary remarks. Making use of the given matrices A, E, N such that $\mathcal{G}_{E,A} > 0$, we introduce the matrices

$$\mathcal{G}_{N,A} := \sum_{k=0}^{\infty} A^{*k}N^*NA^k \quad P := \mathcal{G}_{E,A} - \mathcal{G}_{N,A} = \sum_{k=0}^{\infty} A^{*k}(E^*E - N^*N)A^k. \tag{4.1}$$

Remark 4.1. The matrix P satisfies the Stein equality

$$P - A^*PA = E^*E - N^*N. \tag{4.2}$$

Indeed, the matrix $\mathcal{G}_{N,A}$ satisfies equality $\mathcal{G}_{N,A} - A^*\mathcal{G}_{N,A}A = N^*N$, which being subtracted from (1.8) gives (4.2).

Proposition 4.2. *Let* $S(p) = \sum_{k=0}^{\infty} p^k S_k \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ *and let* $M_S : \mathrm{H}^2 \to \mathrm{H}^2$ *be the multiplication operator defined as in* (2.7)*. Then* S *satisfies the condition*

$$(E^*S)^{\wedge L}(A^*) = \sum_{k=0}^{\infty} A^{*k}E^*S_k = N^* \tag{4.3}$$

if and only if

$$M_S^*\mathbf{e} = \mathbf{n} \quad \text{where} \quad \mathbf{e}(p) = \sum_{k=0}^{\infty} p^k E A^k \quad \text{and} \quad \mathbf{n}(p) = \sum_{k=0}^{\infty} p^k N A^k. \tag{4.4}$$

Proof. We first observe that $\mathbf{e}$ and $\mathbf{n}$ are $\mathbb{H}^{1\times n}$-valued functions whose entries belong to H^2. Moreover, as it is readily seen from (1.7), (4.1) and (4.4),

$$\|\mathbf{e}x\|_{\mathrm{H}^2}^2 = x^*\mathcal{G}_{E,A}x \quad \text{and} \quad \|\mathbf{n}x\|_{\mathrm{H}^2}^2 = x^*\mathcal{G}_{N,A}x \quad \text{for all} \quad x \in \mathbb{H}^n. \tag{4.5}$$

Assuming that equality (4.3) is in force, we also have

$$(E^*M_p^m S)^{\wedge L}(A^*) = \sum_{k=0}^{\infty} A^{*k+m}E^*S_k = A^{*m}N^* \quad \text{for} \quad m = 0, 1, \ldots. \tag{4.6}$$

Therefore, for every $x \in \mathbb{H}^n$, we have

$$\langle M_S^*\mathbf{e}x,\, p^m\rangle_{\mathrm{H}^2} = \langle \mathbf{e}x,\, M_p^m S\rangle_{\mathrm{H}^2} = \sum_{k=0}^{\infty} \overline{S}_k E A^{k+m}x = NA^m x = \langle \mathbf{n}x,\, p^m\rangle_{\mathrm{H}^2}.$$

Since the latter equalities hold for all $x \in \mathbb{H}^n$ and since polynomials are dense in H^2, we conclude (4.4). On the other hand, if (4.4) holds true, then we also have

$$(M_S^*\mathbf{e})(0) = \mathbf{n}(0),$$

and taking adjoints in the latter equality gives (4.3). □

We next characterize solutions to Problem 2 in terms of a positive kernel. This approach has its origins in [20,21,26]. The current proof is adapted from [3,8,10].

Theorem 4.3. *A function $S : \mathbb{B} \to \mathbb{H}$ is a solution to Problem 2 if and only if the following kernel is positive on $\mathbb{B} \times \mathbb{B}$:*

$$\widehat{K}_S(p,q) := \begin{bmatrix} P & (\mathbf{e}(q) - (S \star \mathbf{n})(q))^* \\ \mathbf{e}(p) - (S \star \mathbf{n})(p) & K_S(p,q) \end{bmatrix} \succeq 0, \tag{4.7}$$

where P is given in (4.1) *and where $\mathbf{e}$ and $\mathbf{n}$ are defined in* (4.4)*.*

Proof. Let us assume that S belongs to $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ and satisfies condition (4.3) (or equivalently, condition (4.4)). Then we have from (4.1), (4.5) and (4.4),

$$\begin{aligned} x^*Px &= x^*\mathcal{G}_{E,A}x - x^*\mathcal{G}_{N,A}x \\ &= \|\mathbf{e}x\|_{\mathrm{H}^2}^2 - \|\mathbf{n}x\|_{\mathrm{H}^2}^2 \\ &= \|\mathbf{e}x\|_{\mathrm{H}^2}^2 - \|M_S^*\mathbf{e}x\|_{\mathrm{H}^2}^2 \\ &= \langle (\mathbf{I}_{\mathrm{H}^2} - M_S M_S^*)\mathbf{e}x,\, \mathbf{e}x\rangle_{\mathrm{H}^2} \end{aligned} \tag{4.8}$$

for every $x \in \mathbb{H}^n$. Furthermore,

$$\mathbf{e}x - S \star \mathbf{n}x = (\mathbf{I}_{\mathrm{H}^2} - M_S M_S^*)\mathbf{e}x, \tag{4.9}$$

which together with the reproducing kernel property (2.6) gives

$$\mathbf{e}(p)x - (S \star \mathbf{n})(p)x = \langle (\mathbf{I}_{\mathrm{H}^2} - M_S M_S^*)\mathbf{e}x,\, k_{\mathrm{H}^2}(\cdot, p)\rangle_{\mathrm{H}^2} \tag{4.10}$$

for all $p \in \mathbb{B}$ and $x \in \mathbb{H}^n$. We now use (2.9), (4.8) and (4.10) to compute for a fixed $\alpha \in \mathbb{H}$,

$$\begin{aligned} &\langle (\mathbf{I}_{\mathrm{H}^2} - M_S M_S^*)(\mathbf{e}x + k_{\mathrm{H}^2}(\cdot, q)\alpha), \, \mathbf{e}x + k_{\mathrm{H}^2}(\cdot, p)\alpha \rangle_{\mathrm{H}^2} \\ &\quad = x^* P x + \overline{\alpha}\,(\mathbf{e}(p)x - (S \star \mathbf{n})(p))\, x + x^*(\mathbf{e}(q) - (S \star \mathbf{n})(q))^* \alpha \\ &\qquad + \overline{\alpha} K_S(p,q)\alpha. \end{aligned} \tag{4.11}$$

Since S belongs to $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$, the operator $M_S : \mathrm{H}^2 \to \mathrm{H}^2$ is a contraction (by Theorem 2.2) so that $\mathbf{I}_{\mathrm{H}^2} - M_S M_S^* \geq 0$ and the expression on the right side of (4.11) is a positive (on $\mathbb{B} \times \mathbb{B}$) kernel for any fixed $x \in \mathbb{H}^n$ and $\alpha \in \mathbb{H}$. The latter is equivalent to (4.7).

Conversely, let us assume that (4.7) holds. Then in particular, the kernel $K_S(p,q)$ is positive on $\mathbb{B} \times \mathbb{B}$ and hence, $S \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ by Theorem 2.2. Let us introduce the multiplication operators $T_{\mathbf{e}}$, $T_{\mathbf{n}} : \mathbb{H}^n \to \mathrm{H}^2$ by

$$(T_{\mathbf{e}}x)(p) = \mathbf{e}(p)x \quad \text{and} \quad (T_{\mathbf{n}}x)(p) = \mathbf{n}(p)x.$$

It is seen from (4.5) that

$$T_{\mathbf{e}}^* T_{\mathbf{e}} = \mathcal{G}_{E,A} \quad \text{and} \quad T_{\mathbf{n}}^* T_{\mathbf{n}} = \mathcal{G}_{N,A}. \tag{4.12}$$

Let us consider the operator $\widehat{P} : \begin{bmatrix} \mathbb{H}^n \\ \mathrm{H}^2 \end{bmatrix} \to \begin{bmatrix} \mathbb{H}^n \\ \mathrm{H}^2 \end{bmatrix}$ given by

$$\widehat{P} = \begin{bmatrix} T_{\mathbf{e}}^* T_{\mathbf{e}} - T_{\mathbf{n}}^* T_{\mathbf{n}} & T_{\mathbf{e}}^* - T_{\mathbf{n}}^* M_S^* \\ T_{\mathbf{e}} - M_S T_{\mathbf{n}} & \mathbf{I}_{\mathrm{H}^2} - M_S M_S^* \end{bmatrix} = \begin{bmatrix} P & T_{\mathbf{e}}^* - T_{\mathbf{n}}^* M_S^* \\ T_{\mathbf{e}} - M_S T_{\mathbf{n}} & \mathbf{I}_{\mathrm{H}^2} - M_S M_S^* \end{bmatrix}, \tag{4.13}$$

where the second equality is the consequence of (4.1) and (4.12). The latter operator is related to the kernel (4.7) by the equality

$$\left\langle \widehat{P} f, \, f \right\rangle = \sum_{i,j=1}^{r} \left\langle \widehat{K}_S(p_i, p_j) \begin{bmatrix} x_j \\ \alpha_j \end{bmatrix}, \begin{bmatrix} x_i \\ \alpha_i \end{bmatrix} \right\rangle, \quad \text{where} \quad f = \sum_{j=1}^{r} \begin{bmatrix} x_j \\ k_{\mathrm{H}^2}(\cdot, p_j)\alpha_j \end{bmatrix},$$

holding for any choice of finitely many $\alpha_i \in \mathbb{H}$, $x_i \in \mathbb{H}^n$ and any $p_i \in \mathbb{B}$. Since vectors f of the form as above are dense in $\mathbb{H}^n \oplus \mathrm{H}^2$ and since the kernel $\widehat{K}_S(p,q)$ is positive on $\mathbb{B} \times \mathbb{B}$, it follows from the latter equality that $\widehat{P}$ is positive semidefinite. Observe that $\widehat{P}$ is the Schur complement of the upper left block in the operator

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_{\mathrm{H}^2} & T_{\mathbf{n}} & M_S^* \\ T_{\mathbf{n}}^* & T_{\mathbf{e}}^* T_{\mathbf{e}} & T_{\mathbf{e}}^* \\ M_S & T_{\mathbf{e}} & \mathbf{I}_{\mathrm{H}^2} \end{bmatrix} \tag{4.14}$$

which is therefore, also positive semidefinite. But then the Schur complement of the right bottom block in $\mathbf{P}$ is also positive semidefinite:

$$\begin{bmatrix} \mathbf{I} - M_S^* M_S & T_{\mathbf{n}} - M_S^* T_{\mathbf{e}} \\ T_{\mathbf{n}}^* - T_{\mathbf{e}}^* M_S & T_{\mathbf{e}}^* T_{\mathbf{e}} - T_{\mathbf{e}}^* T_{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} I - M_S^* M_S & T_{\mathbf{n}} - M_S^* T_{\mathbf{e}} \\ T_{\mathbf{n}}^* - T_{\mathbf{e}}^* M_S & 0 \end{bmatrix} \geq 0$$

from which we conclude that $T_{\mathbf{n}} - M_S^* T_{\mathbf{e}} = 0$, i.e., that condition (4.4) (and therefore, condition (4.4) is satisfied). Thus, S is a solution to Problem 2, which completes the proof of the theorem. □

Corollary 4.4. *Problem 2 has a solution if and only if the matrix P (the Pick matrix of the problem) is positive semidefinite.*

Proof. The necessity part follows from Theorem 4.3 since condition (1.11) implies $P \geq 0$. The sufficiency part follows by standard approximation arguments since in case P is positive definite, Problem 2 has a solution as we will see from Theorem 4.5 below. More precisely, consider Problem 2 when replacing N by $\sqrt{\epsilon}N$, with $\epsilon \in (0,1)$. The Stein equation (4.2) becomes

$$P(\epsilon) - A^*P(\epsilon)A = E^*E - \epsilon N^*N.$$

For $\epsilon = 0$, we have $P(0) > 0$ since the pair (E, A) is observable. Furthermore, the function $\epsilon \mapsto P(\epsilon)$ decreases in the sense of Loewner order, and is real analytic in ϵ. It follows that there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ tending to 0 and such that $P(\epsilon_n) > 0$. Consider the corresponding solution S_n of Problem 2. One concludes by using the counterpart of Montel's theorem in the present setting. See [13] for the latter. □

Let us note that

$$\begin{aligned} \mathbf{e}(p) - (S \star \mathbf{n})(p) &= \sum_{k=0}^{\infty} p^k \left(E - S(p)N\right) A^k \\ &= \begin{bmatrix} 1 & -S(p) \end{bmatrix} \star \left(\sum_{k=0}^{\infty} p^k \begin{bmatrix} E \\ N \end{bmatrix} A^k \right). \end{aligned} \tag{4.15}$$

Therefore, all the entries in the kernel inequality (4.7) are defined in terms of given E, N, A and an unknown function S. The description of all functions S satisfying the latter inequality was established in [3] under the assumptions that

1. the right spectrum of A is contained in $\mathbb{B}$ and
2. the unique solution P of the Stein equation (4.2) is positive definite.

Under these assumptions, let us introduce the 2×2 matrix-valued function

$$\Theta(p) = \mathbf{I}_2 + (p-1) \sum_{k=0}^{\infty} p^k \begin{bmatrix} E^* \\ N^* \end{bmatrix} A^{*k} P^{-1} (\mathbf{I}_n - A)^{-1} \begin{bmatrix} E & -N \end{bmatrix} \tag{4.16}$$

which is clearly slice hyperholomorphic in $\mathbb{B}$.

Theorem 4.5. *Let us assume that $P > 0$ and let $\Theta = \left[\begin{smallmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{smallmatrix}\right]$ be defined as in (4.16). Then all solutions S to Problem 2 are given by the formula*

$$S = (\Theta_{11} \star \mathcal{E} + \Theta_{12}) \star (\Theta_{21} \star \mathcal{E} + \Theta_{22})^{-\star} \tag{4.17}$$

with the free parameter $\mathcal{E}$ running through the class $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$.

By Theorem 5.2 in [3], formula (4.17) parametrizes all functions $S \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ such that the kernel (4.7) is positive. But each such function is a solution to Problem 2, by Theorem 4.3.

Theorem 4.6. *Let us assume that the Pick matrix $P \geq 0$ is singular. Then Problem 2 has at most one solution which, if exists, is given by the formula*

$$S(p) = R \star Q(p)^{-\star}, \tag{4.18}$$

where

$$R(p) = \mathbf{e}(p)y = \sum_{k=0}^{\infty} p^k E A^k y \quad \text{and} \quad Q(p) = \mathbf{n}(p)y = \sum_{k=0}^{\infty} p^k N A^k y \tag{4.19}$$

and where $y \in \mathbb{H}^n$ *is any nonzero vector such that* $Py = 0$.

Proof. Let us assume that S is a solution to the problem **NP**. Then the matrix

$$\begin{bmatrix} P & (\mathbf{e}(p) - (S \star \mathbf{n})(p))^* \\ \mathbf{e}(p) - (S \star \mathbf{n})(p) & K_S(p,p) \end{bmatrix}$$

is positive semidefinite for every $p \in \mathbb{B}$. From this positivity and from the equality $Py = 0$ we conclude that $\mathbf{e}y(p) = (S \star \mathbf{n}y)(p)$. Thus any solution S to Problem 2 must satisfy

$$S \star Q(p) = R(p) \quad \text{for all} \quad p \in \mathbb{B}. \tag{4.20}$$

Due to the assumption that the pair (E, A) is observable, not all coefficients of R are zeros and hence, the function R is not vanishing identically. Then it follows from (4.20) that Q is not vanishing identically as well. Therefore, the formula (4.18) holds (first on an open subset of $\mathbb{B}$ and then by continuity on the whole $\mathbb{B}$, since S is assumed to be in $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$). So the solution (if it exists) is unique, and this uniqueness implies in particular, that the representation (4.18) does not depend on the particular choice of $y \in \operatorname{Ker} P$. □

As in the complex case, Nevanlinna–Pick type interpolation problems are of considerable interest for geometric function theory of slice hyperholomorphic functions [16] with further applications to quaternionic dynamics and semigroups of composition operators on H^2. Besides, Problems 1 and 2 considered in this paper are prototypical for interpolation problems in, respectively, reproducing kernel Hilbert spaces of slice hyperholomorphic functions and the classes of contractive multipliers of such Hilbert spaces. All these questions still are largely open.

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