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Integral Equations and Operator Theory

Stability of the Norm-Eigenfunctions of the $p(\cdot)$-Laplacian

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Abstract. We investigate the stability of the Dirichlet eigenfunctions for the $p(\cdot)$-Laplacian under perturbations of the variable-exponent $p(\cdot)$.

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1. Introduction

For a real number $p \in (1, \infty)$ and a bounded domain $\Omega \subset \mathbb{R}^n$, the eigenvalue problem for the p-Laplacian

$$\Delta_p(u) := \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) \tag{1.1}$$

is given by

$$-\Delta_p(u) = \lambda |u|^{p-2} u. \tag{1.2}$$

Aside from being a generalization of the eigenvalue problem for the Laplacian, problem (1.2) can be viewed as the Euler–Lagrange equation for the optimization problem

$$\inf_{0 \neq u \in W_0^{1,p}(\Omega)} \frac{\| |\nabla u| \|_p}{\|u\|_p}; \tag{1.3}$$

indeed, the Fréchet derivative of the norm

$$u \longrightarrow \| |\nabla u| \|_p \tag{1.4}$$

on the Sobolev space $W_0^{1,p}(\Omega)$ is given by

$$(\text{grad}\, \|f\|_p)(x) = \frac{|f(x)|^{p-2} f(x)}{\|f\|_p^{p-1}}. \tag{1.5}$$

The reader is referred to [3] for a more general treatment of such problems. With the introduction of variable exponents Lebesgue spaces by Kováčik and Rákosník ([11]) a frenzy of generalizations to the new setting ensued.

The tempting natural extension that results from the mere replacement of p with $p(x)$ in problem (1.2) gives raise to what is best described as the modular version of the eigenvalue problem. However, due to the lack of homogeneity in the setting of variable p, the modular version is not even related to the optimization problem (1.3). The Euler–Lagrange equation for (1.3) in the variable exponent case relies on Theorem 3.2 (see also [6–8]). In [10], this alternative eigenvalue problem is presented. In this paper we prove that the eigenfunctions are stable with respect to the variable exponent p. For constant p, the question of stability was introduced and answered in [15]: The methods therein are not suitable for the treatment of the variable exponent case. Corresponding problems for the modular eigenvalue problem were studied in [12,14].

2. Modulars and Generalized Lebesgue Spaces

Throughout this paper $\Omega \subset \mathbb{R}^n$, $n \geq 1$ will stand for a bounded Lipschitz domain. In the sequel we will exclusively consider exponents

$$p : \Omega \to \mathbb{R} \ , \ \ p \in C(\overline{\Omega})$$

satisfying

$$1 < p_- = \inf_{\overline{\Omega}} p(x), \quad p_+ = \sup_{\overline{\Omega}} p(x) < \infty;$$

functions in this class will be referred to as admissible exponents. Denote by $L^{p(\cdot)}(\Omega)$ the set of all real-valued, Borel measurable functions on Ω for which

$$\rho_{p(\cdot)}(f) := \int_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

The function ρ_p is a convex monotone modular on $L^{p(\cdot)}(\Omega)$ and

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

defines a norm under which $L^{p(\cdot)}(\Omega)$ is a Banach, reflexive, uniformly convex space (see [17]). It is apparent that the latter coincides with the usual Lebesgue $L^p(\Omega)$ norm when p is constant; accordingly the family $\left(L^{p(\cdot)}(\Omega)\right)$ for varying p will be referred to as the generalized Lebesgue class in Ω. The generalized Sobolev class in Ω can be defined analogously, namely

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$$

endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \||\nabla u|\|_{L^{p(\cdot)}(\Omega)}.$$

The closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ will be denoted by $W_0^{1,p(\cdot)}(\Omega)$ and will be furnished with the norm

$$\|v\|_{W_0^{1,p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|\nabla v(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\}.$$

Theorem 2.1. *Let $p(\cdot)$ be an admissible exponent. Then the embedding*

$$E : W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L_{p(\cdot)}(\Omega)$$

is compact.

Corollary 2.2. (Poincaré's inequality) *There exists a positive constant C such that for all $u \in \overset{\circ}{W}{}^{1,p(\cdot)}(\Omega)$,*

$$\|u\|_{p(\cdot)} \leq C \||\nabla u|\|_{p(\cdot)}.$$

Proof. We refer the reader to [5,8,11] for the details of Theorem 2.1 and its Corollary. □

We highlight the inequalities

$$\min\left\{\rho_{p(\cdot)}^{\frac{1}{p_+}}(w), \rho_{p(\cdot)}^{\frac{1}{p_-}}(w)\right\} \leq \|w\|_{p(\cdot)} \leq \max\left\{\rho_{p(\cdot)}^{\frac{1}{p_+}}(w), \rho_{p(\cdot)}^{\frac{1}{p_-}}(w)\right\} \tag{2.1}$$

valid for any $w \in L^{p(\cdot)}(\Omega)$ (see [12]). The estimate of the norm of the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ resulting from the application of Hölder's inequality turns out to be too coarse for the present analysis; we accordingly include the following version of a more refined estimate first observed in [9, Lemma 4.1], (see also [12]):

Lemma 2.3. *For admissible exponents $p(\cdot)$ and $q(\cdot)$ with $p < q < p + \epsilon$ a.e. in Ω and a Borel-measurable function $f : \Omega \to \mathbb{R}$, one has the inequality:*

$$\int_\Omega |f(x)|^{p(x)}\, dx \leq \epsilon|\Omega| + \epsilon^{-\epsilon} \int_\Omega |f(x)|^{q(x)}\, dx.$$

Corollary 2.4. *If $p(\cdot)$ and $q(\cdot)$ are as above, then estimate*

$$\|E_{p,q}\| \leq \epsilon^{-\epsilon} + \epsilon|\Omega|$$

holds for the norm $\|E_{p,q}\|$ of the embedding

$$E_{p,q} : L^{q(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega).$$

3. Smoothness Properties of the Variable-Exponent Lebesgue Norms

In this section we state without proof the smoothness properties of the Lebesgue spaces with variable exponents.

Theorem 3.1. *Let $p(\cdot)$ be an admissible exponent. The following statements are equivalent:*

(i) *$L^{p(\cdot)}(\Omega)$ is reflexive;*
(ii) *$L^{p(\cdot)}(\Omega)$ and $L^{p'(\cdot)}(\Omega)$ have absolutely continuous norms;*
(iii) *$L^{p(\cdot)}(\Omega)$ is uniformly convex;*

Proof. See [8,17]. □

It is well known (see [8, Lemma 1.1, Theorem 1.2]) that the conditions in Theorem 3.1 imply that the norm $\|\cdot\|_{p(\cdot)}$ is Fréchet differentiable. The next result was first obtained in [6,7].

Theorem 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $p(\cdot)$ be admissible. Then for every $f \in L^{p(\cdot)}(\Omega)\backslash\{0\}$,*

$$(\text{grad}\, \|f\|_{p(\cdot)})(x) = \frac{p(x)\, \|f\|_{p(\cdot)}^{-p(x)}\, |f(x)|^{p(x)-1} \,\text{sgn}\; f(x)}{\int\limits_{\Omega} p(x)\, \|f\|_{p(\cdot)}^{-p(x)-1}\, |f(x)|^{p(x)}\, dx}. \tag{3.1}$$

Proof. Put

$$A(x) = \left(\frac{|f(x)|}{\|f\|}\right)^{p(x)-1} \,\text{sgn}\; f(x),\;\; B = \int\limits_{\Omega} p(x)\, \|f\|_{p(\cdot)}^{-p(x)-1}\, |f(x)|^{p(x)}\, dx.$$

Thus the right-hand side of (3.1) equals $p(x)\, \|f\|_{p(\cdot)}^{-1}\, A(x)/B$. Note that

$$\frac{p_-}{\|f\|_{p(\cdot)}} = \frac{p_-}{\|f\|_{p(\cdot)}} \rho_p \left(f/\, \|f\|_{p(\cdot)}\right) \le B \le \frac{p_+}{\|f\|_{p(\cdot)}}.$$

Moreover,

$$\rho_{p'}\,(A) = \int\limits_{\Omega} \left(\frac{|f(x)|}{\|f\|_{p(\cdot)}}\right)^{p(x)} dx = 1.$$

Hence the right-hand side of (3.1) represents an element of $L^{p'(\cdot)}(\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ and so can be identified with an element of the dual of $L^{p(\cdot)}(\Omega)$, the value of which at f is $\|f\|_p$. The result follows. □

An immediate consequence of Theorem 3.2 is the following:

Corollary 3.3. *The norm on $W_0^{1,p(\cdot)}(\Omega)$,*

$$u \longrightarrow \||\nabla u|\|_{p(\cdot)} = \|u\|_{1,p(\cdot)}$$

is Fréchet differentiable with derivative given by

$$(\text{grad}\, \||f|\|_{1,p(\cdot)})(x) = \frac{p(x)\, \||\nabla f|\|_{p(\cdot)}^{-p(x)}\, ||\nabla f(x)||^{p(x)-1} \,\text{sgn}\; \nabla f(x)}{\int_{\Omega} p(x)\, \||\nabla f|\|_{p(\cdot)}^{-p(x)-1}\, |\nabla f(x)|^{p(x)}\, dx}.$$

As it transpires from Theorem 2.1 and Corollary 2.2, there exists (at least) a function u_0 that minimizes the Rayleigh quotient, that is

$$\inf_{0\neq v \in \overset{\circ}{W}{}^1_{p(\cdot)}(\Omega)} \frac{\||\nabla v|\|_{p(\cdot)}}{\|v\|_{p(\cdot)}} = \frac{\||\nabla u_0|\|_{p(\cdot)}}{\|u_0\|_{p(\cdot)}} \tag{3.2}$$

Notice that u_0 can be chosen satisfying $\||\nabla u_0|\|_{p(\cdot)} = 1$. As shown in [3] (see also [8]), each such extremal function satisfies the Euler–Lagrange equation

$$\int_{\Omega} \text{grad}\left(\|u_0\|_{1,p(\cdot)}\right)(x) \nabla h(x)\, dx = \lambda_p \int_{\Omega} \text{grad}\left(\|u_0\|_{p(\cdot)}\right)(x) h(x)\, dx, \tag{3.3}$$

with $\lambda_p = \frac{1}{\|E\|}$,
where E is the embedding given in Theorem 2.1. In what follows we denote by $\mathcal{E}_{p(\cdot)}$ the eigenvalue problem (that follows from 3.3)

$$-\,\text{div}\left(|\nabla u|^{p(\cdot)-2}\, \nabla u\right) = \lambda_p\, |u|^{p(\cdot)-2}\, u \tag{3.4}$$

which generalizes the well-known eigenvalue problem for the p-Laplacian.

4. Stability of the Eigenfunctions

In this section we turn to the stability of the eigenvalue problem (3.4) under perturbation of the exponent p.

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary. Consider a non-decreasing sequence $(p_j)_j \subseteq C(\overline{\Omega})$ converging uniformly in Ω to its supremum $q \in C(\overline{\Omega})$. Assume further that $n < q_- \leq q_+ < \infty$. For each $j \in \mathbb{N}$ let u_j be the extremal function chosen in (3.2), with $\||\nabla u_j|\|_{p_j(\cdot)} = 1$. Then there exists a subsequence of $(p_j)_j$ (still denoted by $(p_j)_j$) and a function $u \in W_0^{1,q(\cdot)}(\Omega)$ with*

$$\||\nabla u|\|_{q(\cdot)} = 1$$

that minimizes the Rayleigh quotient (3.2), such that $u_j \rightharpoonup u$ in $W_0^{1,p_1(\cdot)}(\Omega)$. Moreover, if λ_i and λ_q denote the first eigenvalues of the problems $\mathcal{E}_{p_i(\cdot)}$ and $\mathcal{E}_{q(\cdot)}$ (see 3.3), respectively one has in addition:

$$\lambda_i \to \lambda_q \ \text{as} \ i \to \infty. \tag{4.1}$$

In particular, if $q_- \geq 2$, the following strong convergence holds:

$$\||\nabla u_j - \nabla u|\|_{q(\cdot)} \to 0 \ \text{as} \ j \to \infty. \tag{4.2}$$

Proof. In view of Lemma 2.3, for $j \leq i$ it holds that

$$\||\nabla u_i|\|_{p_j(\cdot)} \leq \left(\|p_i - p_j\|_\infty^{-\|p_i - p_j\|_\infty} + \|p_i - p_j\|_\infty |\Omega| \right). \tag{4.3}$$

Remark. The condition $q_{>}n$ in the above theorem cannot be removed, even in the case of constant exponent p. In [16] the author constructs a specific domain for which the theorem fails for a constant exponent $p : 1 < p \leq n$.

It follows that the sequence $(u_i)_{i \geq j}$ is bounded in $W_0^{1,p_j(\cdot)}(\Omega)$. By virtue of the compactness result in Theorem 2.1, the sequence $(u_i)_{i \geq j}$ can be assumed to be weakly convergent in $W_0^{1,p_j(\cdot)}(\Omega)$. A straightforward argument yields the validity of the assumption that $(u_i)_i$ is weakly convergent (in $W^{1,p_j(\cdot)}(\Omega)$) to

$$u \in \bigcap_1^\infty W^{1,p_j(\cdot)}(\Omega).$$

Next, we show that actually

$$u \in W^{1,q(\cdot)}(\Omega) \ \text{and} \ \||\nabla u|\|_{q(\cdot)} \leq 1; \tag{4.4}$$

in fact, by definition and in view of the inequalities given in (2.1), assertion (4.4) is an automatic consequence the next Lemma:

Lemma 4.2. *For u as in the previous paragraph,*

$$\int_\Omega |\nabla u|^q \leq 1 \tag{4.5}$$

Proof. For any admissible exponent r, the functional

$$F_r : W_0^{1,r(\cdot)}(\Omega) \longrightarrow [0,\infty) \tag{4.6}$$

$$F_r(v) = \int_\Omega |\nabla v|^r \tag{4.7}$$

is weakly-lower semicontinuous; this fact in conjunction with Fatou's Lemma and Lemma 2.3 yields

$$\begin{aligned} \int_\Omega |\nabla u|^q &\le \liminf_k \int_\Omega |\nabla u|^{p_k} \le \liminf_k \liminf_j \int_\Omega |\nabla u_j|^{p_k} \\ &\le \liminf_k \liminf_j \left(\|p_k - p_j\|_\infty^{-\|p_k-p_j\|_\infty} + \|p_k - p_j\|_\infty |\Omega| \right), \end{aligned} \tag{4.8}$$

from which (4.5) is an immediate consequence, since $p_i \to q$ uniformly in Ω. □

Next, we address the minimality of the Rayleigh quotient for u. Let $v \in W_0^{1,q(\cdot)}(\Omega)$ with $\||\nabla v|\|_{q(\cdot)} \le 1$; furthermore take $(v_k)_k \subset C_0^\infty(\Omega)$ with

$$v_k \to v \text{ and } \nabla v_k \to \nabla v \text{ in } L^{q(\cdot)}(\Omega).$$

For each fixed $k \in \mathbb{N}$, we underline the convergence statements (which follow directly from Lebesgue's dominated convergence theorem) for fixed k, one has

$$\int_\Omega |v_k|^{p_i} \to \int_\Omega |v_k|^q$$

as $i \to \infty$ and

$$\int_\Omega |\nabla v_k|^{p_i} \to \int_\Omega |\nabla v_k|^q \text{ as } i \to \infty. \tag{4.9}$$

A straightforward calculation shows that this implies

$$\|v_k\|_{p_i(\cdot)} \to \|v_k\|_{q(\cdot)}$$

and

$$\||\nabla v_k|\|_{p_i(\cdot)} \to \||\nabla v_k|\|_{q(\cdot)}$$

as $i \to \infty$.

On the other hand, the sequence $(u_i)_i \subset L^{p(\cdot)}(\Omega)$ can be considered to converge to u a.e.; accordingly and using the fact that $q_- > n$, which in particular implies

$$\|u_i\|_\infty \le C$$

for some positive constant C, independent of i, it follows that

$$\int_\Omega |u_i|^{p_i} \to \int_\Omega |u|^q \text{ as } i \to \infty, \tag{4.10}$$

which immediately yields

$$\|u_i\|_{p_i(\cdot)} \to \|u\|_{q(\cdot)} \text{ as } i \to \infty. \tag{4.11}$$

As a byproduct of (4.11) one has assertion (4.1):

$$\lambda_i = \frac{1}{\|u_i\|_{p_i(\cdot)}} \to \frac{1}{\|u\|_{q(\cdot)}} \text{ as } i \to \infty.$$

Hence, for fixed $\epsilon > 0$ and $i \in \mathbb{N}$ large enough, the minimal character of u_i yields

$$\frac{\|v_k\|_{q(\cdot)}}{\||\nabla v_k|\|_{q(\cdot)}} \leq \frac{\|v_k\|_{p_i(\cdot)}}{\||\nabla v_k|\|_{p_i(\cdot)}} + \epsilon \leq \|u_i\|_{p_i(\cdot)} + \epsilon. \tag{4.12}$$

Letting $i \to \infty$ in (4.12) it follows that

$$\frac{\|v_k\|_{q(\cdot)}}{\||\nabla v_k|\|_{q(\cdot)}} \leq \|u\|_{q(\cdot)} \tag{4.13}$$

and letting $k \to \infty$ in (4.13)

$$\frac{\|v\|_{q(\cdot)}}{\||\nabla v|\|_{q(\cdot)}} \leq \|u\|_{q(\cdot)}. \tag{4.14}$$

In particular, since $\||\nabla u|\|_{q(\cdot)} \leq 1$, one can set $u = v$ in the preceding inequality to get

$$\||\nabla u|\|_{q(\cdot)} \geq 1;$$

in all

$$\||\nabla u|\|_{q(\cdot)} = 1. \tag{4.15}$$

We now turn to the proof of (4.2). Invoking the weak-lower semicontinuity of the functional (4.6) it is clear that

$$\begin{aligned}\int_\Omega |\nabla u|^{p_k} &\leq \liminf_j \int_\Omega \left|\frac{\nabla(u+u_j)}{2}\right|^{p_k} \\ &\leq \liminf_j \left(\|p_k - p_j\|_\infty^{-\|p_k-p_j\|_\infty} \int_\Omega \left|\frac{\nabla(u+u_j)}{2}\right|^{p_j} + \|p_k-p_j\|_\infty |\Omega| \right).\end{aligned} \tag{4.16}$$

Convexity yields

$$\int_\Omega \left|\frac{\nabla(u+u_j)}{2}\right|^{p_j} \leq \frac{1}{2}\int_\Omega |\nabla u|^{p_j} + \frac{1}{2}\int_\Omega |\nabla u_j|^{p_j} = \frac{1}{2}\int_\Omega |\nabla u|^{p_j} + \frac{1}{2}, \tag{4.17}$$

Lebesgue's dominated convergence theorem implies furthermore

$$\lim_{j\to\infty} \int_\Omega |\nabla u|^{p_j} = \int_\Omega |\nabla u|^q \leq 1. \tag{4.18}$$

In all

$$\liminf_j \int_\Omega \left|\frac{\nabla(u+u_j)}{2}\right|^{p_j} \leq 1. \tag{4.19}$$

For an arbitrary $\delta > 0$, let I be so large that the conditions $j \geq I, k \geq I$ imply

$$\|p_k - p_j\|_\infty^{-\|p_j-p_k\|_\infty} < 1 + \delta \ , \ \ \|p_k - p_j\|_\infty |\Omega| < \delta$$

and

$$\liminf_j \int_\Omega \left|\frac{\nabla(u+u_j)}{2}\right|^{p_j} < \delta + \inf_{j\geq I} \int_\Omega \left|\frac{\nabla(u+u_j)}{2}\right|^{p_j}, \tag{4.20}$$

then, by virtue of (4.16), if $j, k \geq I$:

$$\int_\Omega |\nabla u|^{p_k} \leq \left((1+\delta)\int_\Omega \left|\frac{\nabla(u+u_j)}{2}\right|^{p_j} + \delta\right) \tag{4.21}$$
$$\leq (1+\delta)\left(\frac{1}{2}\int_\Omega |\nabla u|^{p_j} + \frac{1}{2}\int_\Omega |\nabla u_j|^{p_j}\right) + \delta.$$

The precedding inequality together with (4.19) yields

$$\lim_j \int_\Omega \left|\frac{\nabla(u+u_j)}{2}\right|^{p_j} = 1. \tag{4.22}$$

Using Clarkson's inequality one gets

$$\int_\Omega \left|\frac{\nabla u+\nabla u_j}{2}\right|^{p_j} + \int_\Omega \left|\frac{\nabla u-\nabla u_j}{2}\right|^{p_j} \leq \frac{1}{2}\int_\Omega |\nabla u|^{p_j} + \frac{1}{2}\int_\Omega |\nabla u_j|^{p_j}; \tag{4.23}$$

in conjunction with (4.22) , inequality (4.23) yields

$$\lim_j \int_\Omega \left|\frac{\nabla u-\nabla u_j}{2}\right|^{p_j} = 0,$$

which by virtue of (2.1) implies (4.2). □

We next investigate the spectral variation under the assumption of a non-increasing sequence of variable exponents.

Theorem 4.3. *Let $\Omega \subset \mathbf{R}^n$, be a bounded,Lipschitz domain. Consider a non-increasing sequence $(p_j) \subset C(\overline{\Omega})$ uniformly convergent in Ω to its infimum $p = \inf_j p_j$; assume $1 < p_- \leq p_+ < \infty$. Let λ_i and λ_p denote respectively the first eigenvalues of the problems $\mathcal{E}_{p_i(\cdot)}$ and $\mathcal{E}_{p(\cdot)}$. Let u_i be an eigenfunction corresponding to λ_i. Then, there exists $u \in W_0^{1,p(\cdot)}(\Omega)$ such that*

$$u_j \rightharpoonup u \ \text{ in } W_0^{1,p(\cdot)}(\Omega) \text{ and } u_j \to u \text{ strongly in } L^{p(\cdot)}(\Omega).$$

In addition, it holds that

$$\lambda_i \to \lambda_p, \ \text{ as } i \to \infty. \tag{4.24}$$

In particular, if $p_- \geq 2$, the following strong convergence holds:

$$\||\nabla u_j - \nabla u|\|_{q(\cdot)} \to 0 \ \text{ as } j \to \infty. \tag{4.25}$$

Proof. As was shown above (see 3.2), for each $i \in \mathbb{N}$ the function $u_i \in W_0^{1,p_i(\cdot)}(\Omega)$ that minimizes the *ith*-Rayleigh quotient can be chosen so that $\|u_i\|_{p_i(\cdot)} = 1$, that is:

$$\frac{\||\nabla u_i|\|_{p_i(\cdot)}}{\|u_i\|_{p_i(\cdot)}} = \inf_{0 \neq v \in \overset{\circ}{W}{}^1_{p_i(\cdot)}(\Omega)} \frac{\||\nabla v|\|_{p_i(\cdot)}}{\|v\|_{p_i(\cdot)}}. \tag{4.26}$$

We distinguish three mutually exclusive cases, namely $n \leq p_-$, $p_- < n \leq p_+$ and $p_+ < n$.

To address the case $p_+ < n$, we observe than under this condition, one has in Ω, $\frac{np}{n-p} > \frac{p_-^2}{n-p_+}$. It follows that uniformly in Ω,

$$p + \frac{p_-^2}{n-p_+} < \frac{np}{n-p}.$$

Select $I \in \mathbb{N}$ such that $i \geq I$ implies $p_i < p + \frac{p_-^2}{n-p_+}$ uniformly in Ω.

Sobolev's embedding theorem yields the existence of $u \in L^{\frac{np}{n-p}}(\Omega)$ such that $u_i \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$ and $u_i \to u$ strongly in $L^{\frac{np}{n-p}}(\Omega)$; moreover, since the unit ball is weakly compact in $W_0^{1,p(\cdot)}(\Omega)$,

$$\||\nabla u|\|_{p(\cdot)} \leq 1.$$

A straightforward application of Lebesgue's dominated convergence theorem yields

$$\int_\Omega |u|^{p_i} \to \int_\Omega |u|^p \ \text{ as } \ i \to \infty,$$

which readily implies

$$\|u\|_{p_i(\cdot)} \to \|u\|_{p(\cdot)} \ \text{ as } \ i \to \infty.$$

For $I \leq i \in \mathbb{N}$ it holds the inclusion

$$L^{\frac{np}{n-p}}(\Omega) \hookrightarrow L^{p_i}(\Omega).$$

Thus, provided $i \geq I$:

$$\|u_i\|_{p_i(\cdot)} \leq \|u_i - u\|_{p_i(\cdot)} + \|u\|_{p_i(\cdot)}. \tag{4.27}$$

Select $v \in W_0^{1,p(\cdot)}(\Omega)$ with $\||\nabla v|\|_{p(\cdot)} = 1$ and a sequence $(v_k)_k \subset C_0^\infty(\Omega)$ with $v_k \to v$ in $L^{p(\cdot)}(\Omega)$ and $\nabla v_k \to \nabla v$ in $L^{p(\cdot)}(\Omega)$. It is clear from Lebesgue's dominated convergence theorem that as $i \to \infty$,

$$\int_\Omega |v_k|^{p_i} \longrightarrow \int_\Omega |v_k|^p$$

and

$$\int_\Omega |\nabla v_k|^{p_i} \longrightarrow \int_\Omega |\nabla v_k|^p.$$

It is readily concluded that as $i \to \infty$,

$$\|v_k\|_{p_i(\cdot)} \to \|v_k\|_{p(\cdot)}$$

and

$$\|\nabla v_k\|_{p_i(\cdot)} \to \|\nabla v_k\|_{p(\cdot)}$$

Therefore, for a given $\epsilon > 0$ and k large enough,

$$\begin{aligned} \frac{\||\nabla v|\|_{p(\cdot)}}{\|v\|_{p(\cdot)}} &\geq \frac{\||\nabla v_k|\|_{p(\cdot)}}{\|v_k\|_{p(\cdot)}} - \epsilon \geq \frac{\||\nabla v_k|\|_{p_i(\cdot)}}{\|v_k\|_{p_i(\cdot)}} - \epsilon \\ &\geq \frac{\||\nabla u_i|\|_{p_i(\cdot)}}{\|u_i\|_{p_i(\cdot)}} - \epsilon = \frac{1}{\|u_i\|_{p_i(\cdot)}} - \epsilon; \end{aligned} \tag{4.28}$$

the latter in conjunction with (4.27) implies

$$\frac{\||\nabla v|\|_{p(\cdot)}}{\|v\|_{p(\cdot)}} \geq \frac{1}{\|u_i - u\|_{p_i(\cdot)} + \|u\|_{p_i(\cdot)}} + \epsilon; \tag{4.29}$$

in all

$$\frac{\||\nabla v|\|_{p(\cdot)}}{\|v\|_{p(\cdot)}} \geq \frac{1}{\|u\|_{p(\cdot)}}. \tag{4.30}$$

In particular, (4.30) holds for $v = u$, which immediately yields

$$\||\nabla u|\|_{p(\cdot)} \geq 1.$$

In all, the limit function u minimizes the Rayleigh quotient, i.e, (3.2) is valid taking $u = u_0$.

Consider now the second case, namely $n \in (p_-, p_+]$. As in the previous instance, a straightforward application of Sobolev's embedding theorem yields $u \in W_0^{1,p(\cdot)}(\Omega)$ and a subsequence $(u_i)_i$ convergent to u weakly in $W_0^{1,p(\cdot)}(\Omega)$ and strongly in $L^{p(\cdot)}(\Omega)$. Our first order of business will be to show that $u \in L^{p_I}(\Omega)$ for some $I \in \mathbb{N}$. To that effect we start by considering the function

$$S : [1, n) \longrightarrow \left[\frac{n}{n-1}, \infty\right) \ , \ S(x) = \frac{nx}{n-x}.$$

Select positive numbers δ, ϵ satisfying the conditions

$$\epsilon < \min\left\{\frac{1}{n-1}, \frac{p_-^2}{n-p_-}\right\} \ , \ p_- - \delta > 1 \ \text{ and } \ n + \delta < \frac{n(n-\delta)}{\delta}. \qquad (4.31)$$

Set $q_1 = p_p - \delta$, $r_1 = S^{-1}(S(q_1) + \epsilon))$, choose $q_2 \in (q_1, r_1)$ and $r_2 = S^{-1}(S(q_2) + \epsilon))$. For $j \geq 3$, let $q_j \in (r_{j-2}, r_{j-1})$ and

$$r_j = S^{-1}(S(q_j) + \epsilon)).$$

It is clear that

$$r_j \longrightarrow n \ \text{ as } \ j \longrightarrow \infty;$$

let r_M be the first term in the sequence that exceeds $n - \delta$. The collection

$$\left\{p^{-1}((q_j, r_j)) \ , j = 1, 2, \ldots, M\right\} \bigcup \qquad (4.32)$$
$$\left\{p^{-1}((q_M, r_M)), p^{-1}((n-\delta, n+\delta)), p^{-1}\left((n + \frac{\delta}{2}, \infty),\right)\right\}$$

is an open covering of Ω; for the sake of notational uniformity let's establish the following convention:

$$q_{M+1} = n - \delta \ , \ r_{M+1} = n + \delta \ , \ q_{M+2} = n + \frac{\delta}{2} \ , \ r_{M+2} = \infty$$

and

$$\Omega_k = p^{-1}((q_k, r_k)).$$

Let $(\varphi_k)_{1 \leq k \leq M+2}$ be a (smooth) partition of unity subordinated to the covering $(\Omega_k)_k$. For each $j = 1, 2, \ldots, M+2$ the sequence $(u_i\varphi_j)_i$ converges in $L^{p(\cdot)}(\Omega)$ to $u\varphi_j$. Moreover,

$$(u_i\varphi_j)_i \subset W_0^{1,q_j}(\Omega)$$

and

$$\|u_i\varphi_j\|_{1,q_j,\Omega} \leq C$$

for a positive constant C independent of i. Because of the compactness of the embedding

$$W_0^{1,q_j}(\Omega) \hookrightarrow L^{\frac{nq_j}{n-q_j}}(\Omega)$$

it follows that without loss of generality that for each $j = 1, 2, \dots, M$,

$$(u_i\varphi_j)_i$$

can be considered to be convergent in $L^{\frac{nq_j}{n-q_j}}(\Omega)$, say to v. Clearly, v is supported in Ω_j and denoting by χ_k the indicator function of Ω_k, one has

$$p\chi_j \le \frac{nq_j}{n-q_j}. \tag{4.33}$$

Thus, $v \in L^{p(\cdot)}(\Omega)$, since

$$\int_\Omega |v|^p = \int_\Omega |v|^{p\chi_j} \le \left(\|v\|_{p\chi_j}\right)^\alpha \le C\left(\|v\|_{\frac{nq_j}{n-q_j}}\right)^\alpha < \infty \tag{4.34}$$

for some α independent of i, hence

$$\int_\Omega |u_i\varphi_j - v|^p = \int_\Omega |u_i\varphi_j - v|^{p\chi_j} \le \left(\|u_i\varphi_j - v\|_{p\chi_j}\right)^\alpha \le C\|u_i\varphi_j - v\|_{\frac{nq_j}{n-q_j}}.$$

It follows that $v = u\varphi_j \in L^{\frac{nq_j}{n-q_j}}(\Omega)$.

Likewise, since for any $w \in W_0^{1,p(\cdot)}(\Omega)$ one has

$$1 = \int_\Omega \left|\frac{\nabla(w\varphi_{M+1})}{\|\nabla(w\varphi_{M+1})\|_{p(\cdot)}}\right|^p = \int_\Omega \left|\frac{\nabla(w\varphi_{M+1})}{\|\nabla(w\varphi_{M+1})\|_{p(\cdot)}}\right|^{p\chi_{M+1}}$$

and

$$q_{M+1} < p\chi_{M+1} < r_{M+1} < \frac{nr_{M+1}}{n-r_{M+1}} - \epsilon = \frac{nq_{M+1}}{n-q_{M+1}}$$

it follows that

$$\|\nabla(w\varphi_{M+1})\|_{q_{M+1}(\cdot)} \le C\|\nabla(w\varphi_{M+1})\|_{\chi_{M+1}p(\cdot)} \le C\|\nabla(w\varphi_{M+1})\|_{p(\cdot)}.$$

Thus, for some positive constant C independent of i,

$$\|u_i\varphi_{M+1}\|_{1,n-\delta,\Omega} \le C,$$

whence $(u_i\varphi_{M+1})_i$ can be considered to converge strongly in

$$L^{\frac{n(n-\delta)}{\delta}}(\Omega) \hookrightarrow L^{n+\delta}(\Omega).$$

As in (4.34), a straightforward calculation shows that

$$v \in L^{p(\cdot)}(\Omega);$$

consequently

$$v = u\varphi_{M+1} \in L^{\frac{n(n-\delta)}{\delta}}(\Omega).$$

Along the same lines it can be readily shown that

$$u\varphi_{M+2} \in C(\overline{\Omega}).$$

Let $I \in \mathbb{N}$ be large enough so that uniformly in Ω,

$$p_I < p + \frac{1}{2}\min\left\{\frac{nq_j}{n-q_j} - p\chi_j \;,\; j = 1, 2, \dots M+1\right\}.$$

Then, for $i = 1, 2, \dots, M+2$

$$u\varphi_j \in L^{p_I(\cdot)}(\Omega),$$

in all:

$$u = \sum_{j=1}^{M+2} u\varphi_j \in L^{p_I(\cdot)}(\Omega).$$

Next, we observe that for $i \geq I$,

$$\|u_i\|_{p_i(\cdot)} \leq \|u_i - u\|_{p_i(\cdot)} + \|u\|_{p_i(\cdot)}. \tag{4.35}$$

Since

$$\|u_i - u\|_{p_i(\cdot)} \longrightarrow 0 \text{ as } i \longrightarrow \infty$$

and

$$\|u\|_{p_i(\cdot)} \longrightarrow \|u\|_{p(\cdot)} \text{ as } i \longrightarrow \infty,$$

letting $i \to \infty$ in (4.35) one has

$$\|u_i\|_{p_i(\cdot)} \longrightarrow \|u\|_{p(\cdot)} \text{ as } i \longrightarrow \infty. \tag{4.36}$$

Next, the argument following (4.27) shows that u satisfies the required minimization property and that

$$\||\nabla u|\|_{p(\cdot)} = 1.$$

Finally, (4.36) is obviously valid if one assumes $n \leq p_-$, for then the inclusion $W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact and in this setting, convergence statements refer to uniform convergence in Ω. The strong convergence (4.25) is obtained along the same lines as the corresponding part of Theorem 4.1. This last observation ends the proof of Theorem 4.3. □

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