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Boundary Interpolation for Slice Hyperholomorphic Schur Functions

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Abstract. A boundary Nevanlinna–Pick interpolation problem is posed and solved in the quaternionic setting. Given nonnegative real numbers $\kappa_1, \ldots, \kappa_N$, quaternions $p_1, \ldots, p_N$ all of modulus 1, so that the 2-spheres determined by each point do not intersect and $p_u \neq 1$ for $u = 1, \ldots, N$, and quaternions $s_1, \ldots, s_N$, we wish to find a slice hyperholomorphic Schur function s so that

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} s(rp_u) = s_u \quad \text{for } u = 1, \ldots, N,$$

and

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1 - s(rp_u)\overline{s_u}}{1-r} \leq \kappa_u, \quad \text{for } u = 1, \ldots, N.$$

Our arguments rely on the theory of slice hyperholomorphic functions and reproducing kernel Hilbert spaces.

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1. Introduction

In the paper [1] the Nevanlinna–Pick interpolation problem for slice hyperholomorphic Schur functions has been solved using the FMI (fundamental matrix inequality) method (see [20] for details). By a Schur function we mean a function f which is slice hyperholomorphic on the open unit ball $\mathbb{B}_1$ of the

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quaternions and is bounded in modulus by 1, i.e., $\sup_{p\in\mathbb{B}_1}|f(p)|\leq 1$. In the present paper we solve a boundary interpolation problem for slice hyperholomorphic functions using the reproducing kernel Hilbert space method based on de Branges–Rovnyak spaces. We refer the reader to [2,3,17] for more information on the reproducing kernel Hilbert space approach to interpolation problems.

We state the problem we will solve in this paper and introduce some notation and definitions. Let us denote by $\mathbb{B}_1$ and $\mathbb{H}_1$, the open unit ball and the unit sphere of $\mathbb{H}$, respectively. For a given element $p\in\mathbb{H}$ we denote by $[p]$ the associated 2-sphere:

$$[p]=\left\{qpq^{-1}:q\in\mathbb{H}\setminus\{0\}\right\}.$$

Recall that two quaternions belong to the same sphere if and only if they have the same modulus and the same real part.

Problem 1.1. Given $p_1,\ldots,p_N\in\mathbb{H}_1\setminus\{1\}$ such that $[p_u]\cap[p_v]=\emptyset$ for $u\neq v$ (the interpolation nodes), $s_1,\ldots,s_N\in\mathbb{H}_1$, and $\kappa_1,\ldots,\kappa_N\in[0,\infty)$, find a necessary and sufficient condition for a slice hyperholomorphic Schur function s to exist such that the conditions

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} s(rp_u)=s_u, \tag{1.1}$$

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1-s(rp_u)\overline{s_u}}{1-r}\leq\kappa_u \tag{1.2}$$

hold for $u=1,\ldots N$, and describe the set of all Schur functions satisfying (1.1), (1.2) when this condition is in force.

We note that (1.1), (1.2) imply that

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1-|s(rp_u)|^2}{1-r^2}\leq\kappa_u,\quad u=1,\ldots,N, \tag{1.3}$$

since

$$\frac{1-|s(rp_u)|^2}{1-r^2}=\frac{1-s(rp_u)\overline{s_u}}{(1-r)(1+r)}+(s(rp_u)\overline{s_u})\frac{1-s_u\overline{s(rp_u)}}{(1-r)(1+r)}. \tag{1.4}$$

We also note that the fact that the limits (1.3) will be part of the requirement in the interpolation problem (in the complex case, the corresponding limit is well-known to be non-negative).

As it appears from the statement of Problem 1.1, there is a major difference with the complex case. Here we have to require that not only the interpolation points are distinct, but also the spheres they determine. The fact that this hypothesis is necessary, and cannot be avoided, can be intuitively justified by the fact that the S-spectrum of a matrix, or in general of an operator (see Definition 2.6), consists of spheres (which may reduce to real points). It is important to note that the notion of S-spectrum of a matrix T coincides with the set of right eigenvalues of T, i.e., the set of $\lambda\in\mathbb{H}$ so that $Tx=x\lambda$ for a nonzero vector x.

Another major difference is the lack of a Carathéodory theorem (see e.g. [22, p. 48]) in the quaternionic setting.

Part of the arguments follow the classical case, taking into account the noncommutativity of the quaternions. As we shall see, even though the structure of the proof follows the the arguments from [9], it is necessary to suitably adapt the argument to the quaternionic setting and often the needed modifications are not immediate.

The paper consists of five sections, besides the introduction. In Sect. 2, we recall some basic material on slice hyperholomorphic functions which will be needed in the sequel. Section 3 illustrates the strategy and the various steps we will follow to solve Problem 1.1. Section 4 contains detailed proofs of these steps and Sect. 5 deals with the degenerate case. Section 6 contains an analogue of Carathéodory's theorem in the quaternionic setting.

2. Some Preliminaries

In this section we collect some basic results, which will be used in the sequel. Let $\mathbb{H}$ be the real associative algebra of quaternions with respect to the basis $\{1, i, j, k\}$ satisfying the relations $i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$. A quaternion p is denoted by $p = x_0 + ix_1 + jx_2 + kx_3$, $x_\ell \in \mathbb{R}$, $\ell = 0, \dots, 3$, its conjugate is $\bar{p} = x_0 - ix_1 - jx_2 - kx_3$, and the norm of a quaternion is such that $|p|^2 = p\bar{p}$. A quaternion p can be written as $p = \mathrm{Re}(p) + \underline{p}$ where the real part $\mathrm{Re}(p)$ is x_0 and $\underline{p} = ix_1 + jx_2 + kx_3$. The symbol $\mathbb{S}$ denotes the 2-sphere of purely imaginary unit quaternions, i.e.,

$$\mathbb{S} = \{\underline{p} = ix_1 + jx_2 + kx_3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Note that if $I \in \mathbb{S}$ then $I^2 = -1$. Any nonreal quaternion $p = x_0 + ix_1 + jx_2 + kx_3$ uniquely determines an element $I_p = (ix_1 + jx_2 + kx_3)/|ix_1 + jx_2 + kx_3| \in \mathbb{S}$. If $p = x_0 \in \mathbb{R}$ then $p = x_0 + I0$ for all $I \in \mathbb{S}$. Given $p \in \mathbb{H}$ we can write $p = p_0 + I_p p_1$ and the 2-sphere $[p]$ coincides with the set of all elements of the form $p_0 + Jp_1$ when J varies in $\mathbb{S}$. The set $[p]$ is reduces to the point p if and only if $p \in \mathbb{R}$.

We now recall the definition of a slice hyperholomorphic function, for more details see [15].

Definition 2.1. Let $\Omega \subseteq \mathbb{H}$ be an open set and let $f : \Omega \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I and denote by $x + Iy$ an element on $\mathbb{C}_I$. We say that f is a left slice hyperholomorphic (or slice hyperholomorphic, for short) function in Ω if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

We say that f is a right slice hyperholomorphic function in Ω if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I\right) = 0.$$

Slice hyperholomorphic functions have nice properties on some particular open sets which are defined below.

Definition 2.2. Let Ω be a domain in $\mathbb{H}$. We say that Ω is a slice domain (s-domain for short) if $\Omega \cap \mathbb{R}$ is non empty and if $\Omega \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$. We say that Ω is axially symmetric if, for all $p \in \Omega$, the sphere $[p]$ is contained in Ω.

On an axially symmetric s-domain Ω, a slice hyperholomorphic function satisfies the following formula, which is called the Structure formula or the Representation formula (see [15, Theorem 4.3.2]):

$$f(x+Jy) = \frac{1}{2}\left[f(x+Iy) + f(x-Iy) + JI(f(x-Iy) - f(x+Iy))\right]. \quad (2.1)$$

Formula (2.1) is useful as it allows one to extend a holomorphic map $h: U \subseteq \mathbb{C} \cong \mathbb{C}_I \to \mathbb{H}$ to a slice hyperholomorphic function. Let Ω_U be the axially symmetric completion of U, i.e.,

$$\Omega_U = \bigcup_{J\in\mathbb{S},\, x+Iy\in U} \{x+Jy\}.$$

The left slice hyperholomorphic extension $\mathrm{ext}(h): \Omega_U \subseteq \mathbb{H} \to \mathbb{H}$ of h is the function defined as (see [15]):

$$\mathrm{ext}(h)(x+Jy) = \frac{1}{2}\left[h(x+Iy) + h(x-Iy) + JI(h(x-Iy) - h(x+Iy))\right]. \quad (2.2)$$

It is immediate that $\mathrm{ext}(h+g) = \mathrm{ext}(h) + \mathrm{ext}(g)$ and that if $h(z) = \sum_{n=0}^{\infty} h_n(z)$ then $\mathrm{ext}(h)(z) = \sum_{n=0}^{\infty} \mathrm{ext}(h_n)(z)$.

Two left (resp. right) slice hyperholomorphic functions can be multiplied, on an axially symmetric s-domain, using the so called $\star$-product (resp. $\star_r$-product) in order to obtain another left (resp. right) slice hyperholomorphic function.

Let $f, g: \Omega \subseteq \mathbb{H}$ be slice hyperholomorphic functions. Their restrictions to the complex plane $\mathbb{C}_I$ can be written as $f_I(z) = F(z) + G(z)J$, $g_I(z) = H(z) + L(z)J$ where $J \in \mathbb{S}$, $J \perp I$, i.e. $IJ = -JI$. The functions F, G, H, L are holomorphic functions of the variable $z \in \Omega \cap \mathbb{C}_I$. We have the following:

Definition 2.3. Let f and g be slice hyperholomorphic functions defined on an axially symmetric s-domain $\Omega \subseteq \mathbb{H}$. The $\star$-product of f and g is defined as the unique left slice hyperholomorphic function on Ω whose restriction to the complex plane $\mathbb{C}_I$ is given by

$$\begin{aligned}(f \star g)_I(z) &= (F(z) + G(z)J) \star (H(z) + L(z)J) \\ &= (F(z)H(z) - G(z)\overline{L(\bar{z})}) + (G(z)\overline{H(\bar{z})} + F(z)L(z))J. \quad (2.3)\end{aligned}$$

If f and g are slice hyperholomorphic on a ball with center at the origin, they can be expressed in a power series, i.e. $f(p) = \sum_{n=0}^{\infty} p^n a_n$ and $g(p) = \sum_{n=0}^{\infty} p^n b_n$. Thus $(f \star g)(p) = \sum_{n=0}^{\infty} p^n c_n$, where $c_n = \sum_{r=0}^{n} a_r b_{n-r}$ is obtained by convolution on the coefficients. For the construction of the $\star$-product of right slice hyperholomorphic functions and for more information on the $\star$-product, we refer the reader to [7,15].

Given a slice hyperholomorphic function, it is possible to define its slice hyperholomorphic reciprocal, see [15]. Here we limit ourselves to the case in which f admits the power series expansion $f(p) = \sum_{n=0}^{\infty} p^n a_n$. In this case we set

$$f^c(p) = \sum_{n=0}^{\infty} p^n \bar{a}_n, \qquad f^s(p) = (f^c \star f)(p) = \sum_{n=0}^{\infty} p^n c_n, \quad c_n = \sum_{r=0}^{n} a_r \bar{a}_{n-r},$$

so that the left slice hyperholomorphic reciprocal of f is defined, outside the zeros of f^s, as

$$f^{-\star} := (f^s)^{-1} f^c.$$

In the general case, this formula is still valid when f^s and f^c are suitably defined.

Remark 2.4. Let $k(p, q)$ be a function left slice hyperholomorphic in p and right slice hyperholomorphic in $\bar{q}$. When taking the $\star$-product of a function $f(p)$ slice hyperholomorphic in the variable p with a function $k(p, q)$, we will write $f(p) \star k(p, q)$ meaning that the $\star$-product is taken with respect to the variable p; similarly, the $\star_r$-product of $k(p, q)$ with functions right slice hyperholomorphic in the variable $\bar{q}$ is always taken with respect to $\bar{q}$.

The following proposition is taken from [7, Proposition 4.3], where a proof can be found.

Proposition 2.5. *Let $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ be two reproducing kernel Hilbert spaces of $\mathbb{H}^m$ and $\mathbb{H}^n$-valued slice hyperholomorphic functions in Ω, with reproducing kernels K_1 and K_2, respectively. Let R be an $\mathbb{H}^{n\times m}$-valued function slice-hyperholomorphic in Ω. Then the operator of left $\star$-multiplication*

$$M_R : f \mapsto R \star f$$

is continuous from $\mathcal{H}(K_1)$ into $\mathcal{H}(K_2)$ if and only if the kernel

$$K_2(p, q) - R(p) \star K_1(q, p) \star_r R(q)^*$$

is positive definite in Ω. Furthermore

$$M_R^*(K_2(\cdot, q)d) = K_1(\cdot, q) \star_r R(q)^* d, \quad d \in \mathbb{H}^n. \tag{2.4}$$

Let us recall a few facts on the S-spectrum and on the S-resolvent operator.

Definition 2.6. Let A be a bounded quaternionic linear right operator acting on a quaternionic, two sided, Banach space V. We define the S-spectrum $\sigma_S(A)$ of A as:

$$\sigma_S(A) = \{s \in \mathbb{H} : A^2 - 2\mathrm{Re}\,(s)A + |s|^2\mathcal{I} \text{ is not invertible}\},$$

where $\mathcal{I}$ denotes the identity operator on V. The S-resolvent set $\rho_S(A)$ is defined as $\rho_S(A) = \mathbb{H} \setminus \sigma_S(A)$.

From Definition 2.6 it follows that the S-spectrum consists of spheres (which may reduce to real points).
The definition of S-spectrum arises from the following:

Proposition 2.7. *Let A be a bounded quaternionic right linear operator acting on a quaternionic, two sided, Banach space V. Then, for $\|A\| < |p|$, we have*

$$\sum_{n=0}^{\infty} p^{-1-n} A^n = -(A - \bar{p}\mathcal{I})(A^2 - 2\mathrm{Re}(p)A + |p|^2\mathcal{I})^{-1}. \tag{2.5}$$

Definition 2.8. The operator

$$S_R^{-1}(p, A) := -(A - \bar{p}\mathcal{I})(A^2 - 2\mathrm{Re}(p)A + |p|^2\mathcal{I})^{-1}, \tag{2.6}$$

is called the right S-resolvent operator.

The right S-resolvent operator is obviously defined for $p \in \rho_S(A)$. In the sequel we will be in need of the result below:

Proposition 2.9. *Let V be a two sided quaternionic Banach space and let A be a bounded right linear operator from V into itself. Then, for $|p|\,\|A\| < 1$ we have*

$$\sum_{n=0}^{\infty} p^n A^n = (\mathcal{I} - \bar{p}A)(|p|^2 A^2 - 2\mathrm{Re}(p)A + \mathcal{I})^{-1}. \tag{2.7}$$

Another way to write the operator on the right hand side of (2.7) is to observe that it corresponds to the function one obtains by constructing the right $\star$-reciprocal of the function $f(q) = (1 - pq)$. Upon computing $f^{-\star}(A)$ using the quaternionic functional calculus, see [15], one can write:

$$(\mathcal{I} - pA)^{-\star} = \sum_{n=0}^{\infty} p^n A^n. \tag{2.8}$$

Finally, we mention a result which is a restatement of [4, Proposition 2.22] and which contains an identity that will be crucial in the sequel.

Proposition 2.10. *Let $p \in \mathbb{H}$, $1/p \in \rho_S(A)$ and $(G, A) \in \mathbb{H}^{n\times m} \times \mathbb{H}^{m\times m}$. Then*

$$\sum_{t=0}^{\infty} p^t G A^t = (G - \bar{p}GA)(\mathcal{I}_m - 2\mathrm{Re}(p)A + |p|^2 A^2)^{-1}, \tag{2.9}$$

where $\mathcal{I}_m$ denotes the $m \times m$ identity matrix.

Remark 2.11. We note that if $m = 1$ then A is a quaternion a and the condition $1/p \in \rho_S(A)$ translates to the condition $1/p \notin [a]$.

3. The Main Result and the Strategy

For the convenience of the reader we recall the main steps of the reproducing kernel method. We first introduce some notation. We set

$$A = \mathrm{diag}\,(\overline{p_1}, \ldots, \overline{p_N}) \in \mathbb{H}^{N\times N}, \quad C = \begin{pmatrix} 1 & \cdots & 1 \\ \overline{s_1} & \cdots & \overline{s_N} \end{pmatrix} \in \mathbb{H}^{2\times N}, \tag{3.1}$$

and

$$\mathcal{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Consider the matrix equation

$$P - A^* P A = C^* \mathcal{J} C, \tag{3.2}$$

where the unknown is $P \in \mathbb{H}^{N\times N}$. The off diagonal entries of the matrix equation are uniquely determined by the equation

$$P_{uv} - p_u P_{uv} \overline{p_v} = 1 - s_u \overline{s_v}, \tag{3.3}$$

but, in view of the following lemma the diagonal entries can be arbitrary:

Lemma 3.1. *Let p and q be quaternions of modulus 1. Then, the equation*

$$ph - hq = 0, \tag{3.4}$$

where $h \in \mathbb{H}$, has the only solution $h = 0$ if and only if $\mathrm{Re}(p) \neq \mathrm{Re}(q)$, that is, if and only if $[p] \cap [q] = \emptyset$.

Proof. If (3.4) has a solution $h \neq 0$, then $p = hqh^{-1}$ and so p and q are in the same sphere. So a necessary condition for (3.4) to have only $h = 0$ as solution is that $[p] \cap [q] = \emptyset$. We now show that this condition is also sufficient. Let $p = z_1 + z_2 j$ and $q = w_1 + w_2 j$, where $z_1, z_2, w_1, w_2 \in \mathbb{C}$. Since $\mathrm{Re}(p) \neq \mathrm{Re}(q)$ we have

$$\mathrm{Re}(z_1) \pm i\sqrt{1 - (\mathrm{Re}(z_1))^2} \neq \mathrm{Re}(w_1) \pm i\sqrt{1 - (\mathrm{Re}(w_1))^2}. \tag{3.5}$$

We now introduce the injective ring homomorphism $\chi : \mathbb{H} \to \mathbb{C}^{2\times 2}$ given by

$$\chi(p) = \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}. \tag{3.6}$$

Using the map χ, Eq. (3.4) becomes

$$\chi(p)\chi(h) - \chi(h)\chi(q) = 0. \tag{3.7}$$

The eigenvalues of $\chi(p)$ are the solutions of

$$\lambda^2 - 2(\mathrm{Re}(z_1))\lambda + 1 = 0,$$

that is, $\lambda = \mathrm{Re}(z_1) \pm i\sqrt{1 - (\mathrm{Re}(z_1))^2}$, and similarly for $\chi(q)$. By a well known result on matrix equations (see e.g., Corollary 4.4.7 in [19]), Eq. (3.7) has only the solution $\chi(h) = 0$ if and only if $\lambda - \mu \neq 0$ for all possible choices of eigenvalues of $\chi(p)$ and $\chi(q)$, and this condition holds in view of (3.5). So the only solution of (3.7) is $h = 0$. □

We denote by P the $N \times N$ Hermitian matrix with entries P_{uv} given by (3.3) for $u \neq v$ and with diagonal entries equal to $P_{uu} = \kappa_u$, $u, v = 1, \ldots, N$. When P is invertible we define

$$\begin{aligned} \Theta(p) &= \mathcal{I}_2 - (1 - p) \star C \star (\mathcal{I}_N - pA)^{-\star} P^{-1} (\mathcal{I}_N - A)^{-*} C^* \mathcal{J} \\ &= \begin{pmatrix} a(p) & b(p) \\ c(p) & d(p) \end{pmatrix}. \end{aligned} \tag{3.8}$$

Note that Θ is well defined in $\mathbb{B}_1$ since we assumed that the interpolation nodes p_u are all different from 1. Finally we denote by $\mathcal{M}$ the span of the columns of the function

$$F(p) = C \star (\mathcal{I}_N - pA)^{-\star} = \sum_{t=0}^{\infty} p^t C A^t, \tag{3.9}$$

and endow $\mathcal{M}$ with the Hermitian form

$$[F(p)c, F(p)d]_{\mathcal{M}} = d^*Pc, \quad c, d \in \mathbb{H}^N. \tag{3.10}$$

We prove the following theorem.

Theorem 3.2. (1) *There always exists a Schur function so that* (1.1) *holds.*
(2) *Fix* $\kappa_1, \ldots, \kappa_N \geq 0$ *and assume* $P > 0$. *Any solution of Problem* 1.1 *is of the form*

$$s(p) = (a(p) \star e(p) + b(p)) \star (c(p) \star e(p) + d(p))^{-\star}, \tag{3.11}$$

where a, b, c, d *are as in* (3.8) *and* e *is a slice hyperholomorphic Schur function.*
(3) *Conversely, any function of the form* (3.11) *satisfies* (1.1). *If*

$$\lim_{\substack{r \to 1 \\ r \in (0,1)}} \frac{1 - s(rp_u)\overline{s_u}}{1 - r}, \quad u = 1, \ldots, N, \tag{3.12}$$

exists and is real, then s *satisfies* (1.2).
(4) *If* e *is a unitary constant, then the limit* (3.12) *exists* (*but are not necessarily real*) *and satisfies*

$$\frac{|\beta_u - \overline{p_u}\beta_u\overline{p_u}|^2}{|1 - \overline{p_u}^2|} \leq (\operatorname{Re}\beta_u)\kappa_u, \quad u = 1, \ldots, N. \tag{3.13}$$

The strategy of the proof is as follows:

STEP 1: *The condition* $P \geq 0$ *is necessary for Problem* 1.1 *to have a solution.*

STEP 2: *Assume that* s *is a solution of Problem* 1.1. *Then the map* $M_{(1\ -s)}$ *of left* $\star$*-multiplication by* $\big(1\ -s(p)\big)$ *is a contraction from* $\mathcal{M}$ *into* $\mathcal{H}(s)$, *where* $\mathcal{H}(s)$ *denotes the reproducing kernel Hilbert space of quaternionic valued functions which are hyperholomorphic in the ball* $\mathbb{B}_1$ *and with reproducing kernel*

$$K_s(p, q) = \sum_{t=0}^{\infty} p^t(1 - s(p)\overline{s(q)})\bar{q}^t.$$

STEP 3: *Assume that* s *is a solution of Problem* 1.1 *and that* $P > 0$. *Then,* s *is of the form* (3.11).

STEP 4: *Assume that* $P > 0$. *Then any function of the form* (3.11) *satisfies the interpolation condition* (1.1) *and if, in addition,* (3.12) *is in force, then* s *satisfies* (1.2).

STEP 5: *Assume* e *is a unitary constant. Then the claims in* (4) *hold.*

The proofs of Steps 1–5 are given in Sect. 4. The degenerate case is considered in Sect. 5.

4. Proofs of Steps 1–5

Proof of Step 1. Assume a solution s exists. Since s is a Schur function the kernel $K_s(p,q)$ is positive definite and so for every $r \in (0,1)$ the $N \times N$ matrix $P(r)$ with (u,v) entry equal to

$$P_{uv}(r) = K_s(rp_u, rp_v) = \sum_{t=0}^{\infty} r^{2t} p_u^t (1 - s(rp_u)\overline{s(rp_v)}) p_v^t, \quad u, v = 1, \dots N,$$

is positive. Setting

$$G = (1 - s(rp_u)\overline{s(rp_v)}), \quad p = r^2 p_u, \quad \text{and} \quad A = \overline{p_v}$$

in formula (2.9) we have

$$\begin{aligned} P_{uv}(r) &= \Big((1 - s(rp_u)\overline{s(rp_v)}) - r^2 \overline{p_u}(1 - s(rp_u)\overline{s(rp_v)})\overline{p_v} \Big) \\ &\quad \cdot (1 - 2r^2 \mathrm{Re}(p_u)\overline{p_v} + r^4 \overline{p_v}^2)^{-1}. \end{aligned}$$

Furthermore, we note that $P(r)$ is a solution of the matrix equation

$$P(r) - r^2 A^* P(r) A = C(r)^* \mathcal{J} C(r),$$

where

$$C(r) = \begin{pmatrix} 1 & \cdots & 1 \\ \overline{s(rp_1)} & \cdots & \overline{s(rp_N)} \end{pmatrix},$$

and A is as in (3.1). In fact, with the above notation, the (u,v) element of the matrix $P(r) - r^2 A^* P(r) A$ can be computed as follows:

$$\begin{aligned} &P_{uv}(r) - r^2 p_u P_{uv}(r)\overline{p_v} \\ &\quad = \left((G - r^2 \overline{p_u} G \overline{p_v}) - r^2 p_u \left(G - r^2 \overline{p_u} G \overline{p_v} \right) \overline{p_v} \right) (1 - 2r^2 \mathrm{Re}(p_u)\overline{p_v} + r^4 \overline{p_v}^2)^{-1} \\ &\quad = \left(G - r^2 \overline{p_u} G \overline{p_v} - r^2 p_u G \overline{p_v} + r^4 G \overline{p_v}^2 \right) (1 - 2r^2 \mathrm{Re}(p_u)\overline{p_v} + r^4 \overline{p_v}^2)^{-1} \\ &\quad = G \left(1 - 2r^2 \mathrm{Re}(p_u)\overline{p_v} + r^4 \overline{p_v}^2 \right) (1 - 2r^2 \mathrm{Re}(p_u)\overline{p_v} + r^4 \overline{p_v}^2)^{-1} \\ &\quad = (1 - s(rp_u)\overline{s(rp_v)}) \end{aligned}$$

and so the (u,v) element in the matrix $P(r) - r^2 A^* P(r) A$ equals the (u,v) element in $C(r)^* \mathcal{J} C(r)$ as stated. We now let r tend to 1. Since s is assumed to be a solution of Problem 1.1, we have

$$\lim_{\substack{r \to 1 \\ r \in (0,1)}} K_s(rp_u, rp_u) = \lim_{\substack{r \to 1 \\ r \in (0,1)}} \frac{1 - |s(rp_u)|^2}{1 - r^2} \le \kappa_u, \quad u = 1, \dots N,$$

and

$$\lim_{\substack{r \to 1 \\ r \in (0,1)}} C(r) = C,$$

where C is as in (3.1). Furthermore we note that $1 - 2\mathrm{Re}(p_u)\overline{p_v} + \overline{p_v}^2 \neq 0$ since $1 - 2\mathrm{Re}(p_u)x + x^2$ is the so-called minimal (or companion) polynomial associated with the sphere $[p_u]$ which vanishes exactly at points on the sphere $[p_u]$ and $p_v \notin [p_u]$. This fact can also be obtained directly using Lemma 3.1. Indeed, for indices $u \neq v$, we have

$$1 - 2\mathrm{Re}(p_u)\overline{p_v} + \overline{p_v}^2 = p_u(\overline{p_u} - \overline{p_v}) - (\overline{p_u} - \overline{p_v})\overline{p_v} \neq 0, \tag{4.1}$$

since p_u and p_v (and hence p_u and $\overline{p_v}$) are assumed on different spheres for $u \neq v$. It follows that $\lim_{\substack{r\to 1\\ r\in(0,1)}} P_{uv}(r)$ exists and is in fact equal to P_{uv} for $u \neq v$ by uniqueness of the solution of the equation

$$x - p_u x \overline{p_v} = 0. \tag{4.2}$$

Hence $P \geq 0$ since $P(r) \geq 0$ for all $r \in (0,1)$.

Proof of Step 2. Let s be a solution (if any) of Problem 1.1, let $u \in \{1,\ldots,N\}$, and let $r \in (0,1)$. The functions

$$g_{u,r}(p) = K_s(p, rp_u) = \sum_{t=0}^{\infty} p^t (1 - s(p)\overline{s(rp_u)}) r\overline{p_u}^t$$

belong to $\mathcal{H}(s)$ and have uniformly bounded norms since

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \|g_{u,r}(rp_u)\|^2_{\mathcal{H}(s)} = \lim_{\substack{r\to 1\\ r\in(0,1)}} K_s(rp_u, rp_u) \leq \kappa_u.$$

Thus there is a sequence of numbers $r_0, r_1, \ldots \in (0,1)$ which tends to 1 (without loss of generality we may assume that the sequence is the same for $p_1,\ldots,p_N$) and an element $g_u \in \mathcal{H}(s)$ such that the functions g_{u,r_n} tend weakly to g_u. In a reproducing kernel Hilbert space weak convergence implies pointwise convergence, and so

$$\begin{aligned}
g_u(p) &= \lim_{n\longrightarrow\infty} g_{u,r_n}(p)\\
&= \lim_{n\longrightarrow\infty} \sum_{t=0}^{\infty} r_n^t p^t (1 - s(p)\overline{s(r_n p_u)})\overline{p_u}^t\\
&= \sum_{t=0}^{\infty} p^t (1 - s(p)\overline{s_u})\overline{p_u}^t\\
&= \big(1 \; -s(p)\big) \star f_u(p), \quad \forall p \in \mathbb{B}_1,
\end{aligned}$$

where

$$f_u(p) = \sum_{t=0}^{\infty} p^t \begin{pmatrix} 1 \\ \\ \overline{s_u} \end{pmatrix} \overline{p_u}^t \tag{4.3}$$

denotes the u-th column of the matrix-function $F(p)$ and where the interchange of summation and limit is justified since $|p| < 1$. Hence $M_{\big(1 \; -s\big)}$ sends $\mathcal{M}$ into $\mathcal{H}(s)$. Note that for $Y = (y_{u,v})_{u,v=1}^N$ and $Z = (z_{u,v})_{u,v=1}^N$ we define $Y \star Z$ to be the $N \times N$ matrix whose (u,v) entry is given by $\sum_{t=1}^N y_{u,t} \star z_{t,v}$. To show that this operator is a contraction we first compute the inner product $\langle g_v, g_u\rangle_{\mathcal{H}(s)}$ for $u \neq v$. By the definition of the weak limit and of the reproducing kernel, we can write

$$\begin{aligned}
\langle g_v, g_u\rangle_{\mathcal{H}(s)} &= \lim_{n\longrightarrow\infty} \langle g_v, g_{u,r_n}\rangle_{\mathcal{H}(s)}\\
&= \lim_{n\longrightarrow\infty} g_v(r_n p_u)\\
&= \lim_{n\longrightarrow\infty} \sum_{t=0}^{\infty} r_n^t p_u^t (1 - s(r_n p_u)\overline{s_v})\overline{p_v}^t
\end{aligned}$$

$$\begin{aligned}&= \lim_{n\longrightarrow\infty} ((1-s(r_np_u)\overline{s_v}) - r_n\overline{p_u}(1-s(r_np_u)\overline{s_v})\overline{p_v})\\ &\quad \cdot(1-2r_n\mathrm{Re}(p_u)\overline{p_v} + r_n^2\overline{p_v}^2)^{-1}\\ &= ((1-s_u\overline{s_v}) - \overline{p_u}(1-s_u\overline{s_v})\overline{p_v})\,(1-2\mathrm{Re}(p_u)\overline{p_v}+\overline{p_v}^2)^{-1},\end{aligned}$$

where we have used formula (2.9) and, as in the proof of Step 1 (see (4.1)), the fact that $[p_u] \cap [p_v] = \emptyset$ (recall that we assume here $u \neq v$). We claim that

$$P_{uv} = ((1-s_u\overline{s_v}) - \overline{p_u}(1-s_u\overline{s_v})\overline{p_v})\,(1-2\mathrm{Re}(p_u)\overline{p_v}+\overline{p_v}^2)^{-1}. \tag{4.4}$$

The proof is similar to the argument in the proof of Step 1, and is as follows. Set $h_n = \langle g_v, g_{u,r_n}\rangle_{\mathcal{H}(s)}$. Then

$$h_n - r_np_uh_n\overline{p_v} = 1 - s(r_np_u)\overline{s_v}.$$

Letting $n \to \infty$ we see that $h = \lim_{n\to\infty} h_n$ satisfies Eq. (3.3). By the uniqueness of the solution of this equation we have $h = P_{uv}$. Furthermore, by the property of the weak limit versus the norm,

$$\|g_u\|^2_{\mathcal{H}(s)} \leq \lim_{n\to\infty} \|g_{u,r_n}\|^2_{\mathcal{H}(s)} \leq \kappa_u. \tag{4.5}$$

We can now show that $\|M_{(1\ -s)}\| \leq 1$. Let $c \in \mathbb{H}^N$. Then,

$$\left(M_{(1\ -s)}Fc\right)(p) = \sum_{u=1}^N g_u(p)c_u$$

and we have

$$\begin{aligned}\|(M_{(1\ -s)}Fc\|^2_{\mathcal{H}(s)} &= \sum_{u,v=1}^N \overline{c_u}\left(\langle g_v, g_u\rangle_{\mathcal{H}(s)}\right)c_v\\ &= \sum_{u=1}^N |c_u|^2\|g_u\|^2_{\mathcal{H}(s)} + \sum_{\substack{u,v=1\\ u\neq v}}^N \overline{c_u}\left(\langle g_v, g_u\rangle_{\mathcal{H}(s)}\right)c_v\\ &= \sum_{u=1}^N |c_u|^2\|g_u\|^2_{\mathcal{H}(s)} + \sum_{\substack{u,v=1\\ u\neq v}}^N \overline{c_u}P_{uv}c_v\\ &\leq \sum_{u=1}^N |c_u|^2\kappa_u + \sum_{\substack{u,v=1\\ u\neq v}}^N \overline{c_u}P_{uv}c_v\\ &= c^*Pc\\ &= \|Fc\|^2_{\mathcal{M}},\end{aligned}$$

where we have used (4.4) and (4.5). Thus the $\star$-multiplication by $(1\ -s(p))$ is a contraction from $\mathcal{M}$ into $\mathcal{H}(s)$.

Proof of Step 3. Let Θ be defined by (3.8), and

$$K_\Theta(p,q) = \sum_{t=0}^\infty p^t\left(\mathcal{J} - \Theta(p)\mathcal{J}\Theta(q)^*\right)\overline{q}^t. \tag{4.6}$$

The formula

$$F(p)P^{-1}F(q)^* = K_\Theta(p,q) \tag{4.7}$$

is proved as in the complex case when p and q are real, and is then extended to $p, q \in \mathbb{B}_1$ by a slice hyperholomorphic extension. Using (2.4) we have

$$\begin{aligned}&\left(M^*_{\left(1\ -\ s\right)}K_s(\cdot,q)\right)(p)\\ &\quad=\sum_{t=0}^{\infty}p^n\left(\begin{pmatrix}1\\ -\overline{s(q)}\end{pmatrix}-\Theta(p)\mathcal{J}\Theta(q)^*\star_r\begin{pmatrix}1\\ -\overline{s(q)}\end{pmatrix}\right)\overline{q}^t,\end{aligned}$$

and so

$$\begin{aligned}&(M_{\left(1\ -s\right)}M^*_{\left(1\ -s\right)}K_s(\cdot,q))(p)\\ &\quad= K_s(p,q)-\sum_{t=0}^{\infty}p^t\left(\begin{pmatrix}1\ -s(p)\end{pmatrix}\star\Theta(p)\mathcal{J}\Theta(q)^*\star_r\begin{pmatrix}1\\ -\overline{s(q)}\end{pmatrix}\right)\overline{q}^t\\ &\quad\le K_s(p,q),\end{aligned}$$

and therefore the kernel

$$\begin{aligned}&\sum_{t=0}^{\infty}p^t\left(\begin{pmatrix}1\ -s(p)\end{pmatrix}\star\Theta(p)\mathcal{J}\Theta(q)^*\star_r\begin{pmatrix}1\\ -\overline{s(q)}\end{pmatrix}\right)\overline{q}^t\\ &\qquad=\sum_{t=0}^{\infty}p^t\left(A(p)\overline{A(q)}-B(p)\overline{B(q)}\right)\overline{q}^t\end{aligned}$$

is positive definite in $\mathbb{B}_1$, where

$$A(p)=(a-s\star c)(p)\quad\text{and}\quad B(p)=(b-s\star d)(p).$$

The point $p = 1$ is not an interpolation node, and so Θ is well defined at $p = 1$. From (3.8) we have

$$\Theta(1)=\mathcal{I}_2 \tag{4.8}$$

and so $(ca^{-1})(1) = 0$. Since s is bounded by 1 in modulus in $\mathbb{B}_1$ it follows that $(a-s\star c)\not\equiv 0$, in fact by restricting the the real axis we have $a(x)-s(x)c(x) = (1-s(x)c(x)a(x)^{-1})a(x)$ which is nonzero at least in a real left neighborhood of 1. Thus $e = -(a-s\star c)^{-\star}\star(b-s\star d)$ is defined in $\mathbb{B}_1$, with the possible exception of spheres of poles. Since

$$\sum_{t=0}^{\infty}p^t\left(A(p)\overline{A(q)}-B(p)\overline{B(q)}\right)\overline{q}^t = A(p)\star\left\{\sum_{t=0}^{\infty}p^t(1-e(p)\overline{e(q)})\overline{q}^t\right\}\star_r\overline{A(q)},$$

we have from [5, Proposition 5.3] that the kernel

$$K_e(p,q)=\sum_{t=0}^{\infty}p^t(1-e(p)\overline{e(q)})\overline{q}^t$$

is positive definite in its domain of definition, and thus e extends to a Schur function (see [6] for the latter assertion). From

$$e = -(a - s \star c)^{-\star} \star (b - s \star d)$$

we get $s \star (c \star e + d) = a \star e + b$. To conclude we remark that (4.8) implies that

$$(d^{-1}c)(1) = 0.$$

Thus, as just above $c \star e + d \not\equiv 0$ and we get that s is of the form (3.11).

Proof of Step 4. Assume that s is of the form (3.11). Then the formula

$$\begin{aligned} K_s(p,q) = \begin{pmatrix} 1 & -s(p) \end{pmatrix} \star K_\Theta(p,q) \star_r \begin{pmatrix} 1 \\ -\overline{s(q)} \end{pmatrix} \\ +(a - s \star c)(p) \star K_e(p,q) \star_r \overline{(a - s \star c)(q)} \end{aligned} \tag{4.9}$$

implies that $M_{\begin{pmatrix} 1 & -s \end{pmatrix}}$ is a contraction from $\mathcal{H}(\Theta)$ into $\mathcal{H}(s)$. In particular,

$$g_u(p) = \begin{pmatrix} 1 & -s(p) \end{pmatrix} \star f_u(p) = \sum_{t=0}^{\infty} p^t (1 - s(p)\overline{s_u})\overline{p_u}^t \in \mathcal{H}(s) \tag{4.10}$$

and

$$\|g_u\|^2_{\mathcal{H}(s)} \leq \kappa_u.$$

We want to infer from these facts that s satisfies the interpolation conditions (1.1). We have

$$\begin{aligned} |g_u(rp_u)|^2 &= |\langle g_u(\cdot), K_s(\cdot, rp_u)\rangle_{\mathcal{H}(s)}|^2 \\ &\leq \left(\|g_u\|^2_{\mathcal{H}(s)} \right) \cdot K_s(rp_u, rp_u) \\ &\leq \kappa_u \cdot \frac{1 - |s(rp_u)|^2}{1 - r^2} \\ &\leq \frac{2\kappa_u}{1 - r}. \end{aligned} \tag{4.11}$$

In view of (2.9), we get

$$\begin{aligned} & g_u(rp_u) \\ &= \sum_{t=0}^{\infty} r^t p_u^t (1 - s(rp_u)\overline{s_u})\overline{p_u}^t \\ &= ((1 - s(rp_u)\overline{s_u}) - r\overline{p_u}(1 - s(rp_u)\overline{s_u})\overline{p_u}) \left(1 - 2r\mathrm{Re}(p_u)\overline{p_u} + r^2\overline{p_u}^2\right)^{-1} \\ &= ((1 - s(rp_u)\overline{s_u}) - r\overline{p_u}(1 - s(rp_u)\overline{s_u})\overline{p_u}) \left((1 - r)(1 - r\overline{p_u}^2)\right)^{-1}, \end{aligned} \tag{4.12}$$

and so we have

$$\frac{|(1 - s(rp_u)\overline{s_u}) - r\overline{p_u}(1 - s(rp_u)\overline{s_u})\overline{p_u}|}{|1 - r\overline{p_u}^2|} \leq \sqrt{2\kappa_u} \cdot \sqrt{1 - r}.$$

Let σ_u be a limit, via a subsequence, of $s(rp_u)$ as $r \to 1$, and set $X_u = 1 - \sigma_u\overline{s_u}$. The above inequality implies that $X_u = \overline{p_u}X_u\overline{p_u}$, and so

$$X_u p_u = \overline{p_u} X_u. \tag{4.13}$$

The conjugate of (4.13) is

$$\overline{X_u}p_u = \overline{p_u}\,\overline{X_u}. \tag{4.14}$$

Adding (4.13) and (4.14) we obtain

$$\mathrm{Re}(X_u)p_u = \overline{p_u}\mathrm{Re}(X_u).$$

Since p_u is not real we get that $\mathrm{Re}(X_u) = 0$. Let $X_u = \alpha i + \beta j + \gamma k$, where $\alpha, \beta, \gamma \in \mathbb{R}$. From $\sigma_u\overline{s_u} = 1 - X_u$ we have

$$|\sigma_u\overline{s_u}|^2 = 1 + \alpha^2 + \beta^2 + \gamma^2.$$

Since $\sigma_u \in \mathbb{B}_1$ we have $|\sigma_u\overline{s_u}| \leq 1$ and so $\alpha = \beta = \gamma = 0$. Thus, $X_u = 0$ and $\sigma_u\overline{s_u} = 1$. Hence $\sigma_u = s_u$ and the limit $\lim_{\substack{r\to 1\\ r\in(0,1)}} s(rp_u)$ exists and is equal to s_u, and hence (1.1) is satisfied.

To prove that (1.2) is met we proceed as follows. From (4.11) we have in particular

$$|g_u(rp_u)|^2 \leq \kappa_u \cdot \frac{1 - |s(rp_u)|^2}{1 - r^2},$$

and using (4.12) we obtain:

$$\frac{|X(r) - r\overline{p_u}X(r)\overline{p_u}|^2}{(1-r)^2|1 - r\overline{p_u}^2|^2} \leq \kappa_u \cdot \frac{1 - |s(rp_u)|^2}{1 - r^2}, \tag{4.15}$$

where we have set $X(r) = 1 - s(rp_u)\overline{s_u}$. Assume now that (3.12) is in force and let

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1 - s(rp_u)\overline{s_u}}{1 - r} = \beta_u \in \mathbb{R}. \tag{4.16}$$

Then (4.15) together with (1.4) imply that

$$\beta_u^2 \leq \beta_u\kappa_u,$$

from which we get that $\beta_u \geq 0$ and

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1 - s(rp_u)\overline{s_u}}{1 - r} \leq \kappa_u.$$

Proof of Step 5. We first note that the limits (1.1) hold in view of the previous step. Since $|e| = 1$, equality (4.9) implies that the multiplication operator $M_{(1\ -s)}$ is unitary and so the space $\mathcal{H}(s)$ is finite dimensional. Using [6] we see that s can be written in the form

$$s(p) = H + pG \star (I - pT)^{-\star}F, \tag{4.17}$$

where the block matrix

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix}$$

is such that

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix}\begin{pmatrix} P^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} T & F \\ G & H \end{pmatrix}^* = \begin{pmatrix} P^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

for a uniquely determined positive matrix $P \in \mathbb{H}^{v\times v}$, where $v = \dim \mathcal{H}(s)$. The formula (see [5, formula (8.11)])

$$\sum_{u=0}^{\infty} p^u(1 - s(p)\overline{s(q)})\overline{q}^u = G \star (I - pT)^{-\star} P^{-1} \star_r (I - T^*\overline{q})^{-\star_r} \star_r G^* \quad (4.18)$$

implies then that s is unitary on the unit sphere. Equation (4.17) implies that for every p on the unit sphere the function $r \mapsto s(rp)$ is real analytic for r in a neighborhood of the origin, and so $\lim_{\substack{r\to 1\\ r\in(0,1)}} s(rp)$ exists and is unitary. For $p = p_u$ it follows that the limits (1.2) exist. Then (1.4) leads to

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1 - |s(rp_u)|^2}{1 - r^2} = \operatorname{Re} \beta_u$$

and the conclusion follows then from (4.15).

5. The Degenerate Case

We now consider the case when P is singular. We need first a definition. A finite Blaschke product is a finite $\star$-product of terms of the form

$$b_a(p) = (1 - p\bar{a})^{-\star} \star (a - p)\frac{\bar{a}}{|a|}, \quad (5.1)$$

where $a \in \mathbb{H}$, $|a| < 1$ (see [7]).

The purpose of this section is to prove the following theorem. We denote by r the rank of P and assume that the main $r \times r$ minor of P is invertible. This can be done by rearranging the interpolation points.

Theorem 5.1. *Assume that P is singular. Then Problem 1.1 has at most one solution, and the latter is then a finite Blaschke product. It has a unique solution satisfying (3.13) for $u = 1, \ldots, r$.*

We begin with some preliminary results and definitions.

Definition 5.2. Let f be a slice hyperholomorphic function in a neighborhood Ω of $p = 1$, and let $f(p) = \sum_{t=0}^{\infty}(p-1)^t f_t$ be its power series expansion at $p = 1$. We define

$$R_1 f(p) = \sum_{t=1}^{\infty}(p-1)^{t-1} f_t. \quad (5.2)$$

Denoting by ext the slice hyperholomorphic extension, see (2.2), we have

$$R_1 f(p) = \operatorname{ext}\left(R_1 f|_{p=x}\right). \quad (5.3)$$

Lemma 5.3. *Let $f(p) = F(p)\xi$ where $F(p) = C \star (\mathcal{I}_N - pA)^{-\star}$. Then*

$$R_1 f(p) = F(p)A(\mathcal{I}_N - A)^{-1}\xi. \quad (5.4)$$

Proof. First of all, recall that

$$F(p) = C \star (\mathcal{I}_N - pA)^{-\star} = (C - \bar{p}CA)(I_n - 2\mathrm{Re}(p)A + |p|^2A^2)^{-1},$$

and so

$$F(1) = (C - CA)(\mathcal{I}_N - 2A + A^2)^{-1} = C(\mathcal{I}_N - A)^{-1}.$$

Let us compute

$$\begin{aligned} R_1f(p) &= (p-1)^{-1}(f(p) - f(1)) \\ &= (p-1)^{-1}(C \star (\mathcal{I}_N - pA)^{-\star}\xi - C(\mathcal{I}_N - A)^{-1}\xi) \\ &= C \star (p-1)^{-1}((\mathcal{I}_N - pA)^{-\star} - (\mathcal{I}_N - A)^{-1})\xi \\ &= C \star (p-1)^{-1} \star (\mathcal{I}_N - pA)^{-\star} \star ((\mathcal{I}_N - A) \\ &\quad - (\mathcal{I}_N - pA))(\mathcal{I}_N - A)^{-1}\xi \\ &= C \star (p-1)^{-1} \star (\mathcal{I}_N - pA)^{-\star} \star (p-1)A(\mathcal{I}_N - A)^{-1}\xi \\ &= C \star (\mathcal{I}_N - pA)^{-\star}A(\mathcal{I}_N - A)^{-1}\xi \\ &= F(p)A(\mathcal{I}_N - A)^{-1}\xi. \end{aligned}$$

□

Recalling (3.10), we now prove the following:

Lemma 5.4. *Let $f, g \in \mathcal{M}$. Then*

$$[f, g]_{\mathcal{M}} + [R_1f, g]_{\mathcal{M}} + [f, R_1g]_{\mathcal{M}} = g(1)^*\mathcal{J}f(1). \tag{5.5}$$

Proof. Let $f(p) = F(p)\xi$ and $g(p) = F(p)\eta$ with $\xi, \eta \in \mathbb{H}^N$. We have

$$f(1) = C(\mathcal{I}_N - A)^{-1}\xi \quad \text{and} \quad g(1) = C(\mathcal{I}_N - A)^{-1}\eta.$$

These equations together with (5.4) show that (5.5) is equivalent to

$$P + P(\mathcal{I}_N - A)^{-1}A + A^*(\mathcal{I}_N - A)^{-*}P = (\mathcal{I}_N - A)^{-*}C^*\mathcal{J}C(\mathcal{I}_N - A)^{-1}.$$

Multiplying this equation by $\mathcal{I}_N - A^*$ on the left and by $\mathcal{I}_N - A$ on the right we get the equivalent equation (3.2). □

Remark 5.5. Equation (5.5) corresponds to a special case of a structural identity which characterizes $\mathcal{H}(\Theta)$ spaces in the complex setting. A corresponding identity in the half place case was first introduced by de Branges, see [16], and improved by Rovnyak [21]. Ball introduced the corresponding identity in the setting of the open unit disk and proved the corresponding structure theorem (see [13]). In addition, see e.g., [11, p. 17] for further discussions on this topic.

Proposition 5.6. *Let a and b be slice hyperholomorphic functions defined in an axially symmetric s-domain containing $p = 1$. Then*

$$R_1(a \star b)(p) = (R_1a(p))\, b(1) + (a \star R_1b)(p). \tag{5.6}$$

Proof. By the Identity Principle, see [15, Theorem 4.2.4] the equality holds if and only if it holds for the restrictions to a complex plane $\mathbb{C}_I$ i.e., using the notation in Sect. 2, if and only if

$$(R_1(a \star b))_I(z) = (R_1a(z))_I\, b(1) + (a \star R_1b)_I(z), \quad z \in \mathbb{C}_I. \tag{5.7}$$

Let $J \in \mathbb{S}$ be such that J is orthogonal to I and assume that

$$a_I(z) = F(z) + G(z)J, \quad b_I(z) = H(z) + L(z)J.$$

Let us compute the left-hand side of (5.7), using the fact that $(R_1(a\star b))_I(z) = R_1((a \star b)_I)$ and formula (2.3):

$$\begin{aligned} R_1((a \star b)_I) &= R_1\left(F(z)H(z) - G(z)\overline{L(\bar{z})} + (G(z)\overline{H(\bar{z})} + F(z)L(z))J\right) \\ &= (z-1)^{-1}\Big(F(z)H(z) - G(z)\overline{L(\bar{z})} + (G(z)\overline{H(\bar{z})} + F(z)L(z))J \\ &\quad -F(1)H(1) + G(1)\overline{L(1)} - (G(1)\overline{H(1)} + F(1)L(1))J)\Big). \end{aligned}$$

On the right hand side of (5.7) we have $(R_1 a(z))_I\, b(1) = (R_1 a_I(z))\, b(1)$ which can be written as

$$\begin{aligned} &(R_1 a_I(z))\, b(1) \\ &\quad = \left((z-1)^{-1}(F(z) + G(z)J - F(1) - G(1)J)\right)(H(1) + L(1)J) \\ &\quad = (z-1)^{-1}\Big(F(z)H(1) + F(z)L(1)J + G(z)\overline{H(1)}J - G(z)\overline{L(1)} \\ &\qquad -F(1)H(1) - F(1)L(1)J - G(1)\overline{H(1)}J + G(1)\overline{L(1)}\Big), \end{aligned}$$

and moreover,

$$\begin{aligned} &(a \star R_1 b)_I(z) \\ &\quad = (F(z) + G(z)J) \star \left((z-1)^{-1}(H(z) + L(z)J - H(1) - L(1)J)\right) \\ &\quad = (z-1)^{-1}(F(z) + G(z)J) \star (H(z) + L(z)J - H(1) - L(1)J) \\ &\quad = (z-1)^{-1}(F(z)H(z) - G(z)\overline{L(\bar{z})} + (G(z)\overline{H(\bar{z})} + F(z)L(z))J) \\ &\qquad -F(z)H(1) + G(z)\overline{L(1)} - (G(z)\overline{H(1)} + F(z)L(1))J \end{aligned}$$

from which the equality follows. □

We will also need the following result, well known in the complex case. We refer to [12,24] for more information and to [18] for connections with operator ranges.

Theorem 5.7. *Let $K_1(p,q)$ and $K_2(p,q)$ be two $\mathbb{H}$-valued functions positive definite in a set Ω and assume that the corresponding reproducing kernel Hilbert spaces have a zero intersection. Then the sum*

$$\mathcal{H}(K_1 + K_2) = \mathcal{H}(K_1) + \mathcal{H}(K_2)$$

is orthogonal.

Proof. Let $K = K_1 + K_2$. The linear relation in $\mathcal{H}(K) \times (\mathcal{H}(K_1) \times \mathcal{H}(K_2))$ spanned by the pairs

$$(K(p,q), (K_1(p,q), K_2(p,q))), \quad q \in \Omega,$$

is densely defined and isometric. It therefore extends to the graph of an everywhere defined isometry, which we will call T. See [7, Theorem 7.2].

From

$$\begin{aligned}(T^*(f_1,f_2))(q) &= \langle T^*(f_1,f_2), K(p,q)\rangle_{\mathcal{H}(K)}\\ &= \langle (f_1,f_2), TK(p,q)\rangle_{\mathcal{H}(K_1)\times\mathcal{H}(K_2)}\\ &= \langle f_1, K_1(p,q)\rangle_{\mathcal{H}(K_1)} + \langle f_2, K_2(p,q)\rangle_{\mathcal{H}(K_2)}\\ &= f_1(q)+f_2(q), \quad q\in\Omega,\end{aligned}$$

we see that $\ker T^* = \{0\}$ since $\mathcal{H}(K_1)\cap\mathcal{H}(K_2)=\{0\}$. Thus T is unitary and the result follows easily. □

Proof of Theorem 5.1. We proceed in a number of steps. Recall that $r = \operatorname{rank} P$.

STEP 1: *Assume* $r=0$. *Then* $s_1=\cdots=s_N$ *and Problem* 1.1 *is solvable with the unique solution* s, *where* $s\equiv s_1$ *is the constant unitary function.*

The matrix $P=0$, and Eq. (3.2) imply that $C^*\mathcal{J}C=0$, and so $1-s_u\overline{s_v}=0$ for $u\neq v\in\{1,\ldots,N\}$. Thus $s_1=\cdots=s_N$ and the function $s\equiv s_1$ is clearly a solution. Assume that s is a (possibly different) solution of Problem 1.1. The map $M_{\left(1-s\right)}$ of slice multiplication by $\big(1-s(p)\big)$ is a contraction from $\mathcal{M}$ into $\mathcal{H}(s)$ (see the second step in the proof of Theorem 3.2). Thus

$$\big(1-s(p)\big)\star f_u(p)\equiv 0,\quad u=1,\ldots,N,$$

that is $g_u\equiv 0$, where f_u and g_u have been defined in (4.3) and (4.10) respectively. From (2.9) we have (for $|p|<1$)

$$g_u(p)=((1-s(p)\overline{s_u})-\overline{p}(1-s(p)\overline{s_u})\overline{p_u})\,(1-2\mathrm{Re}(p)\overline{p_u}+|p|^2p_u^2)^{-1},$$

since

$$1-2\mathrm{Re}(p)\overline{p_u}+|p|^2p_u^2\neq 0$$

for $|p|<1$. Hence

$$(1-s(p)\overline{s_u})=\overline{p}(1-s(p)\overline{s_u})\overline{p_u},\quad \forall p\in\mathbb{H}_1.$$

Taking absolute values of both sides of this equality we get $(1-s(p)\overline{s_u})\equiv 0$, and so $s(p)\equiv s_u$. This ends the proof of Step 1.

In the rest of the proof we assume $r>0$. By reindexing the interpolating nodes we can also assume that the principal minor of order r of the matrix P is invertible. Thus the corresponding space is a $\mathcal{H}(\Theta_r)$ space, and we can write

$$\mathcal{M}=\mathcal{H}(\Theta_r)\oplus\Theta_r\star\mathcal{N},$$

since Θ_r is $\star$-invertible.

STEP 2: *The elements of* $\mathcal{N}$ *are slice hyperholomorphic in a neighborhood of* $p=1$ *and* $R_1\mathcal{N}\subset\mathcal{N}$.

We follow the argument in Step 1 in the proof of Theorem 3.1 in [10] (see p. 153). From (5.6) we have

$$(R_1(\Theta_r\star n))(p)=(R_1\Theta_r)(p)n(1)+(\Theta_r\star R_1n)(p). \tag{5.8}$$

To prove that $R_1n\in\mathcal{N}$ we show that

$$[(R_1(\Theta_r\star n))(p)-(R_1\Theta_r)(p)n(1),g]_{\mathcal{M}}=0,\quad \forall g\in\mathcal{H}(\Theta_r). \tag{5.9}$$

Using (5.5) we have

$$\begin{aligned}[(R_1(\Theta_r \star n))(p), g]_{\mathcal{M}} &= g(1)^* \mathcal{J}(R_1(\Theta_r \star n))(1) - [\Theta_r \star n, g]_{\mathcal{M}} \\ &\quad - [\Theta_r \star n, R_1 g]_{\mathcal{M}} \\ &= g(1)^* \mathcal{J}(R_1(\Theta_r \star n))(1)\end{aligned}$$

since

$$[\Theta_r \star n, g]_{\mathcal{M}} = 0 \quad \text{and} \quad [\Theta_r \star n, R_1 g]_{\mathcal{M}} = 0,$$

where the second equality follows from $R_1 g \in \mathcal{M}$. Moreover, for real $p = x$ we have the equality of real analytic functions

$$(R_1\Theta_r)(x) = -K_{\Theta_r}(x, 1)\mathcal{J}\Theta_r(1)^*,$$

and so, by slice hyperholomorphic extension, see [4, Remark 2.18], in a suitable neighborhood of $p = 1$ we have

$$(R_1\Theta_r)(p) = -K_{\Theta_r}(p, 1)\mathcal{J}\Theta_r(1)^*.$$

Note that $\Theta_r(1)$ is the identity. Thus

$$\begin{aligned}[(R_1\Theta_r)(p)n(1), g]_{\mathcal{M}} &= -[K_{\Theta_r}(p, 1)\mathcal{J}\Theta_r(1)^* n(1), g]_{\mathcal{M}} \\ &= -(n(1)^*\Theta_r(1)^* g(1)^*) \\ &= -g(1)^*\Theta_r(1)\mathcal{J}n(1),\end{aligned}$$

and so (5.9) is in force. This ends the proof of the second step.

Endow now $\mathcal{N}$ with the Hermitian form

$$[n_1, n_2]_{\mathcal{N}} = [\Theta_r \star n_1, \Theta_r \star n_2]_{\mathcal{M}}.$$

STEP 3: *There exist matrices* $(G, T) \in \mathbb{H}^{2\times(N-r)} \times \mathbb{H}^{(N-r)\times(N-r)}$ *such that* $\mathcal{N}$ *is spanned by the columns of the function* $F_{\mathcal{N}}(p) = G\star(\mathcal{I}_{N-r} - pT)^{-\star}$ *and moreover for* $\xi \in \mathbb{H}^{N-r}$,

$$F_{\mathcal{N}}\xi \equiv 0 \quad \Longrightarrow \quad \xi = 0.$$

Indeed, we first note that the elements of $\mathcal{N}$ are well defined at $p = 1$ since Θ_r is invertible at $p = 1$ (see also the formulas in [10, Theorem 3.3 (2)]). Let $F_{\mathcal{N}}(p)$ be built from the columns of a basis of $\mathcal{N}$ and note that there exists $B \in \mathbb{H}^{(N-r)\times(N-r)}$ such that

$$R_1 F_{\mathcal{N}} = F_{\mathcal{N}} B.$$

Restricting to $p = x$, where x is real, we have

$$\frac{F(x) - F(1)}{x - 1} = F(x)B,$$

and so

$$F(x)(\mathcal{I}_{N-r} + B - xB) = F(1). \tag{5.10}$$

We claim that $\mathcal{I}_{N-r} + B$ is invertible. Let $\xi \in \mathbb{H}^{N-r}$ be such that $B\xi = -\xi$. Then, (5.10) implies that

$$xF(x)\xi = F(1)\xi, \quad x \in (-1, 1).$$

Thus $F(1)\xi = 0$ (by setting $x = 0$) and so $F(x)\xi = 0$ and so $\xi = 0$. Hence

$$F(x) = F(1)(\mathcal{I}_{N-r} + B)^{-1}(\mathcal{I}_{N-r} - xB(\mathcal{I}_{N-r} + B)^{-1})^{-1},$$

and the result follows.

The following step is [10, Step 2 of proof of Theorem 3.1, p. 154]. The proof uses (5.9) and is similar to the above arguments.

STEP 4: *The space $\mathcal{N}$ is neutral and $G^*\mathcal{J}G = 0$.*

$\mathcal{N}$ is neutral by construction since $r = \operatorname{rank} P$. We first show that the inner product in $\mathcal{N}$ satisfies (5.5). We may proceed as in [10, p. 154] and using (5.5) and (5.8) we have for $n_1, n_2 \in \mathcal{M}$:

$$\begin{aligned}[R_1 n_1, n_2]_{\mathcal{N}} &= [\Theta_r \star R_1 n_1, \Theta_r \star n_2]_{\mathcal{M}} \\ &= [R_1(\Theta_r \star n_1), \Theta_r \star n_2]_{\mathcal{M}} - [(R_1\Theta_r)(n_1(1)), \Theta_r \star n_2]_{\mathcal{M}} \\ &= [R_1(\Theta_r \star n_1), \Theta_r \star n_2]_{\mathcal{M}}\end{aligned}$$

since $(R_1\Theta_r)(n_1(1)) \in \mathcal{H}(\Theta_r)$, and so $[(R_1\Theta_r)(n_1(1)), \Theta_r \star n_2]_{\mathcal{M}} = 0$.

Similarly,

$$\begin{aligned}[n_1, R_1 n_2]_{\mathcal{N}} &= [\Theta_r \star n_1, \Theta_r \star R_1 n_2]_{\mathcal{M}} \\ &= [\Theta_r \star n_1, R_1(\Theta_r \star n_2)]_{\mathcal{M}} - [\Theta_r \star n_1, (R_1\Theta_r)(n_2(1))]_{\mathcal{M}} \\ &= [\Theta_r \star n_1, R_1(\Theta_r \star n_2)]_{\mathcal{M}}.\end{aligned}$$

Thus, with $m_1 = \Theta_r \star n_1$ and $m_2 = \Theta_r \star n_2$, and using again (5.5)

$$\begin{aligned}&[n_1, n_2]_{\mathcal{N}} + [R_1 n_1, n_2]_{\mathcal{N}} + [n_1, R_1 n_2]_{\mathcal{N}} \\ &\quad = [m_1, m_2]_{\mathcal{M}} + [R_1 m_1, m_2]_{\mathcal{M}} + [m_1, R_1 m_2]_{\mathcal{M}} \\ &\quad = m_2(1)^*\mathcal{J}m_1(1) \\ &\quad = n_2(1)\mathcal{J}n_1(1)\end{aligned}$$

since $m_v(1) = (\Theta_r \star n_v)(1) = \Theta_r(1)n_v(1)$ for $v = 1, 2$ and $\Theta_r(1)^*\mathcal{J}\Theta_r(1) = \mathcal{J}$.

Proceeding as in Lemma 5.4 it follows that

$$P_{\mathcal{N}} - T^*P_{\mathcal{N}}T = G^*\mathcal{J}G,$$

and so $G^*\mathcal{J}G = 0$.

STEP 5: *Problem 1.1 has at most one solution.*

Let

$$\Theta_r(p) = \begin{pmatrix} a_r(p) & b_r(p) \\ c_r(p) & d_r(p) \end{pmatrix}.$$

From the study of the nondegenerate case, we know that, under the assumptions that ensure the existence of a solution, any solution is of the form

$$s(p) = (a_r(p) \star e(p) + b_r(p)) \star (c_r(p) \star e(p) + d_r(p))^{-\star}, \tag{5.11}$$

for some Schur function e. Furthermore as in Step 1, for every $n \in \mathcal{N}$ we have

$$\begin{pmatrix} 1 & -s \end{pmatrix} \star \Theta_r \star n \equiv 0.$$

Thus

$$(a_r - s \star c_r) \star \begin{pmatrix} 1 & -e \end{pmatrix} \star n \equiv 0,$$

and so

$$\begin{pmatrix}1 & -e\end{pmatrix} \star n \equiv 0.$$

Since $G^*\mathcal{J}G = 0$ we conclude in the way as in step 1. Indeed, let

$$G = \begin{pmatrix} h_1 & \dots & h_{N-r} \\ k_1 & \dots & k_{N-r} \end{pmatrix}.$$

At least one of the h_u or k_u is different from 0 and $G^*\mathcal{J}G = 0$ implies that

$$\overline{h_u}h_v = \overline{k_u}k_v, \quad \forall u, v = 1, \dots, N-r,$$

and so e is a unitary constant.

We now show that the solution, when it exists, is a finite Blaschke product.

STEP 6: *Let s be given by (5.11). Then the associated space $\mathcal{H}(s)$ is finite dimensional.*

This follows from

$$K_s(p,q) = \begin{pmatrix}1 & -s(p)\end{pmatrix} \star K_{\Theta_r}(p,q) \star_r \begin{pmatrix} 1 \\ \overline{s(q)} \end{pmatrix} + \underbrace{\begin{pmatrix}1 & -s(p)\end{pmatrix} \star \Theta_r(p)\mathcal{J}\Theta_r(q)^* \star_r \begin{pmatrix} 1 \\ \overline{s(q)} \end{pmatrix}}_{\text{is equal to 0 since } |e|=1},$$

where K_{Θ_r} is defined as in (4.6) (with Θ_r in place of Θ). See the proof of step 3 in Theorem 3.2.

STEP 7: *The space $\mathcal{H}(s)$ contains an element of the form*

$$f(p) = d \star (1 - p\overline{a})^{-\star}, \tag{5.12}$$

where $d \in \mathbb{H}$ and $a \in \mathbb{B}_1$.

We first recall that (see [5, Theorem 7.1])

$$\|R_0 f\|^2_{\mathcal{H}(s)} \le \|f\|^2_{\mathcal{H}(s)} - |f(0)|^2, \quad \forall f \in \mathcal{H}(s). \tag{5.13}$$

Here, the space $\mathcal{H}(s)$ is finite dimensional and R_0 invariant. Thus R_0 has a right eigenvector f with eigenvalue $\overline{a}$; see [25, p. 36]. Any eigenvector of R_0 is of the form (5.12), and equation (5.13) implies that

$$\|f\|^2 \le \frac{|f(0)|^2}{1 - |a|^2}. \tag{5.14}$$

We will see at the end of the proof of Step 8 that equality in fact holds in (5.14).

STEP 8: *It holds that $s(a) = 0$.*

From [6, p. 282-283] it follows that the span of f endowed with the norm $\|f\|^2 = \frac{|f(0)|^2}{1-|a|^2}$ is equal to $\mathcal{H}(b_a)$, where b_a is a Blaschke factor, see

(5.1). From (5.14) we get that $\mathcal{H}(b_a)$ is contractively included in $\mathcal{H}(s)$ and from [6, Lemma 5.1] we then have that the kernel

$$K_s(p,q) - K_{b_a}(p,q) = \sum_{t=0}^{\infty} p^t (b_a(p)\overline{b_a(q)} - s(p)\overline{s(q)})\overline{q}^t \tag{5.15}$$

is positive definite in $\mathbb{B}_1$. But $b_a(a) = 0$. Thus, setting $p = q = a$ in (5.15) leads to $s(a) = 0$.

STEP 9: *We can write* $s = b_a \star \sigma_1$, *where* σ_1 *is a Schur function.*

In the argument we make use of the Hardy space $\mathbf{H}_2(\mathbb{B}_1)$ which is the reproducing kernel Hilbert space with reproducing kernel

$$(1 - p\overline{q})^{-\star} = \sum_{t=0}^{\infty} p^t \overline{q}^t.$$

Note that this is the kernel k_s with $s \equiv 0$. For more information on this space we refer to [1,6].

Since a Schur function is bounded in modulus and thus belongs to the space $\mathbf{H}_2(\mathbb{B}_1)$ (see [1]), the representation $s = b_a \star \sigma_1$ with $\sigma_1 \in \mathbf{H}_2(\mathbb{B}_1)$, follows from [7, Proof of Theorem 6.2, p. 109]. To see that σ_1 is a Schur multiplier we note that

$$K_s(p,q) - K_{b_a}(p,q) = b_a(p) \star K_{\sigma_1}(p,q) \star_r \overline{b_a(q)} \tag{5.16}$$

implies that $b_a(p) \star K_{\sigma_1}(p,q) \star_r \overline{b_a(q)}$ is positive definite in $\mathbb{B}_1$ and hence $K_{\sigma_1}(p,q)$ is as well by [5, Proposition 5.3].

STEP 10: *It holds that* $\dim(\mathcal{H}(\sigma_1)) = \dim(\mathcal{H}(s)) - 1$.

The decomposition (5.16) gives the decomposition

$$K_s(p,q) = K_{b_a}(p,q) + b_a(p) \star K_{\sigma_1}(p,q) \star_r \overline{b_a(q)}.$$

The corresponding reproducing kernel spaces do not intersect. Indeed, all elements in the reproducing kernel Hilbert space with reproducing kernel $b_a(p)\star K_{\sigma_1}(p,q)\star_r \overline{b_a(q)}$ vanish at the point a while non zero elements in $\mathcal{H}(b_a)$ do not vanish. So the decomposition is orthogonal in $\mathcal{H}(s)$ by Theorem 5.7, and equality holds in (5.14). The claim on the dimensions follows.

After a finite number of iterations, this procedure leads to a constant σ_ℓ, for some positive integer ℓ. This constant has to be unitary since the corresponding space $\mathcal{H}(\sigma_\ell)$ reduces to $\{0\}$.

STEP 11: *Problem* 1.1 *has a unique solution satisfying* (3.13).

To see this it suffices to use item (4) of Theorem 3.2 with Θ_r instead of Θ and $e = \sigma_\ell$.

From the previous arguments we know that s is of the form (5.11) for a uniquely determined unitary constant e. This s does satisfy the first set of interpolation conditions, but need not satisfy the second set. By Theorem 3.2 point (4), s satisfies (3.13). □

We conclude with two remarks and a corollary.

Remark 5.8. Given a Blaschke factor, the operator of multiplication by b_a is an isometry from $\mathbf{H}_2(\mathbb{B}_1)$ into itself (see [7, Theorem 5.17, p. 106]), and so is the operator of multiplication by a finite Blaschke product B. The degree of the Blaschke product is the dimension of the space $\mathbf{H}_2(\mathbb{B}_1) \ominus B\mathbf{H}_2(\mathbb{B}_1)$. Thus the previous argument shows in fact that $\mathcal{H}(s)$ is isometrically included inside $\mathbf{H}_2(\mathbb{B}_1)$ and that $\mathcal{H}(s) = \mathbf{H}_2(\mathbb{B}_1) \ominus M_s\mathbf{H}_2(\mathbb{B}_1)$.

One can plug a unitary constant e also in the linear fractional transformation (3.11) and the same arguments lead to:

Corollary 5.9. *If Problem 1.1 has a solution, it is a Blaschke product of degree* rank P.

Remark 5.10. The arguments in Steps 5–7 take only into account the fact that the space $\mathcal{H}(\Theta)$ is finite dimensional and that e is a unitary constant. In particular, they also apply in the setting of [1], and in that paper too, the solution of the interpolation problem is a Blaschke product of degree rank P when the Pick matrix is degenerate.

6. An Analogue of Carathéodory's Theorem in the Quaternionic Setting

Recall first that Carathéodory's theorem states the following (see for instance [14, pp. 203-205], [22, p. 48]). We write the result for a radial limit, but the result holds in fact for a non tangential limit.

Theorem 6.1. *Let s be a Schur function and let e^{it_0} be a point on the unit circle such that*

$$\liminf_{\substack{r\to 1\\ r\in(0,1)}} \frac{1-|s(re^{it_0})|}{1-r} < \infty.$$

Then, the limits

$$c = \lim_{\substack{r\to 1\\ r\in(0,1)}} s(re^{it_0}) \quad \text{and} \quad \lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1-s(re^{it_0})\overline{c}}{1-r}$$

exist, and the second one is positive.

This result plays an important role in the classical boundary interpolation problem for Schur functions. See for instance [8, 23].

We prove a related result in the setting of slice-hyperholomorphic functions. The condition (6.2) will hold particular for rational functions s, as is proved using a realization of s (see [6] for the latter).

Theorem 6.2. *Let s be a slice hyperholomorphic Schur function, and assume that at some point p_u of modulus 1 we have*

$$\sup_{r\in(0,1)} \frac{1-|s(rp_u)|^2}{1-r^2} < \infty. \tag{6.1}$$

Assume moreover that the function $r \mapsto s(rp_u)$ has a development in series with respect to the real variable r at $r = 1$:

$$s(rp_u) = s_u + (r-1)a_u + O(r-1)^2. \tag{6.2}$$

Then

$$\lim_{\substack{r\to 1\\ r\in(0,1)}}\sum_{t=0}^{\infty} r^t p_u^t(1-s(r_n p_u)\overline{s_u})\overline{p_u}^t = (a_u\overline{s_u}-\overline{p_u}a_u\overline{s_u}\,\overline{p_u})(1-\overline{p_u}^2)^{-1}\geq 0.$$

Proof. In view of (6.1), the family of functions $K_s(\cdot, rp_u)$ has a weakly convergent subsequence. Since weak convergence implies pointwise convergence the weak limit is readily seen to be the function g_u. Thus

$$0\leq \langle g_u, g_u\rangle_{\mathcal{H}(s)} = \lim_{n\to\infty}\langle g_u, K_s(\cdot, r_n p_u)\rangle_{\mathcal{H}(s)} = \lim_{n\to\infty} g_u(r_n p_u),$$

where $(r_n)_{n\in\mathbb{N}}$ is a sequence of numbers in $(0,1)$ with limit equal to 1. Hence we have that

$$\lim_{n\to\infty}\sum_{t=0}^{\infty} r_n^t p_u^t(1-s(r_n p_u)\overline{s_u})\overline{p_u}^t\geq 0,$$

and thus

$$\lim_{\substack{r\to 1\\ r\in(0,1)}}\sum_{t=0}^{\infty} r^t p_u^t(1-s(rp_u)\overline{s_u})\overline{p_u}^t\geq 0.$$

Using (6.2) and (2.9) we have:

$$\begin{aligned}\sum_{t=0}^{\infty} r^t p_u^t(1-s(rp_u)\overline{s_u})\overline{p_u}^t &= \sum_{t=0}^{\infty} r^t p_u^t((r-1)a_u\overline{s_u}+O(r-1)^2)\overline{p_u}^t\\ &= ((r-1)a_u\overline{s_u}-r\overline{p_u}(r-1)a_u\overline{s_u}\,\overline{p_u})(1-r)^{-1}(1-r\overline{p_u}^2)^{-1}\\ &\quad+\sum_{t=0}^{\infty} r^t p_u^t O(r-1)^2\overline{p_u}^t\\ &= (a_u\overline{s_u}-r\overline{p_u}a_u\overline{s_u}\,\overline{p_u})(1-r\overline{p_u}^2)^{-1}\\ &\quad+\sum_{t=0}^{\infty} r^t p_u^t O(r-1)^2\overline{p_u}^t.\end{aligned}$$

This expression tends to

$$(a_u\overline{s_u}-\overline{p_u}a_u\overline{s_u}\,\overline{p_u})(1-\overline{p_u}^2)^{-1}, \tag{6.3}$$

as $r\to 1$. □

Remark 6.3. The example $s(p)=\frac{1+pa}{2}$, where $a\in\mathbb{B}_1$ is such that $ap_u\neq p_u a$, shows that (6.3) is different, in general, from $a_u\overline{s_u}$.

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