A Fractional Muckenhoupt–Wheeden Theorem and its Consequences

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Abstract. In the 1970s Muckenhoupt and Wheeden made several conjectures relating two weight norm inequalities for the Hardy-Littlewood maximal operator to such inequalities for singular integrals. Using techniques developed for the recent proof of the A_2 conjecture we prove a related pair of conjectures linking the Riesz potential and the fractional maximal operator. As a consequence we are able to prove a number of sharp one and two weight norm inequalities for the Riesz potential.

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1. Introduction

In this paper we prove weighted norm inequalities for the Riesz potential operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy, \qquad 0 < \alpha < n.$$

Our main result is motivated by a pair of conjectures for singular integrals due to Muckenhoupt and Wheeden, and to provide a foundation for our work we first sketch these conjectures and the known results.

In the 1970s Muckenhoupt and Wheeden [22] conjectured that if T is a Calderón-Zygmund singular integral operator, then given a pair of weights (u, v), for $1 < p < \infty$,

$$T : L^p(v) \to L^p(u) \tag{1.1}$$

provided that the Hardy-Littlewood maximal operator satisfies

$$M : L^p(v) \to L^p(u) \tag{1.2}$$

$$M : L^{p'}(u^{1-p'}) \to L^{p'}(v^{1-p'}). \tag{1.3}$$

Further, they conjectured that

$$T : L^p(v) \to L^{p,\infty}(u) \tag{1.4}$$

provided that (1.3) holds. Originally, they made these conjectures for the Hilbert transform, but they were soon extended to Calderón-Zygmund singular integrals. While extremely suggestive and true in many important cases, both of these conjectures are false. A counter-example to the strong type conjecture was found by Reguera and Scurry [29], and this was extended to the weak-type conjecture by the first author, Reznikov and Volberg [7].

However, a version of these conjectures is true in the off-diagonal case. If $1 < p < q < \infty$, then the first author, Martell and Pérez [4] showed that

$$T : L^p(v) \to L^q(u) \tag{1.5}$$

provided that

$$M : L^p(v) \to L^q(u) \tag{1.6}$$

$$M : L^{q'}(u^{1-q'}) \to L^{q'}(v^{1-q'}). \tag{1.7}$$

and that

$$T : L^p(v) \to L^{q,\infty}(u) \tag{1.8}$$

provided that (1.7) holds. In fact, they proved a quantitative version of this result in a slightly different form. Let $\sigma = v^{1-q'}$; then they showed that

$$\|T(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^{q,\infty}(u)} \lesssim \|M(\,\cdot\,u)\|_{L^{q'}(u) \to L^{p'}(\sigma)}$$

and

$$\|T(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} \lesssim \|M(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} + \|M(\,\cdot\,u)\|_{L^{q'}(u) \to L^{p'}(\sigma)}. \tag{1.9}$$

Replacing f by f/σ or f/u yields inequalities in the form given above. This formulation has two advantages. First, the weights do not change under duality. More precisely, if M were a linear, self-adjoint operator, then the inequality gotten from (1.6) by duality would be (1.7). However, in the new formulation, the two norm inequalities on the right-hand side of (1.9) would be dual. Even though the maximal operator is not linear, we will abuse terminology and continue to refer to these as dual inequalities. Second, this formulation makes it easier to consider weights v that are equal to infinity on a set of positive measure, replacing it with a weight that is zero. Hereafter we will formulate all of our weighted norm inequalities in this way.

Our main result is an extension of these off-diagonal results to the case of Riesz potentials with the Hardy-Littlewood maximal operator replaced by the fractional maximal operator of Muckenhoupt and Wheeden [23]:

$$M_\alpha f(x) = \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \fint_Q |f| \, dy, \qquad 0 \le \alpha < n.$$

Theorem 1.1. *Given $0 < \alpha < n$, $1 < p < q < \infty$, and a pair of weights (u, σ), then*

$$\|I_\alpha(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^{q,\infty}(u)} \simeq \|M_\alpha(\,\cdot\,u)\|_{L^{q'}(u) \to L^{p'}(\sigma)}$$

and

$$\|I_\alpha(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} \simeq \|M_\alpha(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} + \|M_\alpha(\,\cdot\,u)\|_{L^{q'}(u) \to L^{p'}(\sigma)}.$$

In both inequalities the constants depend on n, α, p and q.

An open question is whether Theorem 1.1 is true in the case $p = q$. Given the parallels between Riesz potentials and singular integrals this seems doubtful and so we frame the conjecture in the negative.

Conjecture 1.2. *Theorem 1.1 is false when $p = q$: there exists a pair (u, σ) such that $\|M_\alpha(\,\cdot\,u)\|_{L^{p'}(u) \to L^{p'}(\sigma)} < \infty$ but $\|I_\alpha(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^{p,\infty}(u)} = \infty$.*

The remainder of this paper is organized as follows. In Sects. 2 and 3 we give applications of Theorem 1.1 to sharp constant, one weight norm inequalities and to two weight, A_p bump conditions. We will also discuss some conjectures related to 1.2 made by us in an earlier paper [5]. In Sect. 4 we prove Theorem 1.1. Finally, in Sects. 5 and 6 we prove the results from Sects. 2 and 3.

Throughout this paper all notation is standard or will be defined as needed. By a cube we will always mean a cube whose sides are parallel to the coordinate axes. If we write $A \lesssim B$, then $A \leq cB$, where the constant c depends on n, p, q and α. By $A \simeq B$ we mean that $A \lesssim B$ and $B \lesssim A$.

2. Generalized One Weight Inequalities

Theorem 1.1 shows that to prove strong and weak type norm inequalities for the Riesz potential, we need to prove strong type norm inequalities for the fractional maximal operator. We will consider two approaches. In this section we give a generalization of the sharp constant, one weight norm inequalities considered in [5].

Given $1 < p < q < \infty$ and a pair of weights (u, σ), we define

$$A_{p,q}^\alpha(u, \sigma, Q) = |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\fint_Q u \, dx \right)^{\frac{1}{q}} \left(\fint_Q \sigma \, dx \right)^{\frac{1}{p'}}$$

$$= |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{q,Q} \|\sigma^{\frac{1}{p'}}\|_{p',Q}.$$

Note that this functional is symmetric in u and σ:

$$A_{p,q}^\alpha(u, \sigma, Q) = A_{q',p'}^\alpha(\sigma, u, Q).$$

It is well known (cf. [3, p. 115]) that if

$$[u, \sigma]_{A_{p,q}^\alpha} = \sup_Q A_{p,q}^\alpha(u, \sigma, Q) < \infty,$$

then

$$M_\alpha(\,\cdot\,\sigma) : L^p(\sigma) \to L^{q,\infty}(u) \quad \text{and} \quad M_\alpha(\,\cdot\,u) : L^{q'}(\sigma) \to L^{p',\infty}(u).$$

The strong type inequality

$$M_\alpha(\,\cdot\,\sigma) : L^p(\sigma) \to L^q(u)$$

holds in this case if we assume also that the weight σ satisfies a reverse Hölder inequality; equivalently, if we assume that σ is in the Muckenhoupt class A_∞. This class can be defined in several ways. Traditionally (see [9]) we say that $\sigma \in A_\infty$ if

$$[\sigma]_{A_\infty^{\exp}} = \sup_Q A_\infty^{\exp}(\sigma, Q) = \sup_Q \left(\fint_Q \sigma \, dx \right) \exp \left(-\fint_Q \log \sigma \, dx \right) < \infty.$$

This is now sometimes referred to as the exponential A_∞ condition. However, for the purposes of sharp constant estimates, an equivalent definition is very useful: $\sigma \in A_\infty$ if and only if

$$[\sigma]_{A_\infty^M} = \sup_Q A_\infty^M(\sigma, Q) = \sup_Q \frac{1}{\sigma(Q)} \int_Q M(\sigma \chi_Q)(x) \, dx < \infty,$$

where M is the Hardy-Littlewood maximal operator. This equivalent condition was discovered independently by Fujii [8] and Wilson [33,34] (see also [35]). The importance of this condition is that $[\sigma]_{A_\infty^M} \lesssim [\sigma]_{A_\infty^{\exp}}$ and in fact the constant $[\sigma]_{A_\infty^M}$ can be substantially smaller [1,13].

Using Theorem 1.1 we give norm estimates for Riesz potentials in terms of these quantities. Our approach to this problem is based on recent work on the sharp constants for singular integrals. (For the history of these results, see [12,13,17] and the references they contain.) The natural approach when p and q satisfy the Sobolev relationship $1/p - 1/q = \alpha/n$, is to find sharp estimates in terms of $[u,\sigma]_{A_{p,q}^\alpha}$. This case was studied in [14]. Our goal here is to refine these estimates and extend them to general $p < q$. Following the work of Hytönen and Pérez [13], we find sharp constants in terms of $[u,\sigma]_{A_{p,q}^\alpha}$, $[u]_{A_\infty^M}$ and $[\sigma]_{A_\infty^M}$. We also give an alternative approach: following Lerner and the second author [16,18], we prove estimates in terms of a mixed condition that combines the $A_{p,q}^\alpha$ and A_∞ condition:

$$[u,\sigma]_{A_{p,q}^\alpha(u,\sigma) A_\infty^{\exp}(\sigma)^{\frac{1}{q}}} = \sup_Q A_{p,q}^\alpha(u,\sigma,Q) A_\infty^{\exp}(\sigma,Q)^{\frac{1}{q}}.$$

The next result gives both kinds of estimates for the fractional maximal operator; We defer the proof until Sect. 5.

Theorem 2.1. *Given $0 < \alpha < n$ and $1 < p \le q < \infty$, suppose $(u,\sigma) \in A_{p,q}^\alpha$ and $\sigma \in A_\infty$. Then*

$$\|M_\alpha(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} \lesssim [u,\sigma]_{A_{p,q}^\alpha(u,\sigma) A_\infty^{\exp}(\sigma)^{\frac{1}{q}}} \tag{2.1}$$

and

$$\|M_\alpha(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} \lesssim [u,\sigma]_{A_{p,q}} [\sigma]_{A_\infty^M}^{\frac{1}{q}}. \tag{2.2}$$

Hereafter, we will refer to estimates like (2.1) as one supremum estimates, and estimates like (2.2) as two suprema estimates. In general the one supremum estimates are incomparable with the two suprema estimates (see [16, 18] for examples).

As an immediate consequence of Theorems 1.1 and 2.1 we get the following estimates for Riesz potentials.

Theorem 2.2. *Given $0 < \alpha < n$ and $1 < p < q < \infty$, suppose $(u, \sigma) \in A_{p,q}^{\alpha}$ and $u \in A_{\infty}$. Then*

$$\|I_{\alpha}(\cdot\,\sigma)\|_{L^p(\sigma) \to L^{q,\infty}(u)} \lesssim [\sigma, u]_{A_{q',p'}^{\alpha}(\sigma, u) A_{\infty}^{\exp}(u)^{\frac{1}{p'}}} \tag{2.3}$$

and

$$\|I_{\alpha}(\cdot\,\sigma)\|_{L^p(\sigma) \to L^{q,\infty}(u)} \lesssim [\sigma, u]_{A_{q',p'}^{\alpha}} [u]_{A_{\infty}^{M}}^{\frac{1}{p'}}. \tag{2.4}$$

Theorem 2.3. *Given $0 < \alpha < n$ and $1 < p < q < \infty$, suppose $(u, \sigma) \in A_{p,q}^{\alpha}$ and $u, \sigma \in A_{\infty}$. Then*

$$\|I_{\alpha}(\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} \lesssim [u, \sigma]_{A_{p,q}^{\alpha}(u, \sigma) A_{\infty}^{\exp}(\sigma)^{\frac{1}{q}}} + [\sigma, u]_{A_{q',p'}^{\alpha}(\sigma, u) A_{\infty}^{\exp}(u)^{\frac{1}{p'}}} \tag{2.5}$$

and

$$\|I_{\alpha}(\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} \lesssim [u, \sigma]_{A_{p,q}^{\alpha}} \left([u]_{A_{\infty}^{M}}^{\frac{1}{p'}} + [\sigma]_{A_{\infty}^{M}}^{\frac{1}{q}} \right). \tag{2.6}$$

Remark 2.4. We proved inequalities (2.4) and (2.6) in [5] using a more complicated corona decomposition argument. Moreover, we only obtained results for p and q that satisfy the Sobolev relation.

Theorems 2.2 and 2.3 can be thought of as generalizing one weight inequalities for the Riesz potential. The classical one weight norm inequalities for Riesz potentials due to Muckenhoupt and Wheeden [23] were for the case when p and q satisfy the Sobolev relation, and there exists a weight w such that $u = w^q$ and $\sigma = w^{-p'}$. In this case it follows from the $A_{p,q}^{\alpha}$ condition that both u and σ are in A_{∞}. In this case we can restate (2.6) as

$$\|I_{\alpha}\|_{L^p(w^p) \to L^q(w^q)} \lesssim [w^q]_{A_{s(p)}}^{\frac{1}{q}} \left([w^q]_{A_{\infty}^{M}}^{\frac{1}{p'}} + [w^{-p'}]_{A_{\infty}^{M}}^{\frac{1}{q}} \right),$$

where $s(p) = 1 + p/q'$ and we say that a weight v is in the Muckenhoupt class A_p if

$$[v]_{A_p} = \sup_Q A_p(v, Q) = \sup_Q \fint_Q v \, dx \left(\fint_Q v^{1-p'} \, dx \right)^{p-1} < \infty.$$

By interpolation we can give a result that is in some sense an improvement of this inequality. We again defer the proof to Sect. 5.

Theorem 2.5. *Given $0 < \alpha < n$ and $1 < p < n/\alpha$, define q by $1/p - 1/q = \alpha/n$. If $w^q \in A_r$ for some $r < s(p) = 1 + p/q'$, then*

$$\|I_\alpha\|_{L^p(w^p) \to L^q(w^q)} \lesssim [w^q]_{A_r}^{\frac{1}{q}} [w^q]_{A_\infty^M}^{\frac{1}{p'}}. \tag{2.7}$$

The corresponding result for singular integrals was proved in [18]. A weaker version of Theorem 2.5, with the assumption $w^q \in A_r$ replaced by the assumption that $w^q \in A_1$, was recently proved by Recchi [28].

In [18] it was conjectured that for singular integrals, the one supremum estimates corresponding to (2.3) and (2.5) could be improved by replacing $A_\infty^{\exp}(\sigma, Q)$ on the right-hand side with the smaller quantity $A_\infty^M(\sigma, Q)$. They were able to prove a partial result involving an additional log term.

We believe that the corresponding conjecture is true for Riesz potentials. We can prove a partial result in the classical one weight case. To state it we define the one supremum constant needed in this case, for a general weight:

$$[w]_{(A_p)^\beta (A_\infty^M)^\gamma} = \sup_Q A_p(w, Q)^\beta A_\infty^M(w, Q)^\gamma.$$

Theorem 2.6. *Given $0 < \alpha < n$ and $1 < p < n/\alpha$, define q by $1/p - 1/q = \alpha/n$. If $w^q \in A_{s(p)}$, $s(p) = 1 + p/q'$ and $\sigma = w^{-p'}$, then*

$$\|M_\alpha\|_{L^p(w^p) \to L^q(w^q)} \lesssim \Phi\big([w^{-p'}]_{A_{s(q')}}\big)^{\frac{1}{q}} [w^{-p'}]_{(A_{s(q')})^{\frac{1}{p'}} (A_\infty^M)^{\frac{1}{q}}}, \tag{2.8}$$

where $\Phi(t) = 1 + \log(t)$.

The proof of Theorem 2.6 requires a testing condition for the fractional maximal function in [19]. Using this condition it is very similar to the argument in [18]. We sketch the details of the proof in Sect. 5. Once again, as a consequence of Theorem 1.1 we have the following result for the Riesz potential.

Theorem 2.7. *Given $0 < \alpha < n$ and $1 < p < n/\alpha$, define q by $1/p - 1/q = \alpha/n$. If $w^q \in A_{s(p)}$, $s(p) = 1 + p/q'$, then*

$$\|I_\alpha\|_{L^p(w^p) \to L^{q,\infty}(w^q)} \lesssim \Phi\big([w^q]_{A_{s(p)}}\big)^{\frac{1}{p'}} [w^q]_{(A_{s(p)})^{\frac{1}{q}} (A_\infty^M)^{\frac{1}{p'}}} \tag{2.9}$$

and

$$\|I_\alpha\|_{L^p(w^p) \to L^q(w^q)} \lesssim \Phi\big([w^q]_{A_{s(p)}}\big)^{\frac{1}{p'}} [w^q]_{(A_{s(p)})^{\frac{1}{q}} (A_\infty^M)^{\frac{1}{p'}}}$$
$$+ \Phi\big([w^{-p'}]_{A_{s(q')}}\big)^{\frac{1}{q}} [w^{-p'}]_{(A_{s(q')})^{\frac{1}{p'}} (A_\infty^M)^{\frac{1}{q}}}, \tag{2.10}$$

where $\Phi(t) = 1 + \log(t)$.

3. Two Weight Inequalities via A_p Bump Conditions

If we do not assume that $u, \sigma \in A_\infty$, then the $A_{p,q}^\alpha$ condition is no longer sufficient for the strong type inequality for the fractional maximal operators or for the Riesz potentials. The construction is deferred until Sect. 6.

Example 3.1. Given $0 < \alpha < n$ and $1 < p \leq q < \infty$, there exists a pair of weights $(u, \sigma) \in A_{p,q}^{\alpha}$ and a function $f \in L^p(\sigma)$ such that $M_\alpha(f\sigma) \notin L^q(u)$.

Remark 3.2. A similar example for the Hardy-Littlewood maximal operator (i.e., when $\alpha = 0$) was constructed by Muckenhoupt and Wheeden [24]. While the existence of Example 3.1 is part of the folklore of harmonic analysis, to the best of our knowledge one has never been published. It is worth noting that our example is considerably different from the one constructed by Muckenhoupt and Wheeden.

It is possible, however, to replace the $A_{p,q}^{\alpha}$ condition with a stronger one defined using Orlicz norms. This approach to weighted norm inequalities is due to Pérez [25, 27] and was motivated by the original Muckenhoupt–Wheeden conjectures.

To state these results we need to make some preliminary definitions. (For further information, see [3, Section 5.2].) A Young function is a function $\Phi : [0, \infty) \to [0, \infty)$ that is continuous, convex and strictly increasing, $\Phi(0) = 0$ and $\Phi(t)/t \to \infty$ as $t \to \infty$. Define the localized Luxemburg average of f over a cube Q by

$$\|f\|_{\Phi,Q} = \inf \left\{ \lambda > 0 : \fint_Q \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

When $\Phi(t) = t^p$, $1 < p < \infty$, this becomes the L^p norm and we write $\|f\|_{\Phi,Q} = \|f\|_{p,Q}$. The associate function of Φ is the Young function

$$\bar{\Phi}(t) = \sup_{s>0} \{st - \Phi(s)\}.$$

Note that $\bar{\bar{\Phi}} = \Phi$. A Young function Φ satisfies the B_p condition if for some $c > 0$,

$$\int_c^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} < \infty.$$

Important examples of such functions are $\Phi(t) = t^{(rp')'}$, $r > 1$, whose associate function is $\bar{\Phi}(t) = t^{rp'}$, and $\Phi(t) = t^p \log(e + t)^{-1-\epsilon}$, $\epsilon > 0$, which have associate functions $\bar{\Phi}(t) \simeq t^{p'} \log(e + t)^{p'-1+\delta}$, $\delta > 0$. We refer to these associate functions as power bumps and log bumps. The B_p condition is important because it characterizes the L^p boundedness of Orlicz maximal operators, which in turn can be used to prove two weight inequalities. Define

$$M_\Phi f(x) = \sup_{Q \ni x} \|f\|_{\Phi,Q};$$

then Pérez [26] showed that M_Φ is bounded on $L^p(\mathbb{R}^n)$ if and only if $\Phi \in B_p$, and

$$\|M_\Phi\|_{L^p \to L^p} \lesssim \left(\int_c^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} \right)^{1/p}.$$

For our results we need to generalize this to the fractional Orlicz maximal operator. Given $0 < \alpha < n$ and a Young function Φ, define

$$M_{\alpha,\Phi}f(x) = \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \|f\|_{\Phi,Q}.$$

We define the associated fractional B_p condition as follows: given $1 < p < n/\alpha$, let $1/q = 1/p - \alpha/n$. Then $\Phi \in B_p^\alpha$ if

$$\int_c^\infty \frac{\Phi(t)^{q/p}}{t^q} \frac{dt}{t} < \infty.$$

We prove the following result in Sect. 6.

Theorem 3.3. *Given $0 < \alpha < n$ and $1 < p < n/\alpha$, define $1/q = 1/p - \alpha/n$. Then for any $\Phi \in B_p^\alpha$, $M_{\alpha,\Phi} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ and*

$$\|M_{\alpha,\Phi}\|_{L^p \to L^q} \lesssim \left(\int_c^\infty \frac{\Phi(t)^{\frac{q}{p}}}{t^q} \frac{dt}{t} \right)^{\frac{1}{q}}. \tag{3.1}$$

When $\alpha = 0$ the two conditions coincide; if $\alpha > 0$ then the B_p^α condition is weaker. To see this, note that because the measure $\frac{dt}{t}$ on $(0, \infty)$ behaves in some sense like a counting measure, we have

$$\left(\int_c^\infty \frac{\Phi(t)^{\frac{q}{p}}}{t^q} \frac{dt}{t} \right)^{\frac{1}{q}} \lesssim \left(\int_c^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} \right)^{1/p}.$$

Moreover, the Young function

$$\Phi(t) = \frac{t^p}{\log(t)^{(1+\epsilon)\frac{p}{q}}}$$

is in B_p^α for any $\epsilon > 0$ but is in B_p only if $\epsilon > q/p - 1$. Hence, $B_p \subsetneq B_p^\alpha$ if $\alpha > 0$.

To state our results we introduce a new weight condition that is stronger than the $[u, \sigma]_{A_{p,q}^\alpha}$ condition, replacing the average on σ by an Orlicz average: we say that $(u, \sigma) \in A_{p,q,\Phi}^\alpha$ if

$$[u, \sigma]_{A_{p,q,\Phi}^\alpha} = \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\fint_Q u \, dx \right)^{\frac{1}{q}} \|\sigma^{\frac{1}{p'}}\|_{\Phi,Q} < \infty.$$

If we assume that Φ is such that $t^{p'} \leq C\Phi(ct)$, then $[u, \sigma]_{A_{p,q}^\alpha} \lesssim [u, \sigma]_{A_{p,q,\Phi}^\alpha}$. This is always the case if $\bar{\Phi} \in B_p$. Note that this new condition lacks the symmetry of the $A_{p,q}^\alpha$ condition since the Orlicz norm is always applied to the second weight.

This condition was introduced by Pérez [25] (see also [3, Section 5.6]), who used it to prove strong type, two weight norm inequalities for the fractional maximal operator:

$$\|M_\alpha(\cdot \sigma)\|_{L^p(\sigma) \to L^q(u)} \lesssim [u, \sigma]_{A_{p,q,\Phi}^\alpha} \|M_{\bar{\Phi}}\|_{L^p \to L^p}. \tag{3.2}$$

When $p > q$ we improve his result both qualitatively and quantitatively, giving a larger class of Young functions and a sharper constant.

Theorem 3.4. *Given $0 \leq \alpha < n$ and $1 < p \leq q < \infty$, define $\beta = n\left(\frac{1}{p} - \frac{1}{q}\right)$. If $\bar{\Phi} \in B_p^\beta$ and the pair of weights $(u, \sigma) \in A_{p,q,\Phi}^\alpha$, then*

$$\|M_\alpha(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} \lesssim [u, \sigma]_{A_{p,q,\Phi}^\alpha} \|M_{\beta, \bar{\Phi}}\|_{L^p \to L^q}.$$

To see that this constant is sharper when $p > q$, we give two examples. If $\bar{\Phi}(t) = t^p \log(t)^{-(1+\epsilon)}$, $\epsilon > 0$, then a straightforward computation shows that the B_p constant is approximately $\epsilon^{-1/p}$ but the B_p^β constant is approximately $\epsilon^{-1/q}$. If $\bar{\Phi}(t) = t^{(rp')'}$, $r > 1$, then the B_p constant is $(r')^{1/p}$ but the B_p^β constant is $(r')^{1/q}$. (This second example will be applied below.)

As an immediate consequence of Theorems 1.1 and 3.4 we get the corresponding two weight, weak and strong type norm inequalities for Riesz potentials.

Theorem 3.5. *Given $0 < \alpha < n$ and $1 < p < q < \infty$, let $\beta = n\left(\frac{1}{p} - \frac{1}{q}\right)$. If $\bar{\Psi} \in B_{q'}^\beta$ and the pair $(u, \sigma) \in A_{q',p',\Psi}^\alpha$, then*

$$\|I_\alpha(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^{q,\infty}(u)} \lesssim [\sigma, u]_{A_{q',p',\Psi}^\alpha} \|M_{\beta, \bar{\Psi}}\|_{L^{q'} \to L^{p'}}.$$

Theorem 3.6. *Given $0 < \alpha < n$ and $1 < p < q < \infty$, let $\beta = n\left(\frac{1}{p} - \frac{1}{q}\right)$. If $\bar{\Phi} \in B_p^\beta$, $\bar{\Psi} \in B_{q'}^\beta$ and the pair (u, σ) satisfies $(u, \sigma) \in A_{p,q,\Phi}^\alpha$ and $(\sigma, u) \in A_{q',p',\Psi}^\alpha$, then*

$$\|I_\alpha(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} \lesssim [u, \sigma]_{A_{p,q,\Phi}^\alpha} \|M_{\beta, \bar{\Phi}}\|_{L^p \to L^q}$$
$$+ [\sigma, u]_{A_{q',p',\Psi}^\alpha} \|M_{\beta, \bar{\Psi}}\|_{L^{q'} \to L^{p'}}. \tag{3.3}$$

Theorem 3.6 is referred to as a separated bump condition: conditions of this kind were implicit in the work of Pérez and were introduced explicitly for singular integrals in [7] (see below). This condition significantly improves the original, "double bump" result of Pérez [25], who showed that

$$\|I_\alpha(\,\cdot\,\sigma)\|_{L^p(\sigma) \to L^q(u)} \lesssim [u, \sigma]_{A_{p,q,\Psi,\Phi}^\alpha} \|M_{\bar{\Psi}}\|_{L^{q'} \to L^{q'}} \|M_{\bar{\Phi}}\|_{L^p \to L^p}, \tag{3.4}$$

where $\bar{\Psi} \in B_{q'}$, $\bar{\Phi} \in B_p$, and

$$[u, \sigma]_{A_{p,q,\Psi,\Phi}^\alpha} = \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Psi, Q} \|\sigma^{\frac{1}{p'}}\|_{\Phi, Q} < \infty. \tag{3.5}$$

By Hölder's inequality for Orlicz norms we have that this quantity is (up to a constant) larger than the right-hand side of (3.3).

As a corollary to Theorem 3.4 we can give an alternative proof of inequality (2.2), which, again by Theorem 1.1, implies inequalities (2.4) and (2.6). We briefly sketch the argument. If $(u, \sigma) \in A_{p,q}^\alpha$ and $\sigma \in A_\infty$, then Theorem 2.3 in [13]

$$\left(\fint_Q \sigma^r \, dx\right)^{1/r} \leq 2 \fint_Q \sigma \, dx,$$

where

$$r = r(\sigma) = 1 + \frac{1}{c_n[\sigma]_{A_\infty^M}}.$$

Notice that $r' \simeq [\sigma]_{A_\infty^M}$. Let $\Phi(t) = t^{rp'}$; then $\bar{\Phi}(t) = t^{(rp')'}$ and

$$[u,\sigma]_{A_{p,q,\Phi}^\alpha} \lesssim [u,\sigma]_{A_{p,q}^\alpha}.$$

Further, as we noted above

$$\|M_{\beta,\bar{\Phi}}\|_{L^p \to L^q} \lesssim (r')^{\frac{1}{q}} \simeq [\sigma]_{A_\infty^M}^{\frac{1}{q}}.$$

Remark 3.7. If we use the original inequality (3.2) in this argument, we get a worse power of $1/p$ on the constant $[\sigma]_{A_\infty^M}$:

$$\|M_\alpha(\cdot\sigma)\|_{L^p(\sigma) \to L^q(u)} \lesssim [u,\sigma]_{A_{p,q}^\alpha}[\sigma]_{A_\infty^M}^{\frac{1}{p}}.$$

Theorems 3.5 and 3.6 give positive answers for all $1 < p < q < \infty$ to two conjectures we originally made in [5]. There we proved partial results using a more complicated corona decomposition argument. We were forced to assume that Φ and Ψ were log bumps: i.e.,

$$\Phi(t) = t^{p'} \log(e+t)^{p'-1+\delta}, \quad \Psi(t) = t^q \log(e+t)^{q-1+\delta}, \quad \delta > 0,$$

and make the further restriction that $(p'/q')(1 - \alpha/n) \geq 1$ for the weak type inequality and $\min(q/p, p'/q')(1 - \alpha/n) \geq 1$ for the strong type inequality. These conditions hold if p and q satisfy the Sobolev relationship but do not hold if p and q are very close in value.

In [5] we also conjectured that these results hold in the critical exponent case $p = q$. This case is important for its applications in the study of partial differential equations: see [31] and the references it contains. We repeat these conjectures here.

Conjecture 3.8. *Given* $0 < \alpha < n$, $1 < p < \infty$, *and* $\bar{\Psi} \in B_{p'}$, *suppose* $(u,\sigma) \in A_{p',p',\Psi}^\alpha$. *Then*

$$\|I_\alpha(\cdot\sigma)\|_{L^p(\sigma) \to L^{p,\infty}(u)} \lesssim [\sigma,u]_{A_{p',p',\Psi}^\alpha}\|M_{\bar{\Psi}}\|_{L^{p'} \to L^{p'}}.$$

Conjecture 3.9. *Given* $0 < \alpha < n$, $1 < p < \infty$, $\bar{\Psi} \in B_p$ *and* $\bar{\Phi} \in B_p$, *suppose the pair* (u,σ) *satisfies* $(u,\sigma) \in A_{p',p',\Phi}^\alpha$ *and* $(\sigma,u) \in A_{p,p,\Psi}^\alpha$ *then*

$$\|I_\alpha(\cdot\sigma)\|_{L^p(\sigma) \to L^p(u)} \lesssim [u,\sigma]_{A_{p,p,\Phi}^\alpha}\|M_{\bar{\Phi}}\|_{L^p \to L^p} + [\sigma,u]_{A_{p',p',\Psi}^\alpha}\|M_{\bar{\Psi}}\|_{L^{p'} \to L^{p'}}.$$

Very little is known about these conjectures. We do have that Conjecture 3.8 implies Conjecture 3.9, since for all pairs (u,σ) and exponents $1 < p \leq q < \infty$,

$$\|I_\alpha(\cdot\sigma)\|_{L^p(\sigma) \to L^q(u)} \simeq \|I_\alpha(\cdot\sigma)\|_{L^p(\sigma) \to L^{q,\infty}(u)}$$
$$+ \|I_\alpha(\cdot u)\|_{L^{q'}(u) \to L^{p',\infty}(\sigma)}. \tag{3.6}$$

(See [31].) Conjecture 3.8 is known in the special case $\Psi(t) = t^p \log(e + t)^{2p-1+\delta}$: this was proved in [3, Theorem 9.42]. Note that the exponent is much larger than desired: in the case of log bumps we would expect the exponent to be $p - 1 + \delta$.

Remark 3.10. Conjecture 3.9 is the fractional version of the separated bump conjecture for Calderón-Zygmund operators made in [11]:

$$\|T(\cdot\sigma)\|_{L^p(\sigma)\to L^p(u)} \lesssim [u,\sigma]_{A_{p,\Phi}}\|M_{\bar{\Phi}}\|_{L^p\to L^p} + [\sigma,u]_{A_{p',\Psi}}\|M_{\bar{\Psi}}\|_{L^{p'}\to L^{p'}}$$
(3.7)

(where $[u,\sigma]_{A_{p,\Phi}} = [u,\sigma]_{A^0_{p,p,\Phi}}$). A non-quantitative version of this conjecture first appeared in [7]. In this paper they gave a partial result in the scale of log bumps: if

$$\Phi(t) = t^{p'}\log(e+t)^{p'-1+\delta}, \Psi(t) = t^p\log(e+t)^{p-1+\delta}, \qquad \delta > 0,$$

then

$$\|T(\cdot\sigma)\|_{L^p(\sigma)\to L^p(u)} \lesssim [u,\sigma]_{A_{p,\Phi}}\|M_{\bar{\Phi}}\|_{L^p\to L^p}^{p+1} + [\sigma,u]_{A_{p',\Psi}}\|M_{\bar{\Psi}}\|_{L^{p'}\to L^{p'}}^{p'+1}.$$

We conclude this section with an observation. We suspect that the following result, which gives a connection between operator norms for the Riesz potential and a "bilinear" (properly, bisublinear) maximal operator defined by the second author in [20], may be applicable to this problem.

Theorem 3.11. *Given $0 < \alpha < n$ and a dyadic grid $\mathscr{D}$, let X and Y be Banach function spaces. Then*

$$\|I_\alpha\|_{X\to Y} \simeq \|\mathcal{M}_\alpha\|_{X\times Y'\to L^1},$$

where for $f, g \in L^1_{loc}$,

$$\mathcal{M}_\alpha(f,g)(x) = \sup_{Q\ni x} |Q|^{\frac{\alpha}{n}} \fint_Q |f|\,dx \cdot \fint_Q |g|\,dx.$$

Earlier, related estimates for singular integrals were implicit in [2] and the corresponding version of Theorem 3.11 for Calderón-Zygmund operators was proved in [17]. Theorem 3.11 is proved in essentially the same way and we omit the details. Inequality (3.4) follows from Theorem 3.11 and the weighted theory for $\mathcal{M}_\alpha$ developed in [20, Theorem 6.6], but we are unable to prove separated bump results using this approach.

4. Proof of Theorem 1.1

We divide this section into two parts. In the first we gather some results about dyadic Riesz potentials, and in the second give the proof itself.

Dyadic Riesz Potentials

A dyadic grid, usually denoted $\mathscr{D}$, is a collection of cubes in $\mathbb{R}^n$ with the following properties:

(a)   given $Q \in \mathscr{D}$, the side-length satisfies $\ell(Q) = 2^k$ for some $k \in \mathbb{Z}$;
(b)   given $Q, P \in \mathscr{D}$, $Q \cap P$ is either P, Q, or $\varnothing$;
(c)   for a fixed $k \in \mathbb{Z}$ the set $\mathscr{D}_k = \{Q \in \mathscr{D} : \ell(Q) = 2^k\}$ is a partition of $\mathbb{R}^n$.

Given $t \in \{0, 1/3\}^n$ we define the family of dyadic grids

$$\mathscr{D}^t = \{2^{-k}([0,1)^n + m + (-1)^k t) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

When $t = 0$, $\mathscr{D}^0$ is the classic dyadic grid with base point at the origin used in the Calderón-Zygmund decomposition.

Given a dyadic grid $\mathscr{D}$ and $0 < \alpha < n$, we define a dyadic version of I_α:

$$I_\alpha^{\mathscr{D}} f(x) = \sum_{Q \in \mathscr{D}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q f(y) \, dy \cdot \chi_Q(x). \tag{4.1}$$

In [5] we showed that for non-negative functions f,

$$I_\alpha f(x) \lesssim \max_{t \in \{0,1/3\}^n} I_\alpha^{\mathscr{D}^t} f(x). \tag{4.2}$$

Since I_α and $I_\alpha^{\mathscr{D}}$ are positive operators, hereafter we may assume that we are dealing with non-negative functions and can apply these inequalities to reduce to the dyadic case.

To estimate the norm of $I_\alpha^{\mathscr{D}}$, we will use a testing condition due to Lacey, Sawyer and Uriarte-Tuero [15]. To state their result, we need two definitions. First, given a cube $Q_0 \in \mathscr{D}$, for $x \in Q_0$ define the "outer" dyadic Riesz potential

$$I_\alpha^{Q_0} f(x) = \sum_{\substack{Q \in \mathscr{D} \\ Q \supset Q_0}} |Q|^{\frac{\alpha}{n}} \fint_Q f(y) \, dy \cdot \chi_Q(x).$$

Second, given $0 < \alpha < n$, $1 < p < q < \infty$ and a pair of weights (u, σ), define the testing constant

$$[u, \sigma]_{I_\alpha^{\mathrm{out}}, p, q} = \sup_{Q_0} \left(\int_{\mathbb{R}^n} I_\alpha^{Q_0}(\sigma \chi_{Q_0})(x)^q u \, dx \right)^{1/q} \sigma(Q_0)^{-1/p}.$$

Theorem 4.1. *Given $0 < \alpha < n$ and $1 < p < q < \infty$,*

$$\|I_\alpha^{\mathscr{D}}(\cdot \sigma)\|_{L^p(\sigma) \to L^{q,\infty}(u)} \simeq [\sigma, u]_{I_\alpha^{out}, q', p'}$$

and

$$\|I_\alpha^{\mathscr{D}}(\cdot \sigma)\|_{L^p(\sigma) \to L^q(u)} \simeq [u, \sigma]_{I_\alpha^{out}, p, q} + [\sigma, u]_{I_\alpha^{out}, q', p'}.$$

Remark 4.2. In Theorem 4.1 the restriction that $p < q$ is essential; this is the reason for this condition for our paper. In [15] they give a different testing condition that holds when $p = q$, but we have been unable to apply our techniques to get estimates in this case.

Proof of Theorem 1.1

Our argument is broadly similar to the one in [4]. By inequality (4.2) it suffices to fix a dyadic grid $\mathscr{D}$ and obtain norm estimates for $I_\alpha^{\mathscr{D}}$ that are independent of the grid. And by Theorem 4.1 it suffices to estimate the testing constant for the outer Riesz potential. The inequality "$\lesssim$" in Theorem 1.1 is a consequence of the following result.

Theorem 4.3. *Given $0 < \alpha < n$, $1 < p < q < \infty$ and a pair of weights (u, σ), then*

$$[u, \sigma]_{I_\alpha^{out}, p, q} \leq (1 - 2^{\alpha-n})^{-1} \|M_\alpha(\cdot \sigma)\|_{L^p(\sigma) \to L^q(u)}$$

and

$$[\sigma, u]_{I_\alpha^{out}, q', p'} \leq (1 - 2^{\alpha-n})^{-1} \|M_\alpha(\cdot u)\|_{L^{q'}(u) \to L^{p'}(\sigma)}.$$

Proof. We will prove the first inequality; the proof of the second is identical. Fix a cube $Q_0 \in \mathscr{D}$ and for each $k \geq 1$ let $Q_k \in \mathscr{D}$ be the unique cube such that $Q_{k-1} \subset Q_k$ and $|Q_k| = 2^{kn}|Q_0|$. By definition,

$$I_\alpha^{Q_0}(\sigma\chi_{Q_0})(x) = \sum_{Q \supset Q_0} |Q|^{\frac{\alpha}{n}-1} \int_Q \sigma\chi_{Q_0} \, dx \cdot \chi_Q(x)$$

$$= \int_{Q_0} \sigma \, dx \cdot \sum_{k=0}^{\infty} |Q_k|^{\frac{\alpha}{n}-1} \cdot \chi_{Q_k}(x).$$

Clearly, the support of $I_\alpha^{Q_0}(\sigma\chi_{Q_0})$ is $\bigcup_{k=0}^{\infty} Q_k$. If $x \in Q_0$, then

$$I_\alpha^{Q_0}(\sigma\chi_{Q_0})(x) = \int_{Q_0} \sigma \, dx \cdot \sum_{k=0}^{\infty} |Q_k|^{\frac{\alpha}{n}-1} = |Q_0|^{\frac{\alpha}{n}-1} \int_{Q_0} \sigma \, dx \cdot \sum_{k=0}^{\infty} 2^{k(\alpha-n)}$$

$$= (1 - 2^{\alpha-n})^{-1} |Q_0|^{\frac{\alpha}{n}-1} \int_{Q_0} \sigma \, dx \leq (1 - 2^{\alpha-n})^{-1} M_\alpha(\sigma\chi_{Q_0})(x).$$

If $x \in Q_{j+1} \backslash Q_j$ for some $j \geq 0$, then

$$I_\alpha^{Q_0}(\sigma\chi_{Q_0})(x) = \int_{Q_0} \sigma \, dx \cdot \sum_{k=j}^{\infty} |Q_k|^{\frac{\alpha}{n}-1}$$

$$= |Q_0|^{\frac{\alpha}{n}-1} \int_{Q_0} \sigma \, dx \cdot \sum_{k=j}^{\infty} 2^{k(\alpha-n)} = (1 - 2^{\alpha-n})^{-1} 2^{j(\alpha-n)} |Q_0|^{\frac{\alpha}{n}-1} \int_{Q_0} \sigma \, dx$$

$$= (1 - 2^{\alpha-n})^{-1} |Q_j|^{\frac{\alpha}{n}-1} \int_{Q_0} \sigma \, dx \leq (1 - 2^{\alpha-n})^{-1} M_\alpha(\sigma\chi_{Q_0})(x).$$

We therefore have that

$$\int_{\mathbb{R}^n} I_\alpha^{Q_0}(\sigma\chi_{Q_0})^q u \, dx \leq (1 - 2^{\alpha-n})^{-q} \int_{\mathbb{R}^n} M_\alpha(\sigma\chi_{Q_0})^q u \, dx,$$

and the desired inequality follows immediately. $\qquad \square$

Finally, we prove the reverse inequalities in Theorem 1.1. We will prove that

$$\|M_\alpha(\cdot u)\|_{L^{q'}(u) \to L^{p'}(\sigma)} \lesssim \|I_\alpha(\cdot \sigma)\|_{L^p(\sigma) \to L^{q,\infty}(u)};$$

the other estimates are proved in essentially the same way. Fix a cube Q; then

$$\int\limits_Q I_\alpha(f\sigma)u\,dx \leq \|I_\alpha(f\sigma)\|_{L^{q,\infty}(u)}\|\chi_Q\|_{L^{q',1}(u)}$$

$$\leq \|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(u)}\|f\|_{L^p(\sigma)}u(Q)^{1/q'}.$$

Let $f = I_\alpha(u\chi_Q)^{p'-1}\chi_Q$. Then, since I_α is self-adjoint, we have that

$$\left(\int\limits_Q I_\alpha(u\chi_Q)^{p'}\sigma\,dx\right)^{1/p'} \leq \|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(u)}u(Q)^{1/q'}.$$

Since for non-negative functions f, $M_\alpha f(x) \lesssim I_\alpha f(x)$, we have that

$$[\sigma,u]_{M_\alpha,q',p'} = \sup_Q \left(\int\limits_Q M_\alpha(u\chi_Q)^{p'}\sigma\,dx\right)^{1/p'}u(Q)^{-1/q'}$$

$$\lesssim \|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(u)}.$$

But by Sawyer's testing condition for the fractional maximal operator [30],

$$\|M_\alpha(\cdot u)\|_{L^{q'}(u)\to L^{p'}(\sigma)} \simeq [\sigma,u]_{M_\alpha,q',p'}.$$

This completes the proof.

5. Estimates Involving A_∞

In this section we prove Theorems 2.1, 2.5 and 2.6. We first give some further results on dyadic operators, and we then prove each of these theorems in turn.

Dyadic fractional maximal operators

Given a dyadic grid $\mathscr{D}$, define the dyadic fractional maximal operator by

$$M_\alpha^{\mathscr{D}} f(x) = \sup_{Q\in\mathscr{D}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int\limits_Q f(y)\,dy\cdot\chi_Q(x).$$

Essentially the same argument in [5] that gave us (4.2) also lets us prove the corresponding estimate for the fractional maximal operator:

$$M_\alpha f(x) \lesssim \max_{t\in\{0,1/3\}^n} M_\alpha^{\mathscr{D}^t} f(x). \tag{5.1}$$

Therefore, it will suffice to prove norm estimates for $M_\alpha^{\mathscr{D}}$. In fact, we will prove estimates for a linearization of this operator. We begin by defining sparse families of cubes. These are a generalization of an idea closely connected to the Calderón-Zygmund decomposition. Given a dyadic grid $\mathscr{D}$, a subset $\mathcal{S}\subseteq\mathscr{D}$ is sparse if there exists a family of disjoint, "thick" subsets: for every $Q\in\mathcal{S}$ there exists $E_Q\subseteq Q$ such that the family $\{E_Q\}_{Q\in\mathcal{S}}$ is pairwise disjoint and $|E_Q|\geq\frac{1}{2}|Q|$.

Given a sparse family $\mathcal{S}$ define the linear operator

$$L_\alpha^{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} |Q|^{\frac{\alpha}{n}} \fint_Q f \, dx \cdot \chi_{E_Q}(x).$$

Theorem 5.1. *Given $0 \le \alpha < n$ and a dyadic grid $\mathcal{D}$, for every non-negative Function $f \in L_c^\infty(\mathbb{R}^n)$ there exists a sparse family $\mathcal{S} \subset \mathcal{D}$ such that for a.e. x,*

$$M_\alpha^{\mathcal{D}} f(x) \lesssim L_\alpha^{\mathcal{S}} f(x), \tag{5.2}$$

and the implicit constants do not depend on $\mathcal{D}$, $\mathcal{S}$ or f. Consequently, for any Banach function spaces X and Y

$$\|M_\alpha^{\mathcal{D}}\|_{X \to Y} \simeq \sup_{\mathcal{S}} \|L_\alpha^{\mathcal{S}}\|_{X \to Y}$$

and

$$\|M_\alpha\|_{X \to Y} \simeq \sup_{\mathcal{S}, \mathcal{D}} \|L_\alpha^{\mathcal{S}}\|_{X \to Y}.$$

The pointwise inequality (5.2) is well known; the basic idea first appeared in Sawyer [30] (see also [3, Section 9.3]). The norm inequalities follow from this and (5.1).

Proof of Theorem 2.1

For the proof we need three lemmas. The first is an L^p estimate for a dyadic maximal operator defined with respect to an arbitrary measure. Let μ be a positive Borel measure on $\mathbb{R}^n$; given a dyadic grid $\mathcal{D}$ and $0 \le \beta < n$, define

$$M_{\beta,\mu}^{\mathcal{D}} f(x) = \sup_{Q \in \mathcal{D}} \frac{1}{\mu(Q)^{1 - \frac{\beta}{n}}} \int_Q |f| \, d\mu \cdot \chi_Q(x).$$

When $\beta = 0$ we simply write $M_{0,\mu}^{\mathcal{D}} = M_\mu^{\mathcal{D}}$. The operator $M_\beta^{\mathcal{D}}$ satisfies norm inequalities with bounds independent of μ: the proof of the next result can be found in [21].

Lemma 5.2. *Given $0 \le \beta < n$ and $1 < p \le \frac{n}{\beta}$, define $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. If μ is a measure such that $\mu(\mathbb{R}^n) = \infty$, then*

$$\|M_{\beta,\mu}^{\mathcal{D}}\|_{L^p(\mu) \to L^q(\mu)} \le \left(1 + \frac{p'}{q}\right)^{1 - \frac{\beta}{n}}.$$

The second lemma is a weighted Carleson embedding theorem. To state it we make two definitions. Given a sequence of positive numbers $\mathbf{c} = \{c_Q\}_{Q \in \mathcal{D}}$, we say that $\mathbf{c}$ is a Carleson sequence with respect to a measure μ if for each $Q_0 \in \mathcal{D}$,

$$\sum_{Q \subseteq Q_0} c_Q \le C \mu(Q_0). \tag{5.3}$$

The infimum of the constants in inequality (5.3) will be denoted $C(\mathbf{c})$. Also, given a sequence $\mathbf{a} = \{a_Q\}_{Q \in \mathcal{D}}$, define the sequential maximal operator

$$M\mathbf{a}(x) = \sup_{Q \in \mathcal{D}} |a_Q| \cdot \chi_Q(x).$$

The proof of the next result is also standard; for instance, see [13, Theorem 4.5].

Lemma 5.3. *Fix* $1 < p < \infty$ *and a dyadic grid* $\mathscr{D}$. *If* $\mathbf{a} = \{a_Q\}_{Q \in \mathscr{D}}$ *is any sequence and* $\mathbf{c} = \{c_Q\}_{Q \in \mathscr{D}}$ *is a Carleson sequence with respect to a measure* μ, *then*

$$\sum_{Q \in \mathscr{D}} |a_Q|^p \, c_Q \leq C(\mathbf{c}) \int_{\mathbb{R}^n} M\mathbf{a}(x)^p \, d\mu.$$

Finally, define the geometric maximal operator by

$$M_0^{\mathscr{D}} f(x) = \sup_{Q \in \mathscr{D}} \exp \left(\fint_Q \log |f| \, dx \right) \cdot \chi_Q(x).$$

By Jensen's inequality, for any $r > 0$, $M_0^{\mathscr{D}} f(x) \leq M^{\mathscr{D}}(|f|^r)(x)^{1/r}$, so the geometric maximal operator is bounded on L^p, $p > 0$. The sharp constant can be readily computed using this fact: see, for instance, [13, Lemma 2.1]. (For the history of this operator, see [6] and the references it contains.)

Lemma 5.4. *Given* $0 < p < \infty$,

$$\|M_0^{\mathscr{D}} f\|_{L^p(\mathbb{R}^n)} \leq e \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof of Theorem 2.1. Fix a non-negative function f. By Theorem 5.1 it will suffice to get a norm estimate for the linearization

$$L_\alpha^{\mathcal{S}}(f\sigma) = \sum_{Q \in \mathcal{S}} |Q|^{\frac{\alpha}{n}} \fint_Q f\sigma \, dx \cdot \chi_{E_Q},$$

with a constant that is independent of the sparse subset $\mathcal{S} \subset \mathscr{D}$. Since the sets $\{E_Q\}_{Q \in \mathcal{S}}$ are pairwise disjoint,

$$\int_{\mathbb{R}^n} L_\alpha^{\mathcal{S}}(f\sigma)^q u \, dx = \sum_{Q \in \mathcal{S}} \left(|Q|^{\frac{\alpha}{n}} \fint_Q f\sigma \, dx \right)^q u(E_Q)$$

$$\leq \sum_{Q \in \mathcal{S}} \left(|Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\fint_Q u \, dx \right)^{1/q} \left(\fint_Q \sigma \, dx \right)^{1/p'} \int_Q f\sigma \, dx \right)^q \sigma(Q)^{-q/p'}.$$

$$(5.4)$$

Now let $\beta = n\left(\frac{1}{p} - \frac{1}{q} \right)$; then

$$\sigma(Q)^{-q/p'} = \left(\frac{1}{\sigma(Q)^{1 - \frac{\beta}{n}}} \right)^q \sigma(Q).$$

Define the sequences $\mathbf{c} = \{c_Q\}$ and $\mathbf{a} = \{a_Q\}$ by

$$c_Q = \left(|Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\fint_Q u \, dx \right)^{1/q} \left(\fint_Q \sigma \, dx \right)^{1/p'} \right)^q \sigma(Q)$$

and

$$a_Q = \frac{1}{\sigma(Q)^{1-\frac{\beta}{n}}} \int_Q f\sigma \, dx$$

for $Q \in \mathcal{S}$ and zero otherwise.

Then we can rewrite (5.4) as

$$\|L_\alpha^{\mathcal{S}}(f\sigma)\|_{L^q(u)} \le \left(\sum_{Q \in \mathcal{S}} (a_Q)^q c_Q \right)^{1/q}. \tag{5.5}$$

To estimate the right-hand side of (5.5) we will show that **c** is a Carleson sequence with respect to the measure σ and its Carleson constant is bounded by either

$$C(\mathbf{c}) \le 2e[u,\sigma]^q_{A_{p,q}(u,\sigma)A_\infty^{\exp}(\sigma)^{\frac{1}{q}}} \tag{5.6}$$

or

$$C(\mathbf{c}) \le 2[u,\sigma]^q_{A_{p,q}}[\sigma]_{A_\infty^M}. \tag{5.7}$$

Given these estimate we are done: by Lemma 5.3 we have

$$\|L_\alpha^{\mathcal{S}}(f\sigma)\|_{L^q(u)} \le C(\mathbf{c})^{1/q}\|M\mathbf{a}\|_{L^q(\sigma)}.$$

By our choice of the sequence **a**, $M\mathbf{a}(x) = M_{\beta,\sigma}^{\mathscr{D}}f(x)$. Since $\sigma \in A_\infty$, $\sigma(\mathbb{R}^n) = \infty$, so by Lemma 5.2,

$$\|L_\alpha^{\mathcal{S}}(f\sigma)\|_{L^q(u)} \le C(\mathbf{c})^{1/q}\|M_{\beta,\sigma}^{\mathscr{D}}f\|_{L^q(\sigma)} \lesssim C(\mathbf{c})^{1/q}\|f\|_{L^p(\sigma)}.$$

To complete the proof we first prove (5.6). Let $Q_0 \in \mathscr{D}$; then by Lemma 5.4,

$$\sum_{Q \subseteq Q_0} c_Q = \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq Q_0}} \left(|Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\fint_Q u\, dx \right)^{1/q} \left(\fint_Q \sigma\, dx \right)^{1/p'} \right)^q \sigma(Q)$$

$$\le 2[u,\sigma]^q_{A_{p,q}^\alpha(u,\sigma)A_\infty^{\exp}(\sigma)^{\frac{1}{q}}} \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq Q_0}} \exp\left(\fint_Q \log\sigma \right) |E_Q|$$

$$\le 2[u,\sigma]^q_{A_{p,q}^\alpha(u,\sigma)A_\infty^{\exp}(\sigma)^{\frac{1}{q}}} \int_{Q_0} M_0(\chi_{Q_0}\sigma)\, dx$$

$$\le 2e[u,\sigma]^q_{A_{p,q}^\alpha(u,\sigma)A_\infty^{\exp}(\sigma)^{\frac{1}{q}}} \sigma(Q_0).$$

We prove (5.7) in a similar fashion: by the definition of the A_∞^M constant,

$$\sum_{Q \subseteq Q_0} c_Q = \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq Q_0}} \left(|Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\fint_Q u\, dx \right)^{1/q} \left(\fint_Q \sigma\, dx \right)^{1/p'} \right)^q \sigma(Q)$$

$$\leq 2[u, \sigma]_{A_{p,q}^\alpha}^q \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq Q_0}} \frac{\sigma(Q)}{|Q|} |E_Q|$$

$$\leq 2[u, \sigma]_{A_{p,q}^\alpha}^q \int_{Q_0} M(\chi_{Q_0} \sigma)\, dx$$

$$\leq 2[u, \sigma]_{A_{p,q}^\alpha}^q [\sigma]_{A_\infty^M} \sigma(Q_0). \qquad \qquad \square$$

Proof of Theorem 2.5

To apply Theorem 2.2 we first make a few preliminary remarks. Given any α and exponents p and q that satisfy the Sobolev relationship, define

$$s(p) = 1 + \frac{p}{q'} = q \left(1 - \frac{\alpha}{n} \right).$$

Given a weight w, if we let $u = w^q$ and $\sigma = w^{-p'}$, then it is immediate that

$$[u, \sigma]_{A_{p,q}^\alpha} = [w^q]_{A_{s(p)}}^{\frac{1}{q}}.$$

With this notation we can restate (2.4) as

$$\|I_\alpha\|_{L^p(w^p) \to L^{q,\infty}(w^q)} \lesssim [w^q]_{A_{s(p)}}^{\frac{1}{q}} [w^q]_{A_\infty}^{\frac{1}{p'}}. \qquad (5.8)$$

Now fix r and w as in the hypotheses. We will prove (2.7) using interpolation with change of measure. Since $r < s(p)$, there exist p_0 and q_0 such that $1/p_0 - 1/q_0 = \alpha/n$ and

$$r = s(p_0) = q_0 \left(1 - \frac{\alpha}{n} \right).$$

Note that $p_0 < p$ and $q_0 < q$. Define $w_0 = w^{q/q_0}$; then

$$[w_0^{q_0}]_{A_{s(p_0)}} = [w^q]_{A_r},$$

and so by (5.8),

$$\|I_\alpha\|_{L^{p_0}(w_0^{p_0}) \to L^{q_0,\infty}(w^q)} \lesssim [w^q]_{A_r}^{\frac{1}{q_0}} [w^q]_{A_\infty}^{\frac{1}{p_0'}}. \qquad (5.9)$$

Define $p_1 > p$ and $q_1 > q$ by

$$\frac{1/2}{p_0} + \frac{1/2}{p_1} = \frac{1}{p}, \qquad \frac{1/2}{q_0} + \frac{1/2}{q_1} = \frac{1}{q}.$$

Let $w_1 = w^{q/q_1}$. Since $s(p_1) > s(p) > r$, we have that

$$[w_1^{q_1}]_{A_{s(p_1)}} \leq [w^q]_{A_r}.$$

Therefore, if we repeat the above argument, we get that

$$\|I_\alpha\|_{L^{p_1}(w_1^{p_1})\to L^{q_1,\infty}(w^q)} \lesssim [w^q]_{A_r}^{\frac{1}{q_1}}[w^q]_{A_\infty}^{\frac{1}{p_1'}}. \tag{5.10}$$

Given inequalities (5.9) and (5.10), by interpolation with change of measure (Stein and Weiss [32]; also see Grafakos [10, Exercise 1.4.9] for a careful treatment of the constants) we get (2.7).

Proof of Theorem 2.6

Again, by (5.1) it suffices to work with the dyadic operator $M_\alpha^{\mathscr{D}}$. We begin with a testing condition from [19]. Given p and q satisfying the Sobolev relationship, define

$$[u,\sigma]_{M^{\mathscr{D}},s(p)} = \sup_{R\in\mathscr{D}} \frac{\left(\int_R M^{\mathscr{D}}(\sigma\chi_R)^{s(p)} u\,dx\right)^{1/q}}{\sigma(R)^{1/q}}$$

(recall $s(p) = 1 + \frac{q}{p'}$). It was shown in [19, Corollary 4.5] that

$$\|M_\alpha^{\mathscr{D}}(\,\cdot\,\sigma)\|_{L^p(\sigma)\to L^q(u)} \lesssim [u,\sigma]_{M^{\mathscr{D}},s(p)};$$

that is, the two weight norm inequality of the fractional maximal operator is bounded by the testing constant of the Hardy-Littlewood maximal operator.

Now fix $w^q \in A_{s(p)}$ and let $u = w^q$ and $\sigma = w^{-p'}$. We will estimate $[u,\sigma]_{M^{\mathscr{D}},s(p)}$. Fix $R \in \mathscr{D}$. By inequality (5.2) there exists a sparse family $\mathcal{S} \subset R$ such that

$$\int_R M^{\mathscr{D}}(\sigma\chi_Q)^{s(p)} u\,dx \simeq \sum_{Q\in\mathcal{S}} \left(\frac{\sigma(Q)}{|Q|}\right)^{s(p)} u(E_Q) \le \sum_{Q\in\mathcal{S}} \left(\frac{\sigma(Q)}{|Q|}\right)^{1+\frac{q}{p'}} \frac{u(Q)}{|Q|}|Q|.$$

For $a \in \mathbb{Z}$ define

$$\mathcal{S}^a = \left\{Q \in \mathcal{S} : 2^a < \left(\frac{u(Q)}{|Q|}\right)^{\frac{p'}{q}} \frac{\sigma(Q)}{|Q|} \le 2^{a+1}\right\}.$$

Since $w^q \in A_{s(p)}$, the sets $\mathcal{S}^a$ are empty if $a < -1$ or $a > \lfloor\log_2[w^{-p'}]_{A_{s(q')}}\rfloor := K$. (The fact that $u = w^q$ and $\sigma = w^{-p'}$ is essential at this step.) Hence,

$$\sum_{Q\in\mathcal{S}} \left(\frac{\sigma(Q)}{|Q|}\right)^{1+\frac{q}{p'}} \frac{u(Q)}{|Q|}|Q|$$
$$= \sum_{a=-1}^{K} \sum_{Q\in\mathcal{S}^a} \left(\frac{\sigma(Q)}{|Q|}\right)^{1+\frac{q}{p'}} \frac{u(Q)}{|Q|}|Q| \simeq \sum_{a=-1}^{K} 2^{a\frac{q}{p'}} \sum_{Q\in\mathcal{S}^a} \left(\frac{\sigma(Q)}{|Q|}\right) |E_Q|. \tag{5.11}$$

We now analyze the inner sum in (5.11). For each a, $-1 \le a \le K$, let $\mathcal{S}_{\max}^a$ be the collection of maximal cubes with respect to inclusion in $\mathcal{S}^a$. Then

the family $\mathcal{S}^a_{\max}$ is pairwise disjoint and every cube in $\mathcal{S}^a$ is contained in a cube from $\mathcal{S}^a_{\max}$. Thus,

$$\sum_{Q \in \mathcal{S}^a} \left(\frac{\sigma(Q)}{|Q|} \right) |E_Q| = \sum_{Q \in \mathcal{S}^a_{\max}} \sum_{\substack{P \in \mathcal{S}^a \\ P \subset Q}} \left(\frac{\sigma(P)}{|P|} \right) |E_P| \le \sum_{Q \in \mathcal{S}^a_{\max}} \int_Q M(\sigma \chi_Q) \, dx.$$

If we substitute this estimate into (5.11), then we have that

$$\sum_{a=-1}^{K} 2^{a\frac{q}{p'}} \sum_{Q \in \mathcal{S}^a} \left(\frac{\sigma(Q)}{|Q|} \right) |E_Q| \le \sum_{a=-1}^{K} 2^{a\frac{q}{p'}} \sum_{Q \in \mathcal{S}^a_{\max}} \int_Q M(\sigma \chi_Q) \, dx$$

$$\le \sum_{a=-1}^{K} \sum_{Q \in \mathcal{S}^a_{\max}} \left(\frac{\sigma(Q)}{|Q|} \right)^{\frac{q}{p'}} \frac{u(Q)}{|Q|} \int_Q M(\sigma \chi_Q) \, dx$$

$$\le [w^{-p'}]^q_{(A_{s(q')})^{\frac{1}{p'}} (A_\infty^M)^{\frac{1}{q}}} \sum_{a=-1}^{K} \sum_{Q \in \mathcal{S}^a_{\max}} \sigma(Q)$$

$$\le (2+K)[w^{-p'}]^q_{(A_{s(q')})^{\frac{1}{p'}} (A_\infty^M)^{\frac{1}{q}}} \sigma(R).$$

If we combine the above inequalities, we get the desired estimate.

6. Two Weight A_p Bump Conditions

In this section we construct Example 3.1 and prove Theorems 3.3 and 3.4.

Construction of Example 3.1

To construct the desired example, we need to consider two cases. In both cases we will work on the real line.

The simpler case is if $\frac{1}{p} - \frac{1}{q} > \frac{\alpha}{n}$. Note that in this case, by the Lebesgue differentiation theorem, if $(u, \sigma) \in A^\alpha_{p,q}$, then u and σ have disjoint supports. Let $f = \sigma = \chi_{[-2,-1]}$ and let $u = x^t \chi_{[0,\infty)}$, where $t = q(1-\alpha) - 1$. Given any $Q = (a, b)$, $A^\alpha_{p,q}(u, \sigma, Q) = 0$ unless $a < -1$ and $b > 0$. In this case we have that

$$A^\alpha_{p,q}(u, \sigma, Q) \le b^{\alpha + \frac{1}{q} - \frac{1}{p}} \left(\frac{1}{b} \int_0^b x^t \, dx \right)^{\frac{1}{q}} \left(\frac{1}{b} \int_{-2}^{-1} dx \right)^{\frac{1}{p'}}$$

$$\lesssim b^{\alpha + \frac{t+1}{q} - 1}$$

$$= 1.$$

Hence, $(u, \sigma) \in A^\alpha_{p,q}$. On the other hand, for all $x > 1$,

$$M_\alpha(f\sigma)(x) \approx x^{\alpha - 1},$$

and so

$$\int_{\mathbb{R}} M_\alpha(f\sigma)(x)^q u(x) \, dx \ge \int_1^\infty x^{q(\alpha-1)} x^{q(1-\alpha)-1} \, dx = \int_1^\infty \frac{dx}{x} = \infty.$$

Now suppose $\frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n}$. We begin with a general lemma that lets us construct pairs in $A_{p,q}^\alpha$; this is an extension of the technique of factored weights developed in [3, Chapter 6].

Lemma 6.1. *Given $0 < \alpha < n$, suppose $1 < p \leq q < \infty$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n}$. Let w_1, w_2 be locally integrable functions, and define*

$$u = w_1 \big(M_\gamma w_2 \big)^{-\frac{q}{p'}}, \qquad \sigma = w_2 \big(M_\gamma w_1 \big)^{-\frac{p'}{q}},$$

where

$$\gamma = \frac{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}}{\frac{1}{n} \left(1 + \frac{1}{q} - \frac{1}{p} \right)}.$$

Then $(u, \sigma) \in A_{p,q}^\alpha$ and $[u, \sigma]_{A_{p,q}^\alpha} \leq 1$.

Proof. By our assumptions on p, q and α, $0 \leq \gamma \leq \alpha$. Fix a cube Q. Then

$$A_{p,q}^\alpha(u, \sigma, Q) = |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\fint_Q w_1 \big(M_\gamma w_2 \big)^{-\frac{q}{p'}} dx \right)^{\frac{1}{q}} \left(\fint w_2 \big(M_\gamma w_1 \big)^{-\frac{p'}{q}} dx \right)^{\frac{1}{p'}}$$

$$\leq |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\fint w_1 \, dx \right)^{\frac{1}{q'}} \left(|Q|^{\frac{\gamma}{n}} \left(\fint_Q w_2 \, dx \right) \right)^{-\frac{1}{p'}}$$

$$\times \left(\fint w_2 \, dx \right)^{\frac{1}{p'}} \left(|Q|^{\frac{\gamma}{n}} \left(\fint_Q w_1 \, dx \right) \right)^{-\frac{1}{q}}$$

$$= |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p} - \frac{\gamma}{n} \left(1 + \frac{1}{q} - \frac{1}{p} \right)}$$

$$= 1. \qquad \qquad \qquad \qquad \square$$

With $n = 1$, fix γ as in Lemma 6.1. The second step is to construct a set $E \subset [0, \infty)$ such that $M_\gamma(\chi_E)(x) \approx 1$ for $x > 0$. Let

$$E = \bigcup_{j \geq 0} [j, j + (j + 1)^{-\gamma}).$$

Suppose $x \in [k, k+1)$; if $k = 0$, then it is immediate that if we take $Q = [0, 2]$, then $M_\gamma(\chi_E) \geq 3 \cdot 2^{\gamma-2} \approx 1$. If $k \geq 1$, let $Q = [0, x]$; then

$$M_\gamma(\chi_E)(x) \geq x^{\gamma-1} \sum_{0 \leq j \leq \lfloor x \rfloor} (j + 1)^{-\gamma} \geq (k + 1)^{\gamma-1} \sum_{j=0}^{k} (j + 1)^{-\gamma}$$

$$\simeq (k + 1)^{\gamma-1} (k + 1)^{1-\gamma} = 1.$$

It remains to prove the reverse inequality. If $|Q| \leq 1$, then

$$|Q|^{\gamma-1} |Q \cap E| \leq |Q|^\gamma \leq 1,$$

so we only have to consider Q such that $|Q| \geq 1$. In this case, given a Q, let Q' be the smallest interval whose endpoints are integers that contains Q. Then $|Q'| \leq |Q| + 2 \leq 3|Q|$, and so $|Q|^{\gamma-1} |E \cap Q| \approx |Q'|^{\gamma-1} |E \cap Q'|$. Therefore,

without loss of generality, it suffices to consider $Q = [a, a + h + 1]$, a, h non-negative integers. Then

$$|Q|^{\gamma-1}|Q \cap E| = (1+h)^{\gamma-1} \sum_{a \leq j \leq a+h} (j+1)^{-\gamma} \approx (1+h)^{\gamma-1} \int_a^{a+h} (t+1)^{-\gamma} \, dt$$

$$\approx (1+h)^{\gamma-1}\big((a+h+1)^{1-\gamma} - (a+1)^{1-\gamma}\big).$$

To estimate the last term suppose first that $h \leq a$. Then by the mean value theorem the last term is dominated by

$$(1+h)^{\gamma-1}(1+h)(a+1)^{-\gamma} \leq 1.$$

On the other hand, if $h > a$, then the last term is dominated by

$$(1+h)^{\gamma-1}(a+h+1)^{1-\gamma} \leq 2^{1-\gamma} \approx 1.$$

We can now give our desired counter example. Let $w_1 = \chi_E$ and let $w_2 = \chi_{[0,1]}$. Then for all $x \geq 2$,

$$M_\gamma w_1(x) \approx 1, \qquad M_\gamma w_2(x) = \sup_Q |Q|^{\gamma-1} \int_Q w_2 \, dt \approx x^{\gamma-1}.$$

Then by Lemma 6.1, if we set

$$u = w_1(M_\beta w_2)^{-\frac{q}{p'}}, \qquad \sigma = w_2(M_\beta w_1)^{-\frac{p'}{q}},$$

then $(u, \sigma) \in A_{p,q}^\alpha$. Moreover, for $x \geq 2$, we have that

$$u(x) \approx x^{(1-\gamma)\frac{q}{p'}}\chi_E, \qquad \sigma(x) \approx \chi_{[0,1]}.$$

Fix $f \in L^p(\sigma)$: without loss of generality, we may assume $\operatorname{supp}(f) \subset [0, 1]$. Then $f\sigma$ is locally integrable, and for $x \geq 2$ we have that

$$M_\alpha(f\sigma)(x) \geq x^{\alpha-1}\|f\sigma\|_1 \approx x^{\alpha-1}.$$

Therefore, for $x \geq 2$,

$$M_\alpha(f\sigma)(x)^q u(x) \gtrsim x^{(\alpha-1)q}x^{(1-\gamma)\frac{q}{p'}}\chi_E.$$

By the definition of γ,

$$\gamma\left(\frac{1}{q} + \frac{1}{p'}\right) = \gamma\left(1 + \frac{1}{q} - \frac{1}{p}\right) = \alpha + \frac{1}{q} - \frac{1}{p} = \alpha - 1 + \frac{1}{q} + \frac{1}{p'};$$

equivalently,

$$(\gamma - 1)\left(\frac{q}{p'} + 1\right) = q(\alpha - 1),$$

and so

$$(\alpha - 1)q + (1 - \gamma)\frac{q}{p'} = \gamma - 1,$$

Therefore, to show that $M_\alpha(f\sigma) \notin L^q(u)$, it will be enough to prove that

$$\int_2^\infty x^{\gamma-1}\chi_E(x) \, dx = \infty.$$

This is straight-forward:

$$
\int_2^\infty x^{\gamma-1}\chi_E(x)\,dx = \sum_{j=2}^\infty \int_j^{j+(j+1)^{-\gamma}} x^{\gamma-1}\,dx
$$

$$
\geq \sum_{j=2}^\infty (j+(j+1)^{-\gamma})^{\gamma-1}(j+1)^{-\gamma}
$$

$$
\geq \sum_{j=2}^\infty (j+1)^{\gamma-1}(j+1)^{-\gamma}
$$

$$
\geq \sum_{j=2}^\infty (j+1)^{-1} = \infty.
$$

Proof of Theorems 3.3 and 3.4

Proof of Theorem 3.3. Our proof is adapted from the argument for the case $\alpha = 0$ given in [27]. Fix a non-negative function f; by a standard approximation argument we may assume that the support of f is contained in some cube Q. Let $\Phi_p(t) = \Phi(t^{1/p})$; then be a rescaling argument (see [3, p. 98]), $\|f^p\|_{Q,\Phi_p} = \|f\|^p_{Q,\Phi}$, and so

$$
M_{\alpha,\Phi}f(x)^q = M_{p\alpha,\Phi_p}(f^p)(x)^{\frac{q}{p}}.
$$

By [3, Lemma 5.49], we have that

$$
|\{x \in Q : M_{p\alpha,\Phi_p}(f^p)(x) > t\}|^{\frac{n-p\alpha}{n}} \leq C \int_{\{x\in Q:f(x)>t/c\}} \Phi_p\left(\frac{f(x)^p}{t}\right)\,dx.
$$

By the Sobolev relationship, $\frac{n-p\alpha}{n} = \frac{p}{q}$. Therefore, we have that

$$
\left(\int_Q M_{\alpha,\Phi}f(y)^q\,dy\right)^{\frac{1}{q}} = \left(\int_Q M_{p\alpha,\Phi_p}(f^p)(y)^{\frac{q}{p}}\,dy\right)^{\frac{1}{q}}
$$

$$
\simeq \left(\int_0^\infty t^{\frac{q}{p}}|\{x \in Q : M_{p\alpha,\Phi_p}(f^p)(x) > t\}|\frac{dt}{t}\right)^{\frac{1}{q}}
$$

$$
\lesssim \left(\int_0^\infty t^{\frac{q}{p}}\left(\int_{\{x\in Q:f(x)^p>t/c\}} \Phi_p\left(\frac{f(x)^p}{t}\right)\,dx\right)^{\frac{q}{p}}\frac{dt}{t}\right)^{\frac{1}{q}};
$$

by Minkowski's inequality and the change of variables $t \mapsto (f(x)/s)^p$,

$$\lesssim \left(\int_Q \left(\int_0^{cf(x)^p} \Phi \left(\frac{f(x)}{t^{\frac{1}{p}}} \right)^{\frac{q}{p}} t^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

$$= \left(\int_Q \left(\int_c^\infty \Phi(s)^{\frac{q}{p}} \left(\frac{f(x)}{s} \right)^q \frac{ds}{s} \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

$$= \left(\int_c^\infty \frac{\Phi(s)^{\frac{q}{p}}}{s^q} \frac{ds}{s} \right)^{\frac{1}{q}} \left(\int_Q f(x)^p \, dx \right)^{\frac{1}{p}}. \qquad \square$$

Proof of Theorem 3.4. Arguing as we did for inequality (5.1) it suffices to work with dyadic operators. Fix a sparse family $\mathcal{S}$; we will estimate

$$\int_{\mathbb{R}^n} L_\alpha^{\mathcal{S}}(f\sigma)^q u \, dx = \sum_{Q \in \mathcal{S}} \left(|Q|^{\frac{\alpha}{n}} \fint_Q f\sigma \right)^q u(E_Q).$$

Let $\beta = n(\frac{1}{p} - \frac{1}{q})$. Then

$$\sum_{Q \in \mathcal{S}} \left(|Q|^{\frac{\alpha}{n}} \fint_Q f\sigma \right)^q u(E_Q) \leq \sum_{Q \in \mathcal{S}} \left(|Q|^{\frac{\alpha}{n}+\frac{1}{q}} \left(\fint_Q u \, dx \right)^{1/q} \fint_Q f\sigma \, dx \right)^q$$

$$\leq \sum_{Q \in \mathcal{S}} \left(|Q|^{\frac{\alpha}{n}+\frac{1}{q}} \left(\fint_Q u \, dx \right)^{1/q} \|\sigma^{\frac{1}{p'}}\|_{\bar{\Phi},Q} \|f\sigma^{\frac{1}{p}}\|_{\bar{\Phi},Q} \right)^q$$

$$\leq [u,\sigma]_{A_{p,q,\Phi}^\alpha}^q \sum_{Q \in \mathcal{S}} \|f\sigma^{\frac{1}{p}}\|_{\bar{\Phi},Q}^q |Q|^{\frac{q}{p}}$$

$$\lesssim [u,\sigma]_{A_{p,q,\Phi}^\alpha}^q \sum_{Q \in \mathcal{S}} (|Q|^{\frac{\beta}{n}} \|f\sigma^{\frac{1}{p}}\|_{\bar{\Phi},Q})^q |E_Q|$$

$$\leq [u,\sigma]_{A_{p,q,\Phi}^\alpha}^q \int_{\mathbb{R}^n} (M_{\beta,\bar{\Phi}}^{\mathscr{D}} f)^q \, dx$$

$$\leq [u,\sigma]_{A_{p,q,\Phi}^\alpha}^q \|M_{\beta,\bar{\Phi}}\|_{L^p \to L^q}^q \|f\|_{L^p(\sigma)}^q. \qquad \square$$

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