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Integral Equations and Operator Theory

Commutative Toeplitz Banach Algebras on the Ball and Quasi-Nilpotent Group Action

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Abstract. Studying commutative C^*-algebras generated by Toeplitz operators on the unit ball it was proved that, given a maximal commutative subgroup of biholomorphisms of the unit ball, the C^*-algebra generated by Toeplitz operators, whose symbols are invariant under the action of this subgroup, is commutative on each standard weighted Bergman space. There are five different pairwise non-conjugate model classes of such subgroups: *quasi-elliptic, quasi-parabolic, quasi-hyperbolic, nilpotent,* and *quasi-nilpotent*. Recently it was observed in Vasilevski (Integr Equ Oper Theory. 66:141–152, 2010) that there are many other, not geometrically defined, classes of symbols which generate commutative Toeplitz operator algebras on each weighted Bergman space. These classes of symbols were subordinated to the *quasi-elliptic* group, the corresponding commutative operator algebras were *Banach*, and being extended to C^*-algebras they became non-commutative. These results were extended then to the classes of symbols, subordinated to the *quasi-hyperbolic* and *quasi-parabolic* groups. In this paper we prove the analogous commutativity result for Toeplitz operators whose symbols are subordinated to the *quasi-nilpotent* group. At the same time we conjecture that apart from the known C^*-algebra cases there are no more new Banach algebras generated by Toeplitz operators whose symbols are subordinated to the *nilpotent* group and which are commutative on each weighted Bergman space.

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Keywords. Toeplitz operator, Weighted Bergman space, Commutative Banach algebra, Quasi-nilpotent, Quasi-homogeneous.

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1. Introduction

In the present paper (we hope that) we finish the classification of the Banach and C^*-algebras generated by Toeplitz operators that are commutative on each (commonly considered) weighted Bergman space over the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. The short history of this problem is as follows.

The C^*-algebras generated by Toeplitz operators which are commutative on each weighted Bergman space over the unit disk were completely classified in [2]. Under some technical assumption on "richness" of a class of generating symbols the result was as follows. A C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if the corresponding symbols of Toeplitz operators are *constant on cycles of a pencil of hyperbolic geodesics* on the unit disk, or if and only if the corresponding symbols of Toeplitz operators are *invariant under the action of a maximal commutative subgroup* of the Möbius transformations of the unit disk. We note that the commutativity *on each weighted Bergman space* was crucial in the part "only if" of the above result.

Generalizing this result to Toeplitz operators on the unit ball, it was proved in [3,4] that, given a maximal commutative subgroup of biholomorphisms of the unit ball, the C^*-algebra generated by Toeplitz operators, whose symbols are invariant under the action of this subgroup, is commutative on each weighted Bergman space. We note that there are five different pairwise non-conjugate model classes of such subgroups: *quasi-elliptic, quasi-parabolic, quasi-hyperbolic, nilpotent,* and *quasi-nilpotent* (the last one depends on a parameter, giving in total $n+2$ model classes for the n-dimensional unit ball). As a consequence, for the unit ball of dimension n, there are $n+2$ essentially different "model" commutative C^*-algebras, all others are conjugated with one of them via biholomorphisms of the unit ball.

It was firmly expected that the above algebras exhaust all possible algebras of Toeplitz operators which are commutative on each weighted Bergman space. That is, the invariance under the action of a maximal commutative subgroup of biholomorphisms for generating symbols is the only reason for the appearance of Toeplitz operator algebras which are commutative on each weighted Bergman space.

Recently and quite unexpectedly it was observed in [6] that for $n > 1$ there are many other, not geometrically defined, classes of symbols which generate commutative Toeplitz operator algebras on each weighted Bergman space. These classes of symbols were in a sense originated from, or subordinated to the *quasi-elliptic* group, the corresponding commutative operator algebras were *Banach*, and being extended to C^*-algebras they became non-commutative. Moreover, for $n = 1$ all of them collapsed to the commutative C^*-algebra generated by Toeplitz operators with radial symbols (one-dimensional quasi-elliptic case). These results were extended in [1,7] then to the classes of symbols, subordinated to the *quasi-hyperbolic* and *quasi-parabolic* groups, which as well generate via corresponding Toeplitz operators classes of Banach algebras being commutative on each weighted Bergman space. That is, together with [6], these papers cover the multidimensional extensions of the (only) three model cases on the unit disk. The study of the last two model cases of maximal commutative subgroup of biholomorphisms of the unit ball, the *nilpotent*, and *quasi-nilpotent* groups (which appear only for $n > 1$ and $n > 2$, respectively), was left as an important and interesting open question.

After many unsuccessful attempts to find commutative algebras generated by Toeplitz operators and subordinated to the nilpotent group we conjecture that *apart from the known cases there are no more new Banach algebras generated by Toeplitz operators with symbols subordinated to the nilpotent group of biholomorphisms of the unit ball* $\mathbb{B}^n$ *and commutative on each weighted Bergman space.*

At the same time such commutative algebras subordinated to the *quasi-nilpotent* group do exist, and the present paper is devoted to their description. According to our current understanding the only additional source for the appearance of (Banach) Toeplitz operator algebras which are commutative on each weighted Bergman space comes from a torus action on $\mathbb{B}^n$. More precisely, the maximal commutative group of biholomorphisms, to which the symbols are subordinated, must contain the torus $\mathbb{T}^k$, with $k \geq 2$, as a subgroup. In the case of the one-dimensional torus $\mathbb{T}$ the above commutative Toeplitz operator algebras collapse to known commutative C^*-algebras generated by Toeplitz operators whose symbols are invariant under the action of the maximal commutative group of biholomorphisms in question.

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2. Preliminaries

In this section we recall some notation from [4] that are used throughout the text. Let

$$\mathbb{B}^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z|^2 = |z_1|^2 + \ldots + |z_n|^2 < 1\}$$

be the unit ball in $\mathbb{C}^n$. The Siegel domain D_n in $\mathbb{C}^n$, which is an unbounded realization of the unit ball $\mathbb{B}^n$, has the form

$$D_n = \Big\{ z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid \mathrm{Im} z_n - |z'|^2 > 0 \Big\}.$$

Recall that the Cayley transform $\omega : \mathbb{B}^n \to D_n$ maps biholomorphically the unit ball $\mathbb{B}^n$ onto D_n. Let v be the usual Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and fix $\lambda > -1$. Then the standard weighted measure μ_λ on $\mathbb{B}^n$ with weight parameter λ is given by:

$$d\mu_\lambda := c_\lambda (1 - |z|^2)^\lambda dv, \qquad \text{and} \qquad c_\lambda := \frac{\Gamma(n + \lambda + 1)}{\pi^n \Gamma(\lambda + 1)}.$$

Here c_λ is a normalizing constant such that $\mu_\lambda(\mathbb{B}^n) = 1$. On D_n we can consider the corresponding weighted measure $\tilde{\mu}_\lambda$ defined by

$$d\tilde{\mu}_\lambda(\zeta', \zeta_n) = \frac{c_\lambda}{4} \big(\mathrm{Im}\, \zeta_n - |\zeta'|^2\big)^\lambda dv(\zeta', \zeta_n).$$

Let f be a function on $\mathbb{B}^n$, then we put $(\mathcal{U}_\lambda f)(\zeta) := 2^{n+\lambda+1}(1-i\zeta_n)^{-n-\lambda-1} f \circ \omega^{-1}(\zeta)$ where $\zeta \in D_n$. A straightforward calculation shows, cf. [1,4]

Lemma 2.1. *Let $\lambda > -1$, then $\mathcal{U}_\lambda$ defines a unitary transformation of $L_2(\mathbb{B}^n, \mu_\lambda)$ onto $L_2(D_n, \tilde{\mu}_\lambda)$.*

In the following we write $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ and $\mathcal{A}^2_\lambda(D_n)$ for the weighted Bergman spaces of all complex analytic functions in $L_2(\mathbb{B}^n, \mu_\lambda)$ and $L_2(D_n, \tilde{\mu}_\lambda)$, respectively. It is known that by restriction $\mathcal{U}_\lambda$ defines a unitary transformation of $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ onto $\mathcal{A}^2_\lambda(D_n)$.

Let $B_{D_n,\lambda}$ be the Bergman projection of $L_2(D_n, \tilde{\mu}_\lambda)$ onto $\mathcal{A}^2_\lambda(D_n)$. Given a bounded measurable function $f \in L^\infty(D_n)$ we define the *Toeplitz operator* T_f acting on the weighted Bergman space $\mathcal{A}^2_\lambda(D_n)$ in the usual way by

$$T_f := B_{D_n,\lambda} M_f,$$

where M_f denotes the multiplication by f. In this paper we study a class of commutative Banach algebras generated by Toeplitz operators on $\mathcal{A}^2_\lambda(D_n)$. To simplify the notation we will not indicate the dependence of T_f on the weight parameter λ. Note that via the unitary transformation $\mathcal{U}_\lambda$ the results in this paper on Toeplitz operators acting on weighted Bergman spaces over D_n can be directly translated to the corresponding setting of Toeplitz operators on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$.

Put $\mathcal{D} := \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$. Then the map:

$$\kappa : \mathcal{D} \to D_n : (z', u, v) \mapsto (z', u + iv + i|z'|^2)$$

defines a diffeomorphism with inverse $\kappa^{-1}(z', z_n) = (z', \mathrm{Re}\, z_n, \mathrm{Im} z_n - |z'|^2)$. Given a function f on D_n, we define $U_0 f := f \circ \kappa$ to obtain a function $U_0 f$ on $\mathcal{D}$. On the domain $\mathcal{D}$ we consider the measure

$$d\eta_\lambda(z', u, v) := \frac{c_\lambda}{4} v^\lambda dv(z', u, v).$$

We have the following, cf. [1,4]

Lemma 2.2. *The operator U_0 is unitary from $L_2(D_n, \tilde{\mu}_\lambda)$ to $L_2(\mathcal{D}, \eta_\lambda)$ with inverse $U_0^{-1} = U_0^*$ given by $U_0^* f = f \circ \kappa^{-1}$.*

We occasionally omit the dependence of the weight $\lambda > -1$ and put $\mathcal{A}_0(\mathcal{D}) := U_0(\mathcal{A}^2_\lambda(D_n))$ which clearly forms a closed subspace of $L_2(\mathcal{D}, \eta_\lambda)$.

3. Quasi-Nilpotent Group Action and a Decomposition of the Bergman Projection

As was explained in [3,4] the classification of maximal commutative subgroups G of biholomorphisms of D_n or $\mathbb{B}^n$ yields five essentially different types. Corresponding to each type there are commutative Banach or C^*-algebras of Toeplitz operators acting on weighted Bergman spaces. The aim of this paper is to define such algebras in case of the quasi-nilpotent group G of biholomorphisms. We recall the definition.

Let $1 \leq k \leq n-2$. We rather use the notation $z = (z', w', z_n)$ for $z \in D_n$ where $z' \in \mathbb{C}^k$ and $w' \in \mathbb{C}^{n-k-1}$. The *quasi-nilpotent group* $\mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$ acts on D_n, cf. [4], as follows: given $(t, b, h) \in \mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$, we have:

$$\tau_{(t,b,h)} : (z', w', z_n) \mapsto \big(tz', w' + b, z_n + h + 2iw' \cdot b + i|b|^2\big).$$

Note that in the case $k = n-1$ we obtain the *quasi-parabolic group*, while for $k = 0$ the group action is called *nilpotent*.

On the domain $\mathcal{D} = \mathbb{C}^k \times \mathbb{C}^{n-k-1} \times \mathbb{R} \times \mathbb{R}_+$ we use the variables (z', w', u, v) and we represent $L_2(\mathcal{D}, \eta_\lambda)$ in the form:

$$L_2(\mathcal{D}, \eta_\lambda) = L_2(\mathbb{C}^k) \otimes L_2(\mathbb{C}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda). \tag{3.1}$$

Let F be the Fourier transform on $L_2(\mathbb{R})$, and with respect to the decomposition (3.1) consider the unitary operators $U_1 := I \otimes I \otimes F \otimes I$ acting on $L_2(\mathcal{D}, \eta_\lambda)$. With this notation we put $\mathcal{A}_1(\mathcal{D}) := U_1(\mathcal{A}_0(\mathcal{D}))$.

Next, we introduce polar coordinates on $\mathbb{C}^k$ and put $r = (r_1, \ldots, r_k) = (|z_1'|, \ldots, |z_k'|)$. Moreover, in the following we write $x' := \operatorname{Re} w'$ and $y' := \operatorname{Im} w'$. Then one can check that r, y' and $\operatorname{Im} z_n - |w'|^2$ are invariant under the action of the quasi-nilpotent group. Following the ideas in [4] and with $rdr = r_1 dr_1 \cdots r_k dr_k$ we represent $L_2(\mathcal{D}, \eta_\lambda)$ in the form

$$L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{T}^k) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda). \tag{3.2}$$

We define the unitary operator U_2 on $L_2(\mathcal{D}, \eta_\lambda)$ by $U_2 = I \otimes \mathcal{F}_{(k)} \otimes F_{(n-k-1)} \otimes I \otimes I \otimes I$. Here $\mathcal{F}_{(k)} = \mathcal{F} \otimes \cdots \otimes \mathcal{F}$ is the k-dimensional discrete Fourier transform and $F_{(n-k-1)} = F \otimes \cdots \otimes F$ denotes the $(n-k-1)$-dimensional Fourier transform on $L_2(\mathbb{R}^{n-k-1})$. Note that $L_2(\mathcal{D}, \eta_\lambda)$ is isometrically mapped by U_2 onto

$$\ell_2\Big(\mathbb{Z}^k, L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)\Big). \tag{3.3}$$

We put $\mathcal{A}_2(\mathcal{D}) := U_2(\mathcal{A}_1(\mathcal{D}))$ and we write elements in (3.3) as $\{f_\beta(r, x', y', \xi, v)\}_{\beta \in \mathbb{Z}^k}$, where

$$(r, x', y', \xi, v) \in \mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}^{n-k-1} \times \mathbb{R} \times \mathbb{R}_+.$$

Next we recall the definition of the unitary operator U_3 which acts on (3.3) by:

$$U_3 : \{f_\beta(r, x', y', \xi, v)\}_{\beta \in \mathbb{Z}^k} \longmapsto \left\{ f_\beta \left(r, \sqrt{\xi}(x' + y'), \frac{1}{2\sqrt{\xi}}(-x' + y'), \xi, v \right) \right\}_{\beta \in \mathbb{Z}^k}.$$

One immediately checks that the inverse U_3^{-1} has the form

$$U_3^{-1} : \{f_\beta(r, x', y', \xi, v)\}_{\beta \in \mathbb{Z}^k} \longmapsto \left\{ f_\beta \left(r, \frac{x'}{2\sqrt{\xi}} - \sqrt{\xi} y', \frac{x'}{2\sqrt{\xi}} + \sqrt{\xi} y', \xi, v \right) \right\}_{\beta \in \mathbb{Z}^k}.$$

In the following we write $\mathbb{Z}_+ = \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}$ for the non-negative integers. In order to state the main result of Section 8 in [4] we need to introduce the operator R_0, which defines an isometric embedding of $\ell_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))$ into (3.3). It is explicitly given by

$$R_0 : \{c_\beta(x', \xi)\}_{\beta \in \mathbb{Z}_+^k} \longmapsto \left\{ \chi_{\mathbb{Z}_+^k \times \mathbb{R}_+}(\beta, \xi) A_\beta(\xi) r^\beta e^{-\xi(|r|^2+v) - \frac{|y'|^2}{2}} c_\beta(x', \xi) \right\}_{\beta \in \mathbb{Z}^k}$$
$$= \{g_\beta(r, x', y', \xi, v)\}_{\beta \in \mathbb{Z}^k}.$$

Here $\chi_{\mathbb{Z}_+^k \times \mathbb{R}_+}(\beta, \xi)$ denotes the characteristic function of $\mathbb{Z}_+^k \times \mathbb{R}_+$ and $c_\beta(x', \xi)$ is extended by zero for $\xi \in (-\infty, 0)$ and all $x' \in \mathbb{R}^{n-k-1}$. Moreover, we have used the abbreviation

$$A_\beta(\xi) := \pi^{-\frac{n-k-1}{4}} \sqrt{\frac{2^{k+2}}{c_\lambda} \frac{(2\xi)^{|\beta|+\lambda+k+1}}{\beta! \Gamma(\lambda+1)}}. \tag{3.4}$$

The adjoint operator R_0^* is given by:

$$R_0^* : \left\{ f_\beta(r, x', y', \xi, v) \right\}_{\beta \in \mathbb{Z}^k} \longmapsto \left\{ A_\beta(\xi) \right. \tag{3.5}$$
$$\left. \times \int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} r^\beta e^{-\xi(|r|^2+v) - \frac{|y'|^2}{2}} f_\beta(r, x', y', \xi, v) r dr dy' \frac{c_\lambda v^\lambda}{4} dv \right\}_{\beta \in \mathbb{Z}_+^k}.$$

We set $U := U_3 U_2 U_1 U_0$, which gives a unitary operator from $\mathcal{A}_\lambda^2(D_n)$ onto $\mathcal{A}_3(\mathcal{D}) := U_3(\mathcal{A}_2(\mathcal{D}))$. The following result has been proved in [4], Theorem 8.2 and it provides a decomposition of the Bergman projection $B_{D_n,\lambda}$ in form of a certain operator product.

Theorem 3.1. [4] *The operator $R := R_0^* U$ maps $L_2(D_n, \tilde{\mu}_\lambda)$ onto the space $\ell_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))$, and the restriction*

$$R|_{\mathcal{A}_\lambda^2(D_n)} : \mathcal{A}_\lambda^2(D_n) \longrightarrow \ell_2\big(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R})\big)$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : \ell_2\big(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)\big) \longrightarrow \mathcal{A}_\lambda^2(D_n) \subset L_2(D_n, \tilde{\mu}_\lambda)$$

is an isometric isomorphism of $\ell_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))$ onto the subspace $\mathcal{A}_\lambda^2(D_n)$ of $L_2(D_n, \tilde{\mu}_\lambda)$. Furthermore one has:

$$RR^* = I : \ell_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)) \longrightarrow \ell_2\big(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)\big),$$
$$R^* R = B_{D_n,\lambda} : L_2(D_n, \tilde{\mu}_\lambda) \longrightarrow \mathcal{A}_\lambda^2(D_n).$$

4. Toeplitz Operators with Quasi-Homogeneous Symbols

Now, we restrict our attention to bounded measurable symbols on D_n that are invariant or have a certain homogeneity with respect to the quasi-nilpotent group action on D_n.

Definition 4.1. A bounded measurable function $a : D_n \to \mathbb{C}$ is called *quasi-nilpotent* if it has the form $a(z) = a(r, y', \mathrm{Im} z_n - |w'|^2)$. In particular, such a is invariant under the action of the quasi-nilpotent group.

The following theorem was proved in [4].

Theorem 4.2. [4, Theorem 10.4] *Let $a = a(r, y', \mathrm{Im} z_n - |w'|^2)$ be a bounded measurable quasi-nilpotent function on D_n. Then the Toeplitz operator T_a acting on $\mathcal{A}^2_\lambda(D_n)$ is unitary equivalent to the multiplication operator $\gamma_a I = RT_aR^*$ acting on the space $\ell^2(\mathbb{Z}^k_+, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))$. The sequence*

$$\gamma_a = \{\gamma_a(\beta, x', \xi)\}_{\beta \in \mathbb{Z}^k_+} \in \ell^2(\mathbb{Z}^k_+, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))$$

with $(x', \xi) \in \mathbb{R}^{n-k-1} \times \mathbb{R}_+$ is given by:

$$\begin{aligned} \gamma_a(\beta, x', \xi) = 2^k \pi^{-\frac{n-k-1}{2}} \frac{(2\xi)^{|\beta|+\lambda+k+1}}{\beta! \Gamma(\lambda+1)} & \\ \times \int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} a & \left(r, \frac{1}{2\sqrt{\xi}}(-x' + y'), v + |r|^2\right) \\ \times r^\beta & e^{-2\xi(v+|r|^2)-|y'|^2} v^\lambda r dr dy' dv. \end{aligned}$$

We need to prove a similar result for a class of more general symbols. Recall that we use the notation $x' := \mathrm{Re}\, w' \in \mathbb{R}^{n-k-1}$ where $w' \in \mathbb{C}^{n-k-1}$ and let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a tuple in $\mathbb{Z}^m_+$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_m = k$. Similar to [1,6] we divide the coordinates of $z' \in \mathbb{C}^k$ into m groups as follows:

$$z'_{(1)} = (z'_{1,1}, \ldots, z'_{1,\alpha_1}), \quad z'_{(2)} = (z'_{2,1}, \ldots, z'_{2,\alpha_2}), \quad \ldots \quad z'_{(m)} = (z'_{m,1}, \ldots, z'_{m,\alpha_m})$$

and such that $z' = (z'_{(1)}, z'_{(2)}, \ldots, z'_{(m)})$. In the following we will use the same notation also in case of multi-indices $\beta \in \mathbb{Z}^k$ instead of vectors $z' \in \mathbb{C}^k$. By passing to polar coordinates, we write each tuple $z'_{(j)} = (z'_{j,1}, \ldots, z'_{j,\alpha_j}) \in \mathbb{C}^{\alpha_j}$, where $j = 1, \ldots, m$, in the form

$$z'_{(j)} = \mathbf{r}_j \zeta_{(j)} \quad \text{with} \quad \mathbf{r}_j := \sqrt{|z'_{j,1}|^2 + \ldots + |z'_{j,\alpha_j}|^2} \quad \text{and} \quad \zeta_{(j)} \in \mathbb{S}^{2\alpha_j - 1} \subset \mathbb{C}^{\alpha_j}.$$

Here $\mathbb{S}^{2n-1}$ denotes the real $(2n-1)$-dimensional boundary of $\mathbb{B}^n$.

Definition 4.3. Let $a(r, y', \mathrm{Im} z_n - |w'|^2)$ be a quasi-nilpotent function and $\alpha \in \mathbb{Z}^m_+$ as above.

(i) Then a is called "α-quasi-nilpotent quasi-radial" if its radial dependence on r can be expressed as a function of $\mathbf{r}_1, \ldots, \mathbf{r}_m$.

(ii) The function $b(z', w', z_n)$ is called "α-quasi-nilpotent quasi-homogeneous" if it is α-quasi-nilpotent quasi-homogeneous with respect to the variable z', i.e.

$$b(z', w', z_n) = b_0\big(\mathbf{r}_1, \ldots, \mathbf{r}_m, y', \mathrm{Im} z_n - |w'|^2\big) \zeta^p \overline{\zeta}^q, \tag{4.1}$$

where $\zeta = (\zeta_{(1)}, \zeta_{(2)}, \ldots, \zeta_{(m)}) \in \mathbb{S}^{2\alpha_1-1} \times \mathbb{S}^{2\alpha_2-1} \times \ldots \times \mathbb{S}^{2\alpha_m-1}$ and $p, q \in \mathbb{Z}^k_+$ are orthogonal. The pair (p, q) is then called the "degree" of b.

Remark **4.4.** Note that there is a one-to-one correspondence between the set of tuples $\{(p, q) \in \mathbb{Z}^k_+ \times \mathbb{Z}^k_+ \, : \, p \perp q\}$ and $\mathbb{Z}^k$ via $(p, q) \mapsto p - q$.

Consider an α-quasi-nilpotent quasi-homogeneous symbol $b(z', w', z_n)$ as in (4.1) and of degree $(p, q) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k$ with $p \perp q$. Our next aim is to calculate the operator RT_bR^*. On the domain $\mathcal{D} = \mathbb{C}^k \times \mathbb{C}^{n-k-1} \times \mathbb{R} \times \mathbb{R}_+$ we use the variables (z', w', u, v). Moreover, we express z' in polar coordinates $z' = (t_1 r_1, \ldots, t_k r_k)$ where $r_s \geq 0$ and $t_s \in \mathbb{S} = \mathbb{S}^1$ for $s = 1, \ldots, k$. Then we have the relations

$$z_{j,\ell} = \mathbf{r}_j \zeta_{j,\ell} = t_{j,\ell} r_{j,\ell}$$

for $\ell = \{1, \ldots, \alpha_j\}$ and $j = 1, \ldots, m$. It follows that $\zeta_{j,\ell} = t_{j,\ell} r_{j,\ell} \mathbf{r}_j^{-1}$ in the case of $\mathbf{r}_j \neq 0$ and therefore:

$$\zeta^p \overline{\zeta}^q = t^p \overline{t}^q r^{p+q} \prod_{j=1}^m \mathbf{r}_j^{-|p_{(j)}|-|q_{(j)}|}. \tag{4.2}$$

Note that the assignment $z' \mapsto \zeta^p \overline{\zeta}^q$ depends on the initial choice of $\alpha \in \mathbb{Z}_+^m$. Using Theorem 3.1 we can write:

$$\begin{aligned} RT_bR^* &= RB_{D_n,\lambda} b B_{D_n,\lambda} R^* \\ &= R(R^*R) b (R^*R) R^* \\ &= (RR^*) R b R^* (RR^*) \\ &= RbR^* \\ &= R_0^* U_3 U_2 U_1 U_0 b U_0^{-1} U_1^{-1} U_2^{-1} U_3^{-1} R_0 \\ &= R_0^* U_3 U_2 U_1 b_0(\mathbf{r}_1, \ldots, \mathbf{r}_m, y', v + |r|^2) \zeta^p \bar{\zeta}^q U_1^{-1} U_2^{-1} U_3^{-1} R_0 \\ &= R_0^* U_3 U_2 b_0(\mathbf{r}_1, \ldots, \mathbf{r}_m, y', v + |r|^2) \zeta^p \bar{\zeta}^q U_2^{-1} U_3^{-1} R_0. \end{aligned}$$

First we calculate the operator $U_2 b U_2^{-1}$. Let $\{f_\beta(r, x', y', \xi, v)\}_{\beta \in \mathbb{Z}^k}$ be an element in the space (3.3) and write $\mathbf{r} := (\mathbf{r}_1, \ldots, \mathbf{r}_m)$. Since the symbol $b_0(\mathbf{r}, y', v + |r|^2) \zeta^p \bar{\zeta}^q$ is independent of x' we obtain from (4.2) that:

$$\begin{aligned} & U_2 b_0(\mathbf{r}, y', v + |r|^2) \zeta^p \bar{\zeta}^q U_2^{-1} \Big\{ f_\beta(r, x', y', \xi, v) \Big\}_{\beta \in \mathbb{Z}^k} \\ &\quad = \Big\{ b_0(\mathbf{r}, y', v + |r|^2) r^{p+q} \Big(\prod_{j=1}^m \mathbf{r}_j^{-|p_{(j)}|-|q_{(j)}|} \Big) f_{\beta-p+q}(r, x', y', \xi, v) \Big\}_{\beta \in \mathbb{Z}^k}. \end{aligned} \tag{4.3}$$

Combining (4.3) and (3.5) gives:

$$\begin{aligned} RT_bR^* \{c_\beta(x', \xi)\}_{\beta \in \mathbb{Z}_+^k} &= R_0^* U_3 U_2 b U_2^{-1} U_3^{-1} \Big\{ \chi_{\mathbb{Z}_+^k \times \mathbb{R}_+}(\beta, \xi) \\ &\quad A_\beta(\xi) r^\beta e^{-\xi(|r|^2+v) - \frac{|y'|^2}{2}} c_\beta(x', \xi) \Big\}_{\beta \in \mathbb{Z}^k} \\ &= R_0^* U_3 U_2 b U_2^{-1} \Big\{ \chi_{\mathbb{Z}_+^k \times \mathbb{R}_+}(\beta, \xi) A_\beta(\xi) r^\beta \\ &\quad \times e^{-\xi(|r|^2+v) - \frac{1}{2}|\frac{1}{2\sqrt{\xi}} x' + \sqrt{\xi} y'|^2} c_\beta\Big(\frac{1}{2\sqrt{\xi}} x' - \sqrt{\xi} y', \xi \Big) \Big\}_{\beta \in \mathbb{Z}^k} \\ &= R_0^* U_3 \Big\{ \chi_{\mathbb{Z}_+^k \times \mathbb{R}_+}(\beta - p + q, \xi) A_{\beta-p+q}(\xi) r^{\beta+2q} b_0 \\ &\quad \times (\mathbf{r}, y', v + |r|^2) \\ &\quad \times \Big(\prod_{j=1}^m \mathbf{r}_j^{-|p_{(j)}|-|q_{(j)}|} \Big) e^{-\xi(|r|^2+v) - \frac{1}{2}|\frac{1}{2\sqrt{\xi}} x' + \sqrt{\xi} y'|^2} \end{aligned}$$

$$
\begin{aligned}
&\times c_{\beta-p+q}\Big(\frac{1}{2\sqrt{\xi}}x' - \sqrt{\xi}y', \xi\Big)\Big\}_{\beta\in\mathbb{Z}^k}\\
&= R_0^*\Big\{\chi_{\mathbb{Z}_+^k\times\mathbb{R}_+}(\beta-p+q,\xi)A_{\beta-p+q}(\xi)b_0\\
&\times\Big(\mathbf{r}, \frac{-x'+y'}{2\sqrt{\xi}}, v+|r|^2\Big)\\
&\times\Big(\prod_{j=1}^m \mathbf{r}_j^{-|p_{(j)}|-|q_{(j)}|}\Big)r^{\beta+2q}e^{-\xi(|r|^2+v)-\frac{1}{2}|y'|^2}c_{\beta-p+q}\\
&(x',\xi)\Big\}_{\beta\in\mathbb{Z}^k}\\
&= \Big\{A_\beta(\xi)A_{\beta-p+q}(\xi)\chi_{\mathbb{Z}_+^k\times\mathbb{R}_+}(\beta-p+q,\xi)c_{\beta-p+q}(x',\xi)\\
&\times\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} r^{2(\beta+q)}\\
&\times\Big(\prod_{j=1}^m \mathbf{r}_j^{-|p_{(j)}|-|q_{(j)}|}\Big)e^{-2\xi(|r|^2+v)-|y'|^2}\\
&\times b_0\Big(\mathbf{r}, \frac{-x'+y'}{2\sqrt{\xi}}, v+|r|^2\Big)r dr dy'\frac{c_\lambda v^\lambda}{4}dv\Big\}_{\beta\in\mathbb{Z}_+^k}.
\end{aligned}
$$

Now put:

$$
\begin{aligned}
\tilde{\gamma}_{b,p,q}(\beta,x',\xi) := {}& A_\beta(\xi)A_{\beta-p+q}(\xi)\chi_{\mathbb{Z}_+^k\times\mathbb{R}_+}(\beta-p+q,\xi)\\
&\times\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+}\prod_{j=1}^m \mathbf{r}_j^{-|p_{(j)}|-|q_{(j)}|}\\
&\times r^{2(\beta+q)}e^{-2\xi(|r|^2+v)-|y'|^2}b_0\Big(\mathbf{r}, \frac{-x'+y'}{2\sqrt{\xi}}, v+|r|^2\Big)r dr dy'\\
&\times\frac{c_\lambda v^\lambda}{4}dv. \qquad (4.4)
\end{aligned}
$$

Hence, we have proved:

Theorem 4.5. *Let b be defined as in (4.1). The operator RT_bR^* acts on the Hilbert space $\ell_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+))$ by the rule:*

$$
RT_bR^*\{c_\beta(x',\xi)\}_{\beta\in\mathbb{Z}_+^k} = \Big\{\tilde{\gamma}_{b,p,q}\big(\beta,x',\xi\big)\cdot c_{\beta-p+q}\big(x',\xi\big)\Big\}_{\beta\in\mathbb{Z}_+^k}.
$$

Note that, in the case $p=q=0$, Theorem 4.5 reduces to Theorem 4.2.

Example **1.** We calculate RT_bR^* more explicitly in the special case where $b_0\equiv 1$ and we choose $k=m$, i.e. $\alpha=(1,\dots,1)\in\mathbb{Z}_+^k$. Let $(p,q)\in\mathbb{Z}_+^k\times\mathbb{Z}_+^k$ such that $p\perp q$ and put

$$
b(z',w',z_n) = \zeta^p\bar{\zeta}^q = t^p\bar{t}^q.
$$

According to Theorem 4.5 it is sufficient to calculate the functions:

$$
\begin{aligned}
\tilde{\gamma}_{b,p,q}(\beta,x',\xi) = {}& A_\beta(\xi)A_{\beta-p+q}(\xi)\chi_{\mathbb{R}_+}(\xi)\\
&\times\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} r^{2\beta+q-p}e^{-2\xi(|r|^2+v)-|y'|^2}r dr dy'\frac{c_\lambda v^\lambda}{4}dv
\end{aligned}
$$

for all $\beta \in \mathbb{Z}_+^k$ with $\beta - p + q \in \mathbb{Z}_+^k$. We use the identity:

$$\int_{\mathbb{R}^{n-k-1}\times\mathbb{R}_+} e^{-2\xi v-|y'|^2} dy' v^\lambda dv = \pi^{\frac{n-k-1}{2}}\Gamma(\lambda+1)(2\xi)^{-(\lambda+1)},$$

(cf. formula 3.381.4 of [5]) where $\xi > 0$, which together with (3.4) shows that

$$\begin{aligned}\tilde{\gamma}_{b,p,q}(\beta, x', \xi) &= 2^k(2\xi)^{|\beta|+k+\frac{|q|-|p|}{2}} \frac{1}{\sqrt{\beta!(\beta-p+q)!}} \int_{\mathbb{R}_+^k} r^{2\beta+q-p+e} e^{-2\xi|r|^2} dr \\ &= \frac{\prod_{j=1}^k \Gamma(\beta_j + \frac{q_j-p_j}{2}+1)}{\sqrt{\beta!(\beta-p+q)!}}.\end{aligned}$$

In particular, in this case $\tilde{\gamma}_{b,p,q}(\beta, x', \xi)$ is independent of x' and ξ.

5. Commuting Toeplitz Operators

The goal of the present section is to study the commutativity of Toeplitz operators with symbols having certain invariance properties. We will use the above notation. Fix $\alpha \in \mathbb{Z}_+^m$ with $|\alpha| = k$ as before and let $a = a_0(\mathbf{r}_1,\ldots,\mathbf{r}_m, y', \mathrm{Im} z_n - |w'|^2)$ be a bounded measurable α-quasi-nilpotent quasi-radial function on D_n. Consider the symbol:

$$b(z', w', z_n) = b_0(\mathbf{r}_1,\ldots,\mathbf{r}_m, y', \mathrm{Im}\, z_n - |w'|^2)\cdot \zeta^p\bar{\zeta}^q. \tag{5.1}$$

We calculate the operator products $RT_bT_aR^*$ and $RT_aT_bR^*$. According to Theorem 4.5 and Theorem 3.1 we have

$$\begin{aligned}RT_bT_aR^*\{c_\beta\}_{\beta\in\mathbb{Z}_+^k} &= (RT_bR^*)(RT_aR^*)\{c_\beta\}_{\beta\in\mathbb{Z}_+^k} \\ &= (RT_bR^*)\Big\{\tilde{\gamma}_{a,0,0}(\beta,x',\xi)c_\beta(x',\xi)\Big\}_{\beta\in\mathbb{Z}_+^k} \\ &= \Big\{\tilde{\gamma}_{b,p,q}(\beta,x',\xi)\tilde{\gamma}_{a,0,0}(\beta-p+q,x',\xi)c_{\beta-p+q}(x',\xi)\Big\}_{\beta\in\mathbb{Z}_+^k}.\end{aligned} \tag{5.2}$$

On the other hand it follows:

$$\begin{aligned}RT_aT_bR^*\{c_\beta\}_{\beta\in\mathbb{Z}_+^k} &= (RT_aR^*)(RT_bR^*)\{c_\beta\}_{\beta\in\mathbb{Z}_+^k} \\ &= (RT_aR^*)\Big\{\tilde{\gamma}_{b,p,q}(\beta,x',\xi)\cdot c_{\beta-p+q}(x',\xi)\Big\}_{\beta\in\mathbb{Z}_+^k} \\ &= \Big\{\tilde{\gamma}_{a,0,0}(\beta,x',\xi)\tilde{\gamma}_{b,p,q}(\beta,x',\xi)c_{\beta-p+q}(x',\xi)\Big\}_{\beta\in\mathbb{Z}_+^k}.\end{aligned} \tag{5.3}$$

Hence, we conclude from (5.2) and (5.3) that both operators T_a and T_b commute if and only if

$$\tilde{\gamma}_{a,0,0}(\beta,x',\xi) = \tilde{\gamma}_{a,0,0}(\beta-p+q,x',\xi)$$

for all $\beta \in \mathbb{Z}_+^k$. According to (4.4) this is equivalent to:

$$\begin{aligned}&\frac{1}{\beta!}\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} a_0\left(r, \frac{-x'+y'}{2\sqrt{\xi}}, v+|r|^2\right)\\ &\quad\times r^{2\beta}e^{-2\xi(v+|r|^2)-|y'|^2}v^{\lambda}rdrdy'dv\\ &\quad= \frac{(2\xi)^{-|p|+|q|}}{(\beta-p+q)!}\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} a_0\left(r, \frac{-x'+y'}{2\sqrt{\xi}}, v+|r|^2\right)\\ &\quad\quad\times r^{2(\beta-p+q)}e^{-2\xi(v+|r|^2)-|y'|^2}v^{\lambda}rdrdy'dv. \end{aligned}\tag{5.4}$$

Since $a_0(\mathbf{r}, y', \mathrm{Im} z_n - |w'|^2)$ only depends on $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_m)$ we can assume that the above integral has the form:

$$\begin{aligned}&\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} a_0\left(\mathbf{r}, \frac{-x'+y'}{2\sqrt{\xi}}, v+|r|^2\right) r^{2\beta}e^{-2\xi(v+|r|^2)-|y'|^2}\\ &\quad\times v^{\lambda}rdrdy'dv =: (*),\end{aligned}$$

where $\beta \in \mathbb{Z}_+^k$. With $e = (1, 1, \cdots, 1) \in \mathbb{Z}_+^k$ we obtain

$$\begin{aligned}(*) &= \frac{1}{2^k}\int_{\mathbb{R}^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} a_0\left(\mathbf{r}, \frac{-x'+y'}{2\sqrt{\xi}}, v+|r|^2\right)\\ &\quad\times|r^{2\beta}|e^{-2\xi(v+|r|^2)-|y'|^2}v^{\lambda}rdrdy'dv\\ &= \frac{1}{2^k}\int_{\mathbb{R}^{n-k-1}\times\mathbb{R}_+}\int_{\mathbb{R}_+^m\times\mathbb{S}^{\alpha_1-1}\times\cdots\times\mathbb{S}^{\alpha_m-1}} a_0\left(\mathbf{r}, \frac{-x'+y'}{2\sqrt{\xi}}, v+|\mathbf{r}|^2\right)\\ &\quad\times|\rho^{2\beta+e}|\cdot\left(\prod_{j=1}^m \mathbf{r}_j^{2|\beta_{(j)}|+2\alpha_j-1}\right)e^{-2\xi(v+|\mathbf{r}|^2)-|y'|^2}\\ &\quad\times v^{\lambda}d\sigma(\rho_{(1)})\ldots d\sigma(\rho_{(m)})rd\mathbf{r}dy'dv.\end{aligned}$$

In the last integral we wrote $d\sigma(\rho_{(j)})$ for the standard area measure on the sphere $\mathbb{S}^{\alpha_j-1}$. The integral over the m-fold product $\mathbb{S}^{\alpha_1-1}\times\ldots\times\mathbb{S}^{\alpha_m-1}$ can be calculated explicitly by using the following well-known formula:

Lemma 5.1. *Let $d\sigma$ denote the usual surface measure on the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ and let $\theta \in \mathbb{Z}_+^n$. Then*

$$\int_{\mathbb{S}^{n-1}}|y^{\theta}|d\sigma(y) = \frac{2\Gamma\left(\frac{\theta_1+1}{2}\right)\cdot\ldots\cdot\Gamma\left(\frac{\theta_n+1}{2}\right)}{\Gamma\left(\frac{n+|\theta|}{2}\right)}.$$

Using the formula in Lemma 5.1 we define:

$$\begin{aligned}\Theta_\beta &:= \int_{\mathbb{S}^{\alpha_1-1}\times\ldots\times\mathbb{S}^{\alpha_m-1}}|\rho^{2\beta+e}|d\sigma(\rho_{(1)})\ldots d\sigma(\rho_{(m)})\\ &= 2^m\beta!\prod_{j=1}^m\Gamma\left(\frac{\alpha_j+1}{2}+|\beta_{(j)}|\right)^{-1}.\end{aligned}\tag{5.5}$$

This finally gives:

$$(*) = \frac{\Theta_\beta}{2^k} \int_{\mathbb{R}_+^m \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} a_0 \left(\mathbf{r}, \frac{1}{2\sqrt{\xi}}(-x' + y'), v + |\mathbf{r}|^2 \right) \\ \times \mathbf{r}_1^{2|\beta_{(1)}|+2\alpha_1 - 1} \cdot \ldots \cdot \mathbf{r}_m^{2|\beta_{(m)}|+2\alpha_m - 1} e^{-2\xi(v+|\mathbf{r}|^2) - |y'|^2} v^\lambda d\mathbf{r} dy' dv.$$

Note that the last integral does not depend on the full multi-index β but rather on the values $|\beta_{(j)}|$ for $j = 1, \ldots, m$. We denote this integral by $G_a(|\beta_{(1)}|, \ldots, |\beta_{(m)}|)$. Then the commutativity condition (5.4) can be written in the form:

$$\frac{\Theta_\beta}{\beta!} G_a\big(|\beta_{(1)}|, \ldots, |\beta_{(m)}|\big) = (2\xi)^{-|p|+|q|} \frac{\Theta_{\beta-p+q}}{(\beta - p + q)!} \\ \times G_a\big(|\beta_{(1)}| - |p_{(1)}| + |q_{(1)}|, \ldots, |\beta_{(m)}| - |p_{(m)}| + |q_{(m)}|\big).$$

According to the definition (5.5) this is equivalent to:

$$G_a\big(|\beta_{(1)}|, \ldots, |\beta_{(m)}|\big) \prod_{j=1}^m \Gamma \left(\frac{\alpha_j + 1}{2} + |\beta_{(j)}| \right)^{-1} \\ = (2\xi)^{-|p|+|q|} G_a\big(|\beta_{(1)}| - |p_{(1)}| + |q_{(1)}|, \ldots, |\beta_{(m)}| - |p_{(m)}| + |q_{(m)}|\big) \\ \times \prod_{j=1}^m \Gamma \left(\frac{\alpha_j + 1}{2} + |\beta_{(j)}| - |p_{(j)}| + |q_{(j)}| \right)^{-1}.$$

This equality can be only true simultaneously for all α-quasi-nilpotent quasi-radial functions a and all $\beta \in \mathbb{Z}_+^k$ if $|p_{(j)}| = |q_{(j)}|$ for $j = 1, \ldots, m$. Hence, we obtain:

Theorem 5.2. *Let $\alpha \in \mathbb{Z}_+^m$ be given. Then the statements* (a), (b) *and* (c) *below are equivalent:*

(a) *For each α-quasi-nilpotent quasi-radial function $a = a_0(\mathbf{r}, y', \mathrm{Im} z_n - |w'|^2) \in L^\infty(D_n)$ and each α-quasi-nilpotent quasi-homogeneous function*

$$b = b_0(\mathbf{r}_1, \ldots, \mathbf{r}_m, y', \mathrm{Im}\, z_n - |w'|^2) \cdot \zeta^p \bar{\zeta}^q \in L^\infty(D_n) \tag{5.6}$$

of degree $(p, q) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k$ the Toeplitz operators T_a and T_b commute on each weighted Bergman space $\mathcal{A}_\lambda^2(D_n)$.

(b) *The equality $\tilde{\gamma}_{a,0,0}(\beta, x', \xi) = \tilde{\gamma}_{a,0,0}(\beta - p + q, x', \xi)$ holds for all $\beta \in \mathbb{Z}_+^k$ and for each α-quasi-nilpotent quasi-radial functions a.*

(c) *The equality $|p_{(j)}| = |q_{(j)}|$ holds for each $j = 1, \ldots, m$.*

Now, let us assume that $b \in L^\infty(D_n)$ is of the form (5.6). Under the assumption $|p_{(j)}| = |q_{(j)}|$, for each $j = 1, \ldots, m$, we calculate $\tilde{\gamma}_{b,p,q}(\beta, x', \xi)$ in (4.4) more explicitly by reducing the order of integration. Assume that

$\beta - p + q \in \mathbb{Z}_+^k$. Then:

$$\begin{aligned}\tilde{\gamma}_{b,p,q}(\beta, x', \xi) &= A_\beta(\xi) A_{\beta-p+q}(\xi) \chi_{\mathbb{R}_+}(\xi) \int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} r^{2(\beta+q)} \\ &\quad \times \prod_{j=1}^m \mathbf{r}_j^{-|p_{(j)}|-|q_{(j)}|} e^{-2\xi(|r|^2+v)-|\tilde{y}'|^2} b_0 \\ &\quad \times \left(\mathbf{r}, \frac{-x' + \tilde{y}'}{2\sqrt{\xi}}, v + |r|^2\right) r dr d\tilde{y}' \frac{c_\lambda v^\lambda}{4} dv \\ &= \Theta_{\beta+q} A_\beta(\xi) A_{\beta-p+q}(\xi) \chi_{\mathbb{R}_+}(\xi) 2^{-k} \\ &\quad \times \int_{\mathbb{R}_+^m \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} \prod_{j=1}^m \mathbf{r}_j^{2|\beta_{(j)}|+|q_{(j)}|-|p_{(j)}|+2\alpha_j-1} \\ &\quad \times e^{-2\xi(|\mathbf{r}|^2+v)-|\tilde{y}'|^2} b_0\left(\mathbf{r}, \frac{-x' + \tilde{y}'}{2\sqrt{\xi}}, v + |r|^2\right) d\mathbf{r} d\tilde{y}' \frac{c_\lambda v^\lambda}{4} dv \\ &= \frac{\Theta_{\beta+q}}{\Theta_\beta} \frac{A_{\beta-p+q}(\xi)}{A_\beta(\xi)} \cdot D_b(\beta, x', \xi) \\ &= \frac{(\beta+q)!}{\sqrt{\beta!(\beta-p+q)!}} \prod_{j=1}^m \frac{\Gamma\left(\frac{\alpha_j+1}{2} + |\beta_{(j)}|\right)}{\Gamma\left(\frac{\alpha_j+1}{2} + |\beta_{(j)} + q_{(j)}|\right)} \cdot D_b(\beta, x', \xi),\end{aligned}$$

where $D_b(\beta, x', \xi) = \tilde{\gamma}_{b,0,0}(\beta, x', \xi)$, which can be seen by choosing $p = q = 0$ in the above equalities. Hence we have proved:

Proposition 5.3. *Let $\alpha \in \mathbb{Z}_+^m$ be given. Assume that $b \in L^\infty(D_n)$ is of the form (5.6) and let $|p_{(j)}| = |q_{(j)}|$, for each $j = 1, \ldots, m$. Then in the case of $\beta - p + q \in \mathbb{Z}_+^k$ we have*

$$\tilde{\gamma}_{b,p,q}(\beta, x', \xi) = \frac{(\beta+q)!}{\sqrt{\beta!(\beta-p+q)!}} \prod_{j=1}^m \frac{\Gamma\left(\frac{\alpha_j+1}{2} + |\beta_{(j)}|\right)}{\Gamma\left(\frac{\alpha_j+1}{2} + |\beta_{(j)} + q_{(j)}|\right)} \cdot \tilde{\gamma}_{b,0,0}(\beta, x', \xi).$$

In the case of $\beta - p + q \notin \mathbb{Z}_+^k$ we have $\tilde{\gamma}_{b,p,q}(\beta, x', \xi) = 0$. The factor $\tilde{\gamma}_{b,0,0}(\beta, x', \xi)$ can be expressed in the form

$$\begin{aligned}\tilde{\gamma}_{b,0,0}(\beta, x', \xi) &= \Theta_\beta A_\beta^2(\xi) \chi_{\mathbb{R}_+}(\xi) 2^{-k} \int_{\mathbb{R}_+^m \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} \prod_{j=1}^m \mathbf{r}_j^{2|\beta_{(j)}|+2\alpha_j-1} \\ &\quad \times e^{-2\xi(|\mathbf{r}|^2+v)-|y'|^2} b_0\left(\mathbf{r}, \frac{-x' + y'}{2\sqrt{\xi}}, v + |\mathbf{r}|^2\right) d\mathbf{r} dy' \frac{c_\lambda v^\lambda}{4} dv.\end{aligned} \tag{5.7}$$

Let $\alpha \in \mathbb{Z}_+^m$ be given and $(p, q) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k$. From Proposition 5.3 we conclude:

Corollary 5.4. *Let $a = a_0(\mathbf{r}, y', \mathrm{Im} z_n - |w'|^2) \in L^\infty(D_n)$ be an α-quasi-nilpotent quasi-radial function. Under the assumption $|p_{(j)}| = |q_{(j)}|$ for all*

$j = 1, 2, \ldots, m$ we have

$$T_a T_{\zeta^p \bar{\zeta}^q} = T_{\zeta^p \bar{\zeta}^q} T_a = T_{a\zeta^p \bar{\zeta}^q} \tag{5.8}$$

on each weighted Bergman space.

Proof. The first equality in (5.8) is a direct consequence of Theorem 5.2. If $e(z) \equiv 1$ then $T_e = \mathrm{Id}$, and thus $\tilde{\gamma}_{e,0,0}(\beta, x', \xi) \equiv 1$. Hence, Proposition 5.3 implies that in the case of a symbol $b = \zeta^p \bar{\zeta}^q$ with $|p_{(j)}| = |q_{(j)}|$, for all $j = 1, 2, \ldots, m$, one has

$$\tilde{\gamma}_{b,p,q}(\beta, x', \xi) = \frac{(\beta + q)!}{\sqrt{\beta!(\beta - p + q)!}} \prod_{j=1}^{m} \frac{\Gamma\left(\frac{\alpha_j+1}{2} + |\beta_{(j)}|\right)}{\Gamma\left(\frac{\alpha_j+1}{2} + |\beta_{(j)} + q_{(j)}|\right)}, \tag{5.9}$$

whenever $\beta - p + q \in \mathbb{Z}_+^k$ (cf. Example 1 for the choice of $\alpha = (1, \ldots, 1) \in \mathbb{Z}_+^k$ and the case $p_j = q_j$, for $j = 1, \ldots, k$). Moreover, if $\beta - p + q \notin \mathbb{Z}_+^k$, then it holds $\tilde{\gamma}_{b,p,q}(\beta, x', \xi) = 0$. Theorem 5.2, Proposition 5.3 and the assumption that $|p_{(j)}| = |q_{(j)}|$, for all $j = 1, 2, \cdots, m$, imply now that

$$\begin{aligned}\tilde{\gamma}_{ab,p,q}(\beta, x', \xi) &= \tilde{\gamma}_{b,p,q}(\beta, x', \xi) \cdot \tilde{\gamma}_{a,0,0}(\beta, x', \xi) \\ &= \tilde{\gamma}_{b,p,q}(\beta, x', \xi) \cdot \tilde{\gamma}_{a,0,0}(\beta - p + q, x', \xi).\end{aligned}$$

This together with (5.2) and Theorem 4.5 yields the second equality in (5.8). □

6. Commutative Banach Algebras

In this section we define commutative Banach algebras of Toeplitz operators which are induced by the quasi-nilpotent group action. Given a pair of multi-indices $(p, q) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k$, we put

$$\tilde{p}_{(j)} := (0, \ldots, p_{(j)}, 0, \ldots,) \quad \text{and} \quad \tilde{q}_{(j)} := (0, \ldots, 0, q_{(j)}, 0, \ldots, 0)$$

so that $p = \tilde{p}_{(1)} + \tilde{p}_{(2)} + \ldots + \tilde{p}_{(m)}$ and $q = \tilde{q}_{(1)} + \tilde{q}_{(2)} + \ldots + \tilde{q}_{(m)}$. Consider the Toeplitz operators:

$$T_j := T_{\zeta^{\tilde{p}_{(j)}} \bar{\zeta}^{\tilde{q}_{(j)}}}$$

(cf. the notation in Definition 4.3). Now, we can prove that certain products of Toeplitz operators are Toeplitz operators again with the product symbol.

Proposition 6.1. *Let us assume that $|p_{(j)}| = |q_{(j)}|$ for all $j = 1, 2, \ldots, m$. Then the Toeplitz operators T_j commute mutually. Moreover,*

$$\prod_{j=1}^{m} T_j = T_{\zeta^p \bar{\zeta}^q} \tag{6.1}$$

on each weighted Bergman space.

Proof. Let $b_j := \zeta^{\tilde{p}_{(j)}} \bar{\zeta}^{\tilde{q}_{(j)}}$, for $j = 1, \ldots, m$. We only prove the following product rule:

$$T_j T_i = T_{\zeta^{\tilde{p}_{(i)} + \tilde{p}_{(j)}} \bar{\zeta}^{\tilde{q}_{(i)} + \tilde{q}_{(j)}}} \tag{6.2}$$

for $i, j \in \{1, \ldots, m\}$ and $i \neq j$. According to Theorem 4.5 the operator $RT_jT_iR^*$ acts on the sequence space $\ell_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))$ by the rule:

$$\begin{aligned} &RT_jT_iR^*\{c_\beta(x',\xi)\}_{\beta\in\mathbb{Z}_+^k} \\ &= RT_jR^*\Big\{\tilde{\gamma}_{b_i,\tilde{p}_{(i)},\tilde{q}_{(i)}}(\beta,x',\xi)c_{\beta-\tilde{p}_{(i)}+\tilde{q}_{(i)}}(x',\xi)\Big\}_{\beta\in\mathbb{Z}_+^k} \\ &= \Big\{\tilde{\gamma}_{b_j,\tilde{p}_{(j)},\tilde{q}_{(j)}}(\beta,x',\xi)\cdot\tilde{\gamma}_{b_i,\tilde{p}_{(i)},\tilde{q}_{(i)}}(\beta-\tilde{p}_{(j)}+\tilde{q}_{(j)},x',\xi) \\ &\quad \times c_{\beta-\tilde{p}_{(i)}-\tilde{p}_{(j)}+\tilde{q}_{(i)}+\tilde{q}_{(j)}}(x',\xi)\Big\}_{\beta\in\mathbb{Z}_+^k}. \end{aligned}$$

Hence it is clear that (6.2) is equivalent to:

$$\begin{aligned} &\tilde{\gamma}_{b_j,\tilde{p}_{(j)},\tilde{q}_{(j)}}(\beta,x',\xi)\cdot\tilde{\gamma}_{b_i,\tilde{p}_{(i)},\tilde{q}_{(i)}}(\beta-\tilde{p}_{(j)}+\tilde{q}_{(j)},x',\xi) = \\ &\quad = \tilde{\gamma}_{b_ib_j,\tilde{p}_{(i)}+\tilde{p}_{(j)},\tilde{q}_{(i)}+\tilde{q}_{(j)}}(\beta,x',\xi). \end{aligned} \tag{6.3}$$

By (5.9) we have

$$\tilde{\gamma}_{b_j,\tilde{p}_{(j)},\tilde{q}_{(j)}}(\beta,x',\xi) = \frac{(\beta_{(j)}+\tilde{q}_{(j)})!}{\sqrt{\beta_{(j)}!(\beta_{(j)}-\tilde{p}_{(j)}+\tilde{q}_{(j)})!}}\frac{\Gamma(\frac{\alpha_j+1}{2}+|\beta_{(j)}|)}{\Gamma(\frac{\alpha_j+1}{2}+|\beta_{(j)}+\tilde{q}_{(j)}|)},$$

and similar for i replaced by j. Moreover, the function on the right hand side of (6.3) has the explicit form:

$$\begin{aligned} \tilde{\gamma}_{b_ib_j,\tilde{p}_{(i)}+\tilde{p}_{(j)},\tilde{q}_{(i)}+\tilde{q}_{(j)}}(\beta,x',\xi) &= \frac{(\beta+\tilde{q}_{(i)}+\tilde{q}_{(j)})!}{\sqrt{\beta!(\beta-\tilde{p}_{(i)}-\tilde{p}_{(j)}+\tilde{q}_{(i)}+\tilde{q}_{(j)})!}} \\ &\quad \times \prod_{\ell\in\{i,j\}}\frac{\Gamma(\frac{\alpha_\ell+1}{2}+|\beta_{(\ell)}|)}{\Gamma(\frac{\alpha_\ell+1}{2}+|\beta_{(\ell)}+\tilde{q}_{(\ell)}|)}. \end{aligned}$$

Now, (6.3) can be easily checked from these identities. □

Let $\alpha \in \mathbb{Z}_+^m$ with $|\alpha| = k$ as before and consider two α-quasi-nilpotent quasi-homogeneous functions $\varphi_j \in L^\infty(D_n)$ where $j = 1, 2$. We express φ_j, for $j = 1, 2$ in the form

$$\begin{aligned} \varphi_1(z',w',z_n) &= a_1\big(\mathbf{r}_1,\ldots,\mathbf{r}_m,y',\text{Im } z_n-|w'|^2\big)\zeta^p\bar{\zeta}^q, \\ \varphi_2(z',w',z_n) &= a_2\big(\mathbf{r}_1,\ldots,\mathbf{r}_m,y',\text{Im } z_n-|w'|^2\big)\zeta^u\bar{\zeta}^v, \end{aligned}$$

where $(p,q),(u,v) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k$ with $p \perp q$ and $u \perp v$ are the degrees of φ_1 and φ_2, respectively. Moreover, assume that $|p_{(j)}| = |q_{(j)}|$ and $|u_{(j)}| = |v_{(j)}|$, for $j = 1, 2, \ldots, m$.

Theorem 6.2. *The Toeplitz operators T_{φ_1} and T_{φ_2} commute on each weighted Bergman space $\mathcal{A}_\lambda^2(D_n)$ if and only if for each $\ell = 1, 2, \ldots, k$ one of the conditions* (a) – (d) *is fulfilled:*

(a) $p_\ell = q_\ell = 0$
(b) $u_\ell = v_\ell = 0$
(c) $p_\ell = u_\ell = 0$
(d) $q_\ell = v_\ell = 0$.

Proof. Similar to the argument in the proof of Proposition 6.1 it follows that the operators T_{φ_1} and T_{φ_2} commute on $\mathcal{A}^2_\lambda(D_n)$ if and only if for all $\beta \in \mathbb{Z}^k_+$:

$$\begin{aligned}\tilde{\gamma}_{\varphi_1,p,q}(\beta, x', \xi) \cdot \tilde{\gamma}_{\varphi_2,u,v}(\beta - p + q, x', \xi)\\ = \tilde{\gamma}_{\varphi_2,u,v}(\beta, x', \xi) \cdot \tilde{\gamma}_{\varphi_1,p,q}(\beta - u + v, x', \xi).\end{aligned}$$

Since $|p_{(j)}| = |q_{(j)}|$ and $|u_{(j)}| = |v_{(j)}|$ for $j = 1, 2, \cdots, m$ we can use the factorization of $\tilde{\gamma}_{\varphi_1,p,q}(\beta, x', \xi)$ and $\tilde{\gamma}_{\varphi_2,u,v}(\beta, x', \xi)$ in Proposition 5.3:

$$\begin{aligned}\tilde{\gamma}_{\varphi_1,p,q}(\beta, x', \xi) = \Phi_{p,q}(\beta) \cdot \tilde{\gamma}_{\varphi_1,0,0}(\beta, x', \xi),\\ \tilde{\gamma}_{\varphi_2,u,v}(\beta, x', \xi) = \Phi_{u,v}(\beta) \cdot \tilde{\gamma}_{\varphi_2,0,0}(\beta, x', \xi),\end{aligned}$$

where we use the notation:

$$\Phi_{p,q}(\beta) := \frac{(\beta + q)!}{\sqrt{\beta!(\beta - p + q)!}} \prod_{j=1}^{m} \frac{\Gamma(\frac{\alpha_j+1}{2} + |\beta_{(j)}|)}{\Gamma(\frac{\alpha_j+1}{2} + |\beta_{(j)} + q_{(j)}|)}. \tag{6.4}$$

Moreover, it follows from Theorem 5.2 and again by the conditions on (p, q) and (u, v) that

$$\begin{aligned}\tilde{\gamma}_{\varphi_1,0,0}(\beta, x', \xi) = \tilde{\gamma}_{\varphi_1,0,0}(\beta - u + v, x', \xi)\\ \tilde{\gamma}_{\varphi_2,0,0}(\beta, x', \xi) = \tilde{\gamma}_{\varphi_2,0,0}(\beta - p + q, x', \xi).\end{aligned}$$

Therefore we only need to verify that

$$\Phi_{p,q}(\beta) \cdot \Phi_{u,v}(\beta - p + q) = \Phi_{u,v}(\beta) \cdot \Phi_{p,q}(\beta - u + v).$$

By a straightforward calculation this is equivalent to:

$$(\beta + q)! \frac{(\beta - p + q + v)!}{(\beta - p + q)!} = (\beta + v)! \frac{(\beta - u + v + q)!}{(\beta - u + v)!}.$$

Varying β it can be seen that this equality holds if and only if for each $\ell = 1, 2, \ldots, k$ one of the conditions (a) – (d) is fulfilled. □

Let $(p, q) \in \mathbb{Z}^k_+ \times \mathbb{Z}^k_+$ and $\alpha \in \mathbb{Z}^m_+$ such that $|\alpha| = k$. Let $h \in \mathbb{Z}^m_+$ be given with the properties:

(i) $h_j = 0$, if $\alpha_j = 1$,
(ii) $1 \le h_j \le \alpha_j - 1$, if $\alpha_j > 1$. In the case of $\alpha_{j_1} = \alpha_{j_2}$ with $j_1 < j_2$ we assume that $h_{j_1} \le h_{j_2}$.

In the following we assume that $p_{(j)}$ and $q_{(j)}$ for $j = 1, \ldots, m$ are of the particular form

$$p_{(j)} = (p_{j,1}, \ldots, p_{j,h_j}, 0, \ldots, 0) \quad \text{and} \quad q_{(j)} = (0, \ldots, 0, q_{j,h_{j+1}}, \ldots, q_{j,\alpha_j}). \tag{6.5}$$

Below we will use the data α and h to define commutative Banach algebras of Toeplitz operators. The second assumption in (ii) serves to avoid repetition of the unitary equivalent algebras.

Define $\mathcal{R}_\alpha(h)$ to be the linear space generated by all bounded measurable α-quasi-nilpotent quasi-homogeneous functions

$$b(z', w', z_n) = b_0\big(\mathbf{r}_1, \ldots, \mathbf{r}_m, y', \text{Im } z_n - |w'|^2\big) \cdot \zeta^p \bar{\zeta}^q. \tag{6.6}$$

Moreover, in (6.6) we assume that $p_{(j)}$ and $q_{(j)}$ are of the form (6.5) with:

$$p_{j,1} + \ldots + p_{j,h_j} = q_{j,h_{j+1}} + \ldots + q_{j,\alpha_j}.$$

As a corollary to Theorem 6.2 we obtain:

Theorem 6.3. *The Banach algebra generated by Toeplitz operators with symbols from $\mathcal{R}_\alpha(h)$ is commutative.*

Finally, we remark:

(a) For $k > 2$ and $\alpha \neq (1, 1, \ldots, 1)$ the commutative algebras $\mathcal{R}_\alpha(h)$ are just Banach algebras, while the C^*-algebras generated by them are noncommutative.
(b) These algebras are commutative for each weighted Bergman space $\mathcal{A}^2_\lambda(D_n)$ with $\lambda > -1$.
(c) For $k = 0$ (nilpotent case) or $k = 1, 2$ these algebras collapse to the single C^*-algebras which are generated by Toeplitz operators with quasi-nilpotent symbols $b(\mathbf{r}, y', \operatorname{Im} z_n - |z'|)$.

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