Integr. Equ. Oper. Theory 83 (2015), 393–411
DOI 10.1007/s00020-015-2239-0
Published online May 10, 2015

Integral Equations and Operator Theory

Sharp Estimates for Singular Values of Hankel Operators

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Abstract. We consider compact Hankel operators realized in $\ell^2(\mathbb{Z}_+)$ as infinite matrices Γ with matrix elements $h(j+k)$. Roughly speaking, we show that, for all $\alpha > 0$, the singular values s_n of Γ satisfy the bound $s_n = O(n^{-\alpha})$ as $n \to \infty$ provided $h(j) = O(j^{-1}(\log j)^{-\alpha})$ as $j \to \infty$. These estimates on s_n are sharp in the power scale of α. Similar results are obtained for Hankel operators $\mathbf{\Gamma}$ realized in $L^2(\mathbb{R}_+)$ as integral operators with kernels $\mathbf{h}(t+s)$. In this case the estimates of singular values of $\mathbf{\Gamma}$ are determined by the behavior of $\mathbf{h}(t)$ as $t \to 0$ and as $t \to \infty$.

Mathematics Subject Classification. Primary 47B35; Secondary 47B10.

Keywords. Hankel operators, discrete and continuous representations, singular values, Schatten classes.

1. Introduction

1.1. Basic Notions

The theory of Hankel operators exists in two representations: discrete and continuous. In the discrete representation, one starts with a sequence of complex numbers $\{h(j)\}_{j=0}^{\infty}$, and one formally defines the Hankel operator $\Gamma(h)$ in $\ell^2(\mathbb{Z}_+)$ as the "infinite matrix" $\{h(j+k)\}_{j,k=0}^{\infty}$, i.e.

$$(\Gamma(h)u)(j) = \sum_{k=0}^{\infty} h(j+k)u(k), \quad u = (u(0), u(1), \ldots). \tag{1.1}$$

The Nehari theorem says that the Hankel operator $\Gamma(h)$ is bounded on $\ell^2(\mathbb{Z}_+)$ if and only if the symbol of $\Gamma(h)$, defined by

$$\omega(\mu) = \sum_{j=0}^{\infty} h(j)\mu^j, \quad |\mu| = 1, \tag{1.2}$$

belongs to the class $\mathrm{BMO}(\mathbb{T})$ of functions of the bounded mean oscillation on the unit circle $\mathbb{T}$. A simple sufficient condition for the boundedness of $\Gamma(h)$ is the estimate $h(j) = O(j^{-1})$ as $j \to \infty$.

In the continuous representation, one starts with a function $\mathbf{h} \in L^1_{\rm loc}(\mathbb{R}_+)$ ($\mathbb{R}_+ = (0,\infty)$), and the integral Hankel operator $\mathbf{\Gamma}(\mathbf{h})$ in $L^2(\mathbb{R}_+)$ with the kernel $\mathbf{h}$ is given by the formula

$$(\mathbf{\Gamma}(\mathbf{h})\mathbf{u})(t) = \int_0^\infty \mathbf{h}(t+s)\mathbf{u}(s)ds. \tag{1.3}$$

Similarly to the discrete case, the Hankel operator $\mathbf{\Gamma}(\mathbf{h})$ is bounded on $L^2(\mathbb{R}_+)$ if and only if the corresponding symbol (i.e. the Fourier transform of $\mathbf{h}$ understood in the sense of distributions) belongs to the class $\mathrm{BMO}(\mathbb{R})$. A simple sufficient condition for the boundedness of $\mathbf{\Gamma}(\mathbf{h})$ is the estimate

$$|\mathbf{h}(t)| \leq C/t, \quad t > 0.$$

Throughout the paper, we will use the boldface font for objects associated with the continuous representation.

1.2. A Conjecture

Let ω be the symbol (1.2) of a Hankel operator $\Gamma(h)$, and let $\mathbf{S}_p$ be the Schatten class of compact operators (see Sect. 1.4). V. Peller has shown that

$$\Gamma(h) \in \mathbf{S}_p \quad \Leftrightarrow \quad \omega \in B^{1/p}_{pp}(\mathbb{T}), \quad p > 0, \tag{1.4}$$

where $B^{1/p}_{pp}(\mathbb{T})$ is the Besov space; see the book [7] for the proof, the history and references to the relevant papers of other authors. By using the real interpolation between Besov spaces, V. Peller has deduced from (1.4) a necessary and sufficient condition (given by the finiteness of the expression (4.3)) for the estimate

$$s_n(\Gamma(h)) = O(n^{-\alpha}), \quad n \to \infty, \qquad \alpha > 0, \tag{1.5}$$

for the singular values of $\Gamma(h)$; we refer again to the book [7] for the details. This condition is stated in terms of the inclusion of ω into a certain function class of the Besov-Lorentz type denoted in [7, Sect. 6.5] by $\mathfrak{B}^{1/p}_{p,\infty}$ where $p = 1/\alpha$. Similar results exist in the continuous case.

Our aim here is to give a simple sufficient condition for (1.5) directly in terms of the sequence $h(j)$. It is expected that the faster rate of convergence $h(j) \to 0$ as $j \to \infty$ implies the faster rate of convergence of the singular values $s_n(\Gamma(h)) \to 0$ as $n \to \infty$. We show that the correct condition on the decay of $h(j)$ is given in the logarithmic scale. To be more precise, we discuss the following

Conjecture

$$h(j) = O(j^{-1}(\log j)^{-\alpha}) \quad \Rightarrow \quad s_n(\Gamma(h)) = O(n^{-\alpha}), \qquad \alpha > 0. \tag{1.6}$$

Let us consider two special cases that motivate this conjecture.

(i) $\alpha = 0$. It is well known (see, e.g., [7]) that the Hankel operator $\Gamma(h)$ (the Hilbert matrix) corresponding to the sequence

$$h(j) = \frac{1}{j+1}, \qquad j \geq 0, \tag{1.7}$$

is bounded (but not compact). It follows that

$$h(j) = O(1/j), \quad j \to \infty \quad \Rightarrow \quad \Gamma(h) \in \mathcal{B} \tag{1.8}$$

($\mathcal{B}$ is the class of bounded operators).

(ii) $\alpha > 1/2$. A Hankel operator Γ belongs to the Hilbert-Schmidt class $\mathbf{S}_2$ if and only if

$$\sum_{n=1}^{\infty} s_n(\Gamma(h))^2 = \sum_{j=0}^{\infty} (j+1)|h(j)|^2 < \infty. \tag{1.9}$$

Obviously, the series in the r.h.s. converges if $h(j) = O(j^{-1}(\log j)^{-\alpha})$ for some $\alpha > 1/2$, and the series in the l.h.s. converges if $s_n(\Gamma(h)) = O(n^{-\alpha})$ for some $\alpha > 1/2$.

The main purpose of this paper is to show that the above conjecture is partially true. More precisely, we prove that the conjecture is true for $\alpha < 1/2$; for $\alpha \geq 1/2$, we prove that the conclusion of (1.6) becomes true if we assume that the sequence $h(j)$ behaves sufficiently regularly, i.e. if we impose appropriate additional assumptions on the sequence of differences $h(j+1) - h(j)$ and on its higher order iterates. We also obtain analogous results in the continuous case. Precise statements are given in Sect. 2.

Let us comment on the proofs. For $\alpha \geq 1/2$ we deduce our results from Peller's necessary and sufficient condition $\omega \in \mathfrak{B}^{1/p}_{p,\infty}$ for the estimate (1.5). For $\alpha < 1/2$ our approach is more direct and relies on the real interpolation between the cases (i) and (ii) (where α is arbitrarily close to $1/2$) mentioned above.

1.3. Discussion

To a large extent, our aim is to provide technical tools for [8], where we study the asymptotic behavior of eigenvalues of compact self-adjoint Hankel operators. In particular, in [8] we show that for the sequence

$$h(j) = j^{-1}(\log j)^{-\alpha}, \qquad j \geq 2, \qquad \alpha > 0, \tag{1.10}$$

the asymptotics

$$s_n(\Gamma(h)) = v(\alpha) n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty, \tag{1.11}$$

holds with the explicit constant $v(\alpha)$ given by

$$v(\alpha) = 2^{-\alpha}\pi^{1-2\alpha}\left(B\left(\tfrac{1}{2\alpha}, \tfrac{1}{2}\right)\right)^{\alpha}, \tag{1.12}$$

where $B(\cdot, \cdot)$ is the standard Beta function. Clearly, (1.10), (1.11) show that the exponent α in the right-hand side of (1.6) is optimal in the class of Hankel operators we consider.

Our sufficient condition for (1.5) in terms of the sequence h is quite explicit. However, it is far from being necessary because we do not take into account possible oscillations of $h(j)$. In order to illustrate this point, let us observe that in the limit $\alpha \to 0$ our results reduce to the well-known implication (1.8). There are many sequences that fail to satisfy $h(j) = O(j^{-1})$ but such that $\Gamma(h) \in \mathcal{B}$. Consider, for example, $h(j) = n^{-2}$ for $j = n^4$, $n \in \mathbb{N}$, and $h(j) = 0$ otherwise. Obviously, the estimate $h(j) = O(j^{-1})$ fails, but the symbol (1.2) is bounded in the unit disc and hence $\Gamma(h) \in \mathcal{B}$ by the Nehari theorem.

1.4. Schatten Classes

Let us recall some basic information on ideals of compact operators in a Hilbert space (see the books [2,5]). We denote by $\mathcal{B}$ the set of all bounded operators, $\|\cdot\|$ is the operator norm; $\mathbf{S}_\infty$ is the set of all compact operators. Let $\{s_n(\Gamma)\}_{n=1}^\infty$ be the non-increasing sequence of singular values of $\Gamma \in \mathbf{S}_\infty$ (i.e. the eigenvalues of $\sqrt{\Gamma^*\Gamma}$). For $p > 0$, the Schatten class $\mathbf{S}_p$ and the weak Schatten class $\mathbf{S}_{p,\infty}$ of compact operators are defined by the conditions

$$\Gamma \in \mathbf{S}_p \quad \Leftrightarrow \quad \|\Gamma\|_{\mathbf{S}_p}^p := \sum_{n=1}^\infty s_n(\Gamma)^p < \infty$$

and

$$\Gamma \in \mathbf{S}_{p,\infty} \quad \Leftrightarrow \quad \|\Gamma\|_{\mathbf{S}_{p,\infty}} := \sup_{n\geq 1} n^{1/p} s_n(\Gamma) < \infty.$$

The classes $\mathbf{S}_p$ and $\mathbf{S}_{p,\infty}$ are the ideals of the algebra $\mathcal{B}$ with the quasi-norms $\|\cdot\|_{\mathbf{S}_p}$ and $\|\cdot\|_{\mathbf{S}_{p,\infty}}$. The class $\mathbf{S}_{p,\infty}^0$ is the closed linear subspace of $\mathbf{S}_{p,\infty}$ defined by

$$\Gamma \in \mathbf{S}_{p,\infty}^0 \quad \Leftrightarrow \quad \lim_{n\to\infty} n^{1/p} s_n(\Gamma) = 0.$$

Equivalently, $\mathbf{S}_{p,\infty}^0$ may be defined as the closure of the set of all finite rank operators in the quasi-norm $\|\cdot\|_{\mathbf{S}_{p,\infty}}$. We have

$$\mathbf{S}_p \subset \mathbf{S}_{p,\infty}^0 \subset \mathbf{S}_{p,\infty} \subset \mathbf{S}_\infty.$$

1.5. Plan of the Paper

We state our main results in Sect. 2. Their proofs are given in Sects. 3 and 4 for the continuous and discrete cases, respectively. It is convenient to start the proofs with the continuous case because integration by parts is more visual than the corresponding procedure (the Abel transformation for series) in the discrete case.

Throughout the rest of the paper, C (possibly with indices) denotes constants in estimates, and the value of C may change from line to line. Notation $|X|$ means the Lebesgue measure of the set $X \subset \mathbb{T}$ or of $X \subset \mathbb{R}$. We make a standing assumption that the exponents $p > 0$ and $\alpha > 0$ are related by $\alpha = 1/p$.

2. Main Results

2.1. Discrete Representation

Let the Hankel operator $\Gamma(h)$ be defined by formula (1.1) in the space $\ell^2(\mathbb{Z}_+)$. First we justify the conjecture (1.6) for $\alpha < 1/2$. This case turns out to be significantly simpler. Here $p > 2$ and $\mathbf{S}_{p,\infty} \not\subset \mathbf{S}_2$.

Theorem 2.1. *Let $\alpha < 1/2$ and let $\{h(j)\}_{j=0}^{\infty}$ be a sequence of complex numbers such that*

$$h(j) = O(j^{-1}(\log j)^{-\alpha}), \quad j \to \infty. \tag{2.1}$$

Then the singular values of the corresponding Hankel operator $\Gamma(h)$ satisfy the estimate

$$s_n(\Gamma(h)) = O(n^{-\alpha}), \quad n \to \infty. \tag{2.2}$$

Moreover, there is a constant $C(\alpha)$ such that

$$\|\Gamma(h)\|_{\mathbf{S}_{p,\infty}} \leq C(\alpha) \sup_{j\geq 0}(j+1)(\log(j+2))^{\alpha}|h(j)|, \quad p = 1/\alpha.$$

Next, consider the case $\alpha \geq 1/2$. Here, besides (2.1), we require some additional assumptions. For a sequence h, we denote by $h^{(m)}$, $m = 0, 1, 2, \dots$ the sequences of iterated differences. Those are the sequences defined iteratively by setting $h^{(0)}(j) = h(j)$ and

$$h^{(m)}(j) = h^{(m-1)}(j+1) - h^{(m-1)}(j), \quad j \geq 0.$$

The number of times we need to iterate will be determined by the integer

$$M(\alpha) = \begin{cases} [\alpha] + 1, & \text{if } \alpha \geq 1/2, \\ 0, & \text{if } \alpha < 1/2, \end{cases} \tag{2.3}$$

where $[\alpha] = \max\{m \in \mathbb{Z}_+ : m \leq \alpha\}$. We observe that if a sequence h is given explicitly by $h(j) = j^{-1}(\log j)^{-\alpha}$ for all sufficiently large j, then it satisfies

$$h^{(m)}(j) = O(j^{-1-m}(\log j)^{-\alpha}), \quad j \to \infty, \tag{2.4}$$

for all $m \geq 0$.

The following result includes Theorem 2.1 as a particular case.

Theorem 2.2. *Let $\alpha > 0$, and let $M = M(\alpha)$ be defined by (2.3). Let h be a sequence of complex numbers that satisfies (2.4) for all $m = 0, 1, \dots, M$. Then the estimate (2.2) holds, and there is a constant $C(\alpha)$ such that*

$$\|\Gamma(h)\|_{\mathbf{S}_{p,\infty}} \leq C(\alpha) \sum_{m=0}^{M} \sup_{j\geq 0}(j+1)^{1+m}(\log(j+2))^{\alpha}|h^{(m)}(j)|, \quad p = 1/\alpha. \tag{2.5}$$

Theorem 2.3. *If (2.2) holds with o instead of O for all $m = 0, 1, \dots, M$, then we have*

$$s_n(\Gamma(h)) = o(n^{-\alpha}), \quad n \to \infty. \tag{2.6}$$

Theorems 2.1, 2.2 and 2.3 are proven in Sect. 4. As was already mentioned, Theorem 2.1 admits a direct proof based on the real interpolation between the cases $\Gamma(h) \in \mathbf{S}_2$ and $\Gamma(h) \in \mathcal{B}$. In the proof of Theorem 2.2, we proceed from the results of [7] which give necessary and sufficient conditions for $\Gamma(h) \in \mathbf{S}_p$ and hence for $\Gamma(h) \in \mathbf{S}_{p,\infty}$ in terms of the symbol (1.2) of this operator. We prove that under the hypothesis of Theorem 2.2, such conditions are satisfied. Theorem 2.3 is deduced from Theorem 2.2 by simple approximation arguments.

Remark 2.4. 1. As already mentioned above, relations (1.10) and (1.11) show that the exponent α in (2.2) is optimal.

2. Theorem 2.2 is false if no conditions on the iterated differences $h^{(m)}(j)$ are imposed. Further, while our condition on the exponent $M(\alpha)$ is probably not optimal, it is not far from being so. Indeed, Example 4.7 shows that, for $\alpha \geq 2$, one cannot take $M(\alpha) = [\alpha] - 2$ in Theorem 2.2. The same example shows that for $\alpha > 1$ one cannot take $M(\alpha) = 0$. Probably, it is also impossible for $\alpha \in [1/2, 1]$, but we do to have the corresponding counter-example.
3. Some sufficient conditions for the inclusion $\Gamma(h) \in \mathbf{S}_1$, stated in terms of the sequences h, $h^{(1)}$ and $h^{(2)}$ were found in [3]. They are similar in spirit to Theorem 2.2.
4. If $h(j) = O(j^{-\gamma})$ for some $\gamma > 1$ as $j \to \infty$ and if some conditions on the iterated differences $h^{(m)}(j)$ are satisfied, then one can expect that the singular values $s_n(\Gamma(h))$ decay faster than any power of n^{-1} as $n \to \infty$. In fact, H. Widom showed in [9] that for $h(j) = (j+1)^{-\gamma}$, $\gamma > 1$, the corresponding Hankel operator $\Gamma(h)$ is non-negative and its eigenvalues converge to zero as
$$\lambda_n^+(\Gamma(h)) = \exp\left(-\pi\sqrt{2\gamma n} + o(\sqrt{n})\right), \quad n \to \infty.$$
Some additional results in this direction were obtained in [6].

If a sequence $h(j)$ satisfies (2.4) for $m = 0$ and if $\zeta \in \mathbb{T}$, then the sequence $\zeta^j h(j)$ satisfies the same condition; but for $m > 0$ this implication is no longer true. Nevertheless we have the following simple generalization of Theorems 2.2 and 2.3.

Theorem 2.5. *Let the sequences $h_1, h_2, \dots, h_L$ satisfy the hypothesis of Theorem 2.2 (resp. Theorem 2.3), and let $\zeta_\ell \in \mathbb{T}$, $\ell = 1, \dots, L$. Then the estimate (2.2) (resp. (2.6)) holds true for the Hankel operator $\Gamma(h)$ corresponding to the sequence*
$$h(j) = \sum_{\ell=1}^{L} \zeta_\ell^j h_\ell(j), \quad \zeta_\ell \in \mathbb{T}. \tag{2.7}$$

Proof. For a sequence h and for $\zeta \in \mathbb{T}$, we denote by q_ζ the sequence $q_\zeta(j) = \zeta^j h(j)$. Let U_ζ be the unitary operator in $\ell^2(\mathbb{Z}_+)$ given by
$$(U_\zeta f)(j) = \zeta^j f(j), \quad j \geq 0.$$
By inspection we have
$$\Gamma(q_\zeta) = U_\zeta \Gamma(h) U_\zeta \tag{2.8}$$
and therefore $s_n(\Gamma(q_\zeta)) = s_n(\Gamma(h))$ for all n.

Since the classes $\mathbf{S}^0_{p,\infty}$ and $\mathbf{S}_{p,\infty}$ are linear spaces, the estimate (2.2) (resp. (2.6)) for the operators $\Gamma(h_\ell)$ extends to the sum
$$\Gamma(h) = \sum_{\ell=1}^{L} U_{\zeta_\ell} \Gamma(h_\ell) U_{\zeta_\ell}.$$
This concludes the proof. □

Of course, instead of a finite sum in (2.7) one can consider infinite series or integrals.

2.2. Continuous Representation

Now the Hankel operator $\mathbf{\Gamma}(\mathbf{h})$ is defined by formula (1.3) in the space $L^2(\mathbb{R}_+)$.

In the discrete representation, the spectral properties of $\Gamma(h)$ are determined by the asymptotic behaviour of the sequence $h(j)$ as $j \to \infty$. In the continuous representation, the behaviour of the kernel $\mathbf{h}(t)$ for $t \to 0$ and for $t \to \infty$ as well as the local singularities of $\mathbf{h}$ contribute to the spectral properties of the Hankel operator $\mathbf{\Gamma}(\mathbf{h})$. Therefore we impose some local smoothness conditions on $\mathbf{h}(t)$, but our main focus is the behaviour of $\mathbf{h}(t)$ as $t \to 0$ and $t \to \infty$.

Recall (see, e.g., [7]) that the Carleman operator, corresponding to the kernel $\mathbf{h}(t) = 1/t$, is bounded. From here, similarly to (1.8), one easily obtains

$$|\mathbf{h}(t)| \le C/t \quad \Rightarrow \quad \mathbf{\Gamma}(\mathbf{h}) \in \mathcal{B}. \tag{2.9}$$

In the continuous case, the Hilbert-Schmidt condition is given by

$$\sum_{n=1}^{\infty} s_n(\mathbf{\Gamma}(\mathbf{h}))^2 = \int_0^{\infty} t|\mathbf{h}(t)|^2 dt < \infty. \tag{2.10}$$

Of course, this condition is satisfied if we have $\mathbf{h} \in L^2_{\rm loc}(\mathbb{R}_+)$ and $\mathbf{h}(t) = O(t^{-1}|\log t|^{-\alpha})$ for some $\alpha > 1/2$ as $t \to 0$ and for $t \to \infty$.

This suggests that one should consider kernels $\mathbf{h}(t)$ that are logarithmically "smaller" than $1/t$ both for $t \to 0$ and for $t \to \infty$. Indeed, the analogue of the estimate (2.1) in the continuous case is

$$|\mathbf{h}(t)| \le A_0 t^{-1}\langle \log t\rangle^{-\alpha}, \quad t > 0; \tag{2.11}$$

here and in what follows we use the notation $\langle x\rangle = (|x|^2+1)^{1/2}$. For $\alpha < 1/2$, condition (2.11) suffices; for $\alpha \ge 1/2$, we also need additional conditions on the derivatives $\mathbf{h}^{(m)}(t) = (d/dt)^m \mathbf{h}(t)$. The following result is the "continuous analogue" of Theorem 2.2.

Theorem 2.6. *Let $\alpha > 0$ and let $M = M(\alpha)$ be the integer given by (2.3). Let $\mathbf{h}$ be a complex valued function in $L^\infty_{\rm loc}(\mathbb{R}_+)$; if $\alpha \ge 1/2$, suppose also that $\mathbf{h} \in C^M(\mathbb{R}_+)$. Assume that*

$$|\mathbf{h}^{(m)}(t)| \le A_m t^{-1-m}\langle \log t\rangle^{-\alpha}, \quad t > 0, \tag{2.12}$$

with some constants $A_0, \dots, A_M$ for all $m = 0, \dots, M$. Then the singular values of the corresponding Hankel operator $\mathbf{\Gamma}(\mathbf{h})$ satisfy

$$s_n(\mathbf{\Gamma}(\mathbf{h})) = O(n^{-\alpha}), \quad n \to \infty \tag{2.13}$$

and, for some constant $C(\alpha)$,

$$\|\mathbf{\Gamma}(\mathbf{h})\|_{\mathbf{S}_{p,\infty}} \le C(\alpha)(A_0 + \cdots + A_M), \quad p = 1/\alpha.$$

Theorem 2.7. *In addition to the hypothesis of Theorem 2.6, assume that*

$$|\mathbf{h}^{(m)}(t)| = o\left(t^{-1-m}\langle \log t\rangle^{-\alpha}\right) \quad \text{as } t \to 0 \text{ and as } t \to \infty. \tag{2.14}$$

Then

$$s_n(\mathbf{\Gamma}(\mathbf{h})) = o(n^{-\alpha}), \quad n \to \infty. \tag{2.15}$$

Theorems 2.6 and 2.7 are proven in Sect. 3. Their proofs are similar to those in the discrete case. In particular, for $\alpha < 1/2$, Theorem 2.6 admits a direct proof based on the real interpolation between the Hilbert-Schmidt condition (2.10) and the sufficient condition (2.9) for the boundedness of $\mathbf{\Gamma}(\mathbf{h})$. In the case $\alpha \geq 1/2$, we proceed from the results of [7] which give necessary and sufficient conditions for $\mathbf{\Gamma}(\mathbf{h}) \in \mathbf{S}_p$ and hence for $\mathbf{\Gamma}(\mathbf{h}) \in \mathbf{S}_{p,\infty}$. We prove that under the hypothesis of Theorem 2.6, such conditions are satisfied. Theorem 2.7 is deduced from Theorem 2.6 by simple approximation arguments.

Remark 2.8. 1. The exponent α in (2.13) is optimal. Indeed, let $\mathbf{h}(t)$ be a sufficiently smooth real valued function such that for some $\alpha > 0$

$$\mathbf{h}(t) = t^{-1}|\log t|^{-\alpha} \tag{2.16}$$

for all sufficiently small t, and $\mathbf{h}(t) = 0$ for all sufficiently large t. Then it follows from the results of [8] that

$$s_n(\mathbf{\Gamma}(\mathbf{h})) = v(\alpha)n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty, \tag{2.17}$$

where the constant $v(\alpha)$ is given by (1.12). Similarly, if (2.16) holds for all large t and $\mathbf{h}(t) = 0$ for all small t, then again by the results of [8] we obtain (2.17).

2. Some sufficient conditions for the estimate $s_n(\mathbf{\Gamma}(\mathbf{h})) = O(n^{-\alpha})$ in terms of the smoothness of $\mathbf{h}$ were obtained in [4], see, e.g., Corollary 4.6 there. These conditions require that $\mathbf{h}(t)$ vanish very fast as $t \to \infty$ but allow for some singular behaviour as $t \to 0$. These results are somewhat similar to Theorem 2.6 but are less sharp.

For a function $\mathbf{h}(t)$ and for $a \in \mathbb{R}$, let us denote $\mathbf{q}_a(t) = e^{iat}\mathbf{h}(t)$. If $\mathbf{h}$ satisfies (2.12) for some $m > 0$, then $\mathbf{q}_a$ does not necessarily satisfy the same condition. Nevertheless, similarly to the discrete case, the following simple argument allows us to extend our results to $\mathbf{\Gamma}(\mathbf{q}_a)$. Let the unitary operator $\mathbf{U}_a$ in $L^2(\mathbb{R}_+)$ be defined by the formula $(\mathbf{U}_a\mathbf{f})(t) = e^{iat}\mathbf{f}(t)$. The role of (2.8) is now played by the identity

$$\mathbf{\Gamma}(\mathbf{q}_a) = \mathbf{U}_a\mathbf{\Gamma}(\mathbf{h})\mathbf{U}_a.$$

It follows that the singular values of the operators $\mathbf{\Gamma}(\mathbf{q}_a)$ and $\mathbf{\Gamma}(\mathbf{h})$ coincide. Reasoning as in the proof of Theorem 2.5, we obtain the following generalization of Theorems 2.6 and 2.7.

Theorem 2.9. *Let the functions $\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_L$ satisfy the hypothesis of Theorem 2.6 (resp. of Theorem 2.7), and let $a_\ell \in \mathbb{R}$, $\ell = 1, \dots, L$. Then for the Hankel operator $\mathbf{\Gamma}(\mathbf{h})$ with the kernel*

$$\mathbf{h}(t) = \sum_{\ell=1}^{L} e^{ia_\ell t}\mathbf{h}_\ell(t) \tag{2.18}$$

the estimate (2.13) (resp. (2.15)) holds true.

Of course, instead of a finite sum in (2.18) one can consider infinite series or integrals.

Note that the Hankel operators in the discrete and continuous cases are related through the Laguerre transform (see, e.g., [7, Theorem 1.8.9]) or by linking the symbols through a conformal map from the unit disc onto the upper half-plane. In some aspects of the theory of Hankel operators, this relation allows one to transfer results from the discrete case to the continuous one (or vice versa) quite easily. In this paper, technically it is simpler to carry out proofs in each case independently.

3. Continuous Representation

Recall that the Hankel operator $\mathbf{\Gamma}(\mathbf{h})$ is defined by formula (1.3) in the space $L^2(\mathbb{R}_+)$. Here we prove Theorems 2.6 and 2.7.

3.1. The Case $\alpha < 1/2$

We will use weighted L^p classes on $\mathbb{R}_+$ with the weight $\mathbf{v}(t) = 1/t$:

$$\mathbf{g} \in L^p_{\mathbf{v}}(\mathbb{R}_+) \quad \Leftrightarrow \quad \|\mathbf{g}\|^p_{L^p_{\mathbf{v}}} = \int_0^\infty |\mathbf{g}(t)|^p \mathbf{v}(t)dt < \infty, \quad \mathbf{v}(t) = 1/t,$$

and the corresponding weak class

$$\mathbf{g} \in L^{p,\infty}_{\mathbf{v}}(\mathbb{R}_+) \quad \Leftrightarrow \quad \|\mathbf{g}\|^p_{L^{p,\infty}_{\mathbf{v}}} = \sup_{s>0} s^p \int_{t:|\mathbf{g}(t)|>s} \mathbf{v}(t)dt < \infty. \tag{3.1}$$

By definition, for $p = \infty$ the weighted class $L^\infty_{\mathbf{v}}$ coincides with the usual (unweighted) L^∞ class.

Below we will use the real interpolation method (the "K-method"), see, e.g. [1, Sect. 3.1] for the details. A pair (X_0, X_1) of quasi-Banach spaces is called compatible, if both X_0 and X_1 are continuously embedded into the same Hausdorff topological vector space. Real interpolation with the parameters $0 < \theta < 1$ and $1 \leq q \leq \infty$ between a compatible pair of quasi-Banach spaces (X_0, X_1) yields an intermediate quasi-Banach space $(X_0, X_1)_{\theta,q}$. In particular, we have

$$(L^2_{\mathbf{v}}, L^\infty_{\mathbf{v}})_{\theta,\infty} = L^{p,\infty}_{\mathbf{v}}, \quad (\mathbf{S}_2, \mathcal{B})_{\theta,\infty} = \mathbf{S}_{p,\infty}, \quad \theta = 1 - 2/p. \tag{3.2}$$

If (X_0, X_1) and (Y_0, Y_1) are two compatible pairs of quasi-Banach spaces and if T is a bounded linear map from X_0 to Y_0 and from X_1 to Y_1, then the real interpolation method ensures the boundedness of T as a map from $(X_0, X_1)_{\theta,q}$ to $(Y_0, Y_1)_{\theta,q}$.

Lemma 3.1. *Let $\mathbf{v}(t) = 1/t$, and let $\mathbf{h} : \mathbb{R}_+ \to \mathbb{C}$ be a measurable function such that $\mathbf{h}/\mathbf{v} \in L^{p,\infty}_{\mathbf{v}}$ with some $p > 2$. Then $\mathbf{\Gamma}(\mathbf{h}) \in \mathbf{S}_{p,\infty}$ and*

$$\|\mathbf{\Gamma}(\mathbf{h})\|_{\mathbf{S}_{p,\infty}} \leq C_p \|\mathbf{h}/\mathbf{v}\|_{L^{p,\infty}_{\mathbf{v}}}. \tag{3.3}$$

Proof. The case $\mathbf{h} = \mathbf{v}$ corresponds to the Carleman operator, which has the norm π. From here we obtain that if $\mathbf{h}/\mathbf{v} \in L^\infty$, then $\mathbf{\Gamma}(\mathbf{h}) \in \mathcal{B}$, and

$$\|\mathbf{\Gamma}(\mathbf{h})\| \leq \pi \|\mathbf{h}/\mathbf{v}\|_{L^\infty} = \pi \sup_{t>0} t|\mathbf{h}(t)|.$$

On the other hand, we have the Hilbert-Schmidt relation (2.10). Thus, the linear map

$$\mathbf{h}/\mathbf{v} \mapsto \boldsymbol{\Gamma}(\mathbf{h}) \tag{3.4}$$

is bounded from $L^\infty_{\mathbf{v}} = L^\infty$ to $\mathcal{B}$ and from $L^2_{\mathbf{v}}$ to $\mathbf{S}_2$. In view of (3.2), we see that the map (3.4) is bounded from $L^{p,\infty}_{\mathbf{v}}$ to $\mathbf{S}_{p,\infty}$, and the estimate (3.3) holds true. ☐

Proof of Theorem 2.6 for $\alpha \leq 1/2$. Since $|\mathbf{h}(t)/\mathbf{v}(t)| \leq A_0\langle \log t\rangle^{-\alpha}$ and

$$\int_{A_0\langle \log t\rangle^{-\alpha}>s} \mathbf{v}(t)dt = \int_{\langle \log t\rangle<(A_0/s)^p} t^{-1}dt \leq C A_0^p s^{-p}, \quad s>0,$$

it follows from definition (3.1) that $\mathbf{h}/\mathbf{v} \in L^{p,\infty}_{\mathbf{v}}$. Now it remains to use Lemma 3.1. ☐

As a by-product of the above argument, we also obtain

Theorem 3.2. *For all $p \geq 2$, one has*

$$\|\boldsymbol{\Gamma}(\mathbf{h})\|^p_{\mathbf{S}_p} \leq C_p \int_0^\infty t^{p-1}|\mathbf{h}(t)|^p dt. \tag{3.5}$$

Proof. Let us choose the interpolation parameter $q = p$ and use that

$$(L^2_{\mathbf{v}}, L^\infty_{\mathbf{v}})_{\theta,p} = L^{p,p}_{\mathbf{v}} = L^p_{\mathbf{v}}, \quad (\mathbf{S}_2, \mathcal{B})_{\theta,p} = \mathbf{S}_{p,p} = \mathbf{S}_p, \quad \theta = 1-2/p.$$

Then considering again the mapping (3.4), we see that

$$\|\boldsymbol{\Gamma}(\mathbf{h})\|^p_{\mathbf{S}_p} \leq C_p\|\mathbf{h}/\mathbf{v}\|^p_{L^p_{\mathbf{v}}} = C_p \int_0^\infty t^{p-1}|\mathbf{h}(t)|^p dt,$$

as required. ☐

Theorem 3.2 can also be proven by the complex interpolation method which shows that (3.5) holds with $C_p = \pi^{p-2}$.

3.2. The Case $\alpha \geq 1/2$

Let $\mathbf{w} \in C_0^\infty(\mathbb{R}_+)$ be a function with the properties $\mathbf{w} \geq 0$, $\operatorname{supp} \mathbf{w} = [1/2, 2]$ and

$$\sum_{n\in\mathbb{Z}} \mathbf{w}(t/2^n) = 1, \quad \forall t > 0. \tag{3.6}$$

For $n \in \mathbb{Z}$, let $\mathbf{w}_n(t) = \mathbf{w}(t/2^n)$. For a function $\mathbf{h} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and for $n \in \mathbb{Z}$, set

$$\check{\mathbf{h}}_n(x) := \int_0^\infty \mathbf{h}(t)\mathbf{w}_n(t)e^{ixt}dt, \quad x \in \mathbb{R}. \tag{3.7}$$

Theorem 3.3. [7, Theorem 6.7.4]
Let $\mathbf{h} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. The estimate

$$\|\boldsymbol{\Gamma}(\mathbf{h})\|^p_{\mathbf{S}_p} \leq C_p \sum_{n\in\mathbb{Z}} 2^n \int_{-\infty}^\infty |\check{\mathbf{h}}_n(x)|^p dx \tag{3.8}$$

holds, so that $\boldsymbol{\Gamma}(\mathbf{h}) \in \mathbf{S}_p$ if the r.h.s. in (3.8) is finite.

The convergence of the series in (3.8) means that the symbol of the operator $\boldsymbol{\Gamma}(\mathbf{h})$ belongs to the Besov class $B^{1/p}_{pp}(\mathbb{R})$. Further, we have

Theorem 3.4. *Let* $\mathbf{h} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. *Suppose that*

$$|\mathbf{h}|_p^p := \sup_{s>0} s^p \sum_{n\in\mathbb{Z}} 2^n |\{x \in \mathbb{R} : |\, \check{\mathbf{h}}_n(x)| > s\}| < \infty. \tag{3.9}$$

Then $\mathbf{\Gamma}(\mathbf{h}) \in \mathbf{S}_{p,\infty}$ *and*

$$\|\mathbf{\Gamma}(\mathbf{h})\|_{\mathbf{S}_{p,\infty}} \le C_p |\mathbf{h}|_p.$$

In the discrete case (see Theorem 4.4 below), this theorem is proven in [7, Theorem 6.4.4]. In the continuous case, the proof is exactly the same, up to trivial changes in notation. For a given p one chooses some p_0 and p_1 such that $p_0 < p < p_1$ and uses the estimate (3.8) with $p = p_0$ and with $p = p_1$. Then one applies the real interpolation method to these two estimates, choosing the interpolation parameters θ, q such that $1/p = (1-\theta)/p_0 + \theta/p_0$ and $q = \infty$.

The results of [7] also show that if $\mathbf{\Gamma}(\mathbf{h}) \in \mathbf{S}_p$ (resp. if $\mathbf{\Gamma}(\mathbf{h}) \in \mathbf{S}_{p,\infty}$), then the r.h.s. of (3.8) (resp. of (3.9)) is necessary finite, although we will not need these facts.

Our goal is to check that under the assumptions of Theorem 2.6 the expression (3.9) is finite.

Lemma 3.5. *Assume the hypothesis of Theorem 2.6. Then for any* $q > 1/M$ *and for all* $n \in \mathbb{Z}$ *the functions* (3.7) *satisfy the estimates*

$$\|\, \check{\mathbf{h}}_n\|_{L^\infty} \le \int_{2^{n-1}}^{2^{n+1}} |\mathbf{h}(t)|dt, \tag{3.10}$$

$$2^n \|\, \check{\mathbf{h}}_n\|_{L^q}^q \le C_q \left(\sum_{m=0}^{M} \int_{2^{n-1}}^{2^{n+1}} t^m |\mathbf{h}^{(m)}(t)|dt \right)^q \tag{3.11}$$

with a constant C_q *independent of* n.

Proof. The first bound is a direct consequence of the definition (3.7) of $\check{\mathbf{h}}_n$ and of the properties $0 \le \mathbf{w}_n \le 1$ and $\operatorname{supp} \mathbf{w}_n = [2^{n-1}, 2^{n+1}]$. In order to obtain the second bound, we write

$$2^n \|\, \check{\mathbf{h}}_n\|_{L^q}^q = 2^n \int_{|x|\le 2^{-n}} |\, \check{\mathbf{h}}_n(x)|^q dx + 2^n \int_{|x|\ge 2^{-n}} |\, \check{\mathbf{h}}_n(x)|^q dx \tag{3.12}$$

and estimate the two terms in the r.h.s. separately. For the first term, we use (3.10):

$$2^n \int_{|x|\le 2^{-n}} |\, \check{\mathbf{h}}_n(x)|^q dx \le 2\|\, \check{\mathbf{h}}_n\|_{L^\infty}^q \le 2 \left(\int_{2^{n-1}}^{2^{n+1}} |\mathbf{h}(t)|dt \right)^q. \tag{3.13}$$

In order to estimate the second term in the r.h.s. of (3.12), we integrate by parts M times in the definition (3.7) of $\check{\mathbf{h}}_n$:

$$\begin{aligned}\check{\mathbf{h}}_n(x) &= (ix)^{-M}\int_0^\infty \mathbf{h}(t)\mathbf{w}_n(t)(d/dt)^M e^{ixt}dt\\ &= (-ix)^{-M}\int_0^\infty (\mathbf{h}(t)\mathbf{w}_n(t))^{(M)} e^{ixt}dt.\end{aligned} \tag{3.14}$$

Since

$$|\mathbf{w}_n^{(k)}(t)| = 2^{-nk}|\mathbf{w}^{(k)}(t/2^n)| \le C_k 2^{-nk}, \quad k\ge 0, \quad n\in\mathbb{Z}, \tag{3.15}$$

we get

$$\begin{aligned}\left|\int_0^\infty (\mathbf{h}(t)\mathbf{w}_n(t))^{(M)}e^{ixt}dt\right| &\le C_M\sum_{m=0}^M 2^{-n(M-m)}\int_{2^{n-1}}^{2^{n+1}}|\mathbf{h}^{(m)}(t)|dt\\ &\le 2^M C_M 2^{-nM}\sum_{m=0}^M\int_{2^{n-1}}^{2^{n+1}} t^m|\mathbf{h}^{(m)}(t)|dt.\end{aligned} \tag{3.16}$$

Combining (3.14) and (3.16), we see that

$$|\check{\mathbf{h}}_n(x)| \le C_M'|x|^{-M}2^{-nM}\sum_{m=0}^M\int_{2^{n-1}}^{2^{n+1}} t^m|\mathbf{h}^{(m)}(t)|dt$$

whence

$$\begin{aligned}&2^n\int\limits_{|x|\ge 2^{-n}}|\check{\mathbf{h}}_n(x)|^q dx\\ &\quad\le C_M''\left(2^{n-nMq}\int_{|x|\ge 2^{-n}}|x|^{-Mq}dx\right)\left(\sum_{m=0}^M\int\limits_{2^{n-1}}^{2^{n+1}} t^m|\mathbf{h}^{(m)}(t)|dt\right)^q.\end{aligned}$$

Since $Mq>1$, the first factor here equals $2/(Mq-1)$. Putting together the last estimate with (3.13) and using (3.12), we get (3.11). □

Proof of Theorem 2.6 for $\alpha \ge 1/2$. Under the assumption (2.12) for all $m = 0,\dots,M$ we have

$$\begin{aligned}\int_{2^{n-1}}^{2^{n+1}} t^m|\mathbf{h}^{(m)}(t)|dt &\le A_m\int_{2^{n-1}}^{2^{n+1}} t^{-1}\langle\log t\rangle^{-\alpha}dt\\ &= A_m\int_{n-1}^{n+1}\langle x\rangle^{-\alpha}dx \le cA_m\langle n\rangle^{-\alpha};\end{aligned}$$

here we assume that log is the base 2 logarithm, $\log = \log_2$. Fix some $q\in(M^{-1},\alpha^{-1})$; then it follows from (3.10), (3.11) that

$$\|\check{\mathbf{h}}_n\|_{L^\infty} \le CA_0\langle n\rangle^{-\alpha}, \quad n\in\mathbb{Z}, \tag{3.17}$$

$$2^n\|\check{\mathbf{h}}_n\|_{L^q}^q \le C\mathbf{A}^q\langle n\rangle^{-\alpha q}, \quad n\in\mathbb{Z}, \tag{3.18}$$

with some constant C and $\mathbf{A} = A_0+\cdots+A_M$.

Let us now estimate the functional $|\mathbf{h}|_p$ in (3.9). It follows from (3.17) that, for every $s > 0$ and all $n \in \mathbb{Z}$ such that

$$\langle n \rangle > (CA_0)^p s^{-p} =: N(s), \tag{3.19}$$

the inequality $\| \check{\mathbf{h}}_n \|_{L^\infty} < s$ holds. Therefore

$$s^p \sum_{n\in\mathbb{Z}} 2^n |\{x \in \mathbb{R} : |\check{\mathbf{h}}_n(x)| > s\}| = s^p \sum_{\langle n \rangle \leq N(s)} 2^n |\{x \in \mathbb{R} : |\check{\mathbf{h}}_n(x)| > s\}|. \tag{3.20}$$

Using the obvious inequality

$$s^q |\{x \in \mathbb{R} : |\check{\mathbf{h}}_n(x)| > s\}| \leq \| \check{\mathbf{h}}_n \|^q_{L^q}$$

and the bound (3.18), we can estimate the expression (3.20) by

$$s^{p-q} \sum_{\langle n \rangle \leq N(s)} 2^n \| \check{\mathbf{h}}_n \|^q_{L^q} \leq s^{p-q} C \mathbf{A}^q \sum_{\langle n \rangle \leq N(s)} \langle n \rangle^{-\alpha q} \leq s^{p-q} C' \mathbf{A}^q N(s)^{1-\alpha q}$$

(we have taken into account here that $\alpha q < 1$). By virtue of (3.19) this expression is bounded by $C''\mathbf{A}^q$ with a constant C'' that does not depend on s. Therefore it follows from (3.18) that $|h|_p^p \leq C''\mathbf{A}^p$. In view Theorem 3.4, this yields the required result. □

Proof of Theorem 2.7. Suppose first that $\mathbf{h}(t) = 0$ for all small and for all large $t > 0$. Then according to Theorem 2.6 we have $s_n(\mathbf{\Gamma}(\mathbf{h})) = O(n^{-\beta})$ for all β such that $M(\beta) \leq M(\alpha)$. Inspecting the formula (2.3) for $M(\alpha)$, we find that we can always choose $\beta > \alpha$ with $M(\beta) = M(\alpha)$. Thus, we have $s_n(\mathbf{\Gamma}(\mathbf{h})) = O(n^{-\beta}) = o(n^{-\alpha})$ as $n \to \infty$.

Now let us consider the general case. Let $\chi_0, \chi_\infty \in C^\infty(\mathbb{R}_+)$ be such that

$$\chi_0(t) = \begin{cases} 1 & \text{for } t \leq 1/4, \\ 0 & \text{for } t \geq 1/2, \end{cases} \quad \chi_\infty(t) = \begin{cases} 0 & \text{for } t \leq 2, \\ 1 & \text{for } t \geq 4. \end{cases} \tag{3.21}$$

Put

$$\zeta_N(t) = \chi_0(t/N)\chi_\infty(Nt), \quad N \in \mathbb{N},$$

and $\mathbf{h}_N = \mathbf{h}\zeta_N$. As shown by the first step of the proof, $\mathbf{\Gamma}(\mathbf{h}_N) \in \mathbf{S}^0_{p,\infty}$. It remains to prove that

$$\|\mathbf{\Gamma}(\mathbf{h}) - \mathbf{\Gamma}(\mathbf{h}_N)\|_{\mathbf{S}_{p,\infty}} \to 0, \quad N \to \infty. \tag{3.22}$$

According to Theorem 2.6, we need to check that

$$\sup_{t>0} t^{1+m} \langle \log t \rangle^{1/p} \left| \big(\mathbf{h}(t)(1 - \zeta_N(t)\big)^{(m)} \right| \to 0 \qquad N \to \infty, \tag{3.23}$$

for all $m = 0, \ldots, M$. By the construction of ζ_N, we have

$$\sup_{t>0} t^m |(1 - \zeta_N(t))^{(m)}| \leq C_m \quad \text{and} \quad (1 - \zeta_N(t))^{(m)} = 0 \text{ if } t \in (4/N, N/4)$$

for all $m \geq 0$. Therefore our assumption (2.14) on $\mathbf{h}$ implies (3.23) and hence (3.22). □

Remark 3.6. By the result of [4, Theorem 4.9] (see also [10, Example 6.1]), one can construct a bounded kernel $\mathbf{h}(t)$ with one jump discontinuity at some $t = t_0 > 0$ (and vanishing identically for all sufficiently small and all sufficiently large $t > 0$) such that $\boldsymbol{\Gamma}(\mathbf{h}) \in \mathbf{S}_{1,\infty}$ but $\boldsymbol{\Gamma}(\mathbf{h}) \notin \mathbf{S}^0_{1,\infty}$. Similarly, for every $\alpha \in \mathbb{N}$, $\alpha \geq 2$, there exist kernels $\mathbf{h} \in C^{\alpha-2}$, $\mathbf{h} \notin C^{\alpha-1}$, such that $\boldsymbol{\Gamma}(\mathbf{h}) \in \mathbf{S}_{1/\alpha,\infty}$ but $\boldsymbol{\Gamma}(\mathbf{h}) \notin \mathbf{S}^0_{1/\alpha,\infty}$. This shows that, at least for $\alpha \in \mathbb{N}$, $\alpha \geq 2$, the condition $\mathbf{h} \in C^M$ with $M = \alpha - 2$ is not sufficient for the validity of estimate (2.13).

4. Discrete Representation

Recall that the Hankel operator $\Gamma(h)$ is defined by formula (1.1) in the space $\ell^2(\mathbb{Z}_+)$. Here we prove Theorems 2.1, 2.2 and 2.3. The calculations follow closely those of Sect. 3, so we will be brief in places where there is a complete analogy and concentrate only on the points of difference.

4.1. The Case $\alpha < 1/2$

We introduce the weighted ℓ^p class with the weight $v(j) = (j+1)^{-1}$:

$$g \in \ell^p_v \quad \Leftrightarrow \quad \|g\|^p_{\ell^p_v} = \sum_{j=0}^{\infty} |g(j)|^p v(j) < \infty, \quad v(j) = \frac{1}{j+1},$$

and the corresponding weak class

$$g \in \ell^{p,\infty}_v \quad \Leftrightarrow \quad \|g\|^p_{\ell^{p,\infty}_v} = \sup_{s>0} s^p \sum_{j:|g(j)|>s} v(j) < \infty.$$

For a sequence h, we denote by h/v the sequence $\{(j+1)h(j)\}_{j=0}^{\infty}$.

Lemma 4.1. *Let h be a sequence of complex numbers such that $h/v \in \ell^{p,\infty}_v$ for some $p > 2$. Then $\Gamma(h) \in \mathbf{S}_{p,\infty}$ and*

$$\|\Gamma(h)\|_{\mathbf{S}_{p,\infty}} \leq C \|h/v\|_{\ell^{p,\infty}_v}.$$

Proof. As in the continuous case, the result follows by real interpolation between the estimates

$$\|\Gamma(h)\| \leq \pi \|h/v\|_{\ell^\infty} = \pi \|h/v\|_{\ell^\infty_v}$$

(which corresponds to the bound $\|\Gamma(h)\| \leq \pi$ for the Hilbert matrix (1.7)), and the Hilbert-Schmidt relation (1.9). □

Proof of Theorem 2.1. As $|h(j)/v(j)| \leq C(\log(j+2))^{-\alpha}$, the required statement follows from the elementary fact that the sequence $\{(\log(j+2))^{-\alpha}\}_{j=0}^{\infty}$ belongs to the class $\ell^{p,\infty}_v$ for $p = 1/\alpha$. □

Similarly to Theorem 3.2, we also have

Theorem 4.2. *For all $p \geq 2$, one has*

$$\|\Gamma(h)\|^p_{\mathbf{S}_p} \leq \pi^{p-2} \sum_{j=0}^{\infty} (j+1)^{p-1} |h(j)|^p.$$

4.2. The Case $\alpha \geq 1/2$

Here we prove Theorem 2.2 for $0 < p \leq 2$. Let $w \in C_0^\infty(\mathbb{R}_+)$ be a function with the properties $w \geq 0$, $\operatorname{supp} w = [1/2, 2]$ and

$$\sum_{n=0}^{\infty} w(t/2^n) = 1, \quad \forall t \geq 1.$$

Observe that the summation is over $n \in \mathbb{Z}_+$ here, while it is over all $n \in \mathbb{Z}$ in (3.6). Denote $w_n(j) = w(j/2^n)$ for $n \geq 1$ and let w_0 be defined by $w_0(0) = w_0(1) = 1$, $w_0(j) = 0$ for $j \geq 2$. For a sequence of complex numbers $h = \{h(j)\}_{j \geq 0}$, denote by $\check{h}_n$ the polynomial

$$\check{h}_n(\mu) = \sum_{j=0}^{\infty} w_n(j) h(j) \mu^j, \quad \mu \in \mathbb{T}, \quad n \geq 0. \tag{4.1}$$

Let us recall two results due to V. Peller. The first one follows from Theorems 6.1.1, 6.2.1 and 6.3.1 in [7].

Theorem 4.3. *The estimate*

$$\|\Gamma(h)\|_{\mathbf{S}_p}^p \leq C_p \sum_{n=0}^{\infty} 2^n \int_{-\pi}^{\pi} |\check{h}_n(e^{i\theta})|^p d\theta, \quad p > 0, \tag{4.2}$$

holds, so that $\Gamma(h) \in \mathbf{S}_p$ if the r.h.s. in (4.2) is finite.

The next result is deduced from Theorem 4.3 by the real interpolation method using the retract arguments (see, e.g., the book [1, Sect. 6.4]).

Theorem 4.4. [7, Theorem 6.4.4] *Let*

$$|h|_p^p = \sup_{s>0} s^p \sum_{n=0}^{\infty} 2^n |\{\theta \in [-\pi, \pi) : |\check{h}_n(e^{i\theta})| > s\}|. \tag{4.3}$$

Then $\Gamma(h) \in \mathbf{S}_{p,\infty}$ and

$$\|\Gamma(h)\|_{\mathbf{S}_{p,\infty}} \leq C_p |h|_p. \tag{4.4}$$

Remark 4.5. The results of [7] also show that if $\Gamma(h) \in \mathbf{S}_p$ (resp. $\Gamma(h) \in \mathbf{S}_{p,\infty}$), then the r.h.s. of (4.2) (resp. of (4.3)) is necessary finite.

Our goal is to show that under the assumptions of Theorem 2.2 the expression (4.3) is finite. Note that, for the sequence $h(j) = j^{-1}(\log j)^{-\alpha}$, $j \geq 2$, the symbol (1.2) is singular at the point $\mu = 1$. Therefore this point requires a special treatment.

Let us display two elementary identities. The first one is the "summation by parts formula":

$$\sum_{j=0}^{\infty} u(j) v^{(M)}(j) = (-1)^M \sum_{j=0}^{\infty} u^{(M)}(j) v(j+M) \tag{4.5}$$

where it is assumed that at least one of the sequences u or v vanishes for $j = 0, \ldots, M-1$ and for all large j. The second one is the variant of the Leibniz rule for the product $(uv)(j) = u(j)v(j)$:

$$(uv)^{(M)}(j) = \sum_{m=0}^{M} \binom{M}{m} u^{(M-m)}(j+m)v^{(m)}(j). \tag{4.6}$$

Lemma 4.6. *Assume the hypothesis of Theorem 2.2. Then for any $q > 1/M$ and for all $n \in \mathbb{N}$ such that $2^{n-1} \geq M$ one has the estimates*

$$\| \check{h}_n\|_{L^\infty} \leq \sum_{j=2^{n-1}}^{2^{n+1}} |h(j)|, \tag{4.7}$$

$$2^n \| \check{h}_n\|^q_{L^q} \leq C_q \left(\sum_{m=0}^{M} \sum_{j=2^{n-1}-M}^{2^{n+1}} (1+j)^m |h^{(m)}(j)| \right)^q. \tag{4.8}$$

Proof. The first estimate follows from the fact that $0 \leq w_n(j) \leq 1$ for all j and $w_n(j) = 0$ for $j \leq 2^{n-1}$ and for $j \geq 2^{n+1}$. To estimate the L^q norm, we write

$$2^n(2\pi)\| \check{h}_n\|^q_{L^q} = 2^n \int_{|\theta|<2^{-n}} | \check{h}_n(e^{i\theta})|^q d\theta + 2^n \int_{|\theta|\geq 2^{-n}} | \check{h}_n(e^{i\theta})|^q d\theta \tag{4.9}$$

and estimate each term separately. For the first term, we use the estimate (4.7):

$$2^n \int_{|\theta|<2^{-n}} | \check{h}_n(e^{i\theta})|^q d\theta \leq 2\| \check{h}_n\|^q_{L^\infty} \leq 2 \left(\sum_{j=2^{n-1}}^{2^{n+1}} |h(j)| \right)^q. \tag{4.10}$$

In order to estimate the second integral in (4.9), we need to perform a summation by parts calculation. Let us set $\mu(j) = \mu^j$, then the iterated difference is $\mu^{(M)}(j) = (\mu - 1)^M \mu(j)$. Using the definition (4.1) of $\check{h}_n$ and the summation by parts formula (4.5) for the sequences $u(j) = \mu(j)$, $v(j) = w_n(j)h(j)$, we obtain that

$$\begin{aligned} \check{h}_n(\mu) &= (\mu - 1)^{-M} \sum_{j=0}^{\infty} w_n(j)h(j)\mu^{(M)}(j) \\ &= (1-\mu)^{-M} \sum_{j=0}^{\infty} (w_n h)^{(M)}(j)\mu(j+M). \end{aligned} \tag{4.11}$$

Since (cf. (3.15))

$$|w_n^{(k)}(j)| \leq C_k 2^{-nk}, \quad n \geq 2, \quad k \geq 0,$$

it follows from the Leibniz rule (4.6) that

$$|(w_n h)^{(M)}(j)| \leq C_M \sum_{m=0}^{M} 2^{-n(M-m)} |h^{(m)}(j)|.$$

Substituting this into (4.11) and using the fact that $w_n^{(k)}(j) = 0$ for $j \leq 2^{n-1} - k$ and for $j \geq 2^{n+1}$, we obtain the estimate

$$\begin{aligned} |\breve{h}_n(\mu)| &\leq |1-\mu|^{-M} \sum_{j=2^{n-1}-M}^{2^{n+1}} |(w_n h)^{(M)}(j)| \\ &\leq C_M |1-\mu|^{-M} \sum_{m=0}^{M} 2^{-n(M-m)} \sum_{j=2^{n-1}-M}^{2^{n+1}} |h^{(m)}(j)| \\ &\leq C_M |1-\mu|^{-M} 2^{-nM} \sum_{m=0}^{M} \sum_{j=2^{n-1}-M}^{2^{n+1}} (1+j)^m |h^{(m)}(j)|. \end{aligned}$$

From here we get

$$\begin{aligned} 2^n \int_{|\theta|\geq 2^{-n}} |\breve{h}_n(e^{i\theta})|^q d\theta \leq & \left(2^{n-nMq} \int_{|\theta|\geq 2^{-n}} |1-e^{i\theta}|^{-Mq} d\theta \right) \\ & \times \left(C_M \sum_{m=0}^{M} \sum_{j=2^{n-1}-M}^{2^{n+1}} (1+j)^m |h^{(m)}(j)| \right)^q . \end{aligned}$$

Since $Mq > 1$, the first factor here can be estimated by a constant independent of n. Combining this with (4.10), we arrive at (4.8). □

Proof of Theorem 2.2 for $\alpha \geq 1/2$. Denote

$$A_m = \sup_{j\geq 0} (j+1)^{1+m} (\log(j+2))^{\alpha} |h^{(m)}(j)|, \quad m = 0, \dots, M.$$

Substituting these bounds into the estimates (4.7) and (4.8), we obtain

$$\|\breve{h}_n\|_{L^\infty} \leq C A_0 \langle n \rangle^{-\alpha},$$

$$2^n \|\breve{h}_n\|_{L^q}^q \leq C(A_0 + \cdots + A_M)^q \langle n \rangle^{-q\alpha},$$

if $2^{n-1} \geq M$. Using these estimates and arguing exactly as in the proof of Theorem 2.6, we find that

$$|h|_p \leq C(A_0 + \cdots + A_M),$$

and so by (4.4) we are done. □

Proof of Theorem 2.3. Let $\chi_0 \in C^\infty(\mathbb{R}_+)$ be as in (3.21). Define $\zeta_N(j) = \chi_0(j/N)$ and consider the truncated sequence $h_N = h\zeta_N$. Then $\Gamma(h_N)$ is a finite rank operator. Let us show that $\Gamma(h_N)$ converges to $\Gamma(h)$ in the quasi-norm of $\mathbf{S}_{p,\infty}$. Note that

$$\sup_{j\geq 0} (j+1)^m |\zeta_N^{(m)}(j)| \leq C_m, \quad m \geq 0,$$

with constants C_m not depending on N. Therefore it follows from estimate (2.5) and the Leibniz rule (4.6) that

$$\|\Gamma(h) - \Gamma(h_N)\|_{\mathbf{S}_{p,\infty}} \le C(\alpha) \sum_{m=0}^{M} \sup_{j \ge N/4} (1+j)^{1+m} (\log(j+2))^{\alpha} |h^{(m)}(j)|,$$

where $p = 1/\alpha$. Under the assumptions of Theorem 2.3 the r.h.s. here tends to zero as $N \to \infty$. □

Let us finally show that the condition $M(\alpha) = [\alpha] + 1$ in Theorem 2.2 cannot be significantly relaxed.

Example 4.7. Let $\alpha \ge 2$. We will construct a sequence $h(j)$ satisfying condition (2.4) for all $m \le [\alpha] - 2$ but such that estimate (2.2) is violated. For an arbitrary $\gamma \in ([\alpha] - 1, \alpha)$, consider the lacunary sequence

$$h(j) = \begin{cases} 2^{-\gamma n}, & \text{if } j = 2^n,\ n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

In this case the iterated differences $h^{(m)}$ do not decay faster than the sequence h itself. So for all m, we only have

$$h^{(m)}(j) = O(j^{-\gamma}), \quad j \to \infty.$$

Thus the hypothesis (2.4) of Theorem 2.2 is satisfied for all $m < \gamma - 1$ and hence for all $m \le [\alpha] - 2$. On the other hand, for our sequence h the polynomial (4.1) is $\check{h}_n(z) = 2^{-\gamma n} z^{2^n}$. So $|\check{h}_n(e^{i\theta})| = 2^{-\gamma n}$ and the series in the r.h.s. of (4.2) becomes

$$\sum_{n=0}^{\infty} 2^n 2^{-\gamma p n}.$$

This series diverges for $p = 1/\gamma$. Therefore according to Remark 4.5 (the necessity part of [7, Theorem 6.2.1]), we have $\Gamma(h) \notin \mathbf{S}_{1/\gamma}$. Since $\mathbf{S}_{1/\alpha,\infty} \subset \mathbf{S}_{1/\gamma}$ for $\gamma < \alpha$, it follows that $\Gamma(h) \notin \mathbf{S}_{1/\alpha,\infty}$. Thus, one cannot take $M(\alpha) = [\alpha] - 2$ in Theorem 2.2.

The same construction shows that one cannot take $M(\alpha) = 0$ for $1 < \alpha < 2$ (and $1 < \gamma < \alpha$).

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Received: December 1, 2014.
Revised: April 27, 2015.