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Integral Equations and Operator Theory

Estimates on Complex Eigenvalues for Dirac Operators on the Half-Line

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Abstract. We derive bounds on the location of non-embedded eigenvalues of Dirac operators on the half-line with non-Hermitian L^1-potentials. The results are sharp in the non-relativistic or weak-coupling limit. In the massless case, the absence of discrete spectrum is proved under a smallness assumption.

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1. Introduction

The aim of this paper is to obtain estimates for eigenvalues of the Dirac operator

$$D_0 := \begin{pmatrix} mc^2 & -c\frac{\mathrm{d}}{\mathrm{d}x} \\ c\frac{\mathrm{d}}{\mathrm{d}x} & -mc^2 \end{pmatrix} \tag{1.1}$$

on $L^2(\mathbb{R}_+, \mathbb{C}^2)$ subject to separated boundary conditions at zero,

$$\psi_1(0)\cos(\alpha) - \psi_2(0)\sin(\alpha) = 0, \quad \alpha \in [0, \pi/2],^1 \tag{1.2}$$

and perturbed by a matrix-valued (not necessarily Hermitian) potential

$$V \in L^1(\mathbb{R}_+, \mathrm{Mat}(2,\mathbb{C})), \quad \|V\|_1 := \int_0^\infty \|V(x)\|\,\mathrm{d}x,$$

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[1] We exclude the case $\alpha \in (\pi/2, \pi)$ since D_0 has an eigenvalue in the gap $(-mc^2, mc^2)$ in this case, see [11, p.137]).

where the norm in the integral is the operator norm in $\mathbb{C}^2$ (with respect to the Euclidean norm on $\mathbb{C}^2$). Here, we are only concerned with eigenvalues that are not embedded in the spectrum of D_0,

$$\sigma(D_0) = (-\infty, -mc^2] \cup [mc^2, \infty).$$

For the purpose of investigating the non-relativistic limit, we have made the dependence on c (the speed of light) explicit, whereas the reduced Planck constant $\hbar$ is set to unity.

This work is a continuation of [2] where corresponding eigenvalue estimates for Dirac operators on the whole line were established. More precisely, it was shown there that if $v := \|V\|_1/c < 1$, then any eigenvalue $z \in \mathbb{C}\setminus\sigma(D_0)$ of $D_0 + V$ is contained in the union of two closed disks in the left and right half plane with centres $\pm mc^2 x_0$ and radii $mc^2 r_0$, where x_0 and r_0 depend non-linearly on v and diverge as $v \to \infty$ in such a way that the disks cover the entire complex plane minus the imaginary axis. In the non-relativistic limit ($c \to \infty$), the Dirac operator $D_0 + V - mc^2$ converges to the Schrödinger operator $-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ (say, for V a multiple of the identity matrix) in the norm-resolvent sense, and the spectral estimate reduces to the bound in [1]: Any eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$ of the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + V$ satisfies

$$|\lambda|^{1/2} \leq \frac{1}{2} \int_{-\infty}^{\infty} |V(x)|\, \mathrm{d}x. \tag{1.3}$$

Similar estimates for Schrödinger operators on the half-line were established in [4]: Any eigenvalue $\lambda \in \mathbb{C}\setminus[0,\infty)$ of $-\mathrm{d}^2/\mathrm{d}x^2+V$, with boundary condition $\psi'(0) = \sigma\psi(0)$, $\sigma \geq 0$, satisfies (1.3) if the constant 1/2 is replaced by 1; in the case of Dirichlet boundary conditions $\psi(0) = 0$, the sharp estimate

$$|\lambda|^{1/2} \leq \frac{1}{2} g(\cot(\theta/2)) \int_0^{\infty} |V(x)|\, \mathrm{d}x \tag{1.4}$$

holds, where $\lambda = |\lambda|\mathrm{e}^{\mathrm{i}\theta}$ and

$$g(b) := \sup_{y \geq 0} |\mathrm{e}^{\mathrm{i}by} - \mathrm{e}^{-y}| \in [1, 2]. \tag{1.5}$$

Note in particular that (1.3) and (1.4) have the correct semiclassical exponents.

The aim of this note is to obtain corresponding results for the Dirac operator on the half-line. As in [2], an interesting distinction between the massive ($m \neq 0$) and the massless ($m = 0$) Dirac operator occurs: The former behaves like the Schrödinger operator in the non-relativistic limit $c \to \infty$, while the latter ($m = 0$ may be regarded as the "ultra-relativistic" limit) has no complex eigenvalues for sufficiently small L^1-norm of the potential (see [2] for the case of the whole line and Theorem 2.1 for the half-line case). This fact may be expressed by saying that the whole spectrum (which is $\mathbb{R}$ in this case) is non-resonant. This is quite remarkable, considering that the point zero is resonant for the (scalar) relativistic operator $|p|$ on the real line, i.e. there are eigenvalues for arbitrarily small perturbations, even for real-valued potentials. In that case, the absence of eigenvalues for small L^1-norm of the potential would be equivalent (by the variational characterization of eigenvalues and Hölder's inequality)[2] to the boundedness of the resolvent of $|p|$ from L^1 to L^∞, which in turn would be equivalent to the boundedness of the Fourier transform of its symbol; however, the Fourier transform of p.v.$\frac{1}{|p|}$ diverges logarithmically. In contrast, the symbol of the resolvent of the Dirac operator on the line behaves like p.v.$\frac{1}{p}$ (the Hilbert transform), which has a bounded Fourier transform due to cancellations.

The second crucial point is the behaviour of the resolvent $(D_0 - z)^{-1}$ when the spectral parameter z is close to the real axis. For $z = \lambda + \mathrm{i}\epsilon,\ \lambda > 0$, its symbol picks up singularities on the sphere of radius $\lambda^{1/2}$ when $\epsilon \to 0$. In fact, from the well-known formula

$$\lim_{\epsilon\to 0} \frac{1}{x - \mathrm{i}\epsilon} = \mathrm{i}\pi\delta(x) + \text{p.v.}\frac{1}{x}, \tag{1.6}$$

it follows that the (scalar part of) the symbol of $(D_0 - z)^{-1}$ for $m = 0$ has a bounded Fourier transform. We emphasize that in higher dimensions $n \geq 2$ there can be no $L^p \to L^q$ estimate (p and q being dual exponents, i.e. $q = p/(p-1)$) for the resolvent of the Dirac operator that is uniform in the spectral parameter. The reason is that the analogue of (1.6) in higher dimensions implies that $(D_0 - z)^{-1} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded uniformly in $|z| > 1$ if and only if

$$\frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{1}{n} \quad \left(q = \frac{p}{p-1}\right). \tag{1.7}$$

The bound on the left is imposed by the Stein–Tomas restriction theorem, see [10], while the bound on the right is dictated by standard estimates for Bessel potentials of order one, see e.g. [6, Cor. 6.16]. Both conditions are known to be sharp. Unfortunately, this forces $n = 1$. For the Laplacian, the situation is better since the right hand side of (1.7) is then replaced by $2/n$, see [8, Theorem 2.3]. Based on the latter, eigenvalue estimates for multi-dimensional Schrödinger operators with L^p-potentials were established in [3].

2. Main Results

Let D_0 be the Dirac operator in (1.1), with domain consisting of all square integrable functions $\psi \in L^2(\mathbb{R}_+, \mathbb{C}^2)$ that are absolutely continuous on $\mathbb{R}_+$, satisfy (1.2), and such that $D_0\psi \in L^2(\mathbb{R}_+, \mathbb{C}^2)$. In the following, we tacitly assume that the potential V is smooth and has compact support. This assumption allows us to define the sum $D_0 + V$ in an unambiguous way (as an operator sum). However, it is in no way essential, as the attentive reader will gather, and can easily be disposed of. In fact, the assumptions imposed on V in Theorems 2.1–2.3 are sufficient to define a closed extension of $D \supset D_0 + V$

[2] This is an instance of the duality between an isoperimetric eigenvalue problem (minimizing the lowest eigenvalue under the constraint of a fixed Lebesgue norm of the potential) and a critical Sobolev inequality, see [7].

via the resolvent formula (3.4), see [2] and the references therein for details. In particular, the Birman-Schwinger principle remains valid for this extension, by construction. Incidentally, for (3.4) to be well-defined, the existence of a point $z_0 \in \mathbb{C} \setminus \sigma(D_0)$ for which $I + Q(z_0)$ has a bounded inverse, has to be assumed, and once such a point is shown to exist, it will automatically belong to $\mathbb{C} \setminus \sigma(D)$. Hence, from this point of view, Theorems 2.1–2.3 (with V satisfying only the regularity assumptions stated in the respective theorem) yield the existence of a closed extension and the spectral estimates as a byproduct.

Theorem 2.1. *Let $v := \|V\|_1/c < 1/\sqrt{2}$. Then any eigenvalue $z \in \mathbb{C} \setminus \sigma(D_0)$ of $D_0 + V$ subject to the boundary condition (1.2) is contained in the disjoint union of two closed disks with centres $\pm mc^2 x_0$ and radii $mc^2 r_0$, where*

$$x_0 := 1 + \frac{2v^4}{1-2v^2}, \quad r_0 := 2v\frac{1-v^2}{1-2v^2}. \tag{2.1}$$

In particular, the spectrum of the massless Dirac operator $(m = 0)$ with non-Hermitian potential V is $\mathbb{R}$.

Proof. The proof is based on the Birman-Schwinger principle: $z \in \mathbb{C} \setminus \sigma(D_0)$ is an eigenvalue of $D_0 + V$ if and only if -1 is an eigenvalue of the Birman-Schwinger operator

$$Q(z) := |V|^{1/2}(D_0 - z)^{-1}V^{1/2}.$$

Let $z \in \mathbb{C} \setminus \sigma(D_0)$ and define

$$c\kappa(z) := \sqrt{z^2 - (mc^2)^2}, \quad \zeta(z) := \frac{z + mc^2}{c\kappa(z)} \tag{2.2}$$

where the branch of the square root is chosen such that $\operatorname{Im}\kappa(z) > 0$. Let us assume that $\alpha \in (0, \pi/2]$. It can then be checked that

$$\psi_l(x;z) := \begin{pmatrix} \cos(\kappa(z)x) + \zeta(z)\cot(\alpha)\sin(\kappa(z)x) \\ -\zeta(z)^{-1}\sin(\kappa(z)x) + \cot(\alpha)\cos(\kappa(z)x) \end{pmatrix}, \quad \alpha \in (0, \pi/2]. \tag{2.3}$$

is a solution to the differential equation $(D_0 - z)\psi_l(x;z) = 0$ satisfying the boundary condition (1.2). In the case $\alpha = 0$, formally corresponding to $\cot(\alpha) = \infty$, the solution is

$$\psi_l(x;z) = \begin{pmatrix} \zeta(z)\sin(\kappa(z)x) \\ \cos(\kappa(z)x) \end{pmatrix}, \quad \alpha = 0. \tag{2.4}$$

On the other hand,

$$\psi_\infty(x;z) := e^{i\kappa(z)x}\begin{pmatrix} -i\zeta(z) \\ 1 \end{pmatrix} \tag{2.5}$$

is a solution that lies in $L^2(\mathbb{R}_+)$. The resolvent $R_0(z) = (D_0 - z)^{-1}$ is then given by (see e.g. [11, Satz 15.17])

$$c(R_0(z)f)(x) = \frac{1}{W}\left(\psi_\infty(x;z)\int_0^x (\overline{\psi_l(y;z)}, f(y))\,\mathrm{d}y + \psi_l(x;z)\int_x^\infty (\overline{\psi_\infty(y;z)}, f(y))\,\mathrm{d}y\right)$$

where

$$W = \begin{cases} 1 + \mathrm{i}\zeta(z)\cot(\alpha), & \alpha \in (0, \pi/2] \\ \mathrm{i}\zeta(z), & \alpha = 0 \end{cases}$$

is the Wronskian and $(\cdot,\cdot)$ denotes the Hermitian scalar product on $\mathbb{C}^2$ (which we define to be linear in the second variable). The values of the resolvent kernel $R_0(x, y; z)$ are linear maps in $\mathbb{C}^2$, given by

$$cR_0(x,y;z) = \frac{1}{W}\Big(\psi_\infty(x;z)(\overline{\psi_l(y;z)}, \cdot)\theta(x-y) + \psi_l(x;z)(\overline{\psi_\infty(y;z)}, \cdot)\theta(y-x)\Big), \tag{2.6}$$

where θ denotes the characteristic function of $(0,\infty)$. Note that $W \neq 0$ since D_0 has no eigenvalue [11, p.137]; alternatively, this may be seen as follows: By assumption, $\sigma := \cot(\alpha) \geq 0$, and thus the solution $\zeta = \frac{\mathrm{i}}{\sigma}$ of $W = 0$ lies in the (open) upper half plane. However, the function ζ defined in (2.2) takes values in the (open) lower half plane. Indeed, $\zeta(z)$ is a (holomorphic) branch of the square root of $(z + mc^2)/(z - mc^2)$ for $z \in \mathbb{C} \setminus \sigma(D_0)$. The range of the latter is the cut plane $\mathbb{C} \setminus \mathbb{R}_+$, thus any branch of the square root must have values either in the upper or in the lower half plane. One easily checks that $\mathrm{Im}\zeta(0) < 0$.

We now estimate the norm of $cR_0(x, y; z)$ as an operator on $\mathbb{C}^2$. Let us assume that $\alpha \in (0, \pi/2]$, so that ψ_l is given by (2.3); the case $\alpha = 0$ may always be recovered by letting $\cot(\alpha) \to \infty$. We then have (suppressing the z-dependence of κ and ζ)

$$\begin{aligned}\sup_{x\geq y\geq 0}\|cR_0(x,y;z)\|^2 &= \sup_{x\geq y\geq 0}\frac{1}{|W|^2}\|\psi_\infty(x;z)\|^2\|\psi_l(y;z)\|^2 \\ &= \frac{1+|\zeta|^2}{|1+\mathrm{i}\zeta\cot(\alpha)|^2}\sup_{y\geq 0}\mathrm{e}^{-2\mathrm{Im}\,\kappa y}\|\psi_l(y;z)\|^2 \\ &= \frac{|\zeta|+|\zeta|^{-1}}{4}\left(1+|\beta|^2\mathrm{e}^{-4\mathrm{Im}(\kappa)y}\right)\left(|\zeta|+|\zeta|^{-1}\right) \\ &\quad + 2\mathrm{e}^{-2\mathrm{Im}(\kappa)y}\mathrm{Re}\left(\beta\mathrm{e}^{-2\mathrm{iRe}\kappa y}\right)\left(|\zeta|-|\zeta|^{-1}\right)\end{aligned} \tag{2.7}$$

where

$$\beta := \frac{1-\mathrm{i}\zeta\cot(\alpha)}{1+\mathrm{i}\zeta\cot(\alpha)}, \tag{2.8}$$

and where we used (in the second line) that the supremum over x is attained at $x = y$ since $\operatorname{Im}\kappa(z) > 0$. Noticing that $|\beta| \le 1$ (since $\operatorname{Im}(\zeta) < 0$), we find that

$$\sup_{x,y\ge 0} \|cR_0(x,y;z)\|^2 \le \left(|\zeta| + |\zeta|^{-1}\right) \max\{|\zeta|, |\zeta|^{-1}\} = 1 + \max\{|\zeta|^2, |\zeta|^{-2}\}.$$

Using Hölder's inequality, we arrive at

$$\|Q(z)\| \le \sup_{x,y\ge 0} \|cR_0(x,y;z)\|\, v \le \sqrt{1 + \max\{|\zeta|^2, |\zeta|^{-2}\}}\, v. \tag{2.9}$$

By the Birman-Schwinger principle, the left hand side of (2.9) is equal to 1 if z is an eigenvalue. If $m = 0$, then $\zeta(z) = \pm 1$, depending on whether z is in the upper or lower half plane, and hence the right hand side of inequality (2.9) is equal to $\sqrt{2}v$. It follows that z cannot be an eigenvalue if $v < 1/\sqrt{2}$. If $m \ne 0$, then for z in the left half plane the maximum equals $\sqrt{1 + |\zeta(z)|^2}$, while in the right half plane it equals $\sqrt{1 + |\zeta(z)|^{-2}}$. Hence, for every eigenvalue z,

$$|\zeta(z)| \ge \frac{\sqrt{1 - v^2}}{v} =: \rho > 1$$

if z is in the left half plane and $|\zeta(z)| \le \rho^{-1}$ if z is in the right half plane. Since z and $\zeta(z)^2$ are related by the Möbius transformation

$$z = mc^2\, \frac{\zeta^2(z) + 1}{\zeta^2(z) - 1},$$

the domains $\{z \in \mathbb{C} : |\zeta(z)| \ge \rho\}$ and $\{z \in \mathbb{C} : |\zeta(z)| \le \rho^{-1}\}$ are mapped to the two disks in the theorem, see [2] for details on the Möbius transformation. □

From (2.9) one sees that the eigenvalue estimate is equivalent to the inequality

$$\left(4\left(1 + \max\{|\zeta|^2, |\zeta|^{-2}\}\right)\right)^{-1/2} \le \frac{1}{2c}\int_0^\infty \|V(x)\|\, \mathrm{d}x. \tag{2.10}$$

This should be compared to the result of [2] for the whole-line operator, which may also be written as

$$\left(2 + |\zeta|^2 + |\zeta|^{-2}\right)^{-1/2} \le \frac{1}{2c}\int_0^\infty \|V(x)\|\, \mathrm{d}x. \tag{2.11}$$

It is instructive to note that if we replace V by λV, then in the weak coupling limit $\lambda \to 0$, the inequalities (2.10) and (2.11) take the form

$$\left|\frac{z \mp mc^2}{2m}\right|^{1/2} \le \frac{A\lambda}{c}\int_0^\infty \|V(x)\|\, \mathrm{d}x + o(\lambda), \tag{2.12}$$

with $A = 1$ in the case of (2.10) and $A = 1/2$ in case of (2.11), and $\mp$ indicating whether z tends to mc^2 or $-mc^2$ as $\lambda \to 0$. Note that (2.12) has the semiclassical behaviour of a non-relativistic operator, the reason being that the weak-coupling limit is equivalent to the non-relativistic limit: If we subtract (or add, respectively) the rest energy mc^2 (i.e. replace $z \mp mc^2$ by z), we may consider c^{-1} as a small coupling constant (we now fix $\lambda = 1$, whereas before, we considered c fixed). In the limit $c \to \infty$, the Dirac operator converges to the Schrödinger operator with Dirichlet or Neumann boundary conditions, see Sect. 3. On the other hand, for the massless operator (or for large eigenvalues of the massive operator), the inequalities (2.10) and (2.11) reduce to

$$|z|^0 \leq \frac{B}{c} \int_0^\infty \|V(x)\| \, \mathrm{d}x, \tag{2.13}$$

with $B = 1/2$ in the case of (2.10) and $B = 1$ in case of (2.11). Inequality (2.13) has the correct semiclassical behaviour of a relativistic operator. It is an open and interesting question whether there exists a bound on the number of complex eigenvalues of the massless Dirac operator in terms of the right hand side of (2.13).

From the inequality

$$2 \leq \frac{4\left(1 + \max\{|\zeta|^2, |\zeta|^{-2}\}\right)}{2 + |\zeta|^2 + |\zeta|^{-2}} \leq 4$$

it follows that the whole line estimate (2.11) continues to hold for the half-line operators if the constant $1/2$ on the right hand side is replaced by 1. For "Dirichlet boundary conditions" $\psi_1(0) = 0$ or $\psi_2(0) = 0$ this may also be seen from the following argument: suppose $\psi = (\psi_1, \psi_2)^t$ is an eigenfunction of the half-line operator with potential V to an eigenvalue z. Since the parity operator

$$P\psi(x) := \sigma_3 \psi(-x) = \begin{pmatrix} \psi_1(-x) \\ -\psi_2(-x) \end{pmatrix}$$

commutes with D_0, it follows that z is an eigenvalue of the whole-line operator with potential

$$\widetilde{V}(x) := \begin{cases} V(x) & x \geq 0, \\ V(-x) & x < 0, \end{cases}$$

with corresponding eigenfunction

$$\widetilde{\psi}(x) := \begin{cases} \psi(x) & x \geq 0, \\ P\psi(x) & x < 0, \end{cases}$$

and (2.10) follows from the whole-line estimate (2.11) for the operator $D_0 + \widetilde{V}$. In fact, for the massive ($m \neq 0$) Dirac operator with Dirichlet boundary conditions, inequality (2.10) may be refined, in a similar spirit as in [4] for the Schrödinger operator, compare (1.4). We define the functions $G_\mp$ by

$$G_\mp(a, b) := \sqrt{\sup_{y \geq 0} \left[(1 + \mathrm{e}^{-2y}) \mp 2a\mathrm{e}^{-y} \cos(aby)\right]}, \quad a, b \in \mathbb{R}. \tag{2.14}$$

Theorem 2.2. *Let $\alpha \in \{0, \pi/2\}$ and assume that $v = \|V\|_1/c < 1/\sqrt{2}$. Then every eigenvalue $z = mc^2(\zeta^2+1)/(\zeta^2-1)$ of the massive ($m \neq 0$) Dirac operator D_0+V subject to the boundary conditions $\psi_1(0)\cos(\alpha)-\psi_2(0)\sin(\alpha) = 0$ satisfies*

$$\left((|\zeta|+|\zeta|^{-1})G_{\mp}\left(\frac{|\zeta|-|\zeta|^{-1}}{|\zeta|+|\zeta|^{-1}}, \cot(t)\right)\right)^{-1} \leq \frac{1}{2c}\int_0^\infty \|V(x)\|\,\mathrm{d}x,$$

with "−" if $\alpha = 0$ and "+" if $\alpha = \pi/2$, and with $\zeta = |\zeta|\mathrm{e}^{\mathrm{i}t}$, $\pi < t < 2\pi$.

Proof of Theorem 2.2. In the following, we set $a = \frac{|\zeta|-|\zeta|^{-1}}{|\zeta|+|\zeta|^{-1}}$, $b = \cot(t)$. Noting that for $m \neq 0$

$$\kappa = m\sqrt{\left(\frac{\zeta^2+1}{\zeta^2-1}\right)^2 - 1}, \quad \mathrm{Im}(\kappa) > 0,$$

we find that

$$\mathrm{Im}\,\kappa = \frac{2(|\zeta|+|\zeta|^{-1})|\sin(t)|}{(|\zeta|-|\zeta|^{-1})^2\cos^2(t)+(|\zeta|+|\zeta|^{-1})^2\sin^2(t)},$$
$$\mathrm{Re}\,\kappa = -\,\mathrm{sgn}(\sin(t))\,\frac{2(|\zeta|-|\zeta|^{-1})\cos(t)}{(|\zeta|-|\zeta|^{-1})^2\cos^2(t)+(|\zeta|+|\zeta|^{-1})^2\sin^2(t)}.$$

For $\alpha = 0$, we have $\beta = -1$ and for $\alpha = \pi/2$, we have $\beta = +1$ (with β defined in (2.8). Hence, (2.7) implies

$$\begin{aligned}\sup_{x,y\geq 0}\|cR_0(x,y;z)\|^2 &= \frac{|\zeta|+|\zeta|^{-1}}{4}\sup_{y\geq 0}\left[\left(1+\mathrm{e}^{-2y}\right)\left(|\zeta|+|\zeta|^{-1}\right)\right.\\ &\quad \left.\mp 2\mathrm{e}^{-y}\cos\left(\mathrm{Re}(\kappa)\mathrm{Im}(\kappa)^{-1}y\right)\left(|\zeta|-|\zeta|^{-1}\right)\right]\\ &= \frac{\left(|\zeta|+|\zeta|^{-1}\right)^2}{4}G_{\mp}\left(a,b\right)^2.\end{aligned}$$

We thus get

$$1 \leq \|Q(z)\| \leq \frac{\|V\|_1}{c}\frac{\left(|\zeta|+|\zeta|^{-1}\right)}{2}G_{\mp}\left(a,b\right),$$

and the claim follows from the Birman-Schwinger principle like in the proof of Theorem 2.1. □

It follows from Theorem 2.2 that the eigenvalues of D_0+V may only emerge from $\pm mc^2$ as the potential is "turned on". However, if the first moment of the potential is sufficiently small, then the eigenvalues can emerge only from one of those points.

Theorem 2.3. *Let $\alpha \in \{0, \pi/2\}$. If*

$$(2mc)^2\left(\left(\int_0^\infty x\|V(x)\|\,\mathrm{d}x\right)^2 + \left(\int_0^\infty \|V(x)\|\,\mathrm{d}x\right)^2\right) < 1,$$

then the massive ($m \neq 0$) *Dirac operator* $D_0 + V$ *does not have any eigenvalues near* $\pm mc^2$ *(again "+" for* $\alpha = 0$ *and "−" for* $\alpha = \pi/2$*).*

Proof. We only prove the case $\alpha = 0$, the other case is analogous. It follows from (2.4)–(2.6) that

$$\|cR_0(x,y;z)\|^2 = \left[(1+|\zeta|^2)|\sin(\kappa y)|^2 + (1+|\zeta|^{-2})|\cos(\kappa y)|^2\right] \mathrm{e}^{-2\mathrm{Im}(\kappa)x}.$$

Using

$$\sin(\kappa y)\mathrm{e}^{-2\mathrm{Im}(\kappa)x} \le \kappa y \le \kappa x, \quad \cos(\kappa y)\mathrm{e}^{-2\mathrm{Im}(\kappa)x} \le 1,$$

it follows that

$$\|cR_0(x,y;z)\|^2 = (1+|\zeta|^2)\kappa^2 xy + (1+|\zeta|^{-2}),$$

and hence

$$\|Q(z)\|^2 \le \frac{1}{c^2}\left(|z^2-(mc^2)^2| + |z \pm mc^2|^2\right)\left(\int_0^\infty x\|V(x)\|\,\mathrm{d}x\right)^2 + \frac{|z^2-(mc^2)^2| + |z \mp mc^2|^2}{|z^2-(mc^2)^2|}\left(\int_0^\infty \|V(x)\|\,\mathrm{d}x\right)^2.$$

The claim follows again from the Birman-Schwinger principle. □

The eigenvalue inclusion provided by Theorem 2.2 is more intricate than the estimate (1.4) for the Schrödinger operator, because the argument and absolute value still appear simultaneously in the function $G_\mp$ in (2.14), whereas they are separated in (1.4). However, there are special cases when the expression of $G_\mp$ becomes simpler, schematically:

(1) $z \in \mathrm{i}\mathbb{R} \iff |\zeta| = 1 \iff a = 0$;

$$G_\mp(0,b) = \sqrt{2}.$$

(2) $z \in (-mc^2, mc^2) \iff t = -\frac{\pi}{2} \iff b = 0$;

$$G_\mp(a,0) = \max\{2(1 \mp a), 1\}.$$

(3) $z \to \pm mc^2 \iff |\zeta|^{\pm 1} \to \infty \iff a \to \pm 1$;

$$\lim_{a\to 1-} G_-(a,b) = \lim_{a \to -1+} G_+(a,b) = g(b).$$

Here, g is the function (1.5) appearing in the estimate (1.4) for the Schrödinger operator. In case (1) Theorem 2.2 yields no improvement beyond the generic estimate of Theorem 2.1. Case (2) occurs in particular if the potential is Hermitian-valued. Case (3) is of interest in the non-relativistic limit (or the weak coupling limit); we will postpone this to Sect. 3.

Corollary 2.4. *Let* $v := \|V\|_1/c < \sqrt{3}/2$ *with* V *Hermitian-valued. If the boundary conditions* (1.2) *hold with* $\alpha = 0$*, then*

$$\sigma(D_0+V) \subset \left(-\infty, -mc^2\left(1-2v^2\right)\right] \cup \left[mc^2\left(1 - \frac{v^2}{1+\sqrt{1-v^2}}\right), \infty\right).$$

For $\alpha = \pi/2$*, we have*

$$\sigma(D_0+V) \subset \left(-\infty, -mc^2\left(1-\frac{v^2}{1+\sqrt{1-v^2}}\right)\right] \cup \left[mc^2\left(1-2v^2\right), \infty\right).$$

Remark 2.5. Note that these intervals are disjoint so long as $v < \sqrt{3}/2$. The gap closes more slowly from the right than from the left if $\alpha = 0$ and vice versa if $\alpha = \pi/2$; more precisely, e.g. in the first case the end points of the gap are $mc^2\left(1-2v^2\right)$ as opposed to $mc^2\left(1-\frac{1}{2}v^2+O(v^4).\right)$

Proof. We treat the case $\alpha = 0$ only, the case $\alpha = \pi/2$ is analogous. Let z be in the gap of the above half-infinite intervals. Then $\zeta(z)$ lies on the negative imaginary axis, i.e. we have $\cot(t) = 0$ in Theorem 2.2 (case (2) above). Hence, $z \in \mathbb{C} \setminus \sigma(D_0)$ whenever

$$\left((|\zeta|+|\zeta|^{-1})G_-\left(\frac{|\zeta|-|\zeta|^{-1}}{|\zeta|+|\zeta|^{-1}}, 0\right)\right)^{-1} > \frac{v}{2} \tag{2.15}$$

An elementary computation shows that

$$G_-\left(\frac{|\zeta|-|\zeta|^{-1}}{|\zeta|+|\zeta|^{-1}}, 0\right) = \begin{cases} \sqrt{2\left(1-\frac{|\zeta|-|\zeta|^{-1}}{|\zeta|+|\zeta|^{-1}}\right)} & |\zeta| \le \sqrt{3}, \\ 1 & |\zeta| \ge \sqrt{3}. \end{cases}$$

Thus, by (2.15), $z \in \mathbb{C} \setminus \sigma(D_0)$ whenever $|\zeta| \in (\frac{v}{\sqrt{1-v^2}}, \rho)$, where $\rho > \sqrt{3}$ is the larger of the two solutions of the equation $(|\zeta|+|\zeta|^{-1})\frac{v}{2} = 1$. Multiplying the latter by $|\zeta|$ and solving the quadratic equation, then using the relations

$$z = mc^2\frac{|\zeta|^2-1}{|\zeta|^2+1} = mc^2\left(1-\frac{2}{|\zeta|^2+1}\right) = -mc^2\left(1-\frac{2|\zeta|^2}{|\zeta|^2+1}\right),$$

one checks by direct computation that the claimed spectral estimates hold. □

3. The Non-relativistic Limit

The spectral estimates for the Dirac operator on the half-line, Theorems 2.1 and 2.2 reduce to the corresponding bounds for the Schrödinger operator in [4] in the non-relativistic limit $c \to \infty$. Here, e.g. for V a scalar multiple of the identity matrix,

$$\lim_{c\to\infty}(D_0+V+mc^2)^{-1} = 0 \oplus \left(\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V\right)^{-1},$$

$$\lim_{c\to\infty}(D_0+V-mc^2)^{-1} = \left(-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V\right)^{-1} \oplus 0, \tag{3.1}$$

where the limit operators satisfy a Dirichlet or a Neumann condition at zero. For $\alpha \in \{0, \pi/2\}$, and under the assumption that V is relatively D_0-bounded (this of course follows from our global assumption that V is smooth and has compact support), this is a consequence of [9, Theorem 6.1] for abstract Dirac operators. If $\alpha \notin \{0, \pi/2\}$, then D_0 is not an abstract supersymmetric Dirac operator in the sense of [9] because the projections onto the first and second components do not leave the domain of D_0 invariant. Moreover, the proof of Proposition 3.1 shows that V need not be D_0-bounded.

Proposition 3.1. *The limits in* (3.1) *exist in the norm-resolvent sense. In the first case, the nontrivial part of the limit operator satisfies a*

(a) *Dirichlet boundary condition for* $\alpha \in (0, \pi/2]$,
(b) *Neumann boundary condition for* $\alpha = 0$.

In the second case, it satisfies a

(c) *Dirichlet boundary condition for* $\alpha \in [0, \pi/2)$,
(d) *Neumann boundary condition for* $\alpha = \pi/2$.

Proof. Without loss of generality, we assume that $m = 1/2$. We only prove (a) and (b), the proof of (c) and (d) is similar. The resolvent of $D_0 + mc^2$ is given by the formulas (2.3)–(2.6) with the substitution $z \to z - mc^2$ in the expressions for $\kappa(z)$ and $\zeta(z)$ in (2.2). Note that after the substitution, we have that $\kappa = \mathcal{O}(1)$ and $\zeta = \mathcal{O}(c^{-1})$. It is a straightforward computation that the pointwise limit of the resolvent kernel is given by

$$\lim_{c\to\infty} R_0(x, y; z) = 0 \oplus \frac{-1}{2\mathrm{i}\sqrt{-z}} \left(\mathrm{e}^{\mathrm{i}\sqrt{-z}|x-y|} - \mathrm{e}^{\mathrm{i}\sqrt{-z}(x+y)}\right), \quad \alpha \in (0, \pi/2],$$
$$\lim_{c\to\infty} R_0(x, y; z) = 0 \oplus \frac{-1}{2\mathrm{i}\sqrt{-z}} \left(\mathrm{e}^{\mathrm{i}\sqrt{-z}|x-y|} + \mathrm{e}^{\mathrm{i}\sqrt{-z}(x+y)}\right), \quad \alpha = 0. \qquad (3.2)$$

The nontrivial part coincides with the resolvent kernel of the Dirichlet and Neumann Laplacian, respectively.

To prove the convergence in the operator norm on $L^2(\mathbb{R}_+)$, one can use the Schur test, see e.g. [5, Appendix 1]. To this end, one observes that

$$|R_0(x, y; z) - \lim_{c\to\infty} R_0(x, y; z)| \leq Ac^{-1}\mathrm{e}^{-\mathrm{Im}\sqrt{-z}|x-y|}, \quad x, y \in \mathbb{R}_+ \qquad (3.3)$$

for some universal constant $A > 0$; we omit the straightforward details. This proves the claim if $V = 0$. In the general case, the claim follows from the resolvent formula

$$(D_0 + V - z)^{-1} = (D_0 - z)^{-1}$$
$$-(D_0 - z)^{-1}V^{1/2}(I + Q(z))^{-1}|V|^{1/2}(D_0 - z)^{-1} \qquad (3.4)$$

since, upon replacing z by $z - mc^2$ and using the Schur test together with (3.3) again, the right hand side converges to a limit in which D_0 is replaced by the second derivative. □

In view of Proposition 3.1, Theorem 2.2 reduces to [4, Theorem 1.1] in the non-relativistic limit $c \to \infty$. Indeed, Theorem 2.1 implies that, if z is an eigenvalue, then $|\zeta|^{\pm 1} \to \infty$, which is equivalent to $z \to \pm mc^2$. Subtracting mc^2 from $D_0 + V$ amounts to fixing the limit to $+mc^2$. In view of

$$(|\zeta| + |\zeta|^{-1})G_-\left(\frac{|\zeta| - |\zeta|^{-1}}{|\zeta| + |\zeta|^{-1}}, \cot(t)\right) = \left|\frac{2mc^2}{z - mc^2}\right|^{1/2} g(\cot(t)) + o(z - mc^2),$$

we obtain, upon setting $m = \frac{1}{2}$ and replacing z by $z + mc^2$ in Theorem 2.2,

$$|z|^{1/2} \leq \frac{1}{2} g(\cot(\theta/2)) \int_0^\infty |V(x)|\, \mathrm{d}x, \quad z = |z|\mathrm{e}^{\mathrm{i}\theta},$$

in accordance with (1.4).

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