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Integral Equations and Operator Theory

On Some Integral Operators Related to the Poisson Equation

David Kalaj

Abstract. In this paper we estimate various norms of some integral operators related to the Poisson equation defined in a bounded domain in the complex plane with vanishing boundary data.

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1. Introduction and Notation

Throughout the paper Ω is a bounded domain in the complex plane $\mathbf{C}$, whose boundary $\partial\Omega$ is assumed to be of Lipschitz type and $\mathbf{U}$ is the unit disk. By

$$dA(z) = dxdy \quad (z = x + iy),$$

is denoted the Lebesgue area measure in Ω. The main subject of this paper is to discuss a weak solution of the Dirichlet problem

$$\begin{cases} \Delta u = g(z), & z \in \Omega \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{1.1}$$

where $p \geqslant 1, \Delta u$ is the Laplacian and $W_0^{1,p}(\Omega)$ is the space of functions $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$. This is a Poisson's equation. A weakly differentiable function u defined in a domain Ω with $u|_{\partial\Omega} = 0$ and $u \in C(\overline{\Omega})$ is a weak solution of Poisson's equation if the partial derivatives $\partial_x u$ aned $\partial_y u$ are locally integrable in Ω and

$$\int_\Omega [\partial_x u(z)\partial_x v(z) + \partial_y u(z)\partial_y v(z) + g(z)v(z)]\, dA(z) = 0,$$

for all $v \in C_0^1(\Omega)$ (see e.g. [6]).

We recall some basic facts of potential theory in the plane which can be found in [9]. It is well known that for $g \in L^p(\Omega), p \geqslant 1$, the weak solution u of Poisson's equation is given explicitly as the sum of the Newtonian potential

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$$N[g](z) = \frac{1}{2\pi}\int_{\Omega} \log|z-w|g(w)dA(w),$$

and a harmonic function h such that $h|_{\partial\Omega} + N(g)|_{\partial\Omega} \equiv u|_{\partial\Omega}$. A domain Ω has Green's function G_Ω whenever $\mathbf{C}\setminus\Omega$ contains a nondegenerate continuum. We normalize the Green function by $G_\Omega(z,\zeta) = -\log|z-\zeta| + O(1)$ as $z\to\zeta$. In particular, $G_\Omega(z,\zeta) > 0$. If $G_\Omega(z,w)$ is the Green function of the domain Ω, then

$$u(z) = \mathcal{P}_\Omega[g](z) := -\int_{\Omega} G_\Omega(z,w)g(w)dA(w) \tag{1.2}$$

is the explicit solution of (1.1). Here $g\in L^p(\Omega), p\geqslant 1$. In particular if $\Omega = \mathbf{U}$, then the function

$$u(z) = \frac{1}{2\pi}\int_{\mathbf{U}} \log\frac{|z-w|}{|1-\overline{z}w|}g(w)dA(w)$$

is the explicit solution of (1.1) or more generally, if Ω is a simply connected domain and if ψ is a conformal mapping between Ω and the unit disk $\mathbf{U}$ then the solution is given by

$$u(z) = \frac{1}{2\pi}\int_{\Omega} \log\frac{|\psi(z)-\psi(w)|}{|1-\overline{\psi(z)}\psi(w)|}g(w)dA(w). \tag{1.3}$$

For $g\in L^p(\Omega), p\geqslant 1$, *the Cauchy transform and conjugate Cauchy transform for Dirichlet's problem* (see [3, p. 155]) of g are defined by

$$\mathcal{C}_\Omega[g](z) = \partial u(z)$$

and

$$\bar{\mathcal{C}}_\Omega[g](z) = \bar{\partial} u(z).$$

Here we use the notation

$$\partial := \frac{1}{2}\left(\partial_x + \frac{1}{i}\partial_y\right) \text{ and } \bar{\partial} := \frac{1}{2}\left(\partial_x - \frac{1}{i}\partial_y\right).$$

Recall that the norm of an operator $T : X\to Y$ between normed spaces X and Y is defined by

$$\|T\|_{X\to Y} = \sup\{\|Tx\| : \|x\| = 1\}.$$

The space $L^p(\Omega), p\geqslant 1$ is the standard normed Lebesgue space with the norm

$$\|f\|_p := \left(\int_{\Omega} |f(z)|^p dA(z)\right)^{1/p}.$$

For $p = 2$ it is a Hilbert space.

It is well-known that for $p\geqslant 1$, Cauchy transforms

$$\mathcal{C}_\Omega\colon L^p(\Omega)\to L^p(\Omega) \text{ and } \bar{\mathcal{C}}_\Omega\colon L^p(\Omega)\to L^p(\Omega)$$

are bounded operators (the last fact can be deduces from e.g. [6, Lemma 7.12]).

If $u = u_1 + iu_2 : \Omega \to \mathbf{C}$ is a complex valued function, then the Jacobian matrix of a mapping is defined by

$$Du(z) = \begin{pmatrix} \partial_x u_1(z) & \partial_y u_1(z) \\ \partial_x u_2(z) & \partial_y u_2(z) \end{pmatrix}, \qquad z = x + iy.$$

If u is a solution of (1.1), then the from (1.2), the matrix Du satisfies

$$Du(z)h = \int_\Omega (\nabla_z G_\Omega(z, w) \bullet h) \; g(\omega) \, dA(\omega), \quad h \in \mathbf{R}^2 = \mathbf{C}. \tag{1.4}$$

Here $g = (g_1 + ig_2) = (g_1, g_2)$ is a (possibly) complex valued function and $\bullet$ denotes the scalar or inner product. Moreover ∇_z is the gradient with respect to z. Equation (1.4) defines *the differential operator of Dirichlet's problem*

$$\mathcal{D}_\Omega : L^p(\Omega, \mathbf{C}) \to L^p(\Omega, \mathcal{M}_{2,2}), \; \mathcal{D}_\Omega[g] = Du.$$

Here $\mathcal{M}_{2,2}$ is the space of square 2×2 matrices A by the induced norm: $|A| = \max\{|Ah| : |h| = 1\}$.

With respect to the induced norm there holds

$$|Du(z)| = |\partial u(z)| + |\bar\partial u(z)|, \tag{1.5}$$

and this implies that

$$|\mathcal{D}_\Omega[g](z)| = |\mathcal{C}_\Omega[g](z)| + |\bar{\mathcal{C}}_\Omega[g](z)|. \tag{1.6}$$

The formula (1.5) is well-known, but for the completeness we include its proof here. Namely for $h = e^{it}$ we have

$$|Du(z)h| = |\partial u(z)h + \bar\partial u(z)\overline{h}| \leqslant |\partial u(z)| + |\bar\partial u(z)|.$$

On the other hand by choosing $h_\circ = e^{it_\circ}$ such that

$$2t_\circ = \arg\left[\bar\partial u(z)/\partial u(z)\right],$$

provided $\bar\partial u(z) \neq 0$ and $\partial u(z) \neq 0$, we have

$$|Du(z)h_\circ| = |\partial u(z)| + |\bar\partial u(z)|.$$

This and the previous inequality imply (1.5).

Observe that for $g \equiv 1$ and $\Omega = \mathbf{U}$, the solution of (1.1) is $u(z) = \frac{|z|^2 - 1}{4}$ and therefore

$$\|u\|_\infty = \max_{z \in \mathbf{U}} |u(z)| = |u(0)| = \frac{1}{4} = \frac{1}{4}\|g\|_\infty. \tag{1.7}$$

The previous special situation is a motivation for our study. We will study certain norms of operators $\mathcal{P}_\Omega, \mathcal{C}_\Omega, \bar{\mathcal{C}}_\Omega$ and $\mathcal{D}_\Omega$, where Ω is a bounded set of the complex plane.

In what follows we include some background. Suppose that $g \in L^p(\Omega)$, and that $g = 0$ outside Ω. The Cauchy transform $\mathfrak{C}[g]$ of g, is defined by

$$\mathfrak{C}[g](z) = -4\partial_z N[g](z) = \frac{1}{\pi}\int_{\mathbf{\Omega}} \frac{g(z)}{w - z} dA(w).$$

We want to point out the following result of Anderson and Hinkkanen [1]. If $\Omega = \mathbf{U}$, the Cauchy transform $\mathfrak{C}$ restricted to $\mathbf{U}$, satisfies

$$\|\mathfrak{C}\|_{L^2\to L^2} = \frac{2}{\alpha}, \tag{1.8}$$

where $\alpha \approx 2.4048$ is the smallest positive zero of the Bessel function J_0 :

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!^2} \left(\frac{x}{2}\right)^{2k}.$$

In Anderson et al. [2] obtained some non-sharp estimates of the norm of the Cauchy transform $\mathfrak{C}$ in some domain that is not a disk. Later it was proved by Dostanić [5] that the norm of $\mathfrak{C}$ on $L^2(\Omega)$ (where Ω is a bounded domain in $\mathbf{C}$ with piecewise C^1 boundary) is equal to $2/\sqrt{\lambda_1}$, where λ_1 is the smallest eigenvalue of the Dirichlet–Laplacian

$$\begin{cases} -\Delta u = \lambda u, \ z \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}. \tag{1.9}$$

We refer to the additional paper of Dostanić [4] for some L^p estimates for the operator $\mathfrak{C}$. In [7] the author studied the $L^p \to L^p$ and $L^q \to L^\infty$ norms of Cauchy transform with respect to Dirichlet's problem on the unit disk $\mathbf{U}$ and there were obtained some sharp results for $p = 1, 2, \infty$ and $q > 2$.

Together with this introduction, the paper contains two more sections. In Sect. 2 we establish some inequalities concerning the norm $\|\mathcal{P}_\Omega\|_{L^p\to L^\infty}$, where $p \geqslant 1$ and Ω is a domain in the complex plane. The proofs make use of Möbius transformations, subordination principle and Jensen formula. The results are sharp when Ω is a disk in the complex plane. In Sect. 3 we determine the Hilbert norms $\|\mathcal{P}_\Omega\|_{L^2\to L^2}$, $\|\mathcal{C}_\Omega\|_{L^2\to L^2}$ and $\|\mathcal{D}_\Omega\|_{L^2\to L^2}$, provided that Ω is a domain in the complex plane having a piecewise smooth boundary. The proof of the results in Sect. 3 make use of the eigenfunction expansion of a square integrable function f.

2. L^∞ Norm of Solution

We begin with the following lemmas needed in the sequel.

Lemma 2.1. *For $q \geqslant 1$ the function*

$$I_q(z) = \int_{\mathbf{U}} \left|\log \frac{|z-w|}{|1-\bar{z}w|}\right|^q \frac{dA(w)}{2\pi}$$

is equal to

$$I_q(z) = 2^{-1-q}\Gamma(1+q)(1/|z| - |z|)^2 \mathrm{Li}_{q-1}(|z|^2), \tag{2.1}$$

where

$$\mathrm{Li}_s(w) = \sum_{k=1}^{\infty} \frac{w^k}{k^s}$$

is the polylogarithm function.

Proof. For a fixed z, we introduce the change of variables

$$\frac{z-w}{1-\bar{z}w}=a,$$

or, what is the same,

$$w=\frac{z-a}{1-\bar{z}a}.$$

Then

$$dA(w)=\frac{(1-|z|^2)^2}{|1-\bar{z}a|^4}\,dA(a)$$

and

$$I_q(z)=\int_{\mathbf{U}}|\log|a||^q\,\frac{(1-|z|^2)^2}{|1-\bar{z}a|^4}\frac{dA(a)}{2\pi}.$$

Since

$$\frac{1}{|1-\bar{z}a|^4}=\left|\sum_{n=1}^{\infty}na^{n-1}\bar{z}^{n-1}\right|^2,$$

by using polar coordinates $a=re^{it}$ and using Parseval's formula we have

$$\begin{aligned}
I_q(z)&=(1-|z|^2)^2\int_0^1\int_0^{2\pi}(\log 1/r)^q\sum_{n=1}^{\infty}n^2r^{2n-1}|z|^{2n-2}\frac{drdt}{2\pi}\\
&=(1-|z|^2)^2\sum_{n=1}^{\infty}n^2|z|^{2n-2}\int_0^1(\log 1/r)^q\,r^{2n-1}dr\\
&=(1-|z|^2)^2\sum_{n=1}^{\infty}n^2|z|^{2n-2}2^{-1-q}n^{-1-q}\Gamma(1+q)\\
&=\Gamma(1+q)2^{-1-q}(1-|z|^2)^2\sum_{n=1}^{\infty}n^{1-q}|z|^{2n-2}.
\end{aligned}$$

The last expression can be written as (2.1). □

Lemma 2.2. *For $q>1$, the function*

$$f_q(m)=\frac{(1-m)^2}{m}\mathrm{Li}_{q-1}(m),\quad f_q(0)=1$$

is decreasing in $m\in[0,1]$.

Proof. First of all

$$f_q'(m)=\frac{(-1+m)((-1+m)\mathrm{Li}_{q-2}(m)+(1+m)\mathrm{Li}_{q-1}(m))}{m^2}.$$

Further by calculating the Taylor coefficients w.r.t m we have

$$\begin{aligned}
&(-1+m)\mathrm{Li}_{q-2}(m)+(1+m)\mathrm{Li}_{q-1}(m))\\
&\qquad=\sum_{k=1}^{\infty}[2k(2k-1)]^{1-q}\left[(2k)^q-(2k-1)^q\right]m^{k+1}.
\end{aligned}$$

It follows for $0 \leqslant m \leqslant 1$ the inequality $f_q'(m) \leqslant 0$. This implies that f_q is decreasing as desired. □

It follows from Lemmas 2.1 and 2.2 that

Corollary 2.3. *For $q \geqslant 1$ we have*

$$\max_{|z|\leqslant 1} I_q(z) = I_q(0) = \frac{\Gamma(1+q)}{2^{1+q}}.$$

Theorem 2.4. *If u is a solution of equation (1.1) with $\Omega = \mathbf{U}$ is the unit disk, then for $g \in L^p(\mathbf{U})$ we have the following sharp inequality*

$$\|u\|_\infty \leqslant \frac{\Gamma(1+q)^{1/q}}{2^{1/q+1}}\|g\|_p, \tag{2.2}$$

where q is the conjugate of p: $1/p + 1/q = 1$. In other words

$$\|\mathcal{P}_{\mathbf{U}}\|_{L^p\to L^\infty} = \frac{\Gamma(1+q)^{1/q}}{2^{1/q+1}}.$$

Proof. From (1) we have

$$|u(z)| \leqslant \left(\int_{\mathbf{U}} \left|\log\frac{|z-w|}{|1-\overline{z}w|}\right|^q \frac{dA(w)}{2\pi}\right)^{1/q} \left(\int_{\mathbf{U}} |g(w)|^p \frac{dA(w)}{2\pi}\right)^{1/p}. \tag{2.3}$$

The inequality (2.2) follows from Corollary 2.3 and (2.3). The sharpness of the result follows by taking $g(w) = -|\log|1/w||^{q/p}$. The explicit solution is the following positive function

$$u(z) = -\frac{1}{2\pi}\int_{\mathbf{U}} \log\frac{|z-w|}{|1-\overline{z}w|}|\log|1/w||^{q/p} dA(w)$$

and its maximum (the norm $\|u\|_\infty$) is

$$u(0) = \frac{\Gamma(1+q)}{2^{1+q}} = \frac{\Gamma(1+q)^{1/q}}{2^{1/q+1}}\|g\|_p.$$

□

Theorem 2.5. *If $\Omega \subset \mathbf{U}$ is a complex domain then*

$$\|\mathcal{P}_\Omega\|_{L^p\to L^\infty} \leqslant \frac{\Gamma(1+q)^{1/q}}{2^{1/q+1}}. \tag{2.4}$$

The equality is attained if and only if Ω is the unit disk.

Proof. By (1.2) and Hölder inequality we obtain

$$|u(z)| \leqslant \left(\int_{\Omega} |G_\Omega(z,w)|^q \frac{dA(w)}{2\pi}\right)^{1/q} \left(\int_{\Omega} |g(w)|^p \frac{dA(w)}{2\pi}\right)^{1/p}.$$

By subordination principle (see e.g. [9]) for every analytic function $f\colon \Omega \to \mathbf{U}$ and $z, w \in \Omega$ we have

$$G_\Omega(z,w) \leqslant G_{\mathbf{U}}(f(z), f(w)). \tag{2.5}$$

Thus for $z, w \in \Omega$

$$G_\Omega(z,w) \leqslant \log\left|\frac{1-z\overline{w}}{z-w}\right|.$$

This implies that

$$\int_\Omega |G_\Omega(z,w)|^q \frac{dA(w)}{2\pi} \leqslant \int_\Omega \left|\log\frac{|z-w|}{|1-\overline{z}w|}\right|^q \frac{dA(w)}{2\pi}.$$

Since $\Omega \subset \mathbf{U}$ we have

$$\int_\Omega \left|\log\frac{|z-w|}{|1-\overline{z}w|}\right|^q \frac{dA(w)}{2\pi} \leqslant \int_{\mathbf{U}} \left|\log\frac{|z-w|}{|1-\overline{z}w|}\right|^q \frac{dA(w)}{2\pi}. \tag{2.6}$$

This together with (2.3) and Corollary 2.3 yields (2.4). Since (2.5) (or (2.6)) is an equality if only if Ω is the unit disk, the last assertion of the theorem follows at once. □

Corollary 2.6. *If* Ω *is a bounded complex domain with diameter* $\operatorname{diam}(\Omega)$, *then*

$$\|\mathcal{P}_\Omega\|_{L^p\to L^\infty} \leqslant \left(\frac{\operatorname{diam}(\Omega)}{2}\right)^{2-2/p} \frac{\Gamma(1+q)^{1/q}}{2^{1/q+1}}. \tag{2.7}$$

The equality is attained if and only if Ω *is a disk. In other words for every solution* u *to* (1.1) *we have*

$$|u(z)| \leqslant \left(\frac{\operatorname{diam}(\Omega)}{2}\right)^{2-2/p} \frac{\Gamma(1+q)^{1/q}}{2^{1/q+1}} \|g\|_p. \tag{2.8}$$

Proof. The proof follows by making use of the change $z = \frac{\operatorname{diam}(\Omega)}{2} w + b$, to the Poisson equation $\Delta u(z) = g(z)$, and applying Theorem 2.5. Here b is the midpoint of the diameter of Ω. Since the proof is straightforward, the details are omitted. □

2.1. The Refinement of the Case $q = 1$

In the following theorem we obtain a partial refinement of the local estimate (2.8) for the case $q = 1$ provided that $\Omega \subset \mathbf{U}$ is a Jordan domain satisfying certain properties. We will assume that the conformal mapping $\psi\colon \Omega \to \mathbf{U}$ has a conformal extension up to $\mathbf{U}$. In addition we assume that $0 \in \Omega$ is the center of the outscribed circle of $\Omega, \operatorname{diam}(\Omega) = 2$ and $\psi(0) = 0$ (see Remark 2.8 for a nontrivial example of Ω).

Theorem 2.7. *If* $g \in L^\infty(\Omega)$ *and* u *is a solution to* (1.1) *then*

$$|u(z)| \leqslant \min\left\{\frac{1}{4}, \left|\frac{1}{4} - \frac{1}{2}\log\left|\frac{\psi(z)}{z}\right|\right|\right\} \|g\|_\infty, \quad z \in \Omega\setminus\{0\}, \tag{2.9}$$

and

$$|u(0)| \leqslant \min\left\{\frac{1}{4}, \left|\frac{1}{4} - \frac{1}{2}\log|\psi'(0)|\right|\right\} \|g\|_\infty.$$

Proof. Denote by ψ a conformal extension of ψ in $\mathbf{U}$. From (1.3) we have

$$|u(z)| \leqslant \frac{1}{2\pi}\int_{\mathbf{U}}\left|\log\frac{|\psi(z)-\psi(w)|}{|1-\overline{\psi(z)}\psi(w)|}\right| dA(w)\|g\|_\infty.$$

Let

$$\Phi(w) = \frac{\psi(z)-\psi(w)}{1-\overline{\psi(z)}\psi(w)}.$$

Then Φ is an analytic function in the unit disk $\mathbf{U}$. Since z is the only zero of $\Phi(w)$ in the unit disk, by using the Jensen formula we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log|\Phi(re^{it})|dt &= \log|\Phi(0)| + \log\frac{r}{|z|}\\ &= \log\left|\frac{\psi(z)-\psi(0)}{1-\overline{\psi(z)}\psi(0)}\right| + \log\frac{r}{|z|}.\end{aligned}$$

By integrating over $[0,1]$ we have

$$\frac{1}{2\pi}\int_0^1 r\int_0^{2\pi}\log|\Phi(re^{it})|dtdr = \frac{1}{2}\log\left|\frac{\psi(z)-\psi(0)}{1-\overline{\psi(z)}\psi(0)}\right| + \frac{1}{4}\left(-1+2\log\frac{1}{|z|}\right).$$

Thus

$$|u(z)| \leqslant \left|\frac{1}{2}\log\left|\frac{\psi(z)-\psi(0)}{1-\overline{\psi(z)}\psi(0)}\right| + \frac{1}{4}\left(-1+2\log\frac{1}{|z|}\right)\right| \|g\|_\infty.$$

But $\psi(0) = 0$, and therefore

$$|u(z)| \leqslant \left|\frac{1}{4} - \frac{1}{2}\log\left|\frac{\psi(z)}{z}\right|\right| \|g\|_\infty, \quad z \neq 0.$$

On the other hand by Corollary 2.6 we have $|u(z)| \leqslant \frac{1}{4}\|g\|_\infty, \quad z \neq 0$. This concludes the proof. □

Remark 2.8. *By the conditions of Theorem 2.7, there exists a point z_0 on the boundary of Ω such that $|z_0| = 1$. Since $|\psi(z_0)| = 1$, it follows that*

$$\left|\frac{1}{4} - \frac{1}{2}\log\left|\frac{\psi(z_0)}{z_0}\right|\right| = \frac{1}{4}. \tag{2.10}$$

Thus the estimate (2.9) does not provide a better estimate of norm of $\mathcal{P}_\Omega$ than the estimate (2.7). But (2.10) and the Schwarz lemma ($|z| < |\psi(z)|$) implies that near the point z_0 in Ω, the inequality (2.9) is better that the inequality (2.8) for the case $q = 1$. The question arises for which Jordan domains $\Omega \subset \mathbf{U}$ satisfying the conditions of Theorem 2.9 we have

$$\left|\frac{1}{4} - \frac{1}{2}\log\left|\frac{\psi(z)}{z}\right|\right| < \frac{1}{4}, \quad z \in \Omega \setminus \{0\}. \tag{2.11}$$

The last relation is equivalent to

$$0 < \frac{1}{2}\log\left|\frac{\psi(z)}{z}\right| < \frac{1}{2}.$$

Since the right hand inequality is satisfied, because $\Omega \subset \mathbf{U}$ and Schwarz lemma, the relation (2.11) is satisfied provided that $|\psi(z)| < e|z|$.

For example if Ω is the square $\{z : |x| + |y| \leqslant 1\}$, then the mapping

$$\phi(z) = \frac{\Gamma(3/4)}{\sqrt{\pi}\Gamma(5/4)}\,{}_2F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, z^4\right) z$$

is a conformal mapping of the unit disk onto Ω. Here ${}_2F_1$ is the Gauss hypergeometric function. Moreover it has a conformal extension mapping a starlike domain S onto $\mathbf{U}$. For $\psi(z) = \phi^{-1}(z)$ we have

$$1 < \frac{|\psi(z)|}{|z|} \leqslant \frac{\sqrt{\pi}\Gamma(5/4)}{\Gamma(3/4)} \approx 1.31103 < e, \quad z \in \Omega.$$

Thus (2.11) is satisfied.

3. The Hilbert Norm of Solution and of its Gradient

Let $d \geqslant 2$ and let $\Omega \subset \mathbf{R}^d$ be a domain with smooth boundary γ (piecewise smooth if $d = 2$). Let $(\varphi_n)_{n=1}^{\infty}$ be an orthonormal basic of $L^2(\Omega)$ consisting of eigenfunctions of boundary problem (1.9), such that $-\Delta\varphi_n = \lambda_n\varphi_n$ where $\lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_n \ldots$. The functions φ_k are real valued.

We now determine the Hilbert norm of solution.

Theorem 3.1. *The norm of the operator $\mathcal{P}_\Omega : L^2(\Omega, \mathbf{C}) \to L^2(\Omega, \mathbf{C})$ satisfies*

$$\|\mathcal{P}_\Omega\|_{L^2 \to L^2} = \frac{1}{\lambda_1}.$$

In other words

$$\|\mathcal{P}_\Omega[g]\|_2 \leqslant \frac{1}{\lambda_1}\|g\|_2, \quad \textit{for complex valued} \;\; g \in L^2(\Omega). \tag{3.1}$$

Equality is attained in (3.1) for $g(z) = c\varphi_1(z)$, for a.e. $z \in \Omega$, where c is a real constant.

Proof. We start with

Lemma 3.2. *If $f \in L^2(\Omega)$, then under previous notation*

$$\|\mathcal{P}_\Omega[f]\|_2^2 = \sum_{k=1}^{\infty} \frac{\langle f, \varphi_k\rangle^2}{\lambda_k^2}.$$

Proof of Lemma 3.2. Let $a_k = \langle f, \varphi_k\rangle$ and

$$f(z) = \sum_{k=1}^{\infty} a_k\varphi_k(z). \tag{3.2}$$

Here $a_k, k = 1, 2 \ldots$, are complex constants. Then

$$\mathcal{P}_\Omega[f] = \sum_{k=1}^{\infty} a_k\mathcal{P}_\Omega[\varphi_k].$$

Moreover,

$$\mathcal{P}_\Omega[\varphi_k] = \frac{1}{\lambda_k}\mathcal{P}_\Omega[\Delta\varphi_k] = -\frac{1}{\lambda_k}\varphi_k.$$

Since φ_k's make an orthonormal basis we obtain

$$\|\mathcal{P}_\Omega[f]\|^2 = \sum_{k=1}^{\infty}\frac{|a_k|^2}{\lambda_k^2}\int_\Omega |\varphi_k|^2 dA = \sum_{k=1}^{\infty}\frac{|a_k|^2}{\lambda_k^2}.$$

□

According to Jentzsch's theorem, λ_1 is a simple eigenvalue. Thus $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots$. It follows from Lemma 3.5 that

$$\|\mathcal{P}_\Omega[f]\|_2 \leqslant \frac{1}{\lambda_1}\|f\|_2.$$

Thus

$$\|\mathcal{P}_\Omega[f]\|_2 = \frac{1}{\lambda_1}\|f\|_2$$

if and only if $a_n = 0$ for $n \geqslant 2$. □

To determine the Hilbert norm of the Cauchy transform w.r.t Dirichlet's problem, we need the following lemma.

Lemma 3.3. *Under the notation of introduction of this section, for* $n, m \in \mathbf{N}$,

$$\int_\Omega \nabla\varphi_n(z) \bullet \nabla\varphi_m(z) dA(z) = \lambda_n \delta_{mn}.$$

Here $\bullet$ *denotes the inner product.*

Proof. This lemma is proved in [5] for the case $d = 2$. For the completeness we give its detailed proof here. First of all (1.9) is reduced to the following Fredholm integral equation of the second kind with a self-adjoint kernel:

$$u(w) - \lambda\int_\Omega u(z)G_\Omega(w,z)dA(z) = 0, \quad w \in \Omega.$$

Because the volume potential is smooth on the whole space, provided that the density is bounded and $\partial\Omega$ is piecewise smooth in dimension 2 and smooth in larger dimensions, it follows that every solution to (1.9) is smooth up to the boundary ([8]).

Let φ and ψ be scalar functions defined in the region Ω, and suppose that $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and ψ is once continuously differentiable. Then, by the divergence theorem applied to the vector field $\mathbf{F} = \psi\nabla\varphi$, we obtain

$$\int_\Omega (\psi\Delta\varphi + \nabla\varphi \bullet \nabla\psi)\, dA = \oint_{\partial\Omega} \psi\,(\nabla\varphi \bullet \mathbf{n})\, ds \tag{3.3}$$

where $\partial\Omega$ is the boundary of region Ω and $\mathbf{n}$ is the outward pointing unit normal on $\gamma = \partial\Omega$.

By taking $\varphi = \varphi_n$ and $\psi = \varphi_m$ in (3.3), and having in mind the fact that $\Delta\varphi_n = -\lambda_n\varphi_n, \varphi_n|_\gamma \equiv 0$ and $\langle\varphi_n, \varphi_m\rangle = \delta_{nm}$, we obtain that

$$\int_\Omega (\nabla\varphi_n \bullet \nabla\varphi_m)\, dA(z) = \lambda_n \int_\Omega \varphi_n\varphi_m dA(z) = \lambda_n\delta_{nm}.$$

□

We now have:

Theorem 3.4. *Let Ω be a domain in $\mathbf{C}$ with piecewise smooth boundary. The norms of the operators $\bar{\mathcal{C}}_\Omega, \mathcal{C}_\Omega : L^2(\Omega, \mathbf{C}) \to L^2(\Omega, \mathbf{C})$ satisfy*

$$\|\bar{\mathcal{C}}_\Omega\|_{L^2\to L^2} = \|\mathcal{C}_\Omega\|_{L^2\to L^2} = \frac{1}{2\sqrt{\lambda_1}}.$$

In other words

$$\|\bar{\mathcal{C}}_\Omega[g]\|_2, \|\mathcal{C}_\Omega[g]\|_2 \leqslant \frac{1}{2\sqrt{\lambda_1}}\|g\|_2, \quad \textit{for complex valued } \ g \in L^2(\Omega). \quad (3.4)$$

Equality in each case holds in (3.4) if and only if $g(z) = c\varphi_1(z)$, for a.e. $z \in \Omega$, where c is a real constant.

Proof. We begin by the following lemma

Lemma 3.5. *If $f \in L^2(\Omega)$, then under the previous notation*

$$\|\mathcal{C}_\Omega[f]\|_2^2 = \|\bar{\mathcal{C}}_\Omega[f]\|_2^2 = \sum_{k=1}^\infty \frac{\langle f, \varphi_k\rangle^2}{4\lambda_k}.$$

Proof of Lemma 3.5. By using representation (3.2) we have

$$\bar{\mathcal{C}}_\Omega[f] = \sum_{k=1}^\infty a_k\bar{\mathcal{C}}_\Omega[\varphi_k].$$

Moreover, since φ_k is a real valued function, we have that $\nabla\varphi_k = (\varphi_{k_x}, \varphi_{k_y}) = \varphi_{k_x} + i\varphi_{k_y} = 2\bar{\partial}\varphi_k$, treated as two-dimensional vectors. Notice that they are formally different mathematical objects: the first one is a real vector, while the second is a complex number. Having in mind the previous identification, we have

$$\bar{\mathcal{C}}_\Omega[\varphi_k] = -\frac{1}{\lambda_k}\bar{\mathcal{C}}_\Omega[\Delta\varphi_k] = \frac{1}{2\lambda_k}\nabla\varphi_k.$$

According to Lemma 3.3 we obtain

$$\|\bar{\mathcal{C}}_\Omega[f]\|_2^2 = \sum_{k=1}^\infty \frac{|a_k|^2}{4\lambda_k^2}\int_\Omega |\nabla\varphi_k|^2 dA = \sum_{k=1}^\infty \frac{|a_k|^2}{4\lambda_k}.$$

□

Since $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots$, it follows from Lemma 3.5 that

$$\|\bar{\mathcal{C}}_\Omega[f]\|_2 \leqslant \frac{1}{2\sqrt{\lambda_1}}\|f\|_2.$$

Thus

$$\|\bar{\mathcal{C}}_\Omega[f]\|_2 = \frac{1}{2\sqrt{\lambda_1}}\|f\|_2$$

if and only if $a_n = 0$ for $n \geqslant 2$. Similarly the operator $\mathcal{C}_\Omega$ can be treated. □

Theorem 3.4 implies

Corollary 3.6. *For a real valued function $u \in W^{1,2}(\Omega) \cap \mathbb{C}(\overline{\Omega})$ such that $u|_{\partial\Omega} = 0$, we have the following sharp inequality*

$$\|\nabla u\|_2 \leqslant \frac{1}{\sqrt{\lambda_1}}\|\Delta u\|_2,$$

where λ_1 is the smallest eigenvalue of the boundary problem (1.9), i.e.

$$\lambda_1 := \inf\{\lambda > 0 : -\Delta u = \lambda u, u \in W_0^{1,2}(\Omega)\}.$$

As

$$|\mathcal{D}_\Omega[g](z)| = |\mathcal{C}_\Omega[g](z)| + |\bar{\mathcal{C}}_\Omega[g](z)|$$

we obtain that

$$\|\mathcal{D}_\Omega[g]\|_2 \leqslant \frac{1}{2\sqrt{\lambda_1}}\|g\|_2 + \frac{1}{2\sqrt{\lambda_1}}\|g\|_2 = \frac{1}{\sqrt{\lambda_1}}\|g\|_2. \tag{3.5}$$

By the definition of complex partial derivatives, for a real valued function f we have

$$|\partial f| = |\bar{\partial} f| = \frac{1}{2}\sqrt{|\partial_x f|^2 + |\partial_y f|^2} = \frac{1}{2}|\nabla f|.$$

This observation together with (1.5) imply that $|\nabla f| = |Df|$. This equality and Theorem 3.4 imply the following theorem.

Theorem 3.7. *The norm of the differential operator*

$$\mathcal{D}_\Omega : L^2(\Omega, \mathbf{C}) \to L^2(\Omega, \mathcal{M}_{2,2}),\ \mathcal{D}_\Omega[g] = Du,$$

is $1/\sqrt{\lambda_1}$. In other words there holds the sharp inequality

$$\|Du\|_2 \leqslant \frac{1}{\sqrt{\lambda_1}}\|\Delta u\|_2 \tag{3.6}$$

for complex valued functions $u \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ vanishing on the boundary of Ω.

Proof. The relation (3.6) follows from (3.5). By taking $g(z) = \varphi_1(z)$, and having in mind the equation $|\nabla g| = |Dg|$, according to Theorem 3.4 we obtain that (3.6) is sharp. □

3.1. Special Cases

If $\Omega = \mathbf{U}$, then $\lambda_1 = \alpha_1^2$, where $\alpha_1 \approx 2.4048$ is the smallest positive zero of the Bessel function J_0. In this case $\varphi_1 = c|z|J_0(\alpha_1|z|)$ where c is a constant and J_0 is the Bessel function. If $\Omega = [0,\pi]^2$, then $\tilde{\lambda}_{n,m} = n^2 + m^2$ are eigenvalues and $\tilde{\varphi}_{n,m} = \frac{2}{\pi}\sin(nx)\sin(ny)$ are eigenfunctions of boundary problem (1.9), and thus $\lambda_1 = 2$ and $\varphi_1 = \frac{2}{\pi}\sin x \sin y$. See [5] for the last facts. The first eigenvalue λ_1 of the Dirichlet–Laplacian is well-known for the unit ball in $\mathbf{R}^n$, and it seems that the Theorem 3.1 still hold in several dimensional case as well.

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David Kalaj (✉)
Faculty of Natural Sciences and Mathematics
University of Montenegro, Cetinjski put b.b.
81000 Podgorica
Montenegro
e-mail: davidk@ac.me

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