Integr. Equ. Oper. Theory 86 (2016), 41–70
DOI 10.1007/s00020-016-2315-0
Published online September 21, 2016

Integral Equations and Operator Theory

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Spectral Invariance of Non-Smooth Pseudo-Differential Operators

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Abstract. We discuss some spectral invariance results for non-smooth pseudodifferential operators with coefficients in Hölder spaces in this paper. In analogy to the proof in the smooth case of Beals and Ueberberg, c.f. (Duke Math J 44(1):45–57, 1977; Manuscripta Math 61(4):459–475, 1988), we use the characterization of non-smooth pseudodifferential operators to get such a result. The main new difficulties are the limited mapping properties of pseudodifferential operators with non-smooth symbols and the fact, that in general the composition of two non-smooth pseudodifferential operators is not a pseudodifferential operator. In order to improve these spectral invariance results for certain subsets of non-smooth pseudodifferential operators with coefficients in Hölder spaces, we improve the characterization of non-smooth pseudodifferential operators of A. and P., c.f. (Abels and Pfeuffer, Characterization of non-smooth pseudodifferential operators. arXiv:1512.01127, 2015).

Mathematics Subject Classification. 35S05, 47G30.

Keywords. Non-smooth pseudodifferential operators, characterization by mapping properties, spectral invariance.

1. Introduction

A lot of spectral invariance results of pseudodifferential operators are already known for pseudodifferential operators with smooth symbols e.g. of the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. The symbol-class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$ consists of all smooth functions p such that for all $k \in \mathbb{N}_0$

$$|p|^{(m)}_k := \max_{|\alpha|,|\beta| \leq k} \sup_{x,\xi \in \mathbb{R}^n} |\partial^\alpha_\xi \partial^\beta_x p(x,\xi)| \langle \xi \rangle^{-(m-\rho|\alpha|+\delta|\beta|)} < \infty.$$

We would like to thank Prof. Dr. Schrohe for his helpful suggestions of improvements. Moreover, we thank the referee for the careful reading of the manuscript and many valuable suggestions for improvements.

We define the associated pseudodifferential operator via

$$OP(p)u(x) := p(x, D_x)u(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, đ\xi \quad \forall u \in \mathcal{S}(\mathbb{R}^n), \quad x \in \mathbb{R}^n,$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space, i. e., the space of all rapidly decreasing smooth functions. Moreover $\hat{u}$ and $\mathscr{F}[u]$ denote the Fourier transformation of u. Additionally $OPS^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ is the set of all pseudodifferential operators with symbols in the symbol-class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$.

A fundamental result in the theory of pseudodifferential operators allows to conclude that the inverse of a pseudodifferential operator in the set $OPS^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, which is invertible as an operator on $L^2(\mathbb{R}^n)$, is again a pseudodifferential operator in the same symbol-class. This important statement was shown by Beals [5] and Ueberberg [18]. Their proof even showed that the same holds for all Bessel potential spaces $H^s_2(\mathbb{R}^n)$, $s \in \mathbb{R}$, see Definition 2.1 for the definition of these spaces, and that the spectrum is independent of the choice of the space.

Schrohe extended this result for weighted L^p-Sobolev spaces in [15] and together with Leopold even for Besov spaces of variable order of differentiation $B^{s,a}_{p,q}(\mathbb{R}^n)$ in [9]. They verified that the spectrum of smooth pseudodifferential operators in certain symbol-classes is independent of the choice of the weighted L^p-Sobolev space and of the choice of $B^{s,a}_{p,q}(\mathbb{R}^n)$ respectively, cf. [9,15].

There are several other results for spectral invariance of smooth pseudodifferential operators in the literature, cf. e.g. [3,7,8,10,16].

In this paper we show the spectral invariance for non-smooth pseudodifferential operators whose symbols are in the symbol-class $C^\tau_* S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Here

$$C^\tau_*(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{C^\tau_*} := \sup_{j\in\mathbb{N}_0} 2^{j\tau} \|\mathscr{F}^{-1}[\varphi_j \hat{f}]\|_{L^\infty} < \infty \right\},$$

is the so-called Hölder–Zygmund space, where $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of $\mathcal{S}(\mathbb{R}^n)$, $\mathscr{F}^{-1}[u]$ denotes the inverse Fourier transformation of $u \in \mathcal{S}'(\mathbb{R}^n)$ and $(\varphi_j)_{j\in\mathbb{N}_0}$ constitute a dyadic partition of unity. Note that $C^\tau_*(\mathbb{R}^n)$ is equal to the Hölder space $C^\tau(\mathbb{R}^n)$ if $\tau \notin \mathbb{N}$. The symbol-class $C^\tau_* S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ with $m \in \mathbb{R}$, $\tau > 0$, $0 \leq \rho \leq 1$ and $M \in \mathbb{N}_0 \cup \{\infty\}$ is the set of all functions $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that we have for all $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha| \leq M$ and $|\beta| \leq \tau$:

i) $\partial^\beta_x p(x, .) \in C^M(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,
ii) $\partial^\beta_x \partial^\alpha_\xi p \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$,
iii) $\|\partial^\alpha_\xi p(., \xi)\|_{C^\tau_*(\mathbb{R}^n)} \leq C_\alpha \langle\xi\rangle^{m-\rho|\alpha|}$ for all $\xi \in \mathbb{R}^n$.

For a given symbol p of the previous symbol-class we define the associated pseudodifferential operator in the same way as in the smooth case and denote it by $p(x, D_x)$ or $OP(p)$. The set $OPC^\tau_* S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ consists of all pseudodifferential operators with symbols in $C^\tau_* S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$.

In analogy to the proof of the spectral invariance results of Ueberberg in the smooth case, we use the characterization of non-smooth pseudodifferential operators via iterated commutators by using the following set:

Definition 1.1. Let $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Additionally let $\tilde{m} \in \mathbb{N}_0 \cup \{\infty\}$ and $1 < q < \infty$. Then we define $\mathcal{A}^{m,M}_{\rho,0}(\tilde{m}, q)$ as the set of all linear and bounded operators $P : H^m_q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, such that for all $l \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}^n_0$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}^n_0$ with $|\alpha_1| + |\beta_1| = \cdots = |\alpha_l| + |\beta_l| = 1$, $|\alpha| \leq M$ and $|\beta| \leq \tilde{m}$ the iterated commutator of P

$$\mathrm{ad}(-ix)^{\alpha_1}\, \mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\alpha_l}\, \mathrm{ad}(D_x)^{\beta_l} P : H^{m-\rho|\alpha|}_q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

is continuous. Here $\alpha := \alpha_1 + \cdots + \alpha_l$ and $\beta := \beta_1 + \cdots + \beta_l$.

The iterated commutators are defined in Definition 2.1 below. In case $M = \infty$ we write $\mathcal{A}^{m}_{\rho,0}(\tilde{m}, q)$ instead of $\mathcal{A}^{m,\infty}_{\rho,0}(\tilde{m}, q)$. In [2] we showed that every operator in such a characterization set is a non-smooth pseudodifferential operator if certain conditions hold:

Theorem 1.2. *Let $m \in \mathbb{R}$, $1 < q < \infty$, $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$ and $\rho \in \{0, 1\}$. Additionally let $M \in \mathbb{N}_0 \cup \{\infty\}$ and define $\tilde{M} := M - (n+1)$. Assuming $P \in \mathcal{A}^{m,M}_{\rho,0}(\tilde{m}, q)$ and $\tilde{M} \geq 1$ we obtain for $s \in (0, \tilde{m} - n/q]$ with $s \notin \mathbb{N}_0$:*

$$P \in OPC^s S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

We loose some regularity with respect to $\tilde{m}$, here. This loss of regularity can be avoided as we will see in this paper.

In the present paper we proceed as follows: in Sect. 2 we summarize all notations and results needed later on. In particular we introduce the uniformly local Sobolev spaces $W^{m,q}_{uloc}$ and verify some important properties of these spaces, cf. Sect. 2.2. Section 3 is devoted to the definition and the properties of pseudodifferential operators of certain symbol-classes needed in this paper. We begin with pseudodifferential operators with single symbols in Sect. 3.1 while pseudodifferential operators with double symbols are treated in Sect. 3.3.

The main purpose of Sect. 4 is to improve the characterization of non-smooth pseudodifferential operators with coefficients in Hölder spaces, cf. Theorem 1.2. In analogy to the proof of Theorem 1.2 we show the existence of a pointwise convergent subsequence of a bounded set of pseudodifferential operators with coefficients in an uniformly local Sobolev space in Sect. 4.1 and verify a result for the symbol reduction of pseudodifferential operators with coefficients in an uniformly local Sobolev space in Sect. 4.2. With those results at hand, we are able to improve the characterization of non-smooth pseudodifferential operators in Sect. 4.3.

By means of this characterization we show several spectral invariance results for non-smooth pseudodifferential operators in Sect. 5. Section 5.1 is devoted to the inverse of a non-smooth pseudodifferential operator P in the symbol-class $C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$. We show that $P^{-1} \in OPC^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$, where $s < \tau$ is arbitrary. Unfortunately, in contrast to the smooth case, we loose some smoothness of the coefficients. Our next goal is to prove the spectral invariance of non-smooth pseudodifferential operators of the class $C^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$ for sufficiently large N. To be more precise, we arrive at the following statement: The inverse of a non-smooth pseudodifferential operator of the order zero with coefficients in the Hölder space $C^{\tilde{m},\tau}(\mathbb{R}^n)$ is also a non-smooth pseudodifferential operator if its inverse is an element of $\mathscr{L}(H^r_q(\mathbb{R}^n))$ for one $|r| < \tilde{m} + \tau$. This is the topic of Sect. 5.3. Beyond the characterization of non-smooth pseudodifferential operators we also use the technique of approximation with difference quotients for the proof of the above mentioned statement. We introduce this technique in Sect. 5.2. We are able to improve the results of Sect. 5.3 in Sect. 5.4 for certain subclasses of $OPC^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$.

The present paper is based on a part of the PhD-thesis, cf. [13], of the second author in this paper advised by the first author.

2. Preliminaries

We assume $n \in \mathbb{N}$ throughout the whole paper unless otherwise noted. In particular $n \neq 0$. For $x \in \mathbb{R}$ we define

$$x^+ := \max\{0; x\} \quad \text{and} \quad \lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}.$$

Additionally

$$\langle x \rangle := (1 + |x|^2)^{1/2} \quad \text{for all } x \in \mathbb{R}^n \quad \text{and} \quad đ\xi := (2\pi)^{-n} d\xi.$$

Partial derivatives with respect to $x \in \mathbb{R}^n$ scaled with the factor $-i$ are denoted by

$$D^\alpha_x := (-i)^{|\alpha|} \partial^\alpha_x := (-i)^{|\alpha|} \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_n}_{x_n}.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0$ is a *multi-index*. For $j \in \{1, \ldots, n\}$ we define $e_j \in \mathbb{N}^n_0$ as the j-the canonical unit vector, i.e., $(e_j)_k = 1$ if $k = j$ and $(e_j)_k = 0$ else.

Considering two Banach spaces X, Y the set $\mathscr{L}(X, Y)$ contains of all linear and bounded operators $A : X \to Y$. If $X = Y$, we also just write $\mathscr{L}(X)$.

Iterated commutators of linear operators are defined in the usual way:

Definition 2.1. Let $X, Y \in \{\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)\}$ and $P : X \to Y$ be linear. We define $\mathrm{ad}(-ix_j)P : X \to Y$ and $\mathrm{ad}(D_{x_j})P : X \to Y$ for all $j \in \{1, \ldots, n\}$ and $u \in X$ by

$$\begin{aligned} \mathrm{ad}(-ix_j)Pu &:= -ix_j Pu + P\left(ix_j u\right), \\ \mathrm{ad}(D_{x_j})Pu &:= D_{x_j}\left(Pu\right) - P\left(D_{x_j} u\right). \end{aligned}$$

For arbitrary multi-indices $\alpha, \beta \in \mathbb{N}^n_0$ we denote the *iterated commutator* of P as

$$\mathrm{ad}(-ix)^\alpha \mathrm{ad}(D_x)^\beta P := [\mathrm{ad}(-ix_1)]^{\alpha_1} \ldots [\mathrm{ad}(-ix_n)]^{\alpha_n} [\mathrm{ad}(D_{x_1})]^{\beta_1} \ldots [\mathrm{ad}(D_{x_n})]^{\beta_n} P.$$

2.1. Functions on $\mathbb{R}^n$ and Function Spaces

For convenience of the reader we introduce all functions and function spaces needed later on in this subsection. Recall that the *Hölder space* of the order $m \in \mathbb{N}_0$ with Hölder continuity exponent $s \in (0,1)$ is denoted by $C^{m,s}(\mathbb{R}^n)$ and also by $C^{m+s}(\mathbb{R}^n)$. Moreover for $s \in \mathbb{R}$ and $1 < p < \infty$ the set

$$H^s_p(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \langle D_x\rangle^s f \in L^p(\mathbb{R}^n) < \infty\} \tag{2.1}$$

is called *Bessel Potential space*, where $\langle D_x\rangle^s := OP(\langle\xi\rangle^s)$.

For $y \in \mathbb{R}^n$ the translation function $\tau_y(g) : \mathbb{R}^n \to \mathbb{C}$ of $g \in L^1(\mathbb{R}^n)$ is defined as $\tau_y(g)(x) := g(x-y)$ for all $x \in \mathbb{R}^n$.

Moreover a function $f : \mathbb{R}^n \to \mathbb{C}$ is *homogeneous of degree $d \in \mathbb{R}$ (for $|x| \geq 1$)* if $f(rx) = r^d f(x)$ $x \in \mathbb{R}^n$ for all $|x| \geq 1$ and $r \geq 1$.

A frequently used ingredient for verifying several results in this paper is the dyadic partition of unity, i.e. , a partition of unity $(\varphi_j)_{j\in\mathbb{N}_0}$ on $\mathbb{R}^n$ which fulfilles the properties

$$\text{supp } \varphi_0 \subseteq \overline{B_2(0)} \quad \text{and} \quad \text{supp } \varphi_j \subseteq \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$$

for all $j \in \mathbb{N}$. Here $B_r(0)$ denotes the open ball with radius $r > 0$ and center 0. A dyadic partition of unity can be constructed in the following way: We take $\varphi_0 \in C^\infty(\mathbb{R}^n)$ with $\varphi_0(\xi) = 1$ for all $|\xi| \leq 1$ and $\varphi_0(\xi) = 0$ for $|\xi| \geq 2$. Then we set $\varphi_j(\xi) := \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi)$ for all $\xi \in \mathbb{R}^n$ and $j \in \mathbb{N}$.

We also will need the next statement, cf. [2, Lemma 2.1]:

Lemma 2.2. *Let $1 < p < \infty$, $s < 0$ and $m := -\lfloor s \rfloor$. Then for each $f \in H^s_p(\mathbb{R}^n)$ there are functions $g_\alpha \in H^{s-\lfloor s\rfloor}_p(\mathbb{R}^n)$, where $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, such that*

- $f = \sum\limits_{|\alpha|\leq m} \partial_x^\alpha g_\alpha$,
- $\sum\limits_{|\alpha|\leq m} \|g_\alpha\|_{H^{s-\lfloor s\rfloor}_p} \leq C\|f\|_{H^s_p}$,

where C is independent of f, g_α.

2.2. Uniformly Local Sobolev Spaces

Definition 2.3. Let $1 \leq q \leq \infty$, $m \in \mathbb{N}_0$, $U \subseteq \mathbb{R}^n$ be open and X be a Banach space. Then the space of all functions, which belong *uniformly local* to $L^q(U;X)$ or $W^m_q(U;X)$ is denoted by

$$L^q_{uloc}(U;X) := \{f \in L^q_{loc}(U;X) : \|f\|_{L^q_{uloc}(U;X)} < \infty\},$$
$$W^{m,q}_{uloc}(U;X) := \{f \in L^q_{uloc}(U;X) : \partial_x^\alpha f \in L^q_{uloc}(U;X) \text{ for all } |\alpha| \leq m\},$$

respectively, where

$$\|f\|_{L^q_{uloc}(U;X)} := \sup_{x\in U} \|f\|_{L^q(B_1(x)\cap U;X)} \quad \text{for } f \in L^q_{uloc}(U;X),$$
$$\|f\|_{W^{m,q}_{uloc}(U;X)} := \sum_{|\alpha|\leq m} \|\partial_x^\alpha f\|_{L^q_{uloc}(U;X)} \quad \text{for } f \in W^{m,q}_{uloc}(U;X).$$

If $X = \mathbb{C}$, we write $L^q_{uloc}(U)$ instead of $L^q_{uloc}(U;\mathbb{C})$ and $W^{m,q}_{uloc}(U)$ instead of $W^{m,q}_{uloc}(U;\mathbb{C})$. Moreover, we also write $\|.\|_{L^q_{uloc}}$ and $\|.\|_{W^{m,q}_{uloc}}$ instead of $\|.\|_{L^q_{uloc}(\mathbb{R}^n;\mathbb{C})}$ and $\|.\|_{W^{m,q}_{uloc}(\mathbb{R}^n;\mathbb{C})}$.

The spaces $L^q_{uloc}(U;X)$ and $W^{m,q}_{uloc}(U;X)$ are Banach spaces. This can be verified by using the fact that $L^q(U;X)$ is a Banach space.

Lemma 2.4. *Let $1 < q \leq \infty$, $m \in \mathbb{N}_0$, X be a Banach space and $0 < \tau \leq m - n/q$ with $\tau \notin \mathbb{N}$. Then*

$$W^{m,q}_{uloc}(\mathbb{R}^n;X) \hookrightarrow C^\tau(\mathbb{R}^n;X) \quad \textit{is continuous.}$$

Proof. Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \lfloor \tau \rfloor$ be arbitrary. Then we obtain

$$\begin{aligned} &\sup_{x\neq y} \frac{\|\partial_x^\alpha f(x) - \partial_y^\alpha f(y)\|_X}{|x-y|^{\tau - \lfloor \tau \rfloor}} \\ &\quad \leq \sup_{x\in\mathbb{R}^n} \|\partial_x^\alpha f\|_{C^{\tau-\lfloor\tau\rfloor}(\overline{B_1(x)};X)} + 2\|\partial_x^\alpha f\|_{C_b^0(\mathbb{R}^n;X)} \end{aligned} \tag{2.2}$$

by splitting the supremum of the left side in the supremum over $|x-y| > 1$ and the rest. Using inequality (2.2) and $\|f\|_{C_b^{\lfloor\tau\rfloor}(\mathbb{R}^n;X)} \leq \sup_{x\in\mathbb{R}^n} \|f\|_{C^\tau(\overline{B_1(x)};X)}$ for all functions $f \in C^\tau(\mathbb{R}^n;X)$, we get

$$\|f\|_{C^\tau(\mathbb{R}^n;X)} \leq C \sup\nolimits_{x\in\mathbb{R}^n} \|f\|_{C^\tau(\overline{B_1(x)};X)} \quad \text{for all } f \in C^\tau(\mathbb{R}^n;X). \tag{2.3}$$

The use of Corollary 4.3 in [4] and the embeddings (3.1)–(3.3) and (3.6) in [4] yields the continuity of the embedding $W^m_q(B_1(0);X) \hookrightarrow C^\tau(\overline{B_1(0)};X)$. Together with inequality (2.3) we obtain

$$\begin{aligned} \|f\|_{C^\tau(\mathbb{R}^n;X)} &\leq \sup_{x\in\mathbb{R}^n} \|f\|_{C^\tau(\overline{B_1(x)};X)} = \sup_{x\in\mathbb{R}^n} \|f(x+.)\|_{C^\tau(\overline{B_1(0)};X)} \\ &\leq C \sup_{x\in\mathbb{R}^n} \|f(x+.)\|_{W^{m,q}(B_1(0);X)} = C\|f\|_{W^{m,q}_{uloc}(\mathbb{R}^n;X)}. \qquad \square \end{aligned}$$

Now we want to discuss the case $X = L^q_{uloc}(\mathbb{R}^m)$:

Remark 2.5. Let $m, \tilde{m} \in \mathbb{N}_0$, $n_1 \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \tilde{m}$. Assuming $a \in W^{m,q}_{uloc}(\mathbb{R}^n; W^{\tilde{m},q}_{uloc}(\mathbb{R}^n))$, we obtain

1) $L^q_{uloc}(\mathbb{R}^n; L^q_{uloc}(\mathbb{R}^{n_1})) \hookrightarrow L^q_{uloc}(\mathbb{R}^n \times \mathbb{R}^{n_1})$ is continuous $\forall 1 \leq q < \infty$,
2) $\partial_x^\alpha a \in W^{m,q}_{uloc}(\mathbb{R}^n; W^{\tilde{m}-|\alpha|,q}_{uloc}(\mathbb{R}^n_x))$ for all $1 < q < \infty$.

Proof. We obtain the Claim 1) by using $B_1(x,y) \subseteq B_1(x) \times B_1(y)$ for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Claim 2) easily follows from the definition of the distributional derivatives. $\square$

Lemma 2.6. *Let $m, N \in \mathbb{N}_0$ and $1 < q < \infty$. We consider a measurable function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ we have*

- *$\partial_y^\alpha a(.,y) \in W^{m,q}_{uloc}(\mathbb{R}^n)$ for all $y \in \mathbb{R}^n$,*
- *$a(x,.) \in W^{N,q}_{uloc}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,*
- *$\sup_{y\in\mathbb{R}^n} \|\partial_y^\alpha a(.,y)\|_{W^{m,q}_{uloc}(\mathbb{R}^n)} < C_\alpha$ for a constant $C_\alpha > 0$.*

Then $a \in W^{N,q}_{uloc}(\mathbb{R}^n_y; W^{m,q}_{uloc}(\mathbb{R}^n_x))$.

Proof. Because of $a(x,.) \in W^{N,q}_{uloc}(\mathbb{R}^n)$, the α-th weak derivative of a in the sense of $\mathscr{D}'(\mathbb{R}^n_y; W^{m,q}_{uloc}(\mathbb{R}^n_x))$ exists for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$. Hence the claim holds:

$$\|a\|_{W^{N,q}_{uloc}(\mathbb{R}^n_y;W^{m,q}_{uloc}(\mathbb{R}^n_x))} \leq \sum_{|\alpha|\leq N} \sup_{y\in\mathbb{R}^n} \left\{ \int_{B_1(y)} \|\partial_y^\alpha a(.,z)\|^q_{W^{m,q}_{uloc}(\mathbb{R}^n)} dz \right\}^{1/q}$$
$$\leq \sum_{|\alpha|\leq N} C_{\alpha,q} \sup_{y\in\mathbb{R}^n} |B_1(y)|^{1/q} \leq C_{N,q,n}. \qquad \square$$

With all these results at hand, we are able to show:

Lemma 2.7. *Let $1 < q < \infty$ and $\tilde{m}, N \in \mathbb{N}_0$. Furthermore, let $\mathscr{B}$ be the set of all measurable functions $a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that*

- *$\partial_y^\alpha a(.,\xi,y) \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ for all $\xi, y \in \mathbb{R}^n$ and each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$,*
- *$a(x,\xi,.) \in W^{N,q}_{uloc}(\mathbb{R}^n)$ for all $x, \xi \in \mathbb{R}^n$.*

Additionally let $m \in \mathbb{N}_0$ such that for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ we have

$$\sup_{y\in\mathbb{R}^n} \|\partial_y^\alpha a(.,\xi,y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_{\alpha,q}\langle\xi\rangle^m \quad \text{for all } \xi \in \mathbb{R}^n, a \in \mathscr{B}.$$

If we define $b : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ for a fixed but arbitrary $a \in \mathscr{B}$ by

$$b(x,\xi,y) := a(x,\xi,x+y) \quad \text{for all } x,\xi,y \in \mathbb{R}^n,$$

we get for each $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$ and $|\alpha| + |\beta| \leq N$ and for all $\xi \in \mathbb{R}^n$:

$$\|\partial_y^\alpha \partial_x^\beta b(x,\xi,y)\|_{L^q_{uloc}(\mathbb{R}^n_x\times\mathbb{R}^n_y)} < C_{\alpha,q}\langle\xi\rangle^m,$$

where $C_{\alpha,q}$ is independent of $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$.

Proof. Lemma 2.6 and Remark 2.5 imply

$$\partial_y^\alpha \partial_x^\beta a(x,\xi,y) \in W^{N-|\alpha|,q}_{uloc}\left(\mathbb{R}^n_y; W^{\tilde{m}-|\beta|,q}_{uloc}(\mathbb{R}^n_x)\right) \subseteq L^q_{uloc}(\mathbb{R}^n_y; L^q_{uloc}(\mathbb{R}^n_x))$$

for all $\xi \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and $|\beta| \leq \tilde{m}$. Using Tonelli's theorem twice and the change of variable $\hat{y} := z + \tilde{y}$ we obtain for each $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and $|\beta| \leq \tilde{m}$:

$$\int\limits_{B_1(x,y)} \left|\left(\partial_y^\alpha \partial_x^\beta a\right)(z,\xi,z+\tilde{y})\right|^q d(z,\tilde{y}) \leq \int\limits_{B_1(x)\times B_1(y)} \left|\left(\partial_y^\alpha \partial_x^\beta a\right)(z,\xi,z+\tilde{y})\right|^q d(z,\tilde{y})$$
$$= \int\limits_{B_1(x)} \int\limits_{B_1(y+z)} \left|\left(\partial_y^\alpha \partial_x^\beta a\right)(z,\xi,\hat{y})\right|^q d\hat{y}\, dz \leq \int\limits_{B_1(x)} \int\limits_{B_2(y+x)} \left|\partial_{\hat{y}}^\alpha \partial_z^\beta a(z,\xi,\hat{y})\right|^q d\hat{y}\, dz$$
$$= \int\limits_{B_2(y+x)} \int\limits_{B_1(x)} \left|\partial_{\hat{y}}^\alpha \partial_z^\beta a(z,\xi,\hat{y})\right|^q dz\, d\hat{y} \leq \int\limits_{B_2(y+x)} \left\|\partial_{\hat{y}}^\alpha \partial_x^\beta a(x,\xi,\hat{y})\right\|^q_{L^q_{uloc}(\mathbb{R}^n_x)} d\hat{y}$$
$$\leq \int\limits_{B_2(y+x)} \sup_{\hat{y}\in\mathbb{R}^n} \left\|\partial_{\hat{y}}^\alpha a(x,\xi,\hat{y})\right\|^q_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} d\hat{y} \leq C_{\alpha,q,n}\langle\xi\rangle^m,$$

for all $x, \xi, y \in \mathbb{R}^n$ and $a \in \mathscr{B}$. Finally, let $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$ and $|\alpha| + |\beta| \leq N$ be arbitrary. An application of the chain rule and the previous inequality provides:

$$\|\partial_y^\alpha \partial_x^\beta b(x,\xi,y)\|_{L^q_{uloc}(\mathbb{R}^n_x \times \mathbb{R}^n_y)} \leq C_{\alpha,\beta,q,n}\langle \xi \rangle^m \quad \forall \xi \in \mathbb{R}^n \text{ and } a \in \mathscr{B}. \qquad \square$$

Lemma 2.8. *Let $1 < q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Moreover, let $\mathscr{B}$ be a set of measurable functions $a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ with the following property:*

- *$\partial_y^\alpha a(.,\xi,y) \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ for all $\xi, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2\tilde{m}$,*
- *$a(x,\xi,.) \in W^{2\tilde{m},q}_{uloc}(\mathbb{R}^n)$ for all $x, \xi \in \mathbb{R}^n$.*

Additionally we assume that for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2\tilde{m}$ there is a constant $C_{\alpha,q}$ such that

$$\sup_{y \in \mathbb{R}^n} \|\partial_y^\alpha a(.,\xi,y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_{\alpha,q}\langle \xi \rangle^m \quad \text{for all } \xi \in \mathbb{R}^n, a \in \mathscr{B}.$$

Then we have for some $C_{m,q} < \infty$:

$$\sup_{y \in \mathbb{R}^n} \|a(x,\xi,x+y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \leq C_{m,q}\langle \xi \rangle^m \quad \text{for all } \xi \in \mathbb{R}^n, a \in \mathscr{B}.$$

Proof. Using a finite cover of $B_2(0,0)$ with open balls of radius 1 one easily shows for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq 2\tilde{m}$ and $|\alpha| \leq \tilde{m}$:

$$\begin{aligned} &\sup_{x,y \in \mathbb{R}^n} \left[\int_{B_2(x,y)} \left| \partial_{\tilde{y}}^\beta \partial_z^\alpha a(z,\xi,z+\tilde{y}) \right|^q d(\tilde{y},z) \right]^{1/q} \\ &\quad \leq C_q \|\partial_y^\beta \partial_x^\alpha a(x,\xi,x+y)\|_{L^q_{uloc}(\mathbb{R}^n_x \times \mathbb{R}^n_y)} \end{aligned} \tag{2.4}$$

for all $a \in \mathscr{B}, \xi \in \mathbb{R}^n$. Due to Lemma 2.6 and Remark 2.5 we get

$$\partial_y^\alpha \partial_x^\beta a(x,\xi,y) \in W^{2\tilde{m}-|\alpha|,q}_{uloc}\left(\mathbb{R}^n_y; W^{\tilde{m}-|\beta|,q}_{uloc}(\mathbb{R}^n_x)\right) \subseteq L^q_{uloc}(\mathbb{R}^n_y; L^q_{uloc}(\mathbb{R}^n_x))$$

for all $\xi \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq 2\tilde{m}$ and $|\beta| \leq \tilde{m}$. We define $b(x,\xi,y) := a(x,\xi,x+y)$ for all $x,\xi,y \in \mathbb{R}^n$. Using the Sobolev embedding theorem and Tonelli's theorem, we obtain for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \tilde{m}$:

$$\begin{aligned} \int_{B_1(x)} \sup_{\tilde{y} \in B_1(y)} |\partial_z^\alpha b(z,\xi,\tilde{y})|^q \, dz &\leq C_q \int_{B_1(x)} \|\partial_z^\alpha b(z,\xi,.)\|^q_{W^{\tilde{m}}_q(B_1(y))} \, dz \\ &\leq C_q \sum_{|\beta| \leq \tilde{m}} \int_{B_1(x) \times B_1(y)} \left| \partial_{\tilde{y}}^\beta \partial_z^\alpha b(z,\xi,\tilde{y}) \right|^q d(\tilde{y},z) \\ &\leq C_q \sum_{|\beta| \leq \tilde{m}} \int_{B_2(x,y)} \left| \partial_{\tilde{y}}^\beta \partial_z^\alpha b(z,\xi,\tilde{y}) \right|^q d(\tilde{y},z) \end{aligned} \tag{2.5}$$

for all $a \in \mathscr{B}$ and $x, y, \xi \in \mathbb{R}^n$. Therefore (2.5), (2.4) and Lemma 2.7 yield

$$\begin{aligned}
\sup_{y\in\mathbb{R}^n} \|a(x,\xi,x+y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} &\leq \sum_{|\alpha|\leq\tilde{m}} \sup_{y\in\mathbb{R}^n} \|\partial_x^\alpha b(x,\xi,y)\|_{L^q_{uloc}(\mathbb{R}^n_x)} \\
&\leq \sum_{|\alpha|\leq\tilde{m}} \sup_{x,y\in\mathbb{R}^n} \left\{ \int_{B_1(x)} \sup_{\tilde{y}\in B_1(y)} |\partial_z^\alpha b(z,\xi,\tilde{y})|^q \, dz \right\}^{1/q} \\
&\leq C_q \sum_{|\alpha|\leq\tilde{m}} \sum_{|\beta|\leq\tilde{m}} \sup_{x,y\in\mathbb{R}^n} \left\{ \int_{B_2(x,y)} \left|\partial_{\tilde{y}}^\beta \partial_z^\alpha b(z,\xi,\tilde{y})\right|^q d(\tilde{y},z) \right\}^{1/q} \\
&\leq C_q \sum_{|\alpha|\leq\tilde{m}} \sum_{|\beta|\leq\tilde{m}} \left\|\partial_y^\beta \partial_x^\alpha b(x,\xi,y)\right\|_{L^q_{uloc}(\mathbb{R}^n_x\times\mathbb{R}^n_y)} \leq C_{m,q,n}\langle\xi\rangle^m
\end{aligned}$$

for all $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$. □

2.3. Space of Amplitudes and Oscillatory Integrals

In the following we will use *oscillatory integrals* defined by

$$\text{Os-}\iint e^{-iy\cdot\eta} a(y,\eta) dy \,đ\eta := \lim_{\varepsilon\to 0} \iint \chi(\varepsilon y, \varepsilon\eta) e^{-iy\cdot\eta} a(y,\eta) dy\, đ\eta \tag{2.6}$$

for all elements a of the *space of amplitudes* $\mathscr{A}^{m,N}_\tau(\mathbb{R}^n\times\mathbb{R}^n)$, $N \in \mathbb{N}_0 \cup \{\infty\}$, $m, \tau \in \mathbb{R}$, the set of all functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ with the following properties: For all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ we have

i) $\partial_\eta^\alpha \partial_y^\beta a(y,\eta) \in C^0(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$,
ii) $\left|\partial_\eta^\alpha \partial_y^\beta a(y,\eta)\right| \leq C_{\alpha,\beta}(1+|\eta|)^m(1+|y|)^\tau$ for all $y, \eta \in \mathbb{R}^n$.

Here $\chi \in \mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$ with $\chi(0,0) = 1$. If $N = \infty$ we also write $\mathscr{A}^m_\tau(\mathbb{R}^n\times\mathbb{R}^n)$ instead of $\mathscr{A}^{m,\infty}_\tau(\mathbb{R}^n\times\mathbb{R}^n)$.

Now we summarize the properties of the oscillatory integral we need later on. For more details we refer to [2, Subsection 2.3]. In the following we use for all $m \in \mathbb{N}$ the next definition:

$$A^m(D_x,\xi) := \langle\xi\rangle^{-m}\langle D_x\rangle^m \quad \text{if } m \text{ is even},$$

$$A^m(D_x,\xi) := \langle\xi\rangle^{-m-1}\langle D_x\rangle^{m-1} - \sum_{j=1}^n \langle\xi\rangle^{-m}\frac{\xi_j}{\langle\xi\rangle}\langle D_x\rangle^{m-1}D_{x_j} \quad \text{else.}$$

Theorem 2.9. *Let $m, \tau \in \mathbb{R}$ and $N \in \mathbb{N}_0 \cup \{\infty\}$ with $N > n + \tau$. Then the oscillatory integral (2.6) exists for each $a \in \mathscr{A}^{m,N}_\tau(\mathbb{R}^n\times\mathbb{R}^n)$. Additionally for all $l, l', l_0, \tilde{l}_0 \in \mathbb{N}_0$ with $l > n + m$, $\tilde{l}_0 \leq N$ and $N \geq l' > n + \tau$ we have*

$$Os\text{-}\iint e^{-iy\cdot\eta} a(y,\eta)dy\,đ\eta = \iint e^{-iy\cdot\eta} A^{l'}(D_\eta,y)[A^l(D_y,\eta)a(y,\eta)]dy\,đ\eta.$$

$$Os\text{-}\iint e^{-iy\cdot\eta} a(y,\eta)dy\,đ\eta = Os\text{-}\iint e^{-iy\cdot\eta} A^{\tilde{l}_0}(D_\eta,y)A^{l_0}(D_y,\eta)a(y,\eta)dy\,đ\eta$$

Theorem 2.10. *Let $m, \tau \in \mathbb{R}$, $N \in \mathbb{N}_0 \cup \{\infty\}$ and $k \in \mathbb{N}$ with $N > k + \tau$. We set $\tilde{\tau} := \tau$ if $\tau \geq -k$, $\tilde{\tau} := -k - 1/2$ if $\tau \in \mathbb{Z}$ and $\tau < -k$ and $\tilde{\tau} :=$*

$-k-(|\tau|-\lfloor -\tau\rfloor)/2$ else. We define $\hat{\tau} := \tau^+$ if $\tau \geq -k$ and $\hat{\tau} := \tau - \tilde{\tau}$ else. For $a \in \mathscr{A}^{m,N}_\tau(\mathbb{R}^{n+k} \times \mathbb{R}^{n+k})$ we define

$$b(y,\eta) := Os\text{-}\iint e^{-iy'\cdot\eta'} a(y,y',\eta,\eta')dy'đ\eta' \quad \text{for all } y, \quad \eta \in \mathbb{R}^n.$$

Let $M := \max\{m \in \mathbb{N}_0 : N-m \geq l > k+\tilde{\tau}$ for one $l \in \mathbb{N}_0\}$. Then b is an element of $\mathscr{A}^{m^+,M}_{\hat{\tau}}(\mathbb{R}^n \times \mathbb{R}^n)$ and for each $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq M$ we have:

$$\partial_y^\alpha \partial_\eta^\beta b(y,\eta) = Os\text{-}\iint e^{-iy'\cdot\eta'} \partial_y^\alpha \partial_\eta^\beta a(y,y',\eta,\eta')dy'đ\eta' \quad \text{for all } y, \quad \eta \in \mathbb{R}^n.$$

3. Pseudodifferential Operators

Throughout this section we assume $X^{\tilde{m}}_q \in \left\{W^{\tilde{m},q}_{uloc}; H^{\tilde{m}}_q\right\}$ for $1 < q < \infty$, $\tilde{m} \in \mathbb{R}$ if $X^{\tilde{m}}_q = H^{\tilde{m}}_q$ and $\tilde{m} \in \mathbb{N}_0$ else with $\tilde{m} > n/q$ unless otherwise noted.

3.1. Pseudodifferential Operators with Single Symbols

The non-smooth symbol-class with coefficients in $X^{\tilde{m}}_q$ was already introduced in [12]. Analogous to the definition of $C^{\tilde{m},s}S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ we define

Definition 3.1. Let $1 < q < \infty$, $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Then the *symbol-class* $X^{\tilde{m}}_q S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is the set of all $p : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$ such that

i) $\partial_x^\beta p(x,.) \in C^M(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,
ii) $\partial_x^\beta \partial_\xi^\alpha p \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$,
iii) $\partial_\xi^\alpha p(.,\xi) \in X^{\tilde{m}}_q$ and $\|\partial_\xi^\alpha p(.,\xi)\|_{X^{\tilde{m}}_q} \leq C_\alpha \langle\xi\rangle^{m-\rho|\alpha|}$ for all $\xi \in \mathbb{R}^n$

holds for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq M$ and $|\beta| < \tilde{m} - n/q$. The function p is called *(non-smooth) symbol* and m is called *order* of p. If $M = \infty$, the symbols are smooth in ξ. In this case we write $X^{\tilde{m}}_q S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$ instead of $X^{\tilde{m}}_q S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; \infty)$.

We define for all $k \in \mathbb{N}_0$ with $k \leq M$ the semi-norms

$$|p|^{(m)}_k := \sup_{\xi \in \mathbb{R}^n} \max_{|\alpha| \leq k} \|\partial_\xi^\alpha p(.,\xi)\|_{X^{\tilde{m}}_q} \langle\xi\rangle^{-m+\rho|\alpha|}$$

for all $p \in X^{\tilde{m}}_q S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Equipped with the family of semi-norms $(|.|^{(m)}_k)_{k \in \{0,\ldots,M\}}$ the symbol-class $X^{\tilde{m}}_q S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is a Fréchet space.

In applications to partial differential equations, many pseudodifferential operators are classical ones. They are defined in the following way:

Definition 3.2. Let $m \in \mathbb{R}$ and $1 < q < \infty$. Then $p \in X^{\tilde{m}}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a *classical symbol of the order* m if p has an asymptotic expansion

$$p(x,\xi) \sim \sum_{j \in \mathbb{N}_0} p_j(x,\xi),$$

where p_j are homogeneous of degree $m-j$ in ξ (for $|\xi| \geq 1$) for all $j \in \mathbb{N}_0$ in the sense, that for all $N \in \mathbb{N}$ we have

$$p(x,\xi) - \sum_{j<N} p_j(x,\xi) \in X_q^{\tilde{m}} S_{1,0}^{m-N}(\mathbb{R}^n \times \mathbb{R}^n).$$

The set of all classical symbols of the order m is denoted by $X_q^{\tilde{m}} S_{cl}^m(\mathbb{R}^n \times \mathbb{R}^n)$.

For a more general definition we refer to [17]. We obtain due to Lemma 2.4 if $X_q^{\tilde{m}} = W_{uloc}^{\tilde{m},q}$ and the continuous embedding $H_q^{\tilde{m}}(\mathbb{R}^n) \hookrightarrow C^\tau(\mathbb{R}^n)$ else:

Lemma 3.3. *Let* $1 < q < \infty$, $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ *and* $0 \leq \rho \leq 1$. *Assuming* $0 < \tau \leq \tilde{m} - n/q$, $\tau \notin \mathbb{N}$ *we have*

$$X_q^{\tilde{m}} S_{\rho,0}^m(\mathbb{R}^n \times \mathbb{R}^n; M) \subseteq C^\tau S_{\rho,0}^m(\mathbb{R}^n \times \mathbb{R}^n; M).$$

Due to the last lemma we already defined the associated pseudodifferential operator $p(x, D_x)$ to a non-smooth symbol $p \in X$, where X is one of the symbol-classes defined in Definitions 3.1 and 3.2. The set of all non-smooth pseudodifferential operators with symbols in X is denoted by OPX. If $M = \infty$ we write $OPX_q^{\tilde{m}} S_{\rho,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$ instead of $OPX_q^{\tilde{m}} S_{\rho,0}^m(\mathbb{R}^n \times \mathbb{R}^n; \infty)$.

Remark 3.4. Let $M \in \mathbb{N}_0 \cup \{\infty\}$, $m \in \mathbb{R}$ and $0 \leq \rho \leq 1$. Additionally let $Y_q^{\tilde{m}} = X_q^{\tilde{m}}$ or, in case $q = \infty$, let $Y_q^{\tilde{m}+1} = C^{\tilde{m},s}$ with for $\tilde{m} \in \mathbb{N}_0$ and $0 < s < 1$. Let $p \in Y_q^{\tilde{m}} S_{\rho,0}^m(\mathbb{R}^n \times \mathbb{R}^n; M)$. Moreover, let $l \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0^n$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}_0^n$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1, \ldots, n\}$, $|\alpha| \leq M$ and $|\beta| < \tilde{m} - n/q$, where $\alpha := \alpha_1 + \cdots + \alpha_l$ and $\beta := \beta_1 + \cdots + \beta_l$. Then

$$\mathrm{ad}(-ix)^{\alpha_1} \mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\alpha_l} \mathrm{ad}(D_x)^{\beta_l} p(x, D_x)$$

is a pseudodifferential operator with the symbol

$$\partial_\xi^\alpha D_x^\beta p(x,\xi) \in Y_q^{\tilde{m}-|\beta|} S_{\rho,0}^{m-\rho|\alpha|}(\mathbb{R}_x^n \times \mathbb{R}_\xi^n; M - |\alpha|).$$

If $p \in X_q^{\tilde{m}} S_{cl}^m(\mathbb{R}^n \times \mathbb{R}^n)$, then $\partial_\xi^\alpha D_x^\beta p(x,\xi) \in X_q^{\tilde{m}-|\beta|} S_{cl}^{m-|\alpha|}(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$. In case $p \in S_{\rho,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$, we have $\partial_\xi^\alpha D_x^\beta p(x,\xi) \in S_{\rho,0}^{m-\rho|\alpha|}(\mathbb{R}^n \times \mathbb{R}^n)$.

Proposition 3.5. *Let* $1 < p < \infty$ *and* $s \in \mathbb{R}$. *Considering a partition of unity* $(\psi_j)_{j \in \mathbb{Z}^n} \subseteq C_c^\infty(\mathbb{R}^n)$ *with*

- $\psi_0(x) > 0$ *for all* $x \in [0,1]^n$,
- $\psi_j(x) = \psi_0(x - j)$ *for all* $x \in \mathbb{R}^n$ *and* $j \in \mathbb{Z}^n$

we obtain

$$\|f\|_{H_p^s(\mathbb{R}^n)} \simeq \left(\sum_{j \in \mathbb{Z}^n} \|\psi_j f\|_{H_p^s(\mathbb{R}^n)}^p \right)^{1/p}.$$

Proof. The case $s \geq 0$ follows directly from [12, Theorem 1.3]. Therefore let $s < 0$. In order to show

$$\|f\|_{H_p^s(\mathbb{R}^n)} \leq C \left(\sum_{j \in \mathbb{Z}^n} \|\psi_j f\|_{H_p^s(\mathbb{R}^n)}^p \right)^{1/p} \quad \text{for all } f \in H_p^s(\mathbb{R}^n) \tag{3.1}$$

let $f \in H^s_p(\mathbb{R}^n)$ and $g \in H^{-s}_q(\mathbb{R}^n)$ with $1/p + 1/q = 1$ be arbitrary. We define

$$\eta_0 := \sum_{k\in Z} \psi_k, \quad \text{where } Z := \{k \in \mathbb{Z}^n : \text{supp } \psi_0 \cap \text{supp } \psi_k \neq \varnothing\}$$

and $\eta_j(x) := \eta_0(x-j)$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}^n$. Let $\langle .,. \rangle_{H^s_p; H^{-s}_q}$ denote the standard duality pairing of Sobolev spaces. An application of the partition of unity, Hölder's inequality for sequence spaces first and the proposition in the case $-s > 0$ afterwards provides

$$\begin{aligned}
\left|\langle f, g\rangle_{H^s_p;H^{-s}_q}\right| &\leq \sum_{j\in\mathbb{Z}^n} \left|\langle \eta_j\psi_j f, g\rangle_{H^s_p;H^{-s}_q}\right| \leq \sum_{j\in\mathbb{Z}^n} \|\psi_j f\|_{H^s_p} \|\eta_j g\|_{H^{-s}_q} \\
&\leq C_{q,Z} \left(\sum_{j\in\mathbb{Z}^n} \|\psi_j f\|^p_{H^s_p}\right)^{1/p} \left(\sum_{j\in\mathbb{Z}^n} \|\psi_j g\|^q_{H^{-s}_q}\right)^{1/q} \\
&\leq C_{q,Z} \left(\sum_{j\in\mathbb{Z}^n} \|\psi_j f\|^p_{H^s_p}\right)^{1/p} \|g\|_{H^{-s}_q}.
\end{aligned}$$

Consequently we get (3.1) by duality and the previous inequality.
Now we define η_j for every $j \in \mathbb{Z}^n$ as before and $m := -\lfloor s \rfloor$. Additionally we choose an arbitrary $f \in H^s_p(\mathbb{R}^n)$. Because of Lemma 2.2, there are $g_\alpha \in H^{s-\lfloor s\rfloor}_p(\mathbb{R}^n)$, $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq m$, fulfilling $f = \sum_{|\alpha|\leq m} \partial^\alpha_x g_\alpha$ and $\sum_{|\alpha|\leq m} \|g_\alpha\|_{H^{s-\lfloor s\rfloor}_p} \leq C\|f\|_{H^s_p}$. Since $\eta_j \equiv 1$ on supp ψ_j we obtain

$$\begin{aligned}
\|\psi_j f\|^p_{H^s_p} &\leq \sum_{|\alpha|\leq m} \|\psi_j \partial^\alpha_x g_\alpha\|^p_{H^s_p} = \sum_{|\alpha|\leq m} \|\psi_j \partial^\alpha_x \{g_\alpha \eta_j\}\|^p_{H^s_p} \\
&\leq \sum_{|\alpha|\leq m} \left\{\|\partial^\alpha_x\{\psi_j g_\alpha \eta_j\}\|_{H^s_p} + \|[\partial^\alpha_x, \psi_j](g_\alpha\eta_j)\|_{H^s_p}\right\}^p
\end{aligned}$$

Since the commutators $[\partial^\alpha_x, \psi_j]$ form a bounded subset of $OPS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, we find

$$\begin{aligned}
\|\psi_j f\|^p_{H^s_p} &\leq \sum_{k\in Z+j} \sum_{|\alpha|\leq m} \left\{\|\partial^\alpha_x\{\psi_k g_\alpha\}\|_{H^s_p} + C\|g_\alpha\psi_k\|_{H^{s-\lfloor s\rfloor}_p}\right\}^p \\
&\leq C \sum_{k\in Z+j} \sum_{|\alpha|\leq m} \|\psi_k g_\alpha\|^p_{H^{s-\lfloor s\rfloor}_p}.
\end{aligned}$$

Using the case $s \geq 0$ provides:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}^n} \|\psi_j f\|^p_{H^s_p} &\leq C \sum_{j\in\mathbb{Z}^n} \sum_{k\in Z+j} \sum_{|\alpha|\leq m} \|\psi_k g_\alpha\|^p_{H^{s-\lfloor s\rfloor}_p} \leq C \sum_{|\alpha|\leq m} \sum_{j\in\mathbb{Z}^n} \|\psi_j g_\alpha\|^p_{H^{s-\lfloor s\rfloor}_p} \\
&\leq C \sum_{|\alpha|\leq m} \|g_\alpha\|^p_{H^{s-\lfloor s\rfloor}_p} \leq C\|f\|^p_{H^s_p}. \qquad \square
\end{aligned}$$

We will use the following boundedness results for non-smooth pseudodifferential operators, cf. Lemma 3.4 in [2], Theorem 3.7 in [2], Theorem 2.1 in [11], Lemma 2.9 in [11]:

Lemma 3.6. *Let $\tau > 0$ and $X \in \{C^\tau(\mathbb{R}^n), C^\tau_*(\mathbb{R}^n), X^{\tilde{m}}_q\}$. Let $M \in \mathbb{N}_0 \cup \{\infty\}$, $m \in \mathbb{R}$ and $\delta = 0$ in the case $X \notin \{C^\tau, C^\tau_*\}$ and $0 \le \rho, \delta \le 1$ else. For every bounded subset $\mathscr{B} \subseteq XS^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$, $\{p(x, D_x) : p \in \mathscr{B}\}$ is a bounded subset of $\mathscr{L}(\mathcal{S}(\mathbb{R}^n), X)$.*

Theorem 3.7. *Let $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ with $\rho > 0$ and $1 < p < \infty$. Additionally let $\tau > \frac{1-\rho}{1-\delta} \cdot \frac{n}{2}$ if $\rho < 1$ and $\tau > 0$ if $\rho = 1$ respectively. Moreover, let $N \in \mathbb{N} \cup \{\infty\}$ with $N > n/2$ for $2 \le p < \infty$ and $N > n/p$ else, $k_p := (1-\rho)n\,|1/2 - 1/p|$ and let $\mathscr{B} \subseteq C^\tau_* S^{m-k_p}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; N)$ be bounded. Then for each $s \in \mathbb{R}$ with*

$$(1-\rho)\frac{n}{p} - (1-\delta)\tau < s < \tau$$

there is some $C_s > 0$ such that

$$\|a(x, D_x)f\|_{H^s_p} \le C_s \|f\|_{H^{s+m}_p} \quad \text{for all } f \in H^{s+m}_p(\mathbb{R}^n) \quad \text{and} \quad a \in \mathscr{B}.$$

Theorem 3.8. *Let $m \in \mathbb{R}$ and $\tau > \frac{n}{2}$. Moreover, let $a \in C^\tau_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$ where $N \in \mathbb{N} \cup \{\infty\}$ with $N > n/2$. Then for each $s \in \mathbb{R}$ with $n/2 - \tau < s < \tau$ we have*

$$\|a(x, D_x)f\|_{H^s_2} \le C_s \|f\|_{H^{s+m}_2} \quad \text{for all } f \in H^{s+m}_2(\mathbb{R}^n).$$

Theorem 3.9. *Let $m \in \mathbb{R}$, $N > n/2$, $\tau > 0$. Moreover let P be an element of $OPC^\tau_* S^{m-n/2}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$. Then*

$$P : H^{s+m}_2(\mathbb{R}^n) \to H^s_2(\mathbb{R}^n) \quad \text{is continuous for all } -\tau < s < \tau.$$

We can prove similar boundedness results for non-smooth pseudodifferential operators.

Theorem 3.10. *Let $1 < p, q < \infty$ and $m, \tilde{m} \in \mathbb{R}$ with $\tilde{m} > n/q$. Moreover, let $\mathscr{B} \subseteq H^{\tilde{m}}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ be bounded. Then for each $s \in \mathbb{R}$ with*

$$n(1/p + 1/q - 1)^+ - \tilde{m} < s \le \tilde{m} - n(1/q - 1/p)^+$$

we have

$$\|a(x, D_x)f\|_{H^s_p} \le C_s \|f\|_{H^{s+m}_p} \quad \text{for all } f \in H^{s+m}_p(\mathbb{R}^n) \quad \text{and all} \quad a \in \mathscr{B}.$$

Proof. By means of [12, Theorem 2.2], $a(x, D_x) \in \mathscr{L}(H^{s+m}_p(\mathbb{R}^n), H^s_p(\mathbb{R}^n))$ for every $a \in \mathscr{B}$. An application of the Banach–Steinhaus theorem yields the uniform norm estimate; see [2, Lemma 3.5] for more details. □

Note that the last theorem even holds for $0 < p \le \infty$ and $q \in \{1, \infty\}$ if $\sharp\mathscr{B} = 1$, cf. [12, Theorem 2.2]. A similar result holds for pseudodifferential operators of the class $H^{\tilde{m}}_q S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 \le \delta \le 1$, if $\sharp\mathscr{B} = 1$, cf. [12, Theorem 2.2].

As a consequence of the previous theorem we obtain the next result.

Lemma 3.11. *Let $1 < p, q < \infty$ and $m \in \mathbb{R}$ with $m > n/q$. Assuming*

$$n\,(1/p + 1/q - 1)^+ - m < s \le m - n\,(1/q - 1/p)^+,$$

there is some $C_s > 0$ such that

$$\|ab\|_{H^s_p} \le C_s \|a\|_{H^m_q} \|b\|_{H^s_p} \quad \text{for all } a \in H^m_q(\mathbb{R}^n), \quad b \in H^s_p(\mathbb{R}^n). \tag{3.2}$$

Theorem 3.12. *Let $m \in \mathbb{R}$, $1 < p, q < \infty$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > n/q$. We assume that $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is bounded. Then for each $s \in \mathbb{R}$ with*

$$-\tilde{m} + n/q < s \leq \tilde{m} - n(1/q - 1/p)^+ \tag{3.3}$$

there is some $C_s > 0$ such that

$$\|a(x, D_x)f\|_{H^s_p} \leq C_s \|f\|_{H^{s+m}_p} \quad \text{for all } f \in H^{s+m}_p(\mathbb{R}^n) \text{ and } a \in \mathscr{B}.$$

Proof. One argues as in the proof of Theorem 3.10, using [12, Theorem 2.6]. □

We remark that the previous theorem even holds for $0 < p < \infty$ and $q = 1$ if $\sharp\mathscr{B} = 1$ and if $s \in \mathbb{R}$ fulfills

$$n(\max\{1, 1/p\} - 1) - \tilde{m} + n/q < s \leq \tilde{m} - n(1/q - 1/p)^+$$

instead of the assumption (3.3) due to [12, Theorem 2.6].

Next we want to improve the previous statement for classical pseudo-differential operators of the symbol-class $W^{\tilde{m},q}_{uloc} S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$:

Theorem 3.13. *Let $m \in \mathbb{R}$, $1 < p, q < \infty$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > n/q$. Assuming a bounded subset $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc} S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ we get for each $s \in \mathbb{R}$ with*

$$n\,(1/p + 1/q - 1)^+ - \tilde{m} < s \leq \tilde{m} - n\,(1/q - 1/p)^+$$

the existence of a constant $C_s > 0$ such that

$$\|a(x, D_x)f\|_{H^s_p} \leq C_s \|f\|_{H^{s+m}_p} \quad \text{for all } f \in H^{s+m}_p(\mathbb{R}^n),\ a \in \mathscr{B}. \tag{3.4}$$

Proof. Let s and p be as in the assumptions. Since $\{H^s_p(\mathbb{R}^n) : s \in \mathbb{R}\}$ is a microlocalizable set, cf. [1, Theorem 5.20], (3.4) is a direct consequence of [17, Proposition 1.1B] for each fixed $a \in \mathscr{B}$ if the following inequality holds:

$$\|fg\|_{H^s_p} \leq C_s \|f\|_{H^s_p} \|g\|_{W^{\tilde{m},q}_{uloc}} \quad \text{for all } f \in H^s_p(\mathbb{R}^n), g \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n). \tag{3.5}$$

In view of Proposition 3.5 we choose a partition of unity $(\psi_j)_{j \in \mathbb{Z}^n} \subseteq C^\infty_c(\mathbb{R}^n)$ with $\operatorname{supp} \psi_0 \subseteq [-r, r]^n$ for one fixed $r > 1$ and $\psi_j(x) = \psi_0(x - j)$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}^n$. With $Z_j := \{k \in \mathbb{Z}^n : \operatorname{supp} \psi_k \cap \operatorname{supp} \psi_j \neq \varnothing\}$, we define

$$\eta_j(x) := \sum_{k \in Z_j} \psi_k(x) \quad \text{for all } x \in \mathbb{R}^n \text{ and every } j \in \mathbb{Z}^n.$$

Choosing a finite cover $(B_1(x_i))_{i=1}^N$, $N \in \mathbb{N}$, of $\operatorname{supp} \eta_0$ with open balls of radius 1 provides a finite cover $(B_1(x_i - j))_{i=1}^N$ of $\operatorname{supp} \eta_j$. Hence N is independent of $j \in \mathbb{Z}^n$. By means of the Leibniz rule we obtain:

$$\begin{aligned} \|\eta_j g\|^q_{H^{\tilde{m}}_q(\mathbb{R}^n)} &\leq \sum_{|\alpha| \leq \tilde{m}} \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2} \sum_{i=1}^N \int_{B_1(x_i + j)} |\partial_x^{\alpha_2} g(x)|^q dx \\ &\leq C_{\tilde{m}} \|g\|^q_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \end{aligned} \tag{3.6}$$

for all $g \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ and $j \in \mathbb{Z}^n$. Together with Proposition 3.5 and Lemma 3.11 we conclude inequality (3.5):

$$\begin{aligned}\|fg\|^p_{H^s_p} &\leq C_s \sum_{j\in\mathbb{Z}^n} \|(\psi_j f)(\eta_j g)\|^p_{H^s_p} \leq C_{s,\tilde{m}} \sum_{j\in\mathbb{Z}^n} \|\psi_j f\|^p_{H^s_p} \|\eta_j g\|^p_{H^{\tilde{m}}_q} \\ &\leq C_{s,\tilde{m}} \|g\|^p_{W^{\tilde{m},q}_{uloc}} \sum_{j\in\mathbb{Z}^n} \|\psi_j f\|^p_{H^s_p} \leq C_{s,\tilde{m}} \|g\|^p_{W^{\tilde{m},q}_{uloc}} \|f\|^p_{H^s_p}\end{aligned}$$

for all $f \in H^s_p(\mathbb{R}^n)$ and $g \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$. The independence of $a \in \mathscr{B}$ of the constant C_s in (3.4) is again a consequence of the Banach–Steinhaus theorem, cf. [2, Lemma 3.5]. □

3.2. Kernel Representation

The present subsection is devoted to the kernel representation of a non-smooth pseudodifferential operator $p(x, D_x)$, whose symbol is in the class $XS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for a Banach space X with $C^\infty_c(\mathbb{R}^n) \subseteq X \subseteq C^0(\mathbb{R}^n)$, we refer to [17, Chapter 1] for the definition of these symbol-classes. In particular we can choose $X \in \{C^{\tilde{m},\tau}, C^{\tilde{m}+\tau}_*, H^{\tilde{m}}_q, W^{\tilde{m},q}_{uloc}\}$ with $\tilde{m} \in \mathbb{N}_0$, $0 < \tau \leq 1$ and $1 < q < \infty$.

Theorem 3.14. *Let $p \in XS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, where $C^\infty_c(\mathbb{R}^n) \subseteq X \subseteq C^0(\mathbb{R}^n)$ is a Banach space and $m \in \mathbb{R}$. Then there is a function $k : \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\}) \to \mathbb{C}$ such that $k(x, .) \in C^\infty(\mathbb{R}^n\backslash\{0\})$ for all $x \in \mathbb{R}^n$ and*

$$p(x, D_x)u(x) = \int k(x, x-y)u(y)dy \quad \text{for all } x \notin \text{supp } u$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. Moreover, for every $\alpha \in \mathbb{N}^n_0$ and each $N \in \mathbb{N}_0$ the kernel k satisfies

$$\|\partial^\alpha_z k(., z)\|_X \leq \begin{cases} C_{\alpha,N}|z|^{-n-m-|\alpha|}\langle z\rangle^{-N} & \text{if } n+m+|\alpha| > 0, \\ C_{\alpha,N}(1+|\log|z||)\langle z\rangle^{-N} & \text{if } n+m+|\alpha| = 0, \\ C_{\alpha,N}\langle z\rangle^{-N} & \text{if } n+m+|\alpha| < 0. \end{cases}$$

uniformly in $z \in \mathbb{R}^n\backslash\{0\}$.

Proof. We are able to prove the statements in a similar way as in [1, Theorem 5.12]. The main idea of the proof is to decompose

$$p(x, D_x)f = \sum_{j=0}^\infty p(x, D_x)\varphi_j(D_x)f \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n)$$

where $(\varphi_j)_{j\in\mathbb{N}_0}$ is a dyadic partition of unity. The series converges in X due to Lemma 3.6. First of all we construct a kernel k_j of the operator $p_j(x, D_x) := p(x, D_x)\varphi_j(D_x)$ for each $j \in \mathbb{N}_0$. This can be made in the same way as in the smooth case. We just have to use $\|\partial^\alpha_\xi p(., \xi)\|_X \leq C_\alpha\langle\xi\rangle^{m-|\alpha|}$ instead of $|\partial^\alpha_\xi p(., \xi)| \leq C_\alpha\langle\xi\rangle^{m-|\alpha|}$ for all $\alpha \in \mathbb{N}^n_0$ and all $\xi \in \mathbb{R}^n$. Afterwards we use this kernel decompositions to construct the kernel of $p(x, D_x)$ as in the smooth case. By means of $X \subseteq C^0(\mathbb{R}^n)$ we get the absolute and uniform convergence of $k(x, z) = \sum_{j=0}^\infty k_j(x, z)$. □

Remark 3.15. If we even have $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ in the previous theorem, we can show that $k(.,z)$ is smooth for all $z \in \mathbb{R}^n$ while applying Theorem 3.14 with $X = C^\tau_*$ for all $\tau \in \mathbb{R}$. This result already was shown in e.g. [1, Theorem 5.12].

3.3. Double Symbols

Definition 3.16. Let $0 < s \leq 1$, $\tilde{m} \in \mathbb{N}_0$, $1 < q < \infty$ and $m, m' \in \mathbb{R}$. Furthermore, let $N \in \mathbb{N}_0 \cup \{\infty\}$, $0 \leq \rho \leq 1$ and $X \in \{C^{\tilde{m},s}, W^{\tilde{m},q}_{uloc}\}$. Additionally let $\tilde{m} > n/q$ in case $X = W^{\tilde{m},q}_{uloc}$. Then the space of *non-smooth double symbols* $XS^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ is the set of all $p : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'} \to \mathbb{C}$ such that

i) $\partial^\alpha_\xi \partial^{\beta'}_{x'} \partial^{\alpha'}_{\xi'} p(.,\xi,x',\xi') \in X \; \forall \xi, x', \xi' \in \mathbb{R}^n$,
$\partial^\beta_x \partial^\alpha_\xi \partial^{\beta'}_{x'} \partial^{\alpha'}_{\xi'} p \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'})$,

ii) $\|\partial^\alpha_\xi \partial^{\beta'}_{x'} \partial^{\alpha'}_{\xi'} p(.,\xi,x',\xi')\|_X \leq C_{\alpha,\beta',\alpha'} \langle \xi \rangle^{m-\rho|\alpha|} \langle \xi' \rangle^{m'-\rho|\alpha'|} \; \forall \xi, x', \xi' \in \mathbb{R}^n$

and $\beta, \alpha, \beta', \alpha' \in \mathbb{N}^n_0$ with $|\beta| \leq \tilde{m}$ and $|\alpha| \leq N$. In case $N = \infty$ we write $XS^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ instead of $XS^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \infty)$. Furthermore, we define the set of semi-norms $\{|.|^{m,m'}_k : k \in \mathbb{N}_0\}$ by

$$|p|^{m,m'}_k := \max_{\substack{|\alpha| \leq N \\ |\alpha|+|\beta'|+|\alpha'| \leq k}} \sup_{\xi,x',\xi' \in \mathbb{R}^n} \|\partial^\alpha_\xi \partial^{\beta'}_{x'} \partial^{\alpha'}_{\xi'} p(.,\xi,x',\xi')\|_X \langle \xi \rangle^{-(m-\rho|\alpha|)} \langle \xi' \rangle^{-(m'-\rho|\alpha'|)}$$

Because of the previous definition $p \in XS^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N)$ is often called a *non-smooth single symbol*.

Definition 3.17. Let $0 < s \leq 1$, $\tilde{m} \in \mathbb{N}_0$, $1 < q < \infty$ and $m, m' \in \mathbb{R}$. Furthermore, let $N \in \mathbb{N}_0 \cup \{\infty\}$, $0 \leq \rho \leq 1$ and $X \in \{C^{\tilde{m},s}, W^{\tilde{m},q}_{uloc}\}$. Additionally let $\tilde{m} > n/q$ in case $X = W^{\tilde{m},q}_{uloc}$. Assuming $p \in XS^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, we define the pseudodifferential operator $P = p(x, D_x, x', D_{x'})$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$Pu(x) := \text{Os-}\iiiint e^{-i(y\cdot\xi + y'\xi')} p(x,\xi,x+y,\xi') u(x+y+y') dy dy' đ\xi đ\xi'.$$

The existence of the previous oscillatory integral is a consequence of the properties of such integrals. For more details we refer to [13, Lemma 4.64].

The set $OPXS^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ consists of all non-smooth pseudodifferential operators whose double symbols are in the symbol-class $XS^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$. Moreover for $N \in \mathbb{N}_0 \cup \{\infty\}$ and $m \in \mathbb{R}$ we denote the space $XS^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ as the set of all symbols $p \in XS^{m,0}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ which are independent of ξ'. The pseudodifferential operator $p(x, D_x, x')$ is defined by

$$p(x, D_x, x') := p(x, D_x, x', D_{x'}).$$

Additionally $OPXS^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ is the set of all non-smooth pseudodifferential operators whose double symbols are in $XS^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$.

Due to Lemma 2.4 and the definition of the non-smooth symbol-classes we obtain:

Remark 3.18. Let $1 < q < \infty$, $m, m' \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Moreover, let $N \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Assuming $0 < \tau \leq \tilde{m} - n/q$, $\tau \notin \mathbb{N}$, we have

$$W^{\tilde{m},q}_{uloc} S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N) \subseteq C^{\tau} S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N).$$

By means of Lemma 2.8 we get:

Lemma 3.19. *Let $m \in \mathbb{R}$, $1 < q < \infty$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Moreover, let $0 \leq \rho \leq 1$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. Assuming $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc} S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ bounded, for each $\gamma, \delta \in \mathbb{N}_0^n$ with $|\delta| \leq N$ there is some $C_{\tilde{m},q,\gamma,\delta}$ such that*

$$\sup_{y \in \mathbb{R}^n} \|\partial_y^{\gamma} \partial_{\xi}^{\delta} a(x, \xi, x + y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \leq C_{\tilde{m},q,\gamma,\delta} \langle \xi \rangle^{m - \rho|\delta|} \quad \forall a \in \mathscr{B}, \xi \in \mathbb{R}^n.$$

4. Improvement of the Characterization

In this section we show that the operator P of Theorem 1.2 is even an element of $OPW^{\tilde{m},q}_{uloc} S^{m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1)$. The proof of this statement is essentially the same as the one of Theorem 1.2. We just have to replace the results for pseudodifferential operators with coefficients in a Hölder space with analogous ones for pseudodifferential operators with coefficients in an uniformly local Sobolev space.

The main difficulty originates from the symbol reduction of non-smooth double symbols of the class $XS^{m}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M)$ to non-smooth single symbols with coefficients in X, where $X = W^{\tilde{m},q}_{uloc}$. Both cases, $X = C^{\tilde{m},\tau}$ and $X = W^{\tilde{m},q}_{uloc}$ make use of

$$\sup_{y \in \mathbb{R}^n} \|\partial_y^{\gamma} \partial_{\xi}^{\delta} a(., \xi, . + y)\|_X \leq C \langle \xi \rangle^m,$$

where $a \in XS^{m}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M)$ and $\gamma, \delta \in \mathbb{N}_0^n$ with $|\delta| \leq M$. While this estimate directly follows from the definition of the symbol-class in case $X = C^{\tilde{m},\tau}$, this proof turned out to be rather tedious for case $X = W^{\tilde{m},q}_{uloc}$ in Sect. 2.2. The symbol reduction for uniformly local Sobolev spaces is the subject of Sect. 4.2. But let us begin with proving the existence of a pointwise convergent subsequence of a bounded set of $W^{\tilde{m},q}_{uloc} S^{0}_{0,0}$.

4.1. Pointwise Convergence in $W^{\tilde{m},q}_{uloc} S^0_{0,0}$

Theorem 4.1. *Let $M \in \mathbb{N}_0 \cup \{\infty\}$, $\tilde{m} \in \mathbb{N}_0$, $1 < q < \infty$ and $X = W^{\tilde{m},q}_{uloc}$. Furthermore, let $(p_{\varepsilon})_{\varepsilon > 0} \subseteq XS^{0}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ be bounded. Then there is a subsequence $(p_{\varepsilon_l})_{l \in \mathbb{N}} \subseteq (p_{\varepsilon})_{\varepsilon > 0}$ with $\varepsilon_l \to 0$ for $l \to \infty$ and some function $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that*

i) *$p(x, .) \in C^{M-1}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,*
ii) *$\partial_x^{\beta} \partial_{\xi}^{\alpha} p \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_{\xi})$,*
iii) *$\partial_x^{\beta} \partial_{\xi}^{\alpha} p_{\varepsilon_l} \xrightarrow{l \to \infty} \partial_x^{\beta} \partial_{\xi}^{\alpha} p$ uniformly on each compact set of $\mathbb{R}^n \times \mathbb{R}^n$*

for every $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq M - 1$ and $|\beta| < \tilde{m} - n/q$. Moreover,

$$p \in XS^{0}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M - 1).$$

In case $X = C^{\tilde{m},s}$, $0 < s \leq 1$, the previous theorem holds for $q = \infty$ due to [2, Theorem 4.3].

Proof of Theorem 4.1. Let $\tau \leq \tilde{m} - n/q$ with $\tau \notin \mathbb{N}$. On account of Lemma 3.3 we are able to apply Theorem 4.3 in [2] and get the existence of a subsequence $(p_{\varepsilon_l})_{l\in\mathbb{N}} \subseteq (p_\varepsilon)_{\varepsilon>0}$ with $\varepsilon_l \to 0$ for $l \to \infty$ which fulfills the properties $i)$, $ii)$ and $iii)$ for some $p \in C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. It remains to show $p \in W^{\tilde{m},q}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. By means of $i)$ and $ii)$, we just have to check $\partial^\alpha_\xi p(.,\xi) \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ for all $\xi \in \mathbb{R}^n$ and

$$\|\partial^\alpha_\xi p(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_\alpha \quad \text{for all } \xi \in \mathbb{R}^n,$$

for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq M-1$. Let $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq M-1$ and $\xi \in \mathbb{R}^n$ be arbitrary but fixed. Moreover let $z_j \in \left\{ n^{-1/2} z : z \in \mathbb{Z}^n \right\}$ for each $j \in \mathbb{N}$ such that $\mathbb{R}^n = \bigcup_{j\in\mathbb{N}} B_1(z_j)$. The boundedness of $(p_{\varepsilon_l})_{l\in\mathbb{N}}$ as a subset of $W^{\tilde{m},q}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ yields

$$\|\partial^\alpha_\xi p_{\varepsilon_l}(.,\xi)\|_{H^{\tilde{m}}_q(B_1(z_j))} \leq \|\partial^\alpha_\xi p_{\varepsilon_l}(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_\alpha, \tag{4.1}$$

for all $j, l \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$. Let $j \in \mathbb{N}$ be arbitrary but fixed. Because of the reflexivity of $H^{\tilde{m}}_q(B_1(z_j))$ there is a subsequence $(\partial^\alpha_\xi p_{\varepsilon_{l_m}})_{m\in\mathbb{N}}$ of $(\partial^\alpha_\xi p_{\varepsilon_l})_{l\in\mathbb{N}}$ such that

$$\partial^\alpha_\xi p_{\varepsilon_{l_m}}(.,\xi) \rightharpoonup q_{\alpha,\xi,j} \quad \text{in } H^{\tilde{m}}_q(B_1(z_j))$$

for $m \to \infty$. The compact embedding $H^{\tilde{m}}_q(B_1(z_j)) \hookrightarrow\hookrightarrow C^0(\overline{B_1(z_j)})$ even gives us

$$\partial^\alpha_\xi p_{\varepsilon_{l_m}}(.,\xi) \xrightarrow{m\to\infty} q_{\alpha,\xi,j} \quad \text{in } C^0(\overline{B_1(z_j)}).$$

Together with iii) the uniqueness of the limit provides $q_{\alpha,\xi,j} = \partial^\alpha_\xi p(.,\xi)$. Consequently every arbitrary weak convergent subsequence of $(\partial^\alpha_\xi p_{\varepsilon_l})_{l\in\mathbb{N}}$ has the same weak limit. Hence an application of [14, Chapter 3, Lemma 0.3] implies

$$\partial^\alpha_\xi p_{\varepsilon_l}(.,\xi) \rightharpoonup \partial^\alpha_\xi p(.,\xi) \quad \text{in } H^{\tilde{m}}_q(B_1(z_j))$$

for $l \to \infty$. Using the previous weak convergence and (4.1), we get

$$\|\partial^\alpha_\xi p(.,\xi)\|_{H^{\tilde{m}}_q(B_1(z_j))} \leq \liminf_{l\to\infty} \|\partial^\alpha_\xi p_{\varepsilon_l}(.,\xi)\|_{H^{\tilde{m}}_q(B_1(z_j))} \leq C_\alpha \ \forall j \in \mathbb{N} \tag{4.2}$$

for all $\xi \in \mathbb{R}^n$. Since there is an $N \in \mathbb{N}$, independent of $x_0 \in \mathbb{R}^n$, with $B_1(x_0) \subseteq \bigcup_{k=1}^N B_1(z_{j_k})$ for $j_1, \ldots, j_N \in \mathbb{N}$, we get together with (4.2)

$$\sup_{\xi\in\mathbb{R}^n} \|\partial^\alpha_\xi p(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} = \sup_{\xi\in\mathbb{R}^n} \sup_{x_0\in\mathbb{R}^n} \|\partial^\alpha_\xi p(.,\xi)\|_{H^{\tilde{m}}_q(B_1(x_0))} \leq C_\alpha. \qquad \square$$

4.2. Symbol Reduction of Double Symbols in $W^{\tilde{m},q}_{uloc}S^m_{0,0}$

The goal of this section is to prove

Theorem 4.2. *Let $1 < q < \infty$, $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > n/q$, $m \in \mathbb{R}$ and $X = W^{\tilde{m},q}_{uloc}$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ with $N > n$. We define $\tilde{N} := N - (n+1)$. Furthermore, let $\mathscr{B} \subseteq XS^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ be bounded. For $a \in \mathscr{B}$ we define $a_L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by*

$$a_L(x,\xi) := Os\text{-}\iint e^{-iy\cdot\eta} a(x, \eta+\xi, x+y)dy đ\eta$$

for all $x, \xi \in \mathbb{R}^n$. Then $\{a_L : a \in \mathscr{B}\} \subseteq XS^m_{0,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; \tilde{N})$ is bounded and we have

$$a(x, D_x, x')u = a_L(x, D_x)u \quad \text{for all } a \in \mathscr{B} \text{ and } u \in \mathcal{S}(\mathbb{R}^n). \tag{4.3}$$

In case $X = C^{\tilde{m},s}$, $0 < s \leq 1$, the previous theorem holds due to [2, Theorem 4.13 and Theorem 4.15]. For the proof of the general case of Theorem 4.2 the next proposition, cf. [2, Proposition 4.6], is needed.

Proposition 4.3. *Let $m \in \mathbb{R}$ and X be a Banach space with $X \hookrightarrow L^\infty(\mathbb{R}^n)$. Considering an $l_0 \in \mathbb{N}_0$ with $-l_0 < -n$, let $\mathscr{B}$ be a set of functions $r : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ which are smooth with respect to the fourth variable such that for all $l \in \mathbb{N}_0$ there is some $C_l > 0$ such that*

$$\|\langle D_y\rangle^{2l} r(.,\xi,\eta,y)\|_X \leq C_l \langle y\rangle^{-l_0}\langle \xi+\eta\rangle^m \quad \forall \xi,\eta,y \in \mathbb{R}^n, r \in \mathscr{B}. \tag{4.4}$$

Then $\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy \in L^1(\mathbb{R}^n_\eta)$ for all $x, \xi \in \mathbb{R}^n$. If we define

$$I(x,\xi) := \int \left[\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy\right] đ\eta$$

for $x, \xi \in \mathbb{R}^n$ and $r \in \mathscr{B}$ we have for some $C > 0$

$$\|I(.,\xi)\|_X \leq C\langle\xi\rangle^m \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } r \in \mathscr{B}.$$

In the same manner as in the proof of Lemma 4.9 of [2] the previous result enables us to prove the next lemma:

Lemma 4.4. *Let $1 < q < \infty$, $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$ and $m \in \mathbb{R}$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ with $n < N$. We define $\tilde{N} := N - (n+1)$. Moreover, let $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ be bounded. For $a \in \mathscr{B}$ we define $a_L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ as in Theorem 4.2. Then we have for each $\gamma \in \mathbb{N}^n_0$ with $|\gamma| \leq \tilde{N}$*

$$\|\partial^\gamma_\xi a_L(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_\gamma\langle\xi\rangle^m \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } a \in \mathscr{B} \tag{4.5}$$

for some $C_\gamma > 0$.

Proof. Since Theorem 4.2 already holds if X is a Hölder space and with Lemma 3.3 at hand, it remains to show (4.5). Due to $a \in \mathscr{A}^{m,N}_0(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$,

$N-\tilde{N}=k>n$ and Lemma 3.19 we can apply Theorem 2.9 and Proposition 4.3 and get for each $x,\xi\in\mathbb{R}^n$, $a\in\mathscr{B}$ and $l_0\in\mathbb{N}_0$ with $n<l_0\le N$:

$$\begin{aligned}\|a_L(.,\xi)\|_{W^{\tilde{m},q}_{uloc}} &= \left\|\mathrm{Os}\text{-}\iint e^{-iy\cdot\eta}A^{l_0}(D_\eta,y)a(x,\xi+\eta,x+y)dy\,đ\eta\right\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)}\\ &= \left\|\iint e^{-iy\cdot\eta}A^{l_0}(D_\eta,y)a(x,\xi+\eta,x+y)dy\,đ\eta\right\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)}\\ &\le C\langle\xi\rangle^m \quad \text{for all } \xi\in\mathbb{R}^n \text{ and } a\in\mathscr{B}.\end{aligned}$$

For more details concerning the second equality we refer to Proposition 4.8 of [2]. Thus the theorem holds for $\gamma=0$. Now let $\gamma\in\mathbb{N}_0^n$ with $|\gamma|\le\tilde{N}$. Because of $N-\tilde{N}=2k>n$ for some $k\in\mathbb{N}_0$ and $a\in\mathscr{A}_0^{m,N}(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)$, we can apply Theorem 2.10 and get

$$\partial_\xi^\gamma a_L(x,\xi)=\mathrm{Os}\text{-}\iint e^{-iy\cdot\eta}\partial_\xi^\gamma a(x,\eta+\xi,x+y)dy\,đ\eta.$$

The first case, applied on the set $\{\partial_\xi^\gamma a : a\in\mathscr{B}\}$, yields (4.5). □

Proof of Theorem 4.2. On account of Remark 3.18, case X is a Höder space and Lemma 4.4 the claim holds. □

4.3. Characterization of Non-Smooth Pseudodifferential Operators

With all the work done in the last subsections we are able to improve Theorem 1.2.

Theorem 4.5. *Let $m\in\mathbb{R}$, $1<q<\infty$, $\rho\in\{0,1\}$, $\tilde{m}\in\mathbb{N}_0$ with $\tilde{m}>n/q$. Additionally let $M\in\mathbb{N}_0\cup\{\infty\}$ and define $\tilde{M}:=M-(n+1)$. Considering $P\in\mathcal{A}^{m,M}_{\rho,0}(\tilde{m},q)$ and $\tilde{M}\ge1$ we have*

$$P\in OPW^{\tilde{m},q}_{uloc}S^m_{\rho,0}(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M}-1)\cap\mathscr{L}(H^m_q(\mathbb{R}^n),L^q(\mathbb{R}^n)).$$

Proof. Let us assume $\rho=0$ first. The proof of the claim is essentially the same as that one of Theorem 1.2, cf. Subsection 4.4 in [2]. In case $m=0$ we just have to replace the results for pseudodifferential operators with coefficients in Hölder spaces with corresponding ones for pseudodifferential operators with coefficients in uniformly local Sobolev spaces.

For verifying the case $\rho=1$ let $\tau\in(0,\tilde{m}-n/q]$, $\tau\notin\mathbb{N}$. An application of Theorem 1.2 yields

$$P\in OPC^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M}-1)\cap\mathscr{L}(H^m_q(\mathbb{R}^n),L^q(\mathbb{R}^n)).$$

Let $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\le\tilde{M}-1$ be arbitrary. Due to the proof of Theorem 1.2, cf. [2, Theorem 4.22] we know that $\mathrm{ad}(-ix)^\alpha P\in\mathcal{A}^{m-|\alpha|,M-|\alpha|}_{1,0}(\tilde{m},q)$. Hence an application of Theorem 4.5 provides

$$\mathrm{ad}(-ix)^\alpha P\in OPW^{\tilde{m},q}_{uloc}S^{m-|\alpha|}_{0,0}(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M}-|\alpha|-1).$$

On account of the proof of Theorem 1.2 the symbol of $\mathrm{ad}(-ix)^\alpha P$ is $\partial_\xi^\alpha p(x,\xi)$. This implies $P\in OPW^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M}-1)$. □

If we use Theorem 4.5 instead of Theorem 1.2 in the proof of Theorem 5.1 in [2], we can improve the result for the symbol composition of non-smooth pseudodifferential operators:

Theorem 4.6. *Let $m_i \in \mathbb{R}$, $M_i \in \mathbb{N} \cup \{\infty\}$ and $\rho_i \in \{0,1\}$ for $i \in \{1,2\}$. Moreover, let $0 < \tau_i < 1$ and $\tilde{m}_i \in \mathbb{N}_0$ be such that $\tau_i + \tilde{m}_i > (1-\rho_i)n/2 =: k_i$ for $i \in \{1,2\}$. We define $\rho := \min\{\rho_1; \rho_2\}$ and $m := m_1 + m_2 + k_1 + k_2$. Additionally let $\tilde{m}, M \in \mathbb{N}$ and $1 < q < \infty$ be such that*

i) $M \leq \min\{M_i - \max\{n/q; n/2\} : i \in \{1,2\}\}$,
ii) $n/q < \tilde{m} \leq \min\{\tilde{m}_1; \tilde{m}_2\}$,
iii) $\tilde{m} < \tilde{m}_2 + \tau_2 - m_1 - k_1$,
iv) $\rho M + \tilde{m} < \tilde{m}_2 + \tau_2 + m_1 + k_1$,
v) $\tilde{M} \geq 1$, *where* $\tilde{M} := M - (n+1)$,
vi) $q = 2$ *if* $(\rho_1, \rho_2) \neq (1,1)$.

Considering two symbols $p_i \in C^{\tilde{m}_i, \tau_i} S^{m_i}_{\rho_i, 0}(\mathbb{R}^n \times \mathbb{R}^n; M_i)$, $i \in \{1,2\}$, we obtain

$$p_1(x, D_x) p_2(x, D_x) \in OPW^{\tilde{m},q}_{uloc} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1).$$

We are even able to improve this result for non-smooth pseudodifferential operators with coefficients in the uniformly local Sobolev spaces:

Theorem 4.7. *Let $m_i \in \mathbb{R}$ and $1 < q_i < \infty$ for $i \in \{1,2\}$. Additionally let $\tilde{m}_i \in \mathbb{N}_0$ with $\tilde{m}_i > n/q_i$ for $i \in \{1,2\}$. We define $m := m_1 + m_2$. Moreover let $\tilde{m}, M \in \mathbb{N}$ and $1 < q < \infty$ be such that*

i) $n/q < \tilde{m} < \min\{\tilde{m}_1 - n/q_1; \tilde{m}_2 - n/q_2\}$,
ii) $\tilde{m} \leq \tilde{m}_2 - n(1/q_2 - 1/q)^+ - m_1$,
iii) $M + \tilde{m} < \tilde{m}_2 - n/q_2 + m_1$,
iv) $\tilde{M} \geq 1$, *where* $\tilde{M} := M - (n+1)$.

Considering two symbols $p_i \in W^{\tilde{m}_i, q_i}_{uloc} S^{m_i}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $i \in \{1,2\}$, we obtain

$$p_1(x, D_x) p_2(x, D_x) \in OPW^{\tilde{m},q}_{uloc} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1).$$

Proof. Proceed as in the proof of Theorem 5.1 in [2]. We just have to replace Theorem 3.7 with Theorem 3.12 and use Remark 3.4. □

Analogous to the statement of Theorem 4.6 it is possible to verify a similar result for the composition of two pseudodifferential operators of the symbol-class $H^{\tilde{m}}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ by using Theorem 3.10 instead of Theorem 3.7.

We also could consider the composition of two non-smooth pseudodifferential operators whose coefficients are either in a Hölder space, in a Bessel potential space or in an uniformly local Sobolev spaces, however in different spaces. Adapting the proof of Theorem 4.6 one could obtain similar results for these cases.

5. Spectral Invariance

5.1. The Inverse of a Pseudodifferential Operator in the Symbol-Class $C^\tau S^0_{0,0}$

In the present subsection we prove the following theorem:

Theorem 5.1. *Let $\tilde{m} \in \mathbb{N}_0$ and $0 < \tau < 1$. We assume*

$$\hat{m} := \max\{k \in \mathbb{N}_0 : \tilde{m} + \tau - k > n/2\} > n/2.$$

For every $p \in C^{\tilde{m},\tau} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $p(x, D_x)^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$ we get

$$p(x, D_x)^{-1} \in OPW^{\hat{m},2}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n) \subseteq OPC^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$$

for all $s \in (0, \hat{m} - n/2]$ with $s \notin \mathbb{N}$.

Ueberberg proved a similar result for the smooth case, cf. [18, Theorem 4.3]:

Theorem 5.2. *Let $1 < q < \infty$ and $0 \leq \delta \leq \rho \leq 1$ with $\delta < 1$.*

i) *Considering a symbol $p \in S^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ where $p(x, D_x)^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$ we obtain $p(x, D_x)^{-1} \in OPS^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$.*
ii) *Assuming a symbol $p \in S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ where $p(x, D_x)^{-1} \in \mathscr{L}(L^q(\mathbb{R}^n))$ we get $p(x, D_x)^{-1} \in OPS^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$.*

In order to verify Theorem 5.1, we use the main idea of the proof in the smooth case: We want to apply the characterization of pseudodifferential operators. Thus we just have to show the boundedness of certain iterated commutators of $p(x, D_x)^{-1}$. Since we already know that the iterated commutators of $p(x, D_x)$ have these mapping properties, we try to write the iterated commutators of $p(x, D_x)^{-1}$ as a sum and compositions of $p(x, D_x)^{-1}$ and the iterated commutators of $p(x, D_x)$. Unfortunately, non-smooth pseudodifferential operators are in general not bounded as operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ like the smooth ones. Therefore we have to prove the formal identities for the iterated commutators rigorously.

Remark 5.3. (Formal identities for the iterated commutators)
Let $m, s \in \mathbb{R}$, $1 < q < \infty$ and $M, \tilde{m} \in \mathbb{N}_0$ with $\tilde{m} + M \geq 1$. We assume that $P \in \mathscr{L}(H^{s+m}_q, H^s_q)$ with $P^{-1} \in \mathscr{L}(H^s_q, H^{s+m}_q)$ and

$$\operatorname{ad}(-ix)^{\alpha_1} \operatorname{ad}(D_x)^{\beta_1} \dots \operatorname{ad}(-ix)^{\alpha_l} \operatorname{ad}(D_x)^{\beta_l} P \in \mathscr{L}(H^{s+m}_q, H^s_q)$$

for all $l \in \mathbb{N}$, $\alpha_1, \dots, \alpha_l \in \mathbb{N}^n_0$ and $\beta_1, \dots, \beta_l \in \mathbb{N}^n_0$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1, \dots, l\}$, $|\alpha_1| + \dots + |\alpha_l| \leq M$ and $|\beta_1| + \dots + |\beta_l| \leq \tilde{m}$. For $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha + \beta| = 1$ we have $\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta P^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. We consider $|\beta| = 0$ and $\alpha = e_j$ for an arbitrary $j \in \{1, \dots, n\}$ first. On account of $\operatorname{ad}(-ix_j)P \in \mathscr{L}(H^{s+m}_q, H^s_q)$, we know that

$$\operatorname{ad}(-ix_j)Pu = -ix_j Pu + P(ix_j u) \in H^s_q(\mathbb{R}^n) \quad \text{for all } u \in \mathcal{S}(\mathbb{R}^n). \tag{5.1}$$

If $u \in \mathcal{S}(\mathbb{R}^n) \subseteq H^{m+s}_q(\mathbb{R}^n)$, we obtain $P(ix_j u) \in H^s_q(\mathbb{R}^n)$. Together with (5.1) this implies

$$-ix_j Pu \in H^s_q(\mathbb{R}^n) \quad \text{for all } u \in \mathcal{S}(\mathbb{R}^n). \tag{5.2}$$

Now we define $\mathscr{D} := \{Pu : u \in \mathcal{S}(\mathbb{R}^n)\} \subseteq H^s_q(\mathbb{R}^n)$. To show the density of $\mathscr{D}$ in $H^s_q(\mathbb{R}^n)$ we choose an arbitrary $v \in H^s_q(\mathbb{R}^n)$. Due to $P^{-1} \in \mathscr{L}(H^s_q, H^{s+m}_q)$ we have $u := P^{-1}v \in H^{s+m}_q(\mathbb{R}^n)$ and therefore $v = Pu$. Considering a sequence $(u_j)_{j\in\mathbb{N}_0} \subseteq \mathcal{S}(\mathbb{R}^n)$, which converges to u in $H^{s+m}_q(\mathbb{R}^n)$, we define $v_j := Pu_j$ for each $j \in \mathbb{N}_0$. By means of $P \in \mathscr{L}(H^{s+m}_q, H^s_q)$ the sequence $(v_j)_{j\in\mathbb{N}}$ converges to v. This implies the density of $\mathscr{D}$ in $H^s_q(\mathbb{R}^n)$. Next we define $Q : \mathscr{D} \to H^{s+m}_q(\mathbb{R}^n)$ by $Qu := -ix_jP^{-1}u + P^{-1}(ix_ju)$ for all $u \in \mathscr{D}$. Due to (5.2) Q is well-defined and we obtain for all $u \in \mathcal{S}(\mathbb{R}^n)$:

$$Q(Pu) = -ix_ju + P^{-1}(ix_jPu) = -P^{-1}[\mathrm{ad}(-ix_j)P]u. \tag{5.3}$$

Because of $\mathrm{ad}(-ix_j)P \in \mathscr{L}(H^{s+m}_q, H^s_q)$ and $P^{-1} \in \mathscr{L}(H^s_q, H^{s+m}_q)$ we get $\|Q(Pu)\|_{H^{s+m}_q} \leq C\|Pu\|_{H^s_q}$ for all $u \in H^s_q(\mathbb{R}^n)$. Due to the density of $\mathscr{D}$ in $H^s_q(\mathbb{R}^n)$ this implies $Q \in \mathscr{L}(H^s_q, H^{s+m}_q)$. As a direct consequence we obtain

$$\mathrm{ad}(-ix_j)P^{-1} \in \mathscr{L}(H^s_q, H^{s+m}_q)$$

since $Qu = \mathrm{ad}(-ix_j)P^{-1}u$ for all $u \in \mathcal{S}(\mathbb{R}^n)$. Together with $\mathscr{D} \subseteq H^s_q(\mathbb{R}^n)$ and (5.3) we get

$$[\mathrm{ad}(-ix_j)P^{-1}]Pu = -P^{-1}[\mathrm{ad}(-ix_j)P]u \quad \text{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

Due to $[\mathrm{ad}(-ix_j)P^{-1}]P \in \mathscr{L}(H^{s+m}_q)$ and $P^{-1}[\mathrm{ad}(-ix_j)P] \in \mathscr{L}(H^{s+m}_q)$ the previous equality holds for all $u \in H^{s+m}_q(\mathbb{R}^n)$. Since $P \in \mathscr{L}(H^{s+m}_q; H^s_q)$ is surjective, we have for all $v \in H^s_q(\mathbb{R}^n)$:

$$\begin{aligned}\mathrm{ad}(-ix)^\alpha\,\mathrm{ad}(D_x)^\beta P^{-1}v = [\mathrm{ad}(-ix_j)P^{-1}]v &= -P^{-1}[\mathrm{ad}(-ix_j)P]P^{-1}v\\ &= -P^{-1}[\mathrm{ad}(-ix)^\alpha\,\mathrm{ad}(D_x)^\beta P]P^{-1}v.\end{aligned} \tag{5.4}$$

In case $\beta = e_j$, $j \in \{1,\ldots,n\}$ and $|\alpha| = 0$ we get equality (5.4) for all $u \in \mathcal{S}(\mathbb{R}^n)$ in the same way as before. Moreover, let $l \in \mathbb{N}$, $\alpha_1,\ldots,\alpha_l \in \mathbb{N}^n_0$ and $\beta_1,\ldots,\beta_l \in \mathbb{N}^n_0$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1,\ldots,l\}$, $|\alpha_1| + \cdots + |\alpha_l| \leq M$ and $|\beta_1| + \cdots + |\beta_l| \leq \tilde{m}$. Denoting $\alpha := \alpha_1 + \cdots + \alpha_l$ and $\beta := \beta_1 + \cdots + \beta_l$ we get by mathematical induction with respect to l:

$$\mathrm{ad}(-ix)^{\alpha_1}\,\mathrm{ad}(D_x)^{\beta_1}\ldots\mathrm{ad}(-ix)^{\alpha_l}\,\mathrm{ad}(D_x)^{\beta_l}P^{-1} = \sum_{\substack{(\alpha^1_1+\ldots+\alpha^1_l)+\cdots+(\alpha^l_1+\ldots+\alpha^l_l)=\alpha\\ (\beta^1_1+\ldots+\beta^1_l)+\cdots+(\beta^l_1+\ldots+\beta^l_l)=\beta}} R_{\alpha^1_1,\ldots,\alpha^l_l,\beta^1_1,\ldots,\beta^l_l}$$

where

$$\begin{aligned}R_{\alpha^1_1,\ldots,\alpha^l_l,\beta^1_1,\ldots,\beta^l_l} := {}& C_{\alpha^1_1,\ldots,\alpha^l_l,\beta^1_1,\ldots,\beta^l_l}P^{-1}\\ &\circ\left[\mathrm{ad}(-ix)^{\alpha^1_l}\,\mathrm{ad}(D_x)^{\beta^1_l}\ldots\mathrm{ad}(-ix)^{\alpha^1_1}\,\mathrm{ad}(D_x)^{\beta^1_1}P\right]P^{-1}\\ &\circ\cdots\circ\left[\mathrm{ad}(-ix)^{\alpha^l_l}\,\mathrm{ad}(D_x)^{\beta^l_l}\ldots\mathrm{ad}(-ix)^{\alpha^l_1}\,\mathrm{ad}(D_x)^{\beta^l_1}P\right]P^{-1}.\end{aligned}$$

Proof of Theorem 5.1. Let $l \in \mathbb{N}_0$, $\alpha_1,\ldots,\alpha_l \in \mathbb{N}^n_0$ and $\beta_1,\ldots,\beta_l \in \mathbb{N}^n_0$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1,\ldots,n\}$ and $|\beta_1 + \cdots + \beta_l| \leq \hat{m}$ be arbitrary. On account of $p(x,D_x) \in \mathcal{A}^0_{0,0}(\hat{m},2)$, which holds by means of Theorem

3.8, and $P^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$ we can apply Remark 5.3 and get if we define $P := p(x, D_x)$:

$$\mathrm{ad}(-ix)^{\alpha_1} \mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\alpha_l} \mathrm{ad}(D_x)^{\beta_l} P^{-1} = \sum_{\substack{(\alpha_1^1+\ldots+\alpha_l^1)+\cdots+(\alpha_1^l+\ldots+\alpha_l^l)=\alpha \\ (\beta_1^1+\ldots+\beta_l^1)+\cdots+(\beta_1^l+\ldots+\beta_l^l)=\beta}} R_{\alpha_1^1,\ldots,\alpha_l^l,\beta_1^1,\ldots,\beta_l^l}$$

where $R_{\alpha_1^1,\ldots,\alpha_l^l,\beta_1^1,\ldots,\beta_l^l}$ are defined as in Remark 5.3. Since $P \in \mathcal{A}^0_{0,0}(\hat{m}, 2)$ and $P^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$, we obtain $P^{-1} \in \mathcal{A}^0_{0,0}(\hat{m}, 2)$. Considering an s with $0 < s \leq \hat{m} - n/2$ and $s \notin \mathbb{N}$, Theorem 4.5 and Lemma 3.3 yield the claim. $\square$

In the same way, using Theorem 3.9 instead of Theorem 3.8, we can show:

Lemma 5.4. *Let $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/2$ and $0 < \tau < 1$. For every non-smooth symbol $p \in C^{\tilde{m},\tau} S^{-n/2}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $p(x, D_x)^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$ we get*

$$p(x, D_x)^{-1} \in OPW^{\tilde{m},2}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n) \subseteq OPC^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$$

for all $s \in (0, \tilde{m} - n/2]$ with $s \notin \mathbb{N}$.

5.2. Properties of Difference Quotients

Our next aim is to prove the spectral invariance for pseudodifferential operators $P \in OPC^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $\tau > 0$. The proof is again based on the formal identities for the iterated commutators of P^{-1}, cf. Remark 5.3. In this case $\mathrm{ad}(-ix)^\alpha \mathrm{ad}(D_x)^\beta P$, $|\alpha| \neq 0$ are pseudodifferential operators of negative order $-|\alpha|$. Hence the order of the Bessel potential space increases by applying $\mathrm{ad}(-ix)^\alpha \mathrm{ad}(D_x)^\beta P$, $|\alpha| \neq 0$. Therefore $P^{-1} \in \mathscr{L}(L^q(\mathbb{R}^n))$ is not sufficient. We even need $P^{-1} \in \mathscr{L}(H^{-s}_q(\mathbb{R}^n))$ for certain $s \in \mathbb{N}_0$. As we always try to restrict the assumptions as much as possible, we use the tools of difference quotients in order to get $P^{-1} \in \mathscr{L}(H^{-s}_q(\mathbb{R}^n))$ if $P^{-1} \in \mathscr{L}(L^q(\mathbb{R}^n))$ is assumed.

Definition 5.5. Let $h \in \mathbb{R}\backslash\{0\}$ and $j \in \{1, \ldots, n\}$. For $u \in \mathcal{S}'(\mathbb{R}^n)$ define the *difference quotient* of u by

$$\partial^h_{x_j} u := h^{-1}\{u(. + he_j) - u\}.$$

Difference quotients have the following useful properties:

Lemma 5.6. *Let $m \in \mathbb{R}$, $\tilde{m} \in \mathbb{N}$, $0 < \tau < 1$ and $M \in \mathbb{N}_0 \cup \{\infty\}$. Considering $p \in C^{\tilde{m},\tau} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$, we get for all $j \in \{1, \ldots, n\}$:*

i) *$\left\{\partial^h_{x_j} p(x, \xi) : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq C^{\tilde{m}-1,\tau} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is bounded,*

ii) *$[\partial^h_{x_j}, p(x, D_x)]u(x) = \left[\left(\partial^{-h}_{x_j} p\right)(x, D_x)u\right](x + he_j)$ for all $u \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $h \in \mathbb{R}\backslash\{0\}$.*

iii) *Additionally let $M \in \mathbb{N}_0 \cup \{\infty\}$ with $M > n/2$ for $q \geq 2$ and $M > n/q$ else and $s \in \mathbb{R}$ with $|s| < \tilde{m} - 1 + \tau$. Then we have for some $C > 0$:*

$$\|[\partial^h_{x_j}, p(x, D_x)]u\|_{H^s_q} \leq C\|u\|_{H^{s+m}_q} \quad \text{for all } u \in H^{s+m}_q(\mathbb{R}^n), h \in \mathbb{R}\backslash\{0\}.$$

Proof. The fundamental theorem of calculus provides claim i). An elementary calculation, using $\frac{e^{ihe_j\cdot\xi}-1}{h}\hat{u}(\xi) = \widehat{\partial_{x_j}^h u}(\xi)$ shows ii). Finally iii) follows from i), Theorem 3.7 and the density of $\mathcal{S}(\mathbb{R}^n)$ in $H_q^{s+m}(\mathbb{R}^n)$. □

Theorem 5.7. (Difference quotients and weak derivatives)
Let $1 < p < \infty$ and $s \in \mathbb{R}$.

i) *There exists a constant C such that, for all $u \in H_p^{s+1}(\mathbb{R}^n)$, all $h \in \mathbb{R}\backslash\{0\}$ and all $j \in \{1, \dots, n\}$,*
$$\|\partial_{x_j}^h u\|_{H_p^s} \le C\|\partial_{x_j} u\|_{H_p^s}.$$

ii) *Let $u \in H_p^s(\mathbb{R}^n)$ and assume that*
$$\|\partial_{x_j}^h u\|_{H_p^s} \le C \quad \textit{for all } j \in \{1, \dots, n\} \textit{ and } h \in \mathbb{R}\backslash\{0\}.$$
Then $u \in H_p^{s+1}(\mathbb{R}^n)$ and $\|\partial_{x_j} u\|_{H_p^s} \le C$.

Note that assertion ii) is false for $p = 1$ while i) also holds for $p = 1$.

Proof. The proof of i) in case $s = 0$ is essentially the same as that one of Theorem 5.8.3 in [6]. Using Lemma 5.6, the general case then holds, since
$$\|\partial_{x_j}^h u\|_{H_p^s} = \|\partial_{x_j}^h \langle D_x\rangle^s u\|_{L^p} \le C\|\partial_{x_j}\langle D_x\rangle^s u\|_{L^p} = C\|\partial_{x_j} u\|_{H_p^s}$$
for all $h \in \mathbb{R}\backslash\{0\}$ and $u \in H_p^{s+1}(\mathbb{R}^n)$. Similarly to ii) is a consequence of case $s = 0$ and Lemma 5.6. □

The previous theorem allows us to verify the following proposition:

Proposition 5.8. *Let $k \in \mathbb{N}_0$, $r \in \mathbb{R}$ and $1 < q < \infty$. Moreover, let P be an operator, which fulfills for all $s \in \{r, r+1, \dots, r+k\}$ the properties*

i) $P \in \mathscr{L}(H_q^s, H_q^s)$,
ii) $P \in \mathscr{L}(H_q^{r+k+1}, H_q^{r+k+1})$,
iii) $\{[P, \partial_{x_j}^h] : h \in \mathbb{R}\backslash\{0\}\} \subseteq \mathscr{L}(H_q^s, H_q^s)$ *is bounded for all* $j \in \{1, \dots, n\}$,
iv) $P^{-1} \in \mathscr{L}(H_q^r, H_q^r)$.

Then $P^{-1} \in \mathscr{L}(H_q^s, H_q^s)$ for each $s \in \{r, r+1, \dots, r+k+1\}$.

Proof. We prove the claim by mathematical induction with respect to s. In case $s = r$ there is nothing to show. For $s \in \{r, r+1, \dots, r+k\}$ we choose an arbitrary $j \in \{1, \dots, n\}$ and $f \in H_q^{s+1}(\mathbb{R}^n) \subseteq H_q^s(\mathbb{R}^n)$. Due to the induction hypothesis there is a $u \in H_q^s(\mathbb{R}^n)$ with $u = P^{-1}f$. Since $P \in \mathscr{L}(H_q^s, H_q^s)$, we get $Pu \in H_q^s(\mathbb{R}^n)$ and consequently $\partial_{x_j}^h(Pu) \in H_q^s(\mathbb{R}^n)$. Similarly we get $P(\partial_{x_j}^h u) \in H_q^s(\mathbb{R}^n)$. An application of P^{-1} to $P(\partial_{x_j}^h u) = [P, \partial_{x_j}^h]u + \partial_{x_j}^h(Pu)$, the induction hypothesis, the assumptions and Theorem 5.7 i) yield
$$\begin{aligned}\|\partial_{x_j}^h u\|_{H_q^s} &= \|P^{-1}\{[P, \partial_{x_j}^h]u + \partial_{x_j}^h(Pu)\}\|_{H_q^s} \le C\|[P, \partial_{x_j}^h]u\|_{H_q^s} + C\|\partial_{x_j}^h f\|_{H_q^s} \\ &\le C\|u\|_{H_q^s} + C\|\partial_{x_j} f\|_{H_q^s} \le C \quad \text{for all } h \in \mathbb{R}\backslash\{0\}, u \in H_q^s(\mathbb{R}^n).\end{aligned}$$
Therefore Theorem 5.7 ii) provides $u \in H_q^{s+1}(\mathbb{R}^n)$ which proves the surjectivity of the linear, bounded and injective operator $P : H_q^{s+1}(\mathbb{R}^n) \to H_q^{s+1}(\mathbb{R}^n)$. Then P^{-1} is an element of $\mathscr{L}(H_q^{s+1}, H_q^{s+1})$ by means of the bounded inverse theorem. □

By means of the previous proposition we obtain the central result of this subsection:

Theorem 5.9. *Let* $1 < q < \infty$, $0 < \tau < 1$ *and* $\tilde{m} \in \mathbb{N}$. *Additionally let* $N \in \mathbb{N}_0 \cup \{\infty\}$ *with* $N > n/2$ *for* $q \geq 2$ *and* $N > n/q$ *else. We define* $k := \max\{l \in \mathbb{N}_0 : r + l < \tilde{m} + \tau\}$ *for one* $r \in \mathbb{R}$ *with* $|r| < \tilde{m} + \tau$. *Considering* $p \in C^{\tilde{m},\tau} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$, *where* $p(x, D_x)^{-1} \in \mathscr{L}(H^r_q, H^r_q)$, *we obtain*

$$p(x, D_x)^{-1} \in \mathscr{L}(H^s_q, H^s_q) \quad \text{for all } s \in [-r - k, r + k]. \tag{5.5}$$

Proof. On account of Theorem 3.7 and Lemma 5.6 we can apply Proposition 5.8 and get the claim for all $s \in \{r, \ldots, r + k\}$. With $(\partial^h_{x_j})^* = -\partial^{-h}_{x_j}$ at hand we have $[P^*, \partial^h_{x_j}] = [P, \partial^{-h}_{x_j}]^*$. An application of Proposition 5.8 to P^* provides the claim for all $s \in \{-r - k, \ldots, r - 1\}$. Then the claim follows for all $s \in [-r - k, r + k]$ by means of interpolation. □

5.3. Spectral Invariance of Pseudodifferential Operators in $C^{\tilde{m},\tau} S^0_{1,0}$

Theorem 5.10. *Let* $1 < q_0 < \infty$ *and* $0 < \tau < 1$. *Moreover, let* $\tilde{m}, \hat{m} \in \mathbb{N}_0$ *with* $\tilde{m} \geq \hat{m} > n/q_0$ *and* $M \in \mathbb{N}_0$ *with* $n < M \leq \tilde{m} - \hat{m}$. *We define* $\tilde{M} := M - (n + 1)$. *Furthermore, let* $N \in \mathbb{N} \cup \{\infty\}$ *with* $N - M > n/2$ *if* $q_0 \geq 2$ *and* $N - M > n/q_0$ *else. Considering* $p \in C^{\tilde{m},\tau} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$, *where* $p(x, D_x)^{-1} \in \mathscr{L}(H^r_{q_0}, H^r_{q_0})$ *for one* $|r| < \tilde{m} + \tau$, *we get*

$$p(x, D_x)^{-1} \in OPW^{\hat{m},q_0}_{uloc} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1).$$

In case $\tilde{M} - 1 > n/\tilde{q}$ *for some* $1 < \tilde{q} \leq 2$, *we even have*

$$p(x, D_x)^{-1} \in \mathscr{L}(L^q, L^q) \quad \text{for all } q \in [\tilde{q}; \infty) \cup \{q_0\}.$$

Proof. An application of Theorem 5.9 provides the boundedness of

$$p(x, D_x)^{-1} \in \mathscr{L}(H^{-s}_{q_0}, H^{-s}_{q_0}) \quad \text{for all } s \in \{0, \ldots, M\}. \tag{5.6}$$

Let $l \in \mathbb{N}_0$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}^n_0$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}^n_0$ with $|\alpha_j| + |\beta_j| = 1$ for all $j \in \{1, \ldots, l\}$, $|\alpha| \leq M$ and $|\beta| \leq \hat{m}$ where $\alpha := \alpha_1 + \cdots + \alpha_l$ and $\beta := \beta_1 + \cdots + \beta_l$. Then Remark 3.4 and Theorem 3.7 yield for all $s \in \{0, \ldots, M - |\alpha|\}$:

$$\mathrm{ad}(-ix)^{\alpha_l} \mathrm{ad}(D_x)^{\beta_l} \ldots \mathrm{ad}(-ix)^{\alpha_1} \mathrm{ad}(D_x)^{\beta_1} p(x, D_x) \in \mathscr{L}(H^{-s-|\alpha|}_{q_0}, H^{-s}_{q_0}). \tag{5.7}$$

Setting $P := p(x, D_x)$ we get due to Remark 5.3

$$\mathrm{ad}(-ix)^{\alpha_1} \mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\alpha_l} \mathrm{ad}(D_x)^{\beta_l} P^{-1} = \sum_{\substack{(\alpha^1_1+\ldots+\alpha^1_l)+\cdots+(\alpha^l_1+\ldots+\alpha^l_l)=\alpha \\ (\beta^1_1+\ldots+\beta^1_l)+\cdots+(\beta^l_1+\ldots+\beta^l_l)=\beta}} R_{\alpha^1_1,\ldots,\alpha^l_l,\beta^1_1,\ldots,\beta^l_l}$$

where $R_{\alpha^1_1,\ldots,\alpha^l_l,\beta^1_1,\ldots,\beta^l_l}$ is defined as in Remark 3.4. Together with (5.6) and (5.7) this provides $P^{-1} \in \mathcal{A}^{0,M}_{1,0}(\hat{m}, q_0)$. By means of Theorem 4.5 and Lemma 3.3 we get for each $0 < \tilde{\tau} \leq \hat{m} - n/q_0$ with $\tilde{\tau} \notin \mathbb{N}$:

$$\begin{aligned} p(x, D_x)^{-1} &\in OPW^{\hat{m},q_0}_{uloc} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(L^{q_0}(\mathbb{R}^n)) \\ &\subseteq C^{\tilde{\tau}} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1). \end{aligned}$$

Finally, considering $\tilde{M}-1 > n/\tilde{q}$ for some $1 < \tilde{q} \leq 2$ we obtain for $q \in [\tilde{q}, \infty)$ due to Theorem 3.7 the boundedness of $P^{-1} : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$. $\square$

The relation to the spectral invariance of Theorem 5.10 is emphasised in the next corollary which easily can be verified by means of Theorem 5.10. For more details we refer to [13, Corollary 6.12].

Corollary 5.11. *Let the assumptions of Theorem* 5.10 *hold. Additionally we choose an arbitrary but fixed* $\tilde{q} \in (1, 2]$ *fulfilling the conditions of Theorem* 5.10 *and denote*

$$P_{L^q} := p(x, D_x) : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \quad \text{for all } \tilde{q} \leq q < \infty.$$

Then $\sigma(P_{L^q}) = \sigma(P_{L^r})$ *for all* $\tilde{q} \leq q, r < \infty$.

Now one may wonder whether it is possible to prove that $p(x, D_x)^{-1}$ is even an element of $OPW^{\hat{m},q}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ if all assumptions of the Theorem 5.10 are fulfilled and additionally $p(x, D_x) \in OPC^{\hat{m},\tau}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Unfortunately in general this is not the case as we see in the next example:

Example 5.12. Let $1 < q_0 < \infty$ and $\tau > \lfloor n/q_0 \rfloor + n + 4$. Additionally let $p(\xi) \in S^0_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ be such that p is not constant and $p(D_x)^{-1} \in \mathscr{L}(L^{q_0}(\mathbb{R}^n))$. Moreover let $a \in C^\tau(\mathbb{R}^n)$ such that there exists a point u_0 which has no open neighbourhood on which a is smooth and such that there exist two constants $c, C > 0$ with $C > a(x) > c$ for all $x \in \mathbb{R}^n$. Then $T := a(x)p(D_x) \in C^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ fulfills all assumptions of Theorem 5.10 for $M = n+3$ and $\hat{m} := \lfloor \tau \rfloor - (n+3)$. Consequently $T^{-1} \in OPW^{\hat{m},q_0}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M}-1)$, where $\tilde{M} := M - (n+1)$, but $T^{-1} \notin OPC^{\tilde{\tau}}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\tilde{\tau} \in (0, \hat{m} - n/q_0]$. In particular $T^{-1} \notin OPW^{\hat{m},q}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ due to Lemma 3.3.

Proof. In fact, by Theorem 5.2, $p(D_x)^{-1} \in OPS^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover $T^{-1} = p(D_x)^{-1}b(x)$ with $b := 1/a \in C^\tau(\mathbb{R}^n)$. In particular, T^{-1} is bounded in $\mathscr{L}(L^{q_0}(\mathbb{R}^n))$. Now assume that $T^{-1} \in C^{\tilde{\tau}}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. If $k(x-y)$ with $k \in C^\infty(\mathbb{R}^n \backslash \{0\})$ is the kernel of $p(D_x)^{-1}$ in the sense of Theorem 3.14, the kernel of T^{-1} is $\tilde{k}(x, x-y) = k(x-y)b(y)$, i.e.,

$$\tilde{k}(x, z) = k(z)b(x-z) = \frac{k(z)}{a(x-z)}.$$

(for $b \in C^\infty(\mathbb{R}^n)$ this is obvious; the general case follows from convolution of b with the usual mollifier $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$ and then passing to the limit $\varepsilon \to 0$). Recall that $\tilde{k}(x, .) \in C^\infty(\mathbb{R}^n \backslash \{0\})$ for every x. Since p is not constant, there exists a z_0 with $k(z_0) \neq 0$. Choosing x_0 such that $u_0 = x_0 - z_0$, it follows from $k(z) = a(x_0 - z)\tilde{k}(x_0, z)$ that there exists a neighbourhood of u_0 on which a is smooth. This is a contradiction to the choice of a. $\square$

5.4. Spectral Invariance of Pseudodifferential Operators in the Symbol-Class $W^{\tilde{m},q}_{uloc}S^0_{1,0}$

The present subsection serves to improve Theorem 5.10 for non-smooth pseudodifferential operators of the order zero with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$.

Analyzing the proof of Theorem 5.10 we see that we need similar results for pseudodifferential operators whose symbols are in $W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ instead of Theorem 5.9 and Remark 3.4:

Lemma 5.13. *Let $1 < q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > 1 + n/q$. Considering $p \in W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, we get the boundedness of*

$$\left\{\partial^h_{x_j} p(x,\xi) : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq W^{\tilde{m}-1,q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$$

for all $j \in \{1,\ldots,n\}$.

Proof. Let $\psi \in C^\infty_c(\mathbb{R}^n)$ be such that $\operatorname{supp}\psi \subseteq \overline{B_2(0)}$ and $\psi(x) = 1$ for all $x \in B_1(0)$. Assuming an arbitrary $\alpha \in \mathbb{N}^n_0$ we get due to Theorem 5.7 and $p \in W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$:

$$\begin{aligned}
\|\partial^\alpha_\xi \partial^h_{x_j} p(x,\xi)\|_{W^{\tilde{m}-1,q}_{uloc}(\mathbb{R}^n_x)} &\leq \sum_{|\beta|\leq \tilde{m}-1} \sup_{y\in\mathbb{R}^n} \|\partial^h_{x_j}\partial^\beta_x\partial^\alpha_\xi[p(x,\xi)\psi(x-y)]\|_{L^q(\mathbb{R}^n_x)}\\
&\leq C \sum_{|\beta|\leq \tilde{m}-1} \sup_{y\in\mathbb{R}^n} \|\partial_{x_j}\partial^\beta_x\partial^\alpha_\xi[p(x,\xi)\psi(x-y)]\|_{L^q(\mathbb{R}^n_x)}\\
&\leq C \sum_{|\beta|\leq \tilde{m}} \sup_{y\in\mathbb{R}^n} \|\partial^\beta_x\partial^\alpha_\xi p(.,\xi)\|_{L^q(B_2(y))}\\
&\leq C\|\partial^\alpha_\xi p(x,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)}\\
&\leq C\langle\xi\rangle^{m-|\alpha|} \quad \text{for all } \xi \in \mathbb{R}^n, h \in \mathbb{R}\backslash\{0\}.
\end{aligned}$$
□

Lemma 5.14. *Let $1 < \tilde{q}, q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > 1 + n/q$. Assuming $p \in W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ we get for every $j \in \{1,\ldots,n\}$ and all $h \in \mathbb{R}\backslash\{0\}$:*

$$[\partial^h_{x_j}, p(x,D_x)]u(x) = \left(\partial^{-h}_{x_j} p\right)(x,D_x)u(x+he_j) \quad \forall u \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n.$$

Moreover, for all $-\tilde{m} + 1 + n/q < s \leq \tilde{m} - 1 - n(1/q - 1/\tilde{q})^+$ there is a constant C, independent of $h \in \mathbb{R}\backslash\{0\}$, such that

$$\|[\partial^h_{x_j}, p(x,D_x)]u\|_{H^s_{\tilde{q}}} \leq C\|u\|_{H^{s+m}_{\tilde{q}}} \quad \textit{for all } u \in H^{s+m}_{\tilde{q}}(\mathbb{R}^n),$$

where $j \in \{1,\ldots,n\}$.

Proof. Proceed as in the proof of Lemma 5.6, using Lemma 5.13 and Theorem 3.12 instead of Lemma 5.6 and Theorem 3.7. □

Lemma 5.15. *Let $1 < q, \tilde{q} < \infty$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > 1 + n/q$. Considering $p \in W^{\tilde{m},q}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, where the inverse operator $p(x,D_x)^{-1} \in \mathscr{L}(H^r_{\tilde{q}}, H^r_{\tilde{q}})$ for one $-\tilde{m} + n/q < r \leq \tilde{m} - n(1/q - 1/\tilde{q})^+$, we obtain*

$$p(x,D_x)^{-1} \in \mathscr{L}(H^s_{\tilde{q}}, H^s_{\tilde{q}}) \quad \textit{for all } s \in [r-l, r+k].$$

Here k and l are defined by $k := \max\{\tilde{k} \in \mathbb{N}_0 : r + \tilde{k} \leq \tilde{m} - n(1/q - 1/\tilde{q})^+\}$ and $l := \max\{\tilde{l} \in \mathbb{N}_0 : -\tilde{m} + n/q < r - \tilde{l}\}$.

Proof. Replacing Theorem 3.7 with Theorem 3.12 and Lemma 5.6 with Lemma 5.14, the statement follows in the same way as that one of Theorem 5.9. □

Comparing the previous result with that one of Theorem 5.9 the difference lies in the choice of the neighbourhood of r. The previous lemma allows us to improve Theorem 5.10:

Theorem 5.16. *Let $1 < q, q_0 < \infty$, $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > \max\{1 + n/q, n/q_0\}$ and $X(N) = W^{\tilde{m},q}_{uloc} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$ for all $N \in \mathbb{N}_0 \cup \{\infty\}$. Additionally let $\hat{m} \in \mathbb{N}_0$ with $n/q_0 < \hat{m} \leq \max\{r \in \mathbb{N}_0 : r < \tilde{m} - n/q\}$. Moreover, let $M \in \mathbb{N}_0$ with $n < M < \tilde{m} - \hat{m} - n/q$. We define $\tilde{M} := M - (n+1)$. Considering $p \in X(\infty)$, where $p(x, D_x)^{-1} \in \mathscr{L}(H^r_{q_0}, H^r_{q_0})$ for one*

$$- \tilde{m} + n/q < r \leq \tilde{m} - n(1/q - 1/q_0)^+ \tag{5.8}$$

we get

$$p(x, D_x)^{-1} \in OPW^{\hat{m},q_0}_{uloc} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1).$$

In case $\tilde{M} - 1 > n/\tilde{q}$ for one $1 < \tilde{q} \leq 2$, we even have

$$p(x, D_x)^{-1} \in \mathscr{L}(L^{\tilde{q}}, L^{\tilde{q}}) \quad \textit{for all } \tilde{q} \in [\tilde{q}; \infty) \cup \{q_0\}.$$

Proof. Proceed as in the proof of Theorem 5.10 using Lemma 5.15, Remark 3.4 and Theorem 3.7 instead of Theorem 5.9, Remark 3.4 and Theorem 3.12. □

Theorem 5.16 in fact is an improvement for $P \in OPW^{\tilde{m},q}_{uloc} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ because Theorem 5.16 holds for the less strict assumption

$$-\tilde{m} + n/q < r \leq \tilde{m} - n(1/q - 1/q_0)^+.$$

In case $X(N) = H^{\tilde{m}}_q S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$ or $X(N) = W^{\tilde{m},q}_{uloc} S^0_{cl}(\mathbb{R}^n \times \mathbb{R}^n; N)$, $N \in \mathbb{N}_0 \cup \{\infty\}$, the claim of Theorem 5.16 even holds for $M \in \mathbb{N}_0$ with $n < M < \tilde{m} - \hat{m} - n(1/q + 1/q_0 - 1)^+$ if we replace (5.8) with the inequality $n(1/q_0 + 1/q - 1)^+ - \tilde{m} < r \leq \tilde{m} - n(1/q - 1/q_0)^+$. For more details, see [13, Section 6.4].

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Received: January 3, 2016.
Revised: August 31, 2016.