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Reduced Crossed Products Associated with Banach Algebra Dynamical Systems

Lizhong Huang and Fangyan Lu

Abstract. We define reduced crossed products associated with a Banach algebra dynamical system. If the group is amenable, we prove that the reduced crossed product and the crossed product are isometrically isomorphic.

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1. Introduction and Preliminaries

The theory of crossed products of C^*-algebras started with the papers by Turumaru [12] and Zeller-Meier [14]. Their origins can be traced back to statistical mechanics, where crossed products were called covariance algebras, and to the group measure space constructions of Murray and von Neumann. The crossed products of C^*-algebras have attracted a great deal of attention for nearly 60 years, and a large part of the C^*-algebra literature is concerned with crossed products. In 2007, Williams [13] gave a detailed systematic exposition of the recent developments in the theory of crossed products of C^*-algebras.

A crossed product of a C^*-algebra is a C^*-algebra $A \rtimes_\alpha G$ built from a C^*-dynamical system (A, G, α), where A is a C^*-algebra, G is a locally compact group, and α is a strongly continuous representation of G on A as involutive automorphisms. The crossed product construction provides a useful means to construct new examples of C^*-algebras.

Various generalizations of C^*-algebra crossed products are available [5,6,8,9]. Among others, Dirksen et al. [3] constructed and studied Banach algebra crossed products associated with a Banach algebra dynamical system (A, G, α), where A is a Banach algebra, G is a locally compact group, and α is a strongly continuous representation of G on A as automorphisms. Roughly speaking, the Banach algebra crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a completion of $C_c(G, A)$ depending on a class $\mathcal{R}$ of representations. Some fundamental properties and applications are established in a series of papers [1–3].

General Banach algebras lack the convenient rigidity of C^*-algebras where, e.g., morphisms are automatically continuous and even contractive, and this makes the task of developing the basics more laborious than it is for crossed products of C^*-algebras.

Given a C^*-dynamical system, besides the full crossed product C^*-algebra, there is another important crossed product C^*-algebra, namely the reduced crossed product, which was defined by Zeller-Meier for discrete groups in [14] and generalized by Takai [11]. For a C^*-algebra dynamical system, it was proved that the crossed product and the associated reduced crossed product are equal by Zeller-Meier [14] for discrete groups and by Takai [11] for amenable groups. The importance of this theorem in the C^*-dynamical system theory lies in the fact that the reduced crossed product is more concrete and many familiar groups are amenable, such as abelian groups and compact groups. In the present paper, we will define the reduced Banach algebra crossed product and establish this equality in the Banach algebra dynamical system setting, see Theorem 3.4. For this purpose, some properties about regular representations are investigated.

We now introduce some basic definitions and notations, and provide a brief recapitulation of the definition of a Banach algebra crossed product from [3].

Given a Banach algebra A, by $\mathrm{Aut}(A)$ we denote the group of bounded automorphisms of A. For a Banach space X, we use $B(X)$ and $\mathrm{Inv}(X)$ to denote the algebra of all bounded linear operators on X and the group of invertible elements in $B(X)$, respectively.

Suppose X is a Banach space and G is a locally compact group. By $C_c(G, X)$ we denote the space of all continuous compactly supported X-valued functions. By [13, Lemma 1.91], there is a linear map $f \mapsto \int_G f(s)d\mu(s)$ from $C_c(G, X)$ to X which is characterized by

$$\left\langle \int_G f(s)d\mu(s), x^* \right\rangle = \int_G \langle f(s), x^* \rangle d\mu(s)$$

for all $f \in C_c(G, X)$ and $x^* \in X^*$, where μ is a fixed left Haar measure on G and X^* is the topological dual of X. It is not difficult to verify that

$$T \int_G f(s)d\mu(s) = \int_G Tf(s)d\mu(s)$$

for all $f \in C_c(G, X)$ and $T \in B(X)$. Moreover, let $\psi\colon G \to B(X)$ be compactly supported and strongly continuous, and define

$$\int_G \psi(s)d\mu(s) := \left[x \mapsto \int_G \psi(s)x d\mu(s) \right].$$

Then by [3, Proposition 2.19], $\int_G \psi(s)d\mu(s)$ is in $B(X)$ and

$$T \int_G \psi(s)d\mu(s)R = \int_G T\psi(s)Rd\mu(s)$$

for all $T, R \in B(X)$.

Recall that a *Banach algebra dynamical system* is a triple (A, G, α), where A is a Banach algebra, G is a locally compact group, and $\alpha\colon G \to$

$\mathrm{Aut}(A)$ is a strongly continuous representation of G on A. Let (A, G, α) be a Banach algebra dynamical system and let X be a Banach space. Then a *covariant representation* of (A, G, α) on X is a pair (π, U), where π is an algebra homomorphism from A to $B(X)$, and U is a group homomorphism from G to $\mathrm{Inv}(X)$, such that for all $a \in A$ and $s \in G$,

$$\pi(\alpha_s(a)) = U_s \pi(a) U_{s^{-1}}.$$

Here U_s means $U(s)$. The covariant representation (π, U) is called continuous if π is norm bounded and U is strongly continuous. Let $\mathcal{R}$ be a class of continuous covariant representations of (A, G, α). Then $\mathcal{R}$ is called uniformly bounded if there exist a constant $C \geq 0$ and a function $\gamma \colon G \to [0, \infty)$, which is bounded on compact subsets of G, such that for all (π, U) in $\mathcal{R}$, $\| \pi \| \leq C$, and $\| U_r \| \leq \gamma(r)$ for all $r \in G$.

Let (A, G, α) be a Banach algebra dynamical system. For any $f, g \in C_c(G, A)$ and $s \in G$ defining the twisted convolution

$$[f * g](s) := \int_G f(r) \alpha_r(g(r^{-1}s)) d\mu(r)$$

gives $C_c(G, A)$ the structure of an associative algebra. If (π, U) is a continuous covariant representation of (A, G, α) on the Banach space X, then the integrated form $\pi \rtimes U$ of (π, U) is defined by

$$\pi \rtimes U(f) := \int_G \pi(f(s)) U_s d\mu(s) \quad (f \in C_c(G, A)).$$

It is not difficult to verify that $\pi \rtimes U$ is a representation of $C_c(G, A)$ on X.

Now let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations of (A, G, α). Then we can define the algebra seminorm $\sigma^{\mathcal{R}}$ on $C_c(G, A)$ by

$$\sigma^{\mathcal{R}}(f) = \sup_{(\pi, U) \in \mathcal{R}} \|\pi \rtimes U(f)\| \quad (f \in C_c(G, A)),$$

and denote the completion of the quotient $C_c(G, A)/\ker \sigma^{R}$ by $(A \rtimes_\alpha G)^{\mathcal{R}}$, with $\| \cdot \|_{\mathcal{R}}$ denoting the norm induced by $\sigma^{\mathcal{R}}$. The Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ is called the crossed product corresponding to (A, G, α) and $\mathcal{R}$.

2. Regular Representations

We begin with a definition.

Definition 2.1. Let G be a locally compact group, let X be a Banach space, and let $1 \leq p < \infty$. We define the p-norm on $C_c(G, X)$ by

$$\|f\|_p = \left(\int_G \|f(s)\|^p d\mu(s) \right)^{1/p},$$

and define $L^p(G, X)$ as the completion of $C_c(G, X)$ with this norm.

Let (A, G, α) be a Banach algebra dynamical system and let π be a continuous representation of A on a Banach space X. To let the following definitions make sense, we require that π is α-bounded, that is, there is $M > 0$

such that $\|\pi(\alpha_t(a))\| \leq M\|a\|$ for all $a \in A$ and $t \in G$. In C^*-algebra dynamical systems, this requirement is automatically satisfied. We now define the induced algebra representation $\widetilde{\pi}$ of A and the *left regular group representation* Λ of G on the space $L^p(G, X)$ for $1 \leq p < \infty$ by the formulae:

$$[\widetilde{\pi}(a)h](s) := \pi(\alpha_{s^{-1}}(a))h(s) \quad (a \in A, s \in G, h \in L^p(G, X)),$$
$$(\Lambda_r h)(s) := h(r^{-1}s) \quad (r, s \in G, h \in L^p(G, X)).$$

Obviously, the requirement that π is α-bounded guarantees the boundedness of $\widetilde{\pi}$. By [1, Corollary 5.9], $(\widetilde{\pi}, \Lambda)$ is a continuous covariant representation of (A, G, α). We will call $(\widetilde{\pi}, \Lambda)$ the *regular covariant representation* associated with π.

Lemma 2.2. *Let (A, G, α) be a Banach algebra dynamical system, let π be a continuous representation of A on a Banach space X, and let $1 \leq p < \infty$.*

(1) *For all $r \in G$, Λ_r is an invertible isometry on $L^p(G, X)$.*
(2) *Suppose there is $M > 0$ such that $\|\pi(\alpha_t(a))\| \leq M\|a\|$ for all $a \in A$ and $t \in G$. Let $\widetilde{\pi}$ be the induced algebra representation of A associated with π on $L^p(G, X)$. Then $\|\widetilde{\pi}\| \leq M$.*
(3) *Suppose $\|\alpha\| = \sup\{\|\alpha_t\| : t \in G\} < \infty$. Then π is α-bounded. Moreover, if $\widetilde{\pi}$ is the induced algebra representation of A associated with π on $L^p(G, X)$, then $\|\widetilde{\pi}\| \leq \|\alpha\|\|\pi\|$.*
(4) *Suppose (π, U) is a continuous covariant representation of (A, G, α) for some group homomorphism U from G to Inv(X) which satisfies $\|U\| := \sup_{t\in G} \|U_t\| < \infty$. Then π is α-bounded and $\|\pi \circ \alpha_t\| \leq \|U\|^2\|\pi\|$ for all $t \in G$. Moreover, if $\widetilde{\pi}$ is the induced algebra representation of A associated with π on $L^p(G, X)$, then $\|\widetilde{\pi}\| \leq \|U\|^2\|\pi\|$.*

Proof. (1) and (2). They are straightforward verifications.

(3). For all $t \in G$, $\|\pi \circ \alpha_t\| \leq \|\pi\|\|\alpha_t\| \leq \|\alpha\|\|\pi\|$. By (2), $\|\widetilde{\pi}\| \leq \|\alpha\|\|\pi\|$.

(4). The covariance relation $\pi(\alpha_t(a)) = U_t\pi(a)U_t^{-1}$ implies that $\|\pi(\alpha_t(a))\| \leq \|U\|^2\|\pi\|\|a\|$ for all $a \in A$ and $t \in G$, and hence $\|\widetilde{\pi}\| \leq \|U\|^2\|\pi\|$ by (2). □

The following lemma shows that for $f, \xi \in C_c(G, X)$, $\widetilde{\pi} \rtimes \Lambda(f)\xi$ can be calculated pointwise.

Lemma 2.3. *Let (A, G, α) be a Banach algebra dynamical system. Let $\pi: A \to B(X)$ be an α-bounded representation of A on a Banach space X, and let $(\widetilde{\pi}, \Lambda)$ be the associated regular covariant representation on $L^p(G, X)$ for $1 \leq p < \infty$. Then for all $f, \xi \in C_c(G, X)$ and $s \in G$, we have*

$$[\widetilde{\pi} \rtimes \Lambda(f)\xi](s) = \int_G \pi(\alpha_s^{-1}(f(r)))\xi(r^{-1}s)d\mu(r).$$

Proof. It is an obvious modification of [1, Lemma 5.10]. □

The following will be used in the proof of Theorem 3.2. A similar result was proved for C^*-algebra dynamical systems (cf. [10, Theorem 7.7.5]).

Proposition 2.4. *Let (A, G, α) be a Banach algebra dynamical system. Let π be an α-bounded representation of A on a Banach space X, and let $(\widetilde{\pi}, \Lambda)$ be the associated regular representation of (A, G, α) on $L^p(G, X)$ for $1 \leq p < \infty$. Let $f \in C_c(G, X)$. If $\pi(f(r)) \neq 0$ for some $r \in G$, then $\widetilde{\pi} \rtimes \Lambda(f) \neq 0$. In particular, if π is faithful, then the integrated form $\widetilde{\pi} \rtimes \Lambda$ is a faithful representation of $C_c(G, A)$ on $L^p(G, X)$.*

Proof. First we suppose that $\pi(f(e)) \neq 0$. Then there are $x \in X$ and $x^* \in X^*$ such that $\langle \pi(f(e))x, x^* \rangle \neq 0$. By the continuity of f, π and α and arguing as for [3, Proposition 2.3], the map $(r, s) \mapsto \pi(\alpha_r^{-1}(f(s)))$ from $G \times G$ to $B(X)$ is continuous. Thus, we have a neighborhood V of e such that

$$|\langle \pi(\alpha_r^{-1}(f(s)))x - \pi(f(e))x, x^* \rangle| < \frac{|\langle \pi(f(e))x, x^* \rangle|}{2}$$

holds for all $r, s \in V$. Choose $\psi \in C_c^+(G)$ with support contained in a symmetric neighborhood W of e satisfying $W^2 \subseteq V$ such that

$$\int_G \int_G \psi(s^{-1}r)\psi(r) d\mu(r) d\mu(s) = 1.$$

For $\xi \in L^p(G, X)$, a simple computation gives

$$\left| \int_G \langle \psi(s)\xi(s), x^* \rangle d\mu(s) \right| \leq \|x^*\| \|\psi\|_q \|\xi\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. So we can define $\phi \in (L^p(G, X))^*$ by

$$\langle \xi, \phi \rangle = \int_G \langle \psi(s)\xi(s), x^* \rangle d\mu(s) \quad (\xi \in L^p(G, X)).$$

Define $\eta \in C_c(G, X)$ by

$$\eta(s) = \psi(s)x \quad (s \in G).$$

Then by Lemma 2.3,

$$\begin{aligned}
&|\langle \widetilde{\pi} \rtimes \Lambda(f)\eta, \phi \rangle - \langle \pi(f(e))x, x^* \rangle| \\
&= \left| \int_G \int_G \langle \psi(r)\pi(\alpha_r^{-1}(f(s)))\eta(s^{-1}r), x^* \rangle d\mu(r) d\mu(s) - \langle \pi(f(e))x, x^* \rangle \right| \\
&= \left| \int_G \int_G \langle \psi(r)\psi(s^{-1}r)\pi(\alpha_r^{-1}(f(s)))x, x^* \rangle d\mu(r) d\mu(s) - \langle \pi(f(e))x, x^* \rangle \right| \\
&= \left| \int_G \int_G \psi(r)\psi(s^{-1}r)\langle \pi(\alpha_r^{-1}(f(s)))x - \pi(f(e))x, x^* \rangle d\mu(r) d\mu(s) \right| \\
&\leq \int_G \int_G \psi(r)\psi(s^{-1}r)|\langle \pi(\alpha_r^{-1}(f(s)))x - \pi(f(e))x, x^* \rangle| d\mu(r) d\mu(s) \\
&\leq \frac{|\langle \pi(f(e))x, x^* \rangle|}{2}.
\end{aligned}$$

From this we see that $\langle \widetilde{\pi} \rtimes \Lambda(f)\eta, \phi \rangle \neq 0$ and hence $\widetilde{\pi} \rtimes \Lambda(f) \neq 0$.

Now suppose that $f(r) \neq 0$ for some $r \in G$. Define $g(s) := \alpha_{r^{-1}}(f(rs))$. Then $g \in C_c(G, A)$ by [3, Lemma 2.11] and $g(e) \neq 0$, and hence $\widetilde{\pi} \rtimes \Lambda(g) \neq 0$ by the previous result. Compute

$$\begin{aligned}\widetilde{\pi} \rtimes \Lambda(g) &= \int_G \widetilde{\pi}(\alpha_{r^{-1}}(f(rs)))\Lambda_s d\mu(s)\\ &= \int_G \widetilde{\pi}(\alpha_{r^{-1}}(f(s)))\Lambda_{r^{-1}s} d\mu(s)\\ &= \int_G \Lambda_{r^{-1}}\widetilde{\pi}(f(s))\Lambda_s d\mu(s)\\ &= \Lambda_{r^{-1}} \circ \widetilde{\pi} \rtimes \Lambda(f).\end{aligned}$$

This together with the inequality $\widetilde{\pi} \rtimes \Lambda(g) \neq 0$ implies that $\widetilde{\pi} \rtimes \Lambda(f) \neq 0$. □

3. Reduced Crossed Products

In this section, we will define and study the reduced crossed product associated with a Banach algebra dynamical system. Let (A, G, α) be a Banach algebra dynamical system, and suppose that $\mathcal{R}$ is a class of continuous covariant representations of (A, G, α). Then $\mathcal{R}$ is called uniformly α-bounded if there exist a constant $C \geq 0$ and a function $\gamma\colon G \to [0, \infty)$, which is bounded on compact subsets of G, such that for all (π, U) in $\mathcal{R}$, $\| \pi \circ \alpha_r \| \leq C$, and $\| U_r \| \leq \gamma(r)$ for all $r \in G$. Since α_e is the identity on A, the uniform α-boundedness implies the uniform boundedness. Conversely, if $\sup\{\|U_r\|\colon r \in G, (\pi, U) \in \mathcal{R}\} < \infty$, then the uniform boundedness of $\mathcal{R}$ implies the uniform α-boundedness by Lemma 2.2.

Now let $\mathcal{R}$ be a non-empty uniformly α-bounded class of continuous covariant representations of a Banach algebra dynamical system (A, G, α). Let $[1, \infty)^{\mathcal{R}}$ denote the set of all maps from $\mathcal{R}$ to $[1, \infty)$. For $\theta \in [1, \infty)^{\mathcal{R}}$, let $\widetilde{\mathcal{R}}(\theta)$ be the set of all regular covariant representations $(\widetilde{\pi}, \Lambda)$ on $L^{\theta(\pi,U)}(G, X)$ that are associated with π as (π, U, X) varies over $\mathcal{R}$. Let Θ be a non-empty subset of $[1, \infty)^{\mathcal{R}}$ and let $\widetilde{\mathcal{R}}(\Theta) = \cup_{\theta \in \Theta}\widetilde{\mathcal{R}}(\theta)$. Then $\widetilde{\mathcal{R}}(\Theta)$ is a non-empty uniformly bounded class of continuous covariant representations by Lemma 2.2. We call $\widetilde{\mathcal{R}}(\Theta)$ the class of regular representations of (A, G, α) associated with $\mathcal{R}$ and Θ, and call $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}(\Theta)}$ the *reduced crossed product* of (A, G, α) associated with $\mathcal{R}$ and Θ.

Takai [11] proved that for a C^*-algebra dynamical system with the group amenable the crossed product and the associated reduced crossed product are equal. We will establish this theorem in the Banach algebra dynamical system setting. For this, we need a characterization of amenable groups.

Lemma 3.1. [4] *Let G be a locally compact group with Haar measure μ. If G is amenable, then, for every $\varepsilon > 0$ and compact set $K \subset G$ containing the identity element of G, there exists a compact set U with*

$$\mu(U) > 0 \quad \textit{and} \quad \frac{\mu(KU \triangle U)}{\mu(U)} < \varepsilon.$$

Here $KU \triangle U = (KU \backslash U) \cup (U \backslash KU)$ is the symmetric difference of the sets KU and U.

The following is the key step to our main results.

Theorem 3.2. *Let (A, G, α) be a Banach algebra dynamical system and let $1 \leq p < \infty$. Let (π, U) be a continuous covariant representation of (A, G, α) on the Banach space X and suppose*

$$\|U\| := \sup_{r\in G} \|U_r\| < \infty.$$

Let $(\widetilde{\pi}, \Lambda)$ be the regular continuous covariant representation of (A, G, α) associated with π on $L^p(G, X)$. If G is amenable, then

$$\|(\widetilde{\pi} \rtimes \Lambda)(f)\| \geq \frac{1}{\|U\|^2}\|(\pi \rtimes U)(f)\|$$

holds for all $f \in C_c(G, A)$.

Proof. Define a map T from $L^p(G, X)$ to itself by

$$(T\xi)(r) = U_r^{-1}(\xi(r)) \quad (\xi \in L^p(G, X), r \in G).$$

Then T is obviously linear. It is not difficult to verify that T is bijective and $(T^{-1}\xi)(r) = U_r(\xi(r)) \quad (\xi \in L^p(G, X), r \in G)$. Moreover, for $\xi \in L^p(G, X)$, we have

$$\begin{aligned}\|T\xi\|^p &= \int_G \|(T\xi)(r)\|^p d\mu(r)\\ &= \int_G \|U_r^{-1}(\xi(r))\|^p d\mu(r)\\ &\leq \int_G \|U_r^{-1}\|^p \|\xi(r)\|^p d\mu(r)\\ &\leq \|U\|^p\|\xi\|^p.\end{aligned}$$

So $\|T\| \leq \|U\|$. Similarly, $\|T^{-1}\| \leq \|U\|$.

Let $f \in C_c(G, A)$. We will show, given $\varepsilon > 0$, that

$$\|T^{-1}(\widetilde{\pi} \rtimes \Lambda)(f)T\| > \|(\pi \rtimes U)(f)\| - \varepsilon. \tag{3.1}$$

If $\|(\pi \rtimes U)(f)\| < \varepsilon$, (3.1) is obvious. If $\|(\pi \rtimes U)(f)\| = \varepsilon$, then $(\pi \rtimes U)(f) \neq 0$ and hence $\pi(f(r)) \neq 0$ for some $r \in G$. Thus $(\widetilde{\pi} \rtimes \Lambda)(f) \neq 0$ by Proposition 2.4 and hence $T^{-1}(\widetilde{\pi} \rtimes \Lambda)(f)T \neq 0$. So

$$\|T^{-1}(\widetilde{\pi} \rtimes \Lambda)(f)T\| > 0 = \|(\pi \rtimes U)(f)\| - \varepsilon.$$

We now suppose that $\|(\pi \rtimes U)(f)\| > \varepsilon$. Choose $x_0 \in X$, such that $\|x_0\| = 1$ and

$$\|(\pi \rtimes U)(f)x_0\| > \|(\pi \rtimes U)(f)\| - \frac{\varepsilon}{2}.$$

Let

$$\delta = \frac{\|(\pi \rtimes U)(f)\| - \frac{\varepsilon}{2}}{\|(\pi \rtimes U)(f)\| - \varepsilon} - 1.$$

Then $\delta > 0$.

Set $S = \mathrm{supp}(f) \cup \{e\}$. Since $f \in C_c(G, A)$, both S and S^{-1} are compact subsets of the locally compact group G. By Lemma 3.1, there is a compact subset $K \subset G$ such that

$$\mu(K) > 0 \quad \text{and} \quad \mu(S^{-1}K \triangle K) < \delta\mu(K).$$

Since $e \in S^{-1}$, the latter inequality implies $\mu(S^{-1}K) < (1+\delta)\mu(K)$. Since the Haar measure is outer regular on Borel sets, there is an open set V such that $V \supseteq S^{-1}K$ and $\mu(V) < (1+\delta)\mu(K)$. By Urysohn's lemma, there is $\psi \in C_c(G)$ satisfying $0 \leq \psi(s) \leq 1$ for all $s \in G$, $\psi(s) = 1$ for all $s \in S^{-1}K$, and $\psi(s) = 0$ for all $s \notin V$.

Define $\eta \in C_c(G, X)$ by

$$\eta(s) = \psi(s)x_0, \quad s \in G.$$

Then

$$\begin{aligned}\|\eta\| &= \left(\int_G \|\eta(r)\|^p d\mu(r)\right)^{1/p} = \left(\int_V |\psi(r)|^p \|x_0\|^p d\mu(r)\right)^{1/p} \\ &\leq \mu(V)^{1/p} < (1+\delta)^{1/p}\mu(K)^{1/p} \leq (1+\delta)\mu(K)^{1/p}.\end{aligned}$$

By definition, $(T\eta)(s) = U_s^{-1}(\psi(s)x_0) = \psi(s)U_s^{-1}(x_0)$ for $s \in G$. It follows from the strong continuity of U that $T\eta \in C_c(G, X)$.

For $r \in K$, by Lemma 2.3 and noting $\eta(s^{-1}r) = x_0$ if $s \in S$ with $f(s) \neq 0$, we have

$$\begin{aligned}&[T^{-1}(\widetilde{\pi} \rtimes \Lambda)(f)T\eta](r) \\ &\quad = U_r(((\widetilde{\pi} \rtimes \Lambda)(f)T\eta)(r)) \\ &\quad = U_r \int_G \pi(\alpha_r^{-1}(f(s)))[(T\eta)(s^{-1}r)]d\mu(s) \\ &\quad = \int_G U_r\pi(\alpha_r^{-1}(f(s)))[(T\eta)(s^{-1}r)]d\mu(s) \\ &\quad = \int_G \pi((f(s))U_rU_{s^{-1}r}^{-1}(\eta(s^{-1}r))d\mu(s) \\ &\quad = \int_G \pi(f(s))U_s(\eta(s^{-1}r))d\mu(s) \\ &\quad = \int_G \pi(f(s))U_s x_0 d\mu(s) \\ &\quad = (\pi \rtimes U)(f)x_0.\end{aligned}$$

Therefore

$$\begin{aligned}\|T^{-1}(\widetilde{\pi} \rtimes \Lambda)(f)T\eta\|^p &\geq \int_K \|[T^{-1}(\widetilde{\pi} \rtimes \Lambda)(f)T\eta](r)\|^p d\mu(r) \\ &= \int_K \|(\pi \rtimes U)(f)x_0\|^p d\mu(r) \\ &= \|(\pi \rtimes U)(f)x_0\|^p \mu(K) \\ &> (\|(\pi \rtimes U)(f)\| - \frac{\varepsilon}{2})^p \mu(K).\end{aligned}$$

Hence,

$$\begin{aligned}\|T^{-1}(\widetilde{\pi} \rtimes \Lambda)(f)T\| &\geq \frac{\|T^{-1}(\widetilde{\pi} \rtimes \Lambda)(f)T\eta\|}{\|\eta\|} \\ &> \frac{(\|(\pi \rtimes U)(f)\| - \frac{\varepsilon}{2})\ \mu(K)^{1/p}}{(1+\delta)\mu(K)^{1/p}} \\ &= \frac{\|(\pi \rtimes U)(f)\| - \varepsilon}{\|(\pi \rtimes U)(f)\| - \frac{\varepsilon}{2}} \cdot \left(\|(\pi \times U)(f)\| - \frac{\varepsilon}{2}\right) \\ &= \|(\pi \rtimes U)(f)\| - \varepsilon,\end{aligned}$$

establishing (3.1).

Now, Eq. (3.1) together with

$$\|T^{-1}(\widetilde{\pi} \rtimes \Lambda)(f)T\| \leq \|T^{-1}\|\|(\widetilde{\pi} \rtimes \Lambda)(f)\|\|T\| \leq \|U\|^2\|(\widetilde{\pi} \rtimes \Lambda)(f)\|,$$

gives

$$\|(\widetilde{\pi} \rtimes \Lambda)(f)\| > \frac{1}{\|U\|^2}(\|(\pi \rtimes U)(f)\| - \varepsilon).$$

By the arbitrariness of ε, we have

$$\|(\widetilde{\pi} \rtimes \Lambda)(f)\| \geq \frac{1}{\|U\|^2}\|(\pi \rtimes U)(f)\|,$$

as desired. □

Theorem 3.3. *Let (A, G, α) be a Banach algebra dynamical system. Let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations of (A, G, α) and suppose that $\sup\{\|U_r\|\colon (\pi, U) \in \mathcal{R}, r \in G\} < \infty$. Let Θ be a non-empty subset of $[1, \infty)^{\mathcal{R}}$. Let $\widetilde{\mathcal{R}}(\Theta)$ be the uniformly bounded class of regular representations of (A, G, α) associated with $\mathcal{R}$ and Θ. If G is amenable, then $(A \rtimes_\alpha G)^{\mathcal{R}\cup\widetilde{\mathcal{R}}(\Theta)}$ and $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}(\Theta)}$ are isomorphic.*

Proof. Let f be in $C_c(G, A)$. Obviously $\sigma^{\mathcal{R}\cup\widetilde{\mathcal{R}}(\Theta)}(f) = \max\{\sigma^{\mathcal{R}}(f), \sigma^{\widetilde{\mathcal{R}}(\Theta)}(f)\}$. By Theorem 3.2, we have

$$\frac{1}{M^2}\sigma^{\mathcal{R}\cup\widetilde{\mathcal{R}}(\Theta)}(f) \leq \sigma^{\widetilde{\mathcal{R}}(\Theta)}(f) \leq \sigma^{\mathcal{R}\cup\widetilde{\mathcal{R}}(\Theta)}(f),$$

where $M = \sup\{\|U_r\|\colon (\pi, U) \in \mathcal{R}, r \in G\}$. So $\ker \sigma^{\widetilde{\mathcal{R}}(\Theta)} = \ker \sigma^{\mathcal{R}\cup\widetilde{\mathcal{R}}(\Theta)}$ and the norms $\|\cdot\|_{\widetilde{\mathcal{R}}(\Theta)}$ and $\|\cdot\|_{\mathcal{R}\cup\widetilde{\mathcal{R}}(\Theta)}$ are equivalent on $C_c(G, A)/\ker \sigma^{\widetilde{\mathcal{R}}(\Theta)}$. Thus, by the definition of the crossed product, $(A \rtimes_\alpha G)^{\mathcal{R}\cup\widetilde{\mathcal{R}}(\Theta)}$ and $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}(\Theta)}$ are isomorphic. □

For a C^*-algebra dynamical system (B, G, α) and a covariant representation (π, U) of (B, G, α) on a Hilbert space, we know that α_t is an isometry for each $t \in G$, $\|\pi\| \leq 1$ and U_t is an isometry for each $t \in G$. The following is a natural generalization of Takai's result in [11] on C^*-algebra dynamical systems to Banach algebra dynamical systems.

Theorem 3.4. *Let (A, G, α) be a Banach algebra dynamical system, where G is amenable. For $C > 0$, let $\mathcal{R}_{C,iso}$ denote the set of all continuous covariant representations (π, U) of (A, G, α) on Banach spaces such that $\|\pi\| \leq C$ and U_t is an isometry for each $t \in G$. Let Θ be a non-empty subset of $[1,\infty)^{\mathcal{R}_{C,iso}}$ and let $\widetilde{\mathcal{R}}_{C,iso}(\Theta)$ be the class of regular representations of (A, G, α) associated with $\mathcal{R}_{C,iso}$ and Θ. Then the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}_{C,iso}}$ and the reduced crossed product $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}_{C,iso}(\Theta)}$ are isometrically isomorphic.*

Proof. By the hypothesis,

$$\sup\{\|U_t\| \colon (\pi, U) \in \mathcal{R}_{C,\text{iso}}, t \in G\} = 1.$$

This together with Lemma 2.2 implies that $\widetilde{\mathcal{R}}_{C,\text{iso}}(\Theta) \subseteq \mathcal{R}_{C,\text{iso}}$. Hence by Theorem 3.2, we have

$$\sigma^{\mathcal{R}_{C,\text{iso}}}(f) \leq \sigma^{\widetilde{\mathcal{R}}_{C,\text{iso}}(\Theta)}(f) \leq \sigma^{\mathcal{R}_{C,\text{iso}}}(f)$$

for all $f \in C_c(G, A)$. So, $\ker \sigma^{\widetilde{\mathcal{R}}_{C,\text{iso}}(\Theta)} = \ker \sigma^{\mathcal{R}_{C,\text{iso}}}$ and $\|\cdot\|_{\widetilde{\mathcal{R}}_{C,\text{iso}}(\Theta)} = \|\cdot\|_{\mathcal{R}_{C,\text{iso}}}$ on $C_c(G, A)/\ker \sigma^{\mathcal{R}_{C,\text{iso}}}$. By the definitions, $(A \rtimes_\alpha G)^{\mathcal{R}_{C,\text{iso}}}$ and $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}_{C,\text{iso}}(\Theta)}$ are isometrically isomorphic. □

We remark that a similar result was established in [7] for special cases. A Banach algebra dynamical system (A, G, α) is called isometric if α_t is an isometry for each $t \in G$. Given an isometric Banach algebra dynamical system (A, G, α), Li and Xu [7] define the crossed product $A \rtimes_\alpha G$ to be our $(A \rtimes_\alpha G)^{\mathcal{R}_{1,\text{iso}}}$ for which $\mathcal{R}_{1,\text{iso}}$ is the class of all continuous covariant representations (π, U) of (A, G, α) on Banach spaces such that $\|\pi\| \leq 1$ and U_t is an isometry for each $t \in G$. Also, they define the reduced crossed product $A \rtimes_{\alpha,r} G$ to be $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}_{\text{con}}}$ for which

$$\widetilde{\mathcal{R}}_{\text{con}} = \bigcup_{1 \leq p < \infty} \{(\widetilde{\pi}, \Lambda)_p \colon \pi \text{ is a contractive representations of } A\},$$

where $(\widetilde{\pi}, \Lambda)_p$ denotes the regular covariant representation of (A, G, α) on $L^p(G, X)$ associated with the contractive representations π of A on the Banach space X. When G is compact, Li and Xu showed that $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,r} G$ are isometrically isomorphic [7, Theorem 5.1].

We observe that [7, Theorem 5.1] is a consequence of Theorem 3.2 and also holds when G is amenable. To see this, let Θ be a non-empty subset of $[1,\infty)^{\mathcal{R}_{1,\text{iso}}}$. In general, $A \rtimes_{\alpha,r} G$ does not coincide with $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}_{1,\text{iso}}(\Theta)}$. However, when G is amenable, by Theorem 3.2 and noting $\widetilde{\mathcal{R}}_{1,\text{iso}}(\Theta) \subseteq \widetilde{\mathcal{R}}_{\text{con}} \subseteq \mathcal{R}_{1,\text{iso}}$,

$$\sigma^{\mathcal{R}_{1,\text{iso}}}(f) \leq \sigma^{\widetilde{\mathcal{R}}_{1,\text{iso}}(\Theta)}(f) \leq \sigma^{\widetilde{\mathcal{R}}_{\text{con}}}(f) \leq \sigma^{\mathcal{R}_{1,\text{iso}}}(f)$$

for all $f \in C_c(G, A)$, which implies that $A \rtimes_{\alpha,r} G = (A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}_{1,\text{iso}}(\Theta)} = A \rtimes_\alpha G$.

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Lizhong Huang
Department of Mathematics
Shanxi Datong University
Datong 037009
People's Republic of China
e-mail: huanglizhongsx@126.com

Fangyan Lu (✉)
Department of Mathematics
Soochow University
Suzhou 215006
People's Republic of China
e-mail: fylu@suda.edu.cn

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