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Symmetries on Bounded Observables: A Unified Approach Based on Adjacency Preserving Maps

Peter Šemrl

Abstract. Self-adjoint operators represent bounded observables in mathematical foundations of quantum mechanics. The set of all self-adjoint operators can be equipped with several operations and relations having important interpretations in physics. Automorphisms with respect to these relations or operations are called symmetries. Many of them turn out to be real-linear up to a translation. We present a unified approach to the description of the general form of such symmetries based on adjacency preserving maps. We consider also symmetries defined on the set of all positive operators or on the set of all positive invertible operators. In particular, we will see that the structural result for adjacency preserving maps on the set of all positive invertible operators differs a lot from its counterpart on the set of all selfadjoint operators.

1. Introduction

Let H be a complex Hilbert space. We denote by $\mathcal{S}(H)$ the real-linear space of all self-adjoint bounded linear operators on H. They are important in the Hilbert space framework of quantum mechanics as they represent bounded observables. The set $\mathcal{S}(H)$ can be equipped with several relations and operations having important physical meanings. It is then of interest to study automorphisms of $\mathcal{S}(H)$ with respect to these relations and/or operations (see for example [3]). Such transformations can be viewed as certain kinds of symmetries of the underlying quantum system.

Two of the most studied relations on $\mathcal{S}(H)$ are the usual partial order defined by $A \leq B$ if and only if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$, and commutativity (or compatibility in the language of quantum mechanics). In the language of quantum mechanics, the bounded observable A is said to be less or equal to the bounded observable B if the mean value (expectation) of

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A in any state is less or equal to the mean value of B in the same state. And two bounded observables are compatible if and only if they can be measured jointly. Hence, a symmetry on $\mathcal{S}(H)$ with respect to compatibility is a bijective map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ such that for every pair $A, B \in \mathcal{S}(H)$ the operators A and B commute if and only if $\phi(A)$ and $\phi(B)$ commute. And a bijective map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ is a symmetry with respect to the usual partial order if for every pair $A, B \in \mathcal{S}(H)$ we have $A \le B$ if and only if $\phi(A) \le \phi(B)$.

There is an essential difference between the above two types of symmetries. In [15] it was proved that if H is a separable Hilbert space with $\dim H \ge 3$ and $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ a symmetry with respect to compatibility, then there exists an either unitary or antiunitary operator U on H and for every operator $A \in \mathcal{S}(H)$ there is a real valued bounded Borel function f_A on $\sigma(A)$ such that $\phi(A) = U f_A(A) U^*, A \in \mathcal{S}(H)$. Recall that $U : H \to H$ is called an antiunitary operator if it is a bijective conjugate-linear isometry. Molnár [11] proved that for every symmetry $\phi : \mathcal{S}(H) \to \mathcal{S}(H), \dim H \ge 2$, with respect to the usual partial order, there exist an operator $C \in \mathcal{S}(H)$ and an invertible bounded linear or conjugate-linear operator $T : H \to H$ such that $\phi(A) = TAT^* + C, A \in \mathcal{S}(H)$. When studying such maps there is no loss of generality in assuming that $\phi(0) = 0$. Indeed, a map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ is a symmetry with respect to the usual partial order if and only if the same is true for the map $A \mapsto \phi(A) - \phi(0), A \in \mathcal{S}(H)$. Assuming that $\phi(0) = 0$ we get $C = 0$ in the conclusion of Molnár's theorem. In particular, after this harmless normalization the symmetry ϕ with respect to $\le$ is real-linear, while this is not true for symmetries with respect to compatibility.

Quite surprisingly, many symmetries behave in this way, that is, they are real-linear (up to a translation). The results on such symmetries known so far have been proved by ad hoc methods. It is the aim of this paper to present a unified approach to this kind of results. With our method based on adjacency preserving maps we will reprove some known results and even improve some of them. Moreover, several new results will be obtained as rather easy consequences of our main theorems on adjacency preserving maps.

To explain the main idea we will consider symmetries with respect to the usual partial order on $\mathcal{S}(H)$. For the sake of simplicity we will consider only the finite-dimensional case. Thus, we are interested in bijective maps ϕ on $\mathcal{H}_n$, the real linear space of all $n \times n$ hermitian matrices, with the property that $A \le B$ if and only if $\phi(A) \le \phi(B), A, B \in \mathcal{H}_n$. Recall that two matrices $A, B \in \mathcal{H}_n$ are said to be adjacent if $\text{rank}\,(B - A) = 1$. If A and B are adjacent, then $B = A + R$ for some rank one matrix $R \in \mathcal{H}_n$. Every such matrix is of the form $R = tP$, where t is a nonzero real number and P a projection of rank one. Hence, $B - A = tP$ is either positive (when $t > 0$), or negative. Assume that $A \le B$ and take any two matrices $C, D \in [A, A + tP] = \{F \in \mathcal{H}_n \,:\, A \le F \le A + tP\}$. It is easy to verify that $[A, A + tP] = \{A + sP \,:\, 0 \le s \le t\}$. It follows that either $C \le D$, or $D \le C$. To summarize, we have shown that if A and B are adjacent, then A and B are comparable and if C and D are any two matrices from the interval between A and B, then C and D are comparable as well. It turns out that the converse statement is true as well. Thus, we have a characterization of the adjacency relation expressed in terms of the relation $\leq$. And clearly, as ϕ preserves the order $\leq$ in both directions, it must preserve also the adjacency in both directions. So, we have reduced our original problem of describing the general form of bijective maps on $\mathcal{H}_n$ preserving the usual order in both directions to the problem of characterizing bijective maps on $\mathcal{H}_n$ preserving adjacency in both directions. It turns out that such a reduction is possible, and rather easy, when dealing with various symmetries.

We will be interested in symmetries defined on the set of all bounded linear self-adjoint operators on a Hilbert space H, as well as in symmetries acting on positive or invertible positive operators. So, in order to use the above idea for studying symmetries we will first study maps preserving adjacency defined either on $\mathcal{S}(H)$, or on the subset of all positive operators, or on the subset of all invertible positive operators. Results on such maps presented in the next section are of independent interest.

Surprisingly, it turned out that the characterizations of adjacency preserving maps on the three underlying sets, namely the set of all bounded linear self-adjoint operators, the subset of all positive operators, and the subset of all positive invertible operators, are essentially different. The details will be given in the next section. Let us just briefly mention that when dealing with adjacency preserving maps on the set of positive operators we need much stronger assumptions to get the same result as in the case of adjacency preserving maps defined on the set of all self-adjoint operators. Of course, the indispensability of stronger assumptions will be illustrated by counterexamples. To briefly explain the difference between the set of positive invertible operators and the set of all self-adjoint operators note that $A \mapsto A^{-1}$ is a bijective map on the set of all positive invertible operators that preserves adjacency in both directions. Indeed, $B - A$ is an operator of rank one if and only if $B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$ is of rank one. However, the map $A \mapsto A^{-1}$ cannot be extended to the set of all positive operators or to the set of all self-adjoint operators in some natural way (and it will turn out that it cannot be extended in such a way that the adjacency preserving property is not lost). So, the characterization of adjacency preserving maps on the set of positive invertible operators will be essentially different from the other two cases.

Adjacency preserving maps have been already studied in the finite-dimensional case but only on the set of all hermitian matrices. As far as we know the cases of positive matrices and invertible positive matrices have not been treated so far. So, we will open a completely new direction in the study of adjacency preserving maps. Several ideas used in the next section are already known. However, many new ideas are needed, especially because of the differences mentioned above.

The last section will be devoted to applications in the theory of symmetries. We will reduce the problem of describing the general form of various symmetries to the problem of characterizing adjacency preserving maps. Then we will apply the results from the second section to reprove or improve several known results as well as to prove a few new theorems.

Let us fix the notation. By H we will always denote a Hilbert space. The inner product will be denoted by $\langle \cdot, \cdot \rangle$. A self-adjoint bounded linear operator $A : H \to H$ is said to be positive if $A \geq 0$, that is, $\langle Ax, x \rangle \geq 0$ for every $x \in H$. When positive A is invertible we write $A > 0$. Then, of course, $A > B$ means that $A - B$ is a bounded invertible positive operator. The symbols $\mathcal{S}(H), \mathcal{S}(H)^{\geq 0}, \mathcal{S}(H)^{>0}, \mathcal{S}_F(H), \mathcal{S}_F(H)^{\geq 0}$, and $\mathcal{S}_F(H)^{>-I}$ stand for the real-linear space of all self-adjoint bounded linear operators on H, the set of all positive self-adjoint bounded linear operators on H, the set of all invertible positive self-adjoint bounded linear operators on H, the real-linear space of all self-adjoint bounded linear operators on H of finite rank, the set of all positive self-adjoint bounded linear operators on H of finite rank, and the set of all bounded self-adjoint finite rank operators A satisfying $A > -I$, respectively. Let $\mathcal{V}$ be any of the above sets. Operators $A, B \in \mathcal{V}$ are adjacent, $A \sim B$, if $B - A$ is a rank one operator. A map $\phi : \mathcal{V} \to \mathcal{V}$ preserves adjacency if for every pair $A, B \in \mathcal{V}$ we have: $A \sim B \Rightarrow \phi(A) \sim \phi(B)$. The map ϕ preserves adjacency in both directions if for every pair $A, B \in \mathcal{V}$ the operators A and B are adjacent if and only if $\phi(A)$ and $\phi(B)$ are adjacent. Let $x \in H$ be a nonzero vector. By $x \otimes x^*$ we denote the rank one bounded linear operator on H defined by $(x \otimes x^*)z = \langle z, x \rangle x$, $z \in H$. Note that $x \otimes x^*$ is a projection if and only if $\|x\| = 1$. Every bounded linear self-adjoint operator of rank one can be written as $tx \otimes x^*$ for some nonzero real number t and some nonzero vector $x \in H$.

2. Adjacency Preserving Maps on Finite Rank Self-Adjoint Operators

2.1. Statement of Main Results

We start with a simple observation. Let H be an infinite-dimensional Hilbert space. Choose and fix an invertible bounded linear operator $T : H \to H$ and define $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ by $\phi(A) = TAT^*$ whenever A is a finite rank operator and $\phi(A) = A$ otherwise. Since two adjacent operators $A, B \in \mathcal{S}(H)$ are either both of finite rank, or both not of finite rank, the bijective map ϕ preserves adjacency in both directions.

More generally, define an equivalence relation on $\mathcal{S}(H)$ by $A \equiv B$ if and only if $A - B$ is a finite rank operator, $A, B \in \mathcal{S}(H)$. Clearly, $A \equiv B$ if and only if there exists a finite sequence of self-adjoint operators $A = A_0, A_1, \ldots, A_k = B$ such that $A_{j-1} \sim A_j$ for all $j = 1, \ldots, k$. Denote by $[A]$ the equivalence class of $A \in \mathcal{S}(H)$ (note that the equivalence classes are cosets of the additive subgroup $\mathcal{S}_F(H) \subset \mathcal{S}(H)$). If $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ preserves adjacency then

$$\phi([A]) \subset [\phi(A)], \quad A \in \mathcal{S}(H).$$

Indeed, if $B \in [A]$ and $A_0, \ldots, A_k$ is a sequence of self-adjoint operators as above, then $\phi(A) \sim \phi(A_1), \phi(A_1) \sim \phi(A_2), \ldots, \phi(A_{k-1}) \sim \phi(B)$, and consequently, $\phi(B) - \phi(A)$ is a finite rank operator. Hence, ϕ maps each equivalence class into some equivalence class and the behavior of ϕ on different equivalent classes is completely unrelated. Assume that $\phi([A]) \subset [B]$. Define ψ by

$\psi(C) = \phi(A + C) - B, C \in \mathcal{S}_F(H)$. Obviously, ψ maps $\mathcal{S}_F(H)$ into itself and preserves adjacency. So, when studying adjacency preserving maps on $\mathcal{S}(H)$, it is enough to restrict our attention to adjacency preserving maps from $\mathcal{S}_F(H)$ into itself.

We continue by giving some examples of adjacency preserving maps $\phi : \mathcal{S}_F(H) \to \mathcal{S}_F(H)$. Recall that a map $T : H \to H$ is conjugate-linear if it is additive and $T(\lambda x) = \overline{\lambda} Tx, \lambda \in \mathbb{C}, x \in H$. If T is conjugate-linear and bounded, then there exists a unique bounded conjugate-linear map $T^* : H \to H$ such that

$$\langle Tx, y\rangle = \overline{\langle x, T^* y\rangle}$$

for all $x, y \in H$.

Let $T : H \to H$ be an injective linear or conjugate-linear (not necessarily bounded) map. Every $A \in \mathcal{S}_F(H)$ can be written as

$$A = \sum_{j=1}^{k} t_j x_j \otimes x_j^*,$$

where k is a positive integer, $t_1, \ldots, t_k \in \mathbb{R}$, and $x_1, \ldots, x_k \in H$. Set

$$\phi(A) = \phi\left(\sum_{j=1}^{k} t_j x_j \otimes x_j^*\right) = \sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^*.$$

In order to see that $\phi : \mathcal{S}_F(H) \to \mathcal{S}_F(H)$ is well-defined we have to show that

$$\sum_{j=1}^{m} t_j x_j \otimes x_j^* = 0$$

yields

$$\sum_{j=1}^{m} t_j (Tx_j) \otimes (Tx_j)^* = 0.$$

We may assume with no loss of generality that all t_j's and all x_j's are non-zero. After rearranging the vectors $x_1, \ldots, x_m$ we may, and we will assume that $x_1, \ldots, x_r$ are linearly independent vectors and

$$x_p = \sum_{j=1}^{r} \lambda_{pj} x_j,$$

$p = r + 1, \ldots, m$. It follows that

$$\begin{aligned} 0 &= \sum_{j=1}^{r} t_j x_j \otimes x_j^* + \sum_{p=r+1}^{m} \left(t_p \left(\sum_{k=1}^{r} \lambda_{pk} x_k \right) \otimes \left(\sum_{j=1}^{r} \overline{\lambda_{pj}} x_j^* \right) \right) \\ &= \sum_{j=1}^{r} t_j x_j \otimes x_j^* + \sum_{p=r+1}^{m} \sum_{k=1}^{r} \sum_{j=1}^{r} t_p \lambda_{pk} \overline{\lambda_{pj}} x_k \otimes x_j^* \\ &= \sum_{j=1}^{r} \left(t_j x_j + \sum_{k=1}^{r} \sum_{p=r+1}^{m} t_p \lambda_{pk} \overline{\lambda_{pj}} x_k \right) \otimes x_j^*. \end{aligned}$$

This yields

$$t_j x_j + \sum_{k=1}^{r} \sum_{p=r+1}^{m} t_p \lambda_{pk} \overline{\lambda_{pj}} x_k = 0$$

for all $j = 1, \dots, r$. Using the fact that $x_1, \dots, x_r$ are linearly independent we finally arrive at

$$\sum_{p=r+1}^{m} t_p \lambda_{pk} \overline{\lambda_{pj}} = 0, \quad j, k \in \{1, \dots, r\}, \quad j \neq k,$$

and

$$t_j + \sum_{p=r+1}^{m} t_p |\lambda_{pj}|^2 = 0, \quad j \in \{1, \dots, r\}.$$

It is now straightforward to verify that

$$\sum_{j=1}^{m} t_j (Tx_j) \otimes (Tx_j)^* = 0.$$

Thus, ϕ is a well-defined map on $\mathcal{S}_F(H)$. Clearly, $\phi(0) = 0$.

Let $A, B \in \mathcal{S}_F(H)$ be adjacent. Then $A = \sum_{j=1}^{k} t_j x_j \otimes x_j^*$ for some positive integer $k, t_1, \dots, t_k \in \mathbb{R}$, and $x_1, \dots, x_k \in H$. Since A and B are adjacent, we necessarily have $B = (\sum_{j=1}^{k} t_j x_j \otimes x_j^*) + sy \otimes y^*$ for some nonzero $s \in \mathbb{R}$ and nonzero $y \in H$. It follows that $\phi(A)$ and $\phi(B)$ are adjacent as well. Hence, ϕ is an adjacency preserving map and the same is true for the map $A \mapsto -\phi(A)$.

Another example of an adjacency preserving map on $\mathcal{S}_F(H)$ can be obtained in the following way. Let $R \in \mathcal{S}_F(H)$ be a rank one operator and $\rho : \mathcal{S}_F(H) \to \mathbb{R}$ a function with the property that $\rho(A) \neq \rho(B)$ whenever A and B are adjacent. It is then clear that the map $\phi : \mathcal{S}_F(H) \to \mathcal{S}_F(H)$ defined by $\phi(A) = \rho(A)R$ preserves adjacency. Observe that such functions ρ exist. As an example choose $\rho(A) = \operatorname{tr} A$, where $\operatorname{tr} A$ denotes the trace of A. If $A, B \in \mathcal{S}_F(H)$ are adjacent, then $B = A + Q$ for some rank one operator $Q \in \mathcal{S}_F(H)$. Note that $\operatorname{tr} Q \neq 0$. Hence, $\rho(B) - \rho(A) = \operatorname{tr} Q \neq 0$.

Let $C \in \mathcal{S}_F(H)$ be any operator. The translation $A \mapsto A + C, A \in \mathcal{S}_F(H)$, is yet another example of an adjacency preserving map on $\mathcal{S}_F(H)$. When studying adjacency preserving maps $\phi : \mathcal{S}_F(H) \to \mathcal{S}_F(H)$ there is no loss of generality in assuming that $\phi(0) = 0$. Indeed, the map $\phi : \mathcal{S}_F(H) \to \mathcal{S}_F(H)$ preserves adjacency if and only if the map $A \mapsto \phi(A) - \phi(0)$ preserves adjacency. After this harmless normalization the following characterization of adjacency preserving maps on $\mathcal{S}_F(H)$ can be proved.

Theorem 2.1. *Let H be a Hilbert space,* $\dim H \geq 2$, *and let $\phi : \mathcal{S}_F(H) \to \mathcal{S}_F(H)$ be a map satisfying $\phi(0) = 0$. Assume that $\phi(A)$ and $\phi(B), A, B \in \mathcal{S}_F(H)$, are adjacent whenever A and B are adjacent. Then either*

- *there exists a rank one operator $R \in \mathcal{S}(H)$ such that the range of ϕ is contained in the linear span of R; or*

- *there exists an injective linear or conjugate-linear map* $T : H \to H$ *such that*

$$\phi\left(\sum_{j=1}^{k} t_j x_j \otimes x_j^*\right) = \sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^*$$

for every $\sum_{j=1}^{k} t_j x_j \otimes x_j^* \in \mathcal{S}_F(H)$*; or*

- *there exists an injective linear or conjugate-linear map* $T : H \to H$ *such that*

$$\phi\left(\sum_{j=1}^{k} t_j x_j \otimes x_j^*\right) = -\sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^*$$

for every $\sum_{j=1}^{k} t_j x_j \otimes x_j^* \in \mathcal{S}_F(H)$*.*

It should be mentioned here that the special case when $\dim H < \infty$ has been already proved in [7]. The general case is a rather easy consequence.

We now turn to adjacency preserving maps acting on $\mathcal{S}_F(H)^{\geq 0}$. If $T : H \to H$ is an injective linear or conjugate-linear map, then clearly, the mapping

$$\sum_{j=1}^{k} t_j x_j \otimes x_j^* \mapsto \sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^*$$

is well-defined and maps $\mathcal{S}_F(H)^{\geq 0}$ into itself. One would expect that we can get a result analogous to the above one for adjacency preserving mappings $\phi : \mathcal{S}_F(H)^{\geq 0} \to \mathcal{S}_F(H)^{\geq 0}$ with the only essential difference that the third possibility cannot occur. Surprisingly, this is not the case. To see this note that for every $A \geq 0$, the operator $I + A$ is invertible and $(I + A)^{-1} \geq 0$, and consider the map $\xi : \mathcal{S}(H)^{\geq 0} \to \mathcal{S}(H)^{\geq 0}$ defined by $\xi(A) = (I + A)^{-1}, A \in \mathcal{S}(H)^{\geq 0}$. Let us show that ξ preserves adjacency in both directions. For any pair $A, B \in \mathcal{S}(H)^{\geq 0}$ we have

$$\begin{aligned}\xi(A) - \xi(B) &= (I + A)^{-1} - (I + B)^{-1} \\ &= (I + A)^{-1}((I + B) - (I + A))(I + B)^{-1} \\ &= (I + A)^{-1}(B - A)(I + B)^{-1},\end{aligned}$$

and hence, $\xi(A)$ and $\xi(B)$ are adjacent if and only if A and B are adjacent.

Let us now consider the map $\phi : \mathcal{S}(H)^{\geq 0} \to \mathcal{S}(H)$ defined by $\phi(A) = \xi(\xi(A)) - \frac{1}{2}I$. It is well-known that for invertible positive self-adjoint operators A, B we have $A \leq B$ if and only if $B^{-1} \leq A^{-1}$ (to see this observe that $A \leq B$ if and only if $I \leq A^{-1/2}BA^{-1/2}$ which is equivalent to $A^{1/2}B^{-1}A^{1/2} = (A^{-1/2}BA^{-1/2})^{-1} \leq I$; and finally $A^{1/2}B^{-1}A^{1/2} \leq I$ holds if and only if $B^{-1} \leq A^{-1}$). It follows easily that $A \geq 0$ yields $\phi(A) \geq 0$. Hence, ϕ maps $\mathcal{S}(H)^{\geq 0}$ into itself. Moreover, $\phi(0) = 0$. Clearly, $A \in \mathcal{S}(H)^{\geq 0}$ is of finite rank if and only if there is a finite chain of operators $0 = A_0, A_1, \ldots, A_k = A$ such that each two consequentive members of this chain are adjacent. Thus the restriction of ϕ to $\mathcal{S}_F(H)^{\geq 0}$ is actually a map from $\mathcal{S}_F(H)^{\geq 0}$ into itself preserving adjacency in both directions satisfying the additional property that

$\phi(0) = 0$. Hence, we need stronger assumptions as in Theorem 2.1 in order to get the analogous conclusion for adjacency preserving maps on $\mathcal{S}_F(H)^{\geq 0}$. Here is our result on such maps.

Theorem 2.2. *Let H be a Hilbert space,* $\dim H \geq 2$*, and let $\phi : \mathcal{S}_F(H)^{\geq 0} \to \mathcal{S}_F(H)^{\geq 0}$ be a bijective map preserving adjacency in both directions. Then there exists a bijective linear or conjugate-linear map $T : H \to H$ such that*

$$\phi\left(\sum_{j=1}^{k} t_j x_j \otimes x_j^*\right) = \sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^*$$

for every $\sum_{j=1}^{k} t_j x_j \otimes x_j^* \in \mathcal{S}_F(H)^{\geq 0}$.

We have shown that the conclusion of the above statement does not hold in the absence of the bijectivity assumption (even if we add the assumption that the zero operator is mapped into itself). It would be interesting to know whether the same conclusion can be obtained in the presence of the bijectivity assumption but under the weaker assumption that adjacency is preserved in one direction only. Note also that we have not assumed that $\phi(0) = 0$ as this property turns out to be automatically fulfilled under our assumptions.

When dealing with adjacency preserving maps on the remaining set $\mathcal{S}(H)^{>0}$ we enter two problems. Even if we impose the strongest assumptions being natural when dealing with this kind of problems, that is, that $\phi : \mathcal{S}(H)^{>0} \to \mathcal{S}(H)^{>0}$ is both bijective and preserves adjacency in both directions, we cannot conclude that ϕ is additive as in the above two theorems. Another problem is that there are no finite rank operators in $\mathcal{S}(H)^{>0}$ whenever H is infinite-dimensional, and thus, we have to slightly modify our approach in this case. Replacing ϕ by $A \mapsto \phi(I)^{-1/2}\phi(A)\phi(I)^{-1/2}$ we may assume without loss of generality that ϕ is unital, that is, $\phi(I) = I$. We will see later that as far as the applications are considered, this is a harmless normalization. Denote by $\mathcal{S}_F(H)^{>-I}$ the set of all linear finite rank bounded self-adjoint operators on H whose all eigenvalues are $> -1, \mathcal{S}_F(H)^{>-I} = \{A \in \mathcal{S}_F(H) : I + A$ is positive and invertible$\}$. It is not difficult to verify that if $\psi : \mathcal{S}(H)^{>0} \to \mathcal{S}(H)^{>0}$ is a bijective unital map preserving adjacency in both directions, then the map ϕ defined by $\phi(A) = \psi(I + A) - I, A \in \mathcal{S}_F(H)^{>-I}$, is a bijective map from $\mathcal{S}_F(H)^{>-I}$ onto itself preserving adjacency in both directions with $\phi(0) = 0$. Indeed, all we need to observe is that $\psi(I + A)$ is a sum of the identity operator and a finite rank operator; this is true because ψ is unital and preserves adjacency. Before formulating the last result in this subsection we need to recall the definition of an antiunitary operator on a Hilbert space H. A map $U : H \to H$ is called an antiunitary operator if it is a bijective conjugate-linear isometry.

Theorem 2.3. *Let H be a Hilbert space,* $\dim H \geq 2$*, and let $\phi : \mathcal{S}_F(H)^{>-I} \to \mathcal{S}_F(H)^{>-I}$ be a bijective map preserving adjacency in both directions. Assume that $\phi(0) = 0$. Then there exists a unitary or an antiunitary operator $U : H \to H$ such that either*

$$\phi(A) = UAU^*$$

for every $A \in \mathcal{S}_F(H)^{>-I}$*; or*

$$\phi(A) = U(I + A)^{-1}U^* - I$$

for every $A \in \mathcal{S}_F(H)^{>-I}$.

2.2. Preliminary Results

Throughout this section H will be a Hilbert space with $\dim H \geq 2$ and $\mathcal{V}$ will denote any of the three sets $\mathcal{S}_F(H), \mathcal{S}_F(H)^{\geq 0}, \mathcal{S}_F(H)^{>-I}$. We define the arithmetic distance on $\mathcal{S}_F(H)$ by $d(A, B) = \operatorname{rank}(A-B), A, B \in \mathcal{S}_F(H)$. Thus, $A, B \in \mathcal{S}_F(H)$ are adjacent if and only if $d(A, B) = 1$. It is easy to verify that $\mathcal{S}_F(H)$ equipped with the arithmetic distance d is a metric space. All we need to verify is that the rank is subadditive, that is, $\operatorname{rank}(A+B) \leq \operatorname{rank} A + \operatorname{rank} B, A, B \in \mathcal{S}_F(H)$, which is true because $\operatorname{Im}(A + B) \subset \operatorname{Im} A + \operatorname{Im} B$.

Our first goal is to prove the following lemma.

Lemma 2.4. *Let* $\phi : \mathcal{V} \to \mathcal{V}$ *be an adjacency preserving map. Then it is a contraction with respect to the arithmetic distance, that is,*

$$d(\phi(A), \phi(B)) \leq d(A, B)$$

for all $A, B \in \mathcal{V}$.

Proof. All we need to show is that for every pair $A, B \in \mathcal{V}$ we can find a sequence of operators $A = A_0, A_1, \ldots, A_r = B$, such that $A_j \in \mathcal{V}, j = 0, 1, \ldots, r, A_{j-1}$ and A_j are adjacent for all $j = 1, \ldots, r$. Here, r denotes the arithmetic distance between A and $B, r = d(A, B)$. Assume for a moment that we have already proved this. Then

$$\begin{aligned} d(\phi(A), \phi(B)) &\leq d(\phi(A_0), \phi(A_1)) + d(\phi(A_1), \phi(A_2)) + \cdots + d(\phi(A_{r-1}), \phi(A_r)) \\ &= 1+1 + \cdots + 1 = r = d(A, B). \end{aligned}$$

Hence, assume that $A, B \in \mathcal{V}$ and $d(A, B) = r$. We need to prove the existence of a sequence of operators as described above. We will do this by induction on r. The cases $r = 0$ and $r = 1$ are trivial. So, assume that $r > 1$ and that $A = B + R$ with $\operatorname{rank} R = r$ (observe, that R does not belong to $\mathcal{V}$ in general). Let $R = R_1 - R_2$, where $R_1, R_2 \in \mathcal{S}_F(H)^{\geq 0}$ and $r = r_1 + r_2$ with $r_j = \operatorname{rank} R_j, j = 1, 2$ (such a decomposition is a trivial consequence of the spectral theorem for hermitian matrices). At least one of the operators R_1 and R_2 is nonzero. We will consider only the case when R_2 is nonzero (the case when R_1 is nonzero can be treated in exactly the same way with the roles of A and B interchanged). Then we can write $R_2 = T_1 + T_2$, where $T_1, T_2 \in \mathcal{S}_F(H)^{\geq 0}, \operatorname{rank} T_1 = 1$, and $\operatorname{rank} T_2 = r_2 - 1$. Clearly, A is adjacent to $A+T_1$ and $A+T_1 \in \mathcal{V}$ because $A+T_1 \geq A$. Moreover, $d(A+T_1, B) = r-1$. We can now complete the proof using the induction hypothesis. □

In particular, if $\phi : \mathcal{V} \to \mathcal{V}$ is a bijective map preserving adjacency in both directions, then we have

$$d(\phi(A), \phi(B)) = d(A, B)$$

for every pair $A, B \in \mathcal{V}$.

We will often use the following well-known and simple fact. We will include a short proof for the sake of completeness.

Lemma 2.5. *Let $A, B \in \mathcal{S}_F(H)$ and assume that* $\operatorname{rank}(A+B) = \operatorname{rank} A + \operatorname{rank} B$*. Then*

$$\operatorname{Im}(A+B) = \operatorname{Im} A \oplus \operatorname{Im} B.$$

Proof. Clearly, $\operatorname{Im}(A+B) \subset \operatorname{Im} A + \operatorname{Im} B$. To complete the proof we only need to compare the dimensions of the above three subspaces. □

Here are two simple consequences.

Corollary 2.6. *Let m, n be positive integers and $P, Q \in \mathcal{S}_F(H)$ operators such that $\operatorname{rank} P = m+n, \operatorname{rank} Q = n$, and $d(P,Q) = m$. Assume that P is a projection. Then Q and $P-Q$ are orthogonal projections, that is, $PQ = QP = Q$.*

Proof. It follows from $d(P,Q) = m$ that $R = P - Q$ is an operator of rank m. By Lemma 2.5 we have $\operatorname{Im} P = \operatorname{Im} Q \oplus \operatorname{Im} R$. Let $x \in \operatorname{Ker} P$. Then $Px = 0 = Qx + Rx$, and since the image of P is a direct sum of $\operatorname{Im} Q$ and $\operatorname{Im} R$, we conclude that $Qx = Rx = 0$. For any vector $x \in \operatorname{Im} Q \subset \operatorname{Im} P$ we have $Px = x = Qx + Rx$. It follows that $Rx = 0$ and $Qx = x$. Similarly, $R = P - Q$ acts like the identity on $\operatorname{Im} R$ while $Qx = 0$ for every $x \in \operatorname{Im} R$. The conclusion of the corollary follows easily. □

Corollary 2.7. *Let m, n be positive integers and $A, B \in \mathcal{S}_F(H)$ operators such that $\operatorname{rank} A = m+n, \operatorname{rank} B = n$, and $d(A,B) = m$. If $A \geq 0$, then $B \geq 0$. If $A \leq 0$, then $B \leq 0$.*

Proof. If $A \geq 0$, then we can find an invertible bounded linear operator $T : H \to H$ such that TAT^* is a projection. Applying the previous corollary for the pair of operators TAT^* and TBT^*, we conclude that TBT^* is a projection as well, yielding that $B \geq 0$. To prove the second part of the statement we replace A and B by $-A$ and $-B$, respectively. □

Another simple observation that will be frequently used in our proofs is the following one. Let $A, B \in \mathcal{S}_F(H)$ be rank one operators. Then A and B are adjacent if and only if $A = tB$ for some real number $t \neq 1$. Indeed, if $A = rx \otimes x^*$ and $B = sy \otimes y^*, r, s \in \mathbb{R} \setminus \{0\}$, and $x, y \in H$ are linearly independent, then it is straightforward to check that $rx \otimes x^* - sy \otimes y^*$ is an operator of rank two.

For any two adjacent operators $A, B \in \mathcal{S}_F(H)$ the line $l(A,B)$ joining A and B is defined to be the set consisting of A, B, and all $C \in \mathcal{S}_F(H)$ which are adjacent to both A and B. It is clear that $C \in l(A,B), C \notin \{A, B\}$, if and only if $C - A$ is adjacent to both 0 and $B - A$, that is, $C - A$ is a rank one operator adjacent to the rank one operator $B - A$. By the previous paragraph we have $l(A,B) = \{A + t(B-A) : t \in \mathbb{R}\}$.

If $\phi : \mathcal{S}_F(H) \to \mathcal{S}_F(H)$ is an adjacency preserving map and $A, B \in \mathcal{S}_F(H)$ is an adjacent pair of operators, then clearly $\phi(l(A,B)) \subset l(\phi(A), \phi(B))$ and the restriction of ϕ to the line $l(A,B)$ is injective.

We continue with a series of elementary and rather simple statements.

Lemma 2.8. *Let $A, B \in \mathcal{S}_F(H)$ with* $\operatorname{rank} A = 1$*. Assume that* $\operatorname{rank}(A + tB) = 1$ *for every real number t. Then $B = 0$.*

Proof. We have $\operatorname{rank} B \leq 1$, since otherwise $\operatorname{rank}(A + tB) > 1$ for large real numbers t. Thus, B is either 0, or a rank one operator. We must show that the second possibility cannot occur. Assume on the contrary that B is of rank one. Then, since $\operatorname{rank}(A+B) = 1$, we have $B = rA$ for some nonzero real number r. It follows that $A + (-\frac{1}{r})B = 0$, a contradiction. □

Lemma 2.9. *Let $k \geq 3, t_1, \ldots, t_k \in \mathbb{R} \setminus \{0\}$, and let $x_1, \ldots, x_k \in H$ be orthonormal vectors. Assume that $A \in \mathcal{S}_F(H)$ is an operator of rank k such that*

$$d(A, t_j x_j \otimes x_j^*) = k - 1, \quad j = 1, \ldots, k,$$

and

$$A \sim \sum_{m \neq j} t_m x_m \otimes x_m^*, \quad j = 1, \ldots, k.$$

Then $A = \sum_{m=1}^{k} t_m x_m \otimes x_m^*$.

Proof. We have $\operatorname{rank} A = k = 1+(k-1) = \operatorname{rank}(t_j x_j \otimes x_j^*) + \operatorname{rank}(A - t_j x_j \otimes x_j^*), j = 1, \ldots, k$. Then by Lemma 2.5, $x_j \in \operatorname{Im} A, j = 1, \ldots, k$. It follows that $\operatorname{Im} A = \operatorname{span}\{x_1, \ldots, x_k\}$. Thus, we can consider only the restrictions of all the considered operators to the subspace $\operatorname{span}\{x_1, \ldots, x_k\}$. We can identify these restrictions with $k \times k$ hermitian matrices representing these operators with respect to the orthonormal basis $\{x_1, \ldots, x_k\}$. The result now follows from [17, Lemma 2.6]. □

Lemma 2.10. *Let $A, B \in \mathcal{S}_F(H)$ be operators such that* $\operatorname{rank} A = \operatorname{rank} B$ *and* $A \sim B$*. Then* $\operatorname{Im} A = \operatorname{Im} B$*.*

Proof. Let $\operatorname{rank} A = \operatorname{rank} B = r$. Then we can write $A = \sum_{j=1}^{r} t_j x_j \otimes x_j^*$, where the t_j's are nonzero real numbers, and $\{x_1, \ldots, x_r\}$ is an orthonormal set of vectors whose linear span is equal to $\operatorname{Im} A$. As B is adjacent to A we have $B = \sum_{j=1}^{r} t_j x_j \otimes x_j^* + sy \otimes y^*$ for some nonzero $s \in \mathbb{R}$ and some nonzero $y \in H$. If y was linearly independent of $x_1, \ldots, x_r$, then we would have $\operatorname{rank} B = r+1$, a contradiction. Thus, $y \in \operatorname{span}\{x_1, \ldots, x_r\}$, and consequently, $\operatorname{Im} B \subset \operatorname{Im} A$. By symmetry, we have $\operatorname{Im} A = \operatorname{Im} B$. □

Lemma 2.11. *Let $A, B \in \mathcal{S}_F(H)$ be an adjacent pair of operators. Then either all elements of the line $l(A, B)$ have the same rank, or there exist $D \in l(A, B)$ and a positive integer r such that* $\operatorname{rank} C = r$ *for every $C \in l(A, B) \setminus \{D\}$ and* $\operatorname{rank} D = r - 1$*.*

Proof. As each $C \in l(A, B), C \neq A$, is adjacent to A, we have $\operatorname{rank} C \leq \operatorname{rank} A + 1$. Set $r = \max\{\operatorname{rank} C : C \in l(A, B)\}$. Choose $G \in l(A, B)$ such that $\operatorname{rank} G = r$. Clearly, $l(A, B) = \{G + tP : t \in \mathbb{R}\}$ for some projection P of rank one. Moreover, $\operatorname{Im} P \subset \operatorname{Im} G$, since otherwise we would have $\operatorname{rank}(G + tP) > \operatorname{rank} G$ for every nonzero real number t. Hence, all operators from our line are equal to zero on the orthogonal complement of $\operatorname{Im} G$. And therefore, we can consider only their restrictions to the r-dimensional subspace $\operatorname{Im} G$. This means that we can identify them with $r \times r$ hermitian matrices. When identifying operators with matrices we are free to choose any orthonormal basis of $\operatorname{Im} G$. We choose the basis in such a way that P is represented by the matrix E_{11}, that is, the $r\times r$ matrix with all entries equal to zero but the $(1,1)$-entry that is equal to 1. Then $\det(G+tP) = \det(G+tE_{11})$ is obviously a polynomial $p(t)$ of degree at most one. Clearly, $p(0) \neq 0$. So, if p is a constant polynomial, then all members of $l(A,B)$ have rank r, and if p is of degree one, then there exists $t_0 \in \mathbb{R}$ such that $G+tP$ is of rank r for all $t \neq t_0$ and $det(G+t_0P) = 0$. Obviously, $\operatorname{rank}(G+t_0P) = r-1$. □

Lemma 2.12. *Let nonzero $A, B \in \mathcal{S}_F(H), A \neq B$, be operators such that* $\operatorname{Im} A = \operatorname{Im} B$. *Then there exist a positive integer m and a sequence of operators $A = A_0, A_1, \ldots, A_m = B$ such that* $\operatorname{Im} A_0 = \operatorname{Im} A_1 = \ldots = \operatorname{Im} A_m$, *all the pairs $A_{k-1}, A_k, k = 1, \ldots, m$, are adjacent, and for every $k = 1, \ldots, m$ there exists $C_k \in l(A_{k-1}, A_k)$ such that* $\operatorname{rank} C_k = \operatorname{rank} A - 1$.

Proof. Once again we can consider the restrictions of A and B to $\operatorname{Im} A$ and identify these restrictions with invertible matrices. Then the result follows from [6, Lemmas 2.5 and 2.6]. □

Let $A \in \mathcal{S}_F(H)^{>-I}$ be an operator of rank 2. Then A has two nonzero eigenvalues (we count eigenvalues with their multiplicities) and thus, we have two possibilities: either both nonzero eigenvalues have the same sign, or they have the opposite signs. We would like to characterize these two cases using the adjacency relation. For this purpose we define J_A to be the set of all rank one operators $R \in \mathcal{S}_F(H)^{>-I}$ with the property that there exists a real number t such that $tR \in \mathcal{S}_F(H)^{>-I}$ and $d(tR, A) = 1$. Clearly, if $R \in J_A$ and $sR \in \mathcal{S}_F(H)^{>-I}$ for some nonzero real s, then sR belongs to J_A as well.

We first claim that if $A \in \mathcal{S}_F(H)^{>-I}$ is an operator of rank 2, then

$$J_A \subset \{R \in \mathcal{S}_F(H)^{>-I} : \operatorname{rank} R = 1 \quad \text{and} \quad \operatorname{Im} R \subset \operatorname{Im} A\}. \tag{1}$$

Indeed, let R be a rank one operator in $\mathcal{S}_F(H)^{>-I}$ such that tR and A are adjacent for some real number t. Clearly, t is nonzero, as $d(A,0) = 2$. Then $A = tR + S$ for some rank one self-adjoint operator S. Hence, $\operatorname{rank} A = \operatorname{rank}(tR) + \operatorname{rank} S$, and by Lemma 2.5 it follows that $\operatorname{Im} R = \operatorname{Im}(tR) \subset \operatorname{Im} A$, as desired.

In our next step we will prove that if $A \in \mathcal{S}_F(H)^{>-I}$ is an operator of rank 2 with both nonzero eigenvalues of the same sign, then we have actually

$$J_A = \{R \in \mathcal{S}_F(H)^{>-I} : \operatorname{rank} R = 1 \quad \text{and} \quad \operatorname{Im} R \subset \operatorname{Im} A\},$$

while in the case that the nonzero eigenvalues of A have different signs, the inclusion (1) is proper, that is,

$$J_A \neq \{R \in \mathcal{S}_F(H)^{>-I} : \operatorname{rank} R = 1 \quad \text{and} \quad \operatorname{Im} R \subset \operatorname{Im} A\}.$$

Assume first that $A \in \mathcal{S}_F(H)^{>-I}$ is an operator of rank 2 with both nonzero eigenvalues positive or both nonzero eigenvalues negative and that $R \in \mathcal{S}_F(H)^{>-I}$ is a rank one operator with $\operatorname{Im} R \subset \operatorname{Im} A$. We have to prove that $R \in J_A$. We already know that there is no loss of generality in assuming that R is a projection. Then we need to find a real number $t > -1$ such that $A - tR$ is of rank one. As $\operatorname{Im} R \subset \operatorname{Im} A$ we may consider only the restrictions of A and

R to the two-dimensional subspace $\operatorname{Im} A$. Once we consider these restrictions we may identify operators with 2×2 matrices. As we are free to choose any orthonormal basis of $\operatorname{Im} A$ we may further assume that A is a diagonal matrix $A = \operatorname{diag}(a_1, a_2)$ with either both a_1, a_2 positive, or $-1 < a_1, a_2 < 0$. We have to find a real number $t > -1$ such that $\operatorname{rank}(A - tR) = 1$. As 2×2 matrix A is invertible, this is the same as

$$\operatorname{rank}(I - tA^{-1}R) = 1. \tag{2}$$

Now, $tA^{-1}R, t \neq 0$, is a rank one matrix and it is easy to check that a rank one matrix is adjacent to the 2×2 identity matrix if and only if this rank one matrix is an idempotent. Further, a rank one matrix is an idempotent if and only if it has trace 1. As R is a projection, we have $r_{11} + r_{22} = 1$, where r_{11} and r_{22} are the diagonal entries of R. And of course, $r_{11}, r_{22} \geq 0$. To conclude, we have (2) if and only if

$$t(a_1^{-1} r_{11} + a_2^{-1} r_{22}) = 1.$$

If both a_1 and a_2 are positive, then $a_1^{-1} r_{11} + a_2^{-1} r_{22} > 0$, and consequently, A is adjacent to tR for $t = (a_1^{-1} r_{11} + a_2^{-1} r_{22})^{-1} > 0$. Similarly, if both a_1, a_2 belong to the open interval $(-1, 0)$, then A is adjacent to tR for the real number t satisfying $t^{-1} = a_1^{-1} r_{11} + a_2^{-1} r_{22} \in (-\infty, -1)$. Hence, $t \in (-1, 0)$, as desired.

Next, we assume that $A \in \mathcal{S}_F(H)^{>-I}$ is an operator of rank 2 with the nonzero eigenvalues having different signs and we have to find a rank one projection R such that $\operatorname{Im} R \subset \operatorname{Im} A$ but $d(A, tR) \neq 1$ for all real $t > -1$. Once again we may assume that A is a diagonal 2×2 matrix with diagonal entries $a_1, a_2, a_1 > 0, -1 < a_2 < 0$. Set

$$p = \frac{a_1}{a_1 - a_2}.$$

Then clearly, $0 < p < 1$. It follows that

$$R = \begin{bmatrix} p & \sqrt{p(1-p)} \\ \sqrt{p(1-p)} & 1 - p \end{bmatrix}$$

is a projection of rank one and one can easily verify that $A^{-1}R$ is a trace zero matrix. Consequently, $\operatorname{tr}(tA^{-1}R) = 0$ for all real numbers t, which yields that $I - tA^{-1}R$ is an invertible 2×2 matrix for each real t, and consequently, $d(A, tR) = 2$ for all real numbers t. We have found a rank one projection R with $\operatorname{Im} R \subset \operatorname{Im} A$ such that $R \notin J_A$.

Hence, we have proved the following result.

Lemma 2.13. *Let $A \in \mathcal{S}_F(H)^{>-I}$ be a rank two operator. Then the following two conditions are equivalent:*

- *A has one positive and one negative eigenvalue,*
- *there exists $B \in \mathcal{S}_F(H)^{>-I}$ of rank two such that $J_A \subset J_B$ and $J_A \neq J_B$.*

Lemma 2.14. *Let $A \in \mathcal{S}_F(H)$ be an operator of rank two with one positive eigenvalue and one negative eigenvalue. Denote by $\mathcal{P}$ the set of all rank one projections P with $\operatorname{Im} P \subset \operatorname{Im} A$. Then there exists a nonempty open subset*

$\mathcal{U} \subset \mathcal{P}$ such that for every projection $P \in \mathcal{U}$ there exists a positive real number t_P such that A and $t_P P$ are adjacent.

Proof. When proving this simple linear algebra result there is no loss of generality in replacing A and projections from $\mathcal{P}$ with their restrictions to the two-dimensional subspace $\mathrm{Im}\, A$. In other words, we may, and we will assume that H is two-dimensional. Then A is invertible and $\mathcal{P}$ is the set of all rank one projections on H. As A has one positive eigenvalue there exists a projection P such that $\mathrm{tr}\,(A^{-1}P)$ is positive (we may take P to be the spectral projection corresponding to the positive eigenvalue of A). Thus, $\mathcal{U} = \{Q \in \mathcal{P} \,:\, \mathrm{tr}\,(A^{-1}Q) > 0\}$ is a nonempty open subset of $\mathcal{P}$. For every $Q \in \mathcal{U}$ the rank one operator

$$\frac{1}{\mathrm{tr}\,(A^{-1}Q)} A^{-1}Q$$

has trace one, and consequently, this operator is an idempotent of rank one. It follows that the operator

$$A - \frac{1}{\mathrm{tr}\,(A^{-1}Q)} Q = A \left(I - \frac{1}{\mathrm{tr}\,(A^{-1}Q)} A^{-1}Q \right)$$

has rank one, that is, A and $\frac{1}{\mathrm{tr}\,(A^{-1}Q)}Q$ are adjacent. □

Lemma 2.15. *Let $A, B \in \mathcal{S}_F(H)$ be operators of rank two such that $\mathrm{Im}\, A = \mathrm{Im}\, B$. Denote by $\mathcal{P}$ the set of all rank one projections P with $\mathrm{Im}\, P \subset \mathrm{Im}\, A = \mathrm{Im}\, B$. Assume that there exists a nonempty open subset $\mathcal{U} \subset \mathcal{P}$ such that for every projection $P \in \mathcal{U}$ there exists a nonzero real number t_P such that A and $t_P P$ are adjacent and B and $t_P P$ are adjacent. Then $A = B$.*

Proof. As in the previous lemma we may assume that $\dim H = 2$. Then A and B are invertible. Our assumption is that for every projection $P \in \mathcal{U}$ there exists a nonzero real number t_P such that

$$\mathrm{rank}\,(A - t_P P) = 1 = \mathrm{rank}\,(B - t_P P),$$

or equivalently,

$$\mathrm{rank}\,(I - t_P A^{-1}P) = 1 = \mathrm{rank}\,(I - t_P B^{-1}P).$$

Note that $t_P A^{-1}P$ is a rank one operator (not necessarily self-adjoint). As it is adjacent to the identity operator its trace must be equal to one. Thus, for every projection $P \in \mathcal{U}$ there exists a nonzero real number t_P such that $\mathrm{tr}\,(t_P A^{-1}P) = 1 = \mathrm{tr}\,(t_P B^{-1}P)$ which yields

$$\mathrm{tr}\,(A^{-1}P) = \frac{1}{t_P} = \mathrm{tr}\,(B^{-1}P).$$

If we write projections $P \in \mathcal{U}$ as $P = u \otimes u^*$ where u is a vector of norm one (uniquely determined up to a multiplication with complex numbers of modulus one), we see that there exists an open nonempty subset $\mathcal{Z}$ of the unit sphere of H such that

$$\mathrm{tr}\,(A^{-1}u \otimes u^*) = \mathrm{tr}\,(B^{-1}u \otimes u^*)$$

for every $u \in \mathcal{Z}$, or equivalently

$$\langle A^{-1}u, u\rangle = \langle B^{-1}u, u\rangle$$

for every $u \in \mathcal{Z}$. Hence $\langle (A^{-1} - B^{-1})u, u\rangle = 0$ for every $u \in \mathcal{Z}$, and since $\mathcal{Z}$ is an open nonempty subset of the unit sphere, this equality actually holds for every $u \in H$. It follows that $A^{-1} = B^{-1}$, and consequently, $A = B$. $\square$

2.3. Proofs of the Main Results

The main tool in the proof of Theorem 2.1 is the following finite-dimensional result from [7].

Theorem 2.16. [7] *Let K and L be finite-dimensional Hilbert spaces with $\dim K \geq 2$. Assume that $\phi : \mathcal{S}(K) \to \mathcal{S}(L)$ is an adjacency preserving map such that $\phi(0) = 0$. Then either there exist a rank one operator $R \in \mathcal{S}(L)$ and a function $\rho : \mathcal{S}(K) \to \mathbb{R}$ such that*

$$\phi(A) = \rho(A)R$$

for every $A \in \mathcal{S}(K)$; or $\dim L \geq \dim K$ and there exist $c \in \{-1, 1\}$ and an injective linear or conjugate-linear map $T : K \to L$ such that

$$\phi(A) = cTAT^*$$

for every $A \in \mathcal{S}(K)$.

Obviously, in this theorem we can omit the assumption that L is finite-dimensional if we add the condition that the range of ϕ is contained in some finite-dimensional subspace of $\mathcal{S}_F(L)$.

Proof of Theorem 2.1. All we need to do is to show that for every projection $P \in \mathcal{S}_F(H)$ there exists a projection $Q \in \mathcal{S}_F(H)$ such that $\phi(P\mathcal{S}_F(H)P) \subset Q\mathcal{S}_F(H)Q$. Indeed, assume for a moment that we have already proved this statement. Then, by Theorem 2.16, for every projection $P \in \mathcal{S}_F(H)$ either there exists a nonzero vector $x \in H$ such that $\phi(P\mathcal{S}_F(H)P) \subset \operatorname{span}\{x \otimes x^*\}$, or there exist $c \in \{-1, 1\}$ and an injective linear or conjugate-linear map $T : \operatorname{Im} P \to H$ such that $\phi(\sum_{j=1}^k t_j x_j \otimes x_j^*) = c \sum_{j=1}^k t_j Tx_j \otimes (Tx_j)^*$ for every positive integer k, all real numbers $t_1, \ldots, t_k$, and all vectors $x_1, \ldots, x_k \in \operatorname{Im} P$.

Assume first that there exists a projection P of rank two such that the first possibility holds and let $A \in \mathcal{S}_F(H)$ be any operator. Then we can find a projection $R \in \mathcal{S}_F(H)$ such that $P \leq R$ and $\operatorname{Im} A \subset \operatorname{Im} R$. We can then apply Theorem 2.16 to the restriction of ϕ to $R\mathcal{S}_F(H)R$. Then, because the restriction ϕ to $P\mathcal{S}_F(H)P \subset R\mathcal{S}_F(H)R$ maps a rank two operator into an operator of rank at most one, the restriction of ϕ to $R\mathcal{S}_F(H)R$ maps all operators from $R\mathcal{S}_F(H)R$ into a linear span of some rank one operator. As $\phi(P\mathcal{S}_F(H)P) \subset \operatorname{span}\{x \otimes x^*\}$, we have necessarily $\phi(R\mathcal{S}_F(H)R) \subset \operatorname{span}\{x \otimes x^*\}$. In particular, $\phi(A) \in \operatorname{span}\{x \otimes x^*\}$. But A was an arbitrary self-adjoint finite rank operator on H, and therefore, $\phi(\mathcal{S}_F(H)) \subset \operatorname{span}\{x \otimes x^*\}$. So, we are done in this case.

It remains to consider the case when we have the second possibility for every projection $P \in \mathcal{S}_F(H)$. It follows that we may assume with no loss of generality that $\dim H = \infty$. Indeed, if $\dim H < \infty$ we just set $P = I$ and the desired conclusion follows immediately.

Assume now that for a certain finite-dimensional subspace $L \subset H, \dim L \geq 2$, there exist real numbers $c_1, c_2 \in \{-1, 1\}$ and injective linear or conjugate-linear maps $T_1, T_2 : L \to H$ such that

$$c_1 \sum_{j=1}^{k} t_j T_1 x_j \otimes (T_1 x_j)^* = c_2 \sum_{j=1}^{k} t_j T_2 x_j \otimes (T_2 x_j)^*$$

for every positive integer k, all real numbers $t_1, \ldots, t_k$, and all vectors $x_1, \ldots, x_k \in L$. We claim that then $c_1 = c_2$, either both T_1 and T_2 are linear, or both T_1 and T_2 are conjugate-linear, and $T_1 = \mu T_2$ for some complex number μ of modulus one. Take any nonzero $x \in L$. Then $c_1 T_1 x \otimes (T_1 x)^*$ is positive if $c_1 = 1$, and negative if $c_1 = -1$. Because $c_1 T_1 x \otimes (T_1 x)^* = c_2 T_2 x \otimes (T_2 x)^*$, we have $c_1 = c_2$. It then follows from $T_1 x \otimes (T_1 x)^* = T_2 x \otimes (T_2 x)^*$ that $T_1 x$ and $T_2 x$ are linearly dependent. Thus, for every $x \in L$ there exists a nonzero complex number λ_x such that $T_1 x = \lambda_x T_2 x$. Assume that $x, y \in L$ are linearly independent. Then

$$\begin{aligned} \lambda_{x+y} T_2 x + \lambda_{x+y} T_2 y &= \lambda_{x+y} T_2(x + y) = T_1(x + y) = T_1 x + T_1 y \\ &= \lambda_x T_2 x + \lambda_y T_2 y, \end{aligned}$$

and since T_2 is injective, $T_2 x$ and $T_2 y$ are linearly independent. It follows that $\lambda_x = \lambda_{x+y} = \lambda_y$ whenever x and y are linearly independent. If x and y are linearly dependent, then we can find $z \in L$ linearly independent of x and y, and consequently, $\lambda_x = \lambda_z = \lambda_y$ in this case as well. Hence, λ_x is independent of x: $\lambda_x = \mu, x \in L \setminus \{0\}$. Thus, $T_1 = \mu T_2$, and therefore, both T_1 and T_2 are linear; or they are both conjugate-linear. Finally, from $T_1 x \otimes (T_1 x)^* = T_2 x \otimes (T_2 x)^*$ and $T_1 = \mu T_2$ we conclude that $|\mu| = 1$.

Let us now choose and fix a subspace $L \subset H$ with $\dim L \geq 2$, and a nonzero vector $x_0 \in L$. We know that there exist $c_L \in \{-1, 1\}$ and an injective linear or conjugate-linear map $T_L : L \to H$ such that $\phi(\sum_{j=1}^{k} t_j x_j \otimes x_j^*) = c_L \sum_{j=1}^{k} t_j T_L x_j \otimes (T_L x_j)^*$ for every positive integer k, all real numbers $t_1, \ldots, t_k$, and all vectors $x_1, \ldots, x_k \in L$. Let further $M \subset H$ be any subspace such that $L \subset M$. Then there exist $c_M \in \{-1, 1\}$ and an injective linear or conjugate-linear map $T_M : M \to H$ such that $\phi(\sum_{j=1}^{k} t_j x_j \otimes x_j^*) = c_M \sum_{j=1}^{k} t_j T_M x_j \otimes (T_M x_j)^*$ for every positive integer k, all real numbers $t_1, \ldots, t_k$, and all vectors $x_1, \ldots, x_k \in M$. Of course, T_M is uniquely determined only up to a multiplicative constant μ of modulus one. But if we additionally require that $T_L x_0 = T_M x_0$, then T_M is uniquely determined. When using the symbol T_M we will always mean this uniquely determined operator.

We define a map $T : H \to H$ in the following way. For any $x \in H$ choose a subspace $M \subset H$ such that $L \subset M$ and $x \in M$ and then define $Tx = T_M x$. Using all the facts that were proved so far it is trivial to verify that T is well-defined, linear or conjugate-linear, injective, and $\phi(\sum_{j=1}^{k} t_j x_j \otimes x_j^*) = c_L \sum_{j=1}^{k} t_j (T x_j) \otimes (T x_j)^*$ for every $\sum_{j=1}^{k} t_j x_j \otimes x_j^* \in \mathcal{S}_F(H)$.

Thus we need to prove that for every projection $P \in \mathcal{S}_F(H)$ there exists a projection $Q \in \mathcal{S}_F(H)$ such that $\phi(P\mathcal{S}_F(H)P) \subset Q\mathcal{S}_F(H)Q$. We will verify this by induction on rank P. If rank $P = 1$, then $P\mathcal{S}_F(H)P = \{tP : t \in \mathbb{R}\}$. As $\phi(P)$ and $\phi(tP), t \neq 1$, are adjacent rank one operators, they have the same one-dimensional image. We are done in the case when rank $P = 1$.

Assume now that $r > 1$ and that the desired conclusion holds for all projections $P \in \mathcal{S}_F(H)$ of rank at most $r-1$. Let $P \in \mathcal{S}_F(H)$ be a projection of rank r. If $R \in \mathcal{S}_F(H)$ is any projection of rank $r-1$ satisfying $R \leq P$, then by the induction hypothesis there exists a projection $Q_R \in \mathcal{S}_F(H)$ such that $\phi(R\mathcal{S}_F(H)R) \subset Q_R\mathcal{S}_F(H)Q_R$. Hence, by Theorem 2.16, either there exists a nonzero vector $x \in H$ such that $\phi(R\mathcal{S}_F(H)R) \subset \operatorname{span}\{x \otimes x^*\}$, or there exist $c_R \in \{-1, 1\}$ and an injective linear or conjugate-linear map $T_R : \operatorname{Im} R \to H$ such that $\phi(\sum_{j=1}^k t_j x_j \otimes x_j^*) = c_R \sum_{j=1}^k t_j (T_R x_j) \otimes (T_R x_j)^*$ whenever $x_1, \ldots, x_k \in \operatorname{Im} R$. In the first case we will say that the restriction of ϕ to $R\mathcal{S}_F(H)R$ is degenerate, while in the second case we will speak of a non-degenerate map $\phi_{|R\mathcal{S}_F(H)R}$.

We have to treat separately the special case when $r = 2$. Let us start with the case when there is an operator $A \in P\mathcal{S}_F(H)P$ such that $\phi(A)$ is of rank two. Denote by $Q \in \mathcal{S}_F(H)$ the rank two projection onto $\operatorname{Im} \phi(A)$. Let $B \in P\mathcal{S}_F(H)P$ be any other operator of rank two. Then we can apply Lemma 2.12. Let $A_0, A_1, \ldots, A_m$ and $C_1, \ldots, C_m$ be as in the conclusion of this lemma. The mapping ϕ maps line $l(A_0, A_1)$ injectively into line $l(\phi(A_0), \phi(A_1))$. We know that $\phi(A_0)$ is of rank two and $\phi(C_1) \in l(\phi(A_0), \phi(A_1))$ is of rank one. It follows from Lemma 2.11 that $\phi(A_1)$ is of rank two as well. Moreover, using Lemma 2.10 we conclude that $\operatorname{Im} \phi(A_0) = \operatorname{Im} \phi(A_1)$. Repeating the same argument for the pair A_1, A_2, then for the pair $A_2, A_3, \ldots$, and finally for the pair A_{m-1}, A_m we end up with $\operatorname{Im} \phi(B) = \operatorname{Im} \phi(A)$. We have shown that each $B \in P\mathcal{S}_F(H)P$ of rank two is mapped into an operator of rank two that is contained in $Q\mathcal{S}_F(H)Q$. Let now $B \in P\mathcal{S}_F(H)P$ be any rank one operator. Then there exists a rank one operator $C \in P\mathcal{S}_F(H)P$ such that $B+C \in P\mathcal{S}_F(H)P$ is of rank two. Clearly, $\operatorname{rank} \phi(B) = 1 = d(\phi(B), \phi(B+C))$. It follows that

$$\operatorname{rank} \phi(B+C) = \operatorname{rank} \phi(B) + \operatorname{rank} (\phi(B+C) - \phi(B))$$

and by Lemma 2.5, $\operatorname{Im} \phi(B) \subset \operatorname{Im} \phi(B+C) = \operatorname{Im} Q$. Thus, $\phi(B) \in Q\mathcal{S}_F(H)Q$, as desired.

We continue with the possibility that there are $A, B \in P\mathcal{S}_F(H)P$ such that $d(\phi(A), \phi(B)) = 2$. Then we define a new map $\psi : \mathcal{S}_F(H) \to \mathcal{S}_F(H)$ with $\psi(C) = \phi(C + A) - \phi(A)$. Clearly, ψ preserves adjacency and $\psi(0) = 0$. Moreover, $B - A \in P\mathcal{S}_F(H)P$ and $\psi(B - A)$ is of rank two. By the previous paragraph we can find a projection R of rank two such that $\psi(P\mathcal{S}_F(H)P) \subset R\mathcal{S}_F(H)R$, and consequently, $\phi(P\mathcal{S}_F(H)P) \subset Q\mathcal{S}_F(H)Q$, where Q is a finite rank projection whose image contains both $\operatorname{Im} R$ and $\operatorname{Im} \phi(A)$. We are done in this subcase and to complete the proof in the case when $r = 2$ we need to consider the remaining case when $d(\phi(A), \phi(B)) \leq 1$ for all $A, B \in P\mathcal{S}_F(H)P$. In particular, every member of $\phi(P\mathcal{S}_F(H)P)$ is of rank at most one and any two different elements of $\phi(P\mathcal{S}_F(H)P)$ are adjacent.

It follows that $\phi(P\mathcal{S}_F(H)P) \subset \operatorname{span}\{x \otimes x^*\}$ for some nonzero $x \in H$, or equivalently, $\phi(P\mathcal{S}_F(H)P) \subset Q\mathcal{S}_F(H)Q$, where Q is a projection of rank one with $\operatorname{Im} Q = \operatorname{span}\{x\}$.

So form now on we will assume that $r > 2$. We will distinguish two cases. We start with the case when the restriction of ϕ to $R\mathcal{S}_F(H)R$ is degenerate for every projection $R \leq P$ with $\operatorname{rank} R = r - 1$. In other words, for every such projection R there exists a rank one projection $x \otimes x^* \in \mathcal{S}_F(H)$ such that $\phi(R\mathcal{S}_F(H)R)$ is contained in the linear span of $x \otimes x^*$. It is not difficult to verify that $x \otimes x^*$ is independent of R. Indeed, let $R_1, R_2 \leq P$ be projections of rank $r-1$ and $\phi(R_j\mathcal{S}_F(H)R_j) \subset \operatorname{span}\{x_j \otimes x_j^*\}$, $j = 1, 2$. Then we can find a rank one projection W such that $W \in R_j\mathcal{S}_F(H)R_j$, $j = 1, 2$. Since $\phi(W)$ is adjacent to 0 we have $\phi(W) = t_1 x_1 \otimes x_1^* = t_2 x_2 \otimes x_2^*$ for some nonzero t_1, t_2. It follows that $x_1 \otimes x_1^* = x_2 \otimes x_2^*$, as desired.

We have thus shown that there exists a rank one projection $x \otimes x^*$ such that $\phi(A) \in \operatorname{span}\{x \otimes x^*\}$ for every $A \in P\mathcal{S}_F(H)P$ of rank at most $r - 1$.

Since every $A \in P\mathcal{S}_F(H)P$ of rank r is adjacent to some operator from $P\mathcal{S}_F(H)P$ of rank $r - 1$ we have $\operatorname{rank}\phi(A) \leq 2$ for every $A \in P\mathcal{S}_F(H)P$.

Assume first that every A from $P\mathcal{S}_F(H)P$ of rank r is sent by ϕ into an operator of rank at most one. We will show that if $\phi(A) \neq 0$, then $\phi(A) \in \operatorname{span}\{x \otimes x^*\}$. Assume on the contrary that $\phi(\sum_{j=1}^{r} t_j P_j) = Z \notin \operatorname{span}\{x \otimes x^*\}$, where the P_j's are pairwise orthogonal rank one projections, the t_j's are nonzero real numbers and $Z \in \mathcal{S}_F(H)$ is a rank one operator. We know that $\phi(\sum_{j=1}^{r-1} t_j P_j)$ is contained in the linear span of $x \otimes x^*$ and is adjacent to Z. Hence, $\phi(\sum_{j=1}^{r-1} t_j P_j) = 0$. Then the line $\{\sum_{j=1}^{r-1} t_j P_j + tP_n \,:\, t \in \mathbb{R}\}$ is mapped by ϕ injectively into the linear span of Z. Thus, there exists a real number $s, s \neq t_n$, such that $\phi(\sum_{j=1}^{r-1} t_j P_j + sP_n) = pZ$ with $p \neq 0$. We know that both $\phi(\sum_{j=2}^{r-1} t_j P_j + t_n P_n)$ and $\phi(\sum_{j=2}^{r-1} t_j P_j + sP_n)$ belong to the linear span of $x \otimes x^*$. The first one is adjacent to Z and the second one to pZ. Thus, $\phi(\sum_{j=2}^{r-1} t_j P_j + t_n P_n) = \phi(\sum_{j=2}^{r-1} t_j P_j + sP_n) = 0$, contradicting the fact that $\sum_{j=2}^{r-1} t_j P_j + t_n P_n$ and $\sum_{j=2}^{r-1} t_j P_j + sP_n$ are adjacent.

The other possibility we have to treat is that there exists an operator $A \in P\mathcal{S}_F(H)P$ of rank r such that $\operatorname{rank}\phi(A) = 2$. Clearly, we have $x \in \operatorname{Im}\phi(A)$ in this case. We will show that this possibility cannot occur thus proving that if the restriction of ϕ to $R\mathcal{S}_F(H)R$ is degenerate for every projection $R \leq P$ with $\operatorname{rank} R = r-1$, then the restriction of ϕ to $P\mathcal{S}_F(H)P$ is degenerate as well. Indeed, if $\operatorname{rank}\phi(A) = 2$ for some rank r operator $A \in P\mathcal{S}_F(H)P$, then we see in exactly the same way as above (in the case when $r = 2$) that $\operatorname{Im}\phi(B) = \operatorname{Im}\phi(A)$ for every $B \in P\mathcal{S}_F(H)P$ of rank r. Thus, by Theorem 2.16 the restriction of ϕ to $P\mathcal{S}_F(H)P$ is either degenerate, or it preserves rank of every operator contradicting the fact that $\operatorname{rank}\phi(A) = 2$.

It remains to consider the case when the restriction of ϕ to $R\mathcal{S}_F(H)R$ is non-degenerate for some projection $R \leq P$ with $\operatorname{rank} R = r-1$. We will show that in this case the restriction of ϕ to $P\mathcal{S}_F(H)P$ is real-linear. Once we prove the real-linearity, the desired conclusion that $\phi(P\mathcal{S}_F(H)P) \subset Q\mathcal{S}_F(H)Q$ for some finite rank projection Q follows trivially. Our first goal is to verify that

$\phi(tA) = t\phi(A)$ for every real number t and every $A \in P\mathcal{S}_F(H)P$ of rank one. Indeed, this is true if $A \in R\mathcal{S}_F(H)R$. If $A \notin R\mathcal{S}_F(H)R$, then we can find an operator $A_1 \in R\mathcal{S}_F(H)R$ of rank one such that rank one operators $\phi(A)$ and $\phi(A_1)$ are linearly independent. There exists a rank two projection P_1 such that $A, A_1 \in P_1\mathcal{S}_F(H)P_1$. Clearly, the restriction of ϕ to $P_1\mathcal{S}_F(H)P_1$ is non-degenerate, and therefore real-linear. Consequently, $\phi(tA) = t\phi(A), t \in \mathbb{R}$, in this case as well.

Let $T \in P\mathcal{S}_F(H)P$ be any operator. Define a map $\phi_T : P\mathcal{S}_F(H)P \to \mathcal{S}_F(H)$ by $\phi_T(X) = \phi(T+X) - \phi(T), X \in P\mathcal{S}_F(H)P$. Obviously, ϕ_T is an adjacency preserving map satisfying $\phi_T(0) = 0$. We will show that ϕ_T is of the same type as the restriction of ϕ to $P\mathcal{S}_F(H)P$, that is, there exists a projection $R_1 \leq P$ of rank $r-1$ such that the restriction of ϕ_T to $R_1\mathcal{S}_F(H)R_1$ is non-degenerate. Indeed, if this was not the case, then by the previous step ϕ_T would be degenerate. Then we would have

$$\phi(Y) = \phi(T + (Y - T)) = \phi(T) + \phi_T(Y - T) = \phi(T) + \rho(Y)y \otimes y^*,$$
$$Y \in P\mathcal{S}_F(H)P,$$

for some function $\rho : P\mathcal{S}_F(H)P \to \mathbb{R}$ and some nonzero vector y. Because $\phi(0) = 0$ this would imply $\phi(T) \in \operatorname{span}\{y \otimes y^*\}$ and therefore, the restriction of ϕ to $P\mathcal{S}_F(H)P$ would be degenerate, a contradiction.

We have thus proved that ϕ_T is of the same type as the restriction of ϕ to $P\mathcal{S}_F(H)P$. By the previous step, $\phi_T(tA) = t\phi_T(A)$ for every $t \in \mathbb{R}$, every rank one operator $A \in P\mathcal{S}_F(H)P$, and every $T \in P\mathcal{S}_F(H)P$. Equivalently,

$$\phi(T + tA) = \phi(T) + t(\phi(T + A) - \phi(T)), \quad t \in \mathbb{R}, \tag{3}$$

for every $T \in P\mathcal{S}_F(H)P$ and every rank one operator $A \in P\mathcal{S}_F(H)P$.

In order to prove the real-linearity of the restriction of ϕ to $P\mathcal{S}_F(H)P$, it is enough to show that

$$\phi(A_1 + \cdots + A_p) = \phi(A_1) + \cdots + \phi(A_p)$$

for every positive integer p and arbitrary rank one operators $A_1, \ldots, A_p \in P\mathcal{S}_F(H)P$. We will prove this by induction on p. Assume that the statement holds true for p and we want to prove it for $p+1$. Let $A_1, \ldots, A_{p+1} \in P\mathcal{S}_F(H)P$ be any rank one operators. Using (3) we see that for every real t we have

$$\phi(A_1 + \cdots + A_p + tA_{p+1}) = \phi(A_1 + \cdots + A_p)$$
$$+t[\phi(A_1 + \cdots + A_{p+1}) - \phi(A_1 + \cdots + A_p)].$$

Applying the induction hypothesis we get

$$\phi(A_1 + \cdots + A_p + tA_{p+1}) = \phi(A_1) + \cdots + \phi(A_p)$$
$$+t[\phi(A_1 + \cdots + A_{p+1}) - \phi(A_1) - \cdots - \phi(A_p)].$$

Now, for every real t the operator $\phi(A_1 + \cdots + A_p + tA_{p+1})$ is adjacent to $\phi(A_2 + \cdots + A_p + tA_{p+1}) = \phi(A_2) + \cdots + \phi(A_p) + t\phi(A_{p+1})$. Hence, the operator

$$\begin{aligned}&\phi(A_1)+\cdots+\phi(A_p)+t[\phi(A_1+\cdots+A_{p+1})-\phi(A_1)-\cdots-\phi(A_p)]\\&\quad-\phi(A_2)-\cdots-\phi(A_p)-t\phi(A_{p+1})\\&\quad=\phi(A_1)+t[\phi(A_1+\cdots+A_{p+1})-\phi(A_1)-\cdots-\phi(A_p)-\phi(A_{p+1})]\end{aligned}$$

is of rank one for every real number t. It follows from Lemma 2.8 that

$$\phi(A_1+\cdots+A_{p+1})-\phi(A_1)-\cdots-\phi(A_p)-\phi(A_{p+1})=0,$$

as desired. □

When proving Theorem 2.2 the first step will be to prove that bijective maps acting on the set of all finite rank self-adjoint bounded linear positive operators, which preserve adjacency in both directions, send the zero operator into itself. This fact will be a direct consequence of the following characterization of the zero operator.

Proposition 2.17. *Let H be a Hilbert space, $\dim H \geq 2$, and $A \in \mathcal{S}_F(H)^{\geq 0}$. Then the following are equivalent:*

- $A = 0$,
- *for every pair of operators $B, C \in \mathcal{S}_F(H)^{\geq 0}$ satisfying $d(A,B) = d(C,B) = 2$ and $d(A,C) = 1$ there exists $D \in \mathcal{S}_F(H)^{\geq 0}$ such that $d(D,A) = d(D,B) = d(D,C) = 1$.*

Proof. Assume first that $A = 0$ and that $B, C \in \mathcal{S}_F(H)^{\geq 0}$ satisfy $d(A,B) = d(C,B) = 2$ and $d(A,C) = 1$. Hence, B is a positive operator of rank two and after replacing A, B, and C by TAT^*, TBT^*, and TCT^*, respectively, where $T : H \to H$ is an appropriate bounded bijective linear operator, we may assume with no loss of generality that $A = 0, H$ is an orthogonal direct sum $H = H_1 \oplus H_2$ with $\dim H_1 = 2$, and the matrix representation of B with respect to this direct sum decomposition is

$$B = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where I stands for the 2×2 identity matrix. Let e_1, e_2 be an orthonormal basis of H_1. As C is of rank one and positive we have $C = x \otimes x^*$ for some nonzero $x \in H$. We claim that $x \in H_1$. Otherwise it would follow from $B - C = e_1 \otimes e_1^* + e_2 \otimes e_2^* - x \otimes x^*$ and the fact that e_1, e_2, x are linearly independent that $\operatorname{rank}(B - C) = 3$, a contradiction. Hence, with respect to the above direct sum decomposition of H we have

$$C = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix},$$

where the $*$ stands for some positive rank one 2×2 matrix, and after replacing A, B, and C by UAU^*, UBU^*, and UCU^*, respectively, where U is an appropriate unitary operator on H we may, and we will assume that

$$C = \begin{bmatrix} \begin{bmatrix} t_0 & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix},$$

where t_0 is a positive real number. It follows from $d(B,C) = 2$ that $t_0 \neq 1$. It is now straightforward to verify that the operator

$$D = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

has all the desired properties.

To prove the other direction assume that $A \in \mathcal{S}_F(H)^{\geq 0}$ satisfies the second condition in our proposition. It is then clear that for every bijective bounded linear operator $T : H \to H$ the operator TAT^* satisfies this condition as well. We have to show that $A = 0$. Assume, on the contrary, that $A \neq 0$. We will distinguish two cases.

The first case we will treat is that $\operatorname{rank} A \geq 2$. We may replace A by a congruent operator TAT^*. Here, of course, $T : H \to H$ is a bounded bijective linear operator. So, we may, and we will assume that H is an orthogonal direct sum $H = H_1 \oplus H_2$ with $\dim H_1 = 2$, and the matrix representation of A with respect to this direct sum decomposition is

$$A = \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix},$$

where I stands for the 2×2 identity matrix and $W : H_2 \to H_2$ is a bounded linear positive finite rank operator acting on H_2. Let B and C be bounded linear positive finite rank operators on H with the matrix representations

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & W \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & W \end{bmatrix},$$

where

$$B_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_1 = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

We obviously have $d(A,B) = d(C,B) = 2$ and $d(A,C) = 1$. Assume that there exists a bounded linear finite rank positive operator

$$D = \begin{bmatrix} D_1 & Z \\ Z^* & U \end{bmatrix},$$

which is adjacent to A, B, and C. In the sequel we will use the following trivial inequalities: $1 = d(D,A) \geq d(D_1,I), 1 = d(D,B) \geq d(D_1,B_1)$, and $1 = d(D,C) \geq d(D_1,C_1)$. If $D_1 = I$, then $d(D,B) \geq d(I,B_1) = 2$, a contradiction. If $D_1 = B_1$ then $d(D,C) \geq d(B_1,C_1) = 2$, a contradiction. Similarly, $D_1 \neq C_1$. It follows that $D_1 \sim I$ and $D_1 \sim B_1$ and $D_1 \sim C_1$. Note that

$$C_1 = I + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

and therefore, D_1 belongs to the line joining I and C_1. In other words, there exists a real number $t \geq -1$ such that

$$D_1 = I + t \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Since 2×2 matrices D_1 and B_1 are adjacent, we have $\det(D_1 - B_1) = 0$. A straightforward computation yields that

$$\det(D_1 - B_1) = \left(\frac{t}{2} - 1\right)\left(\frac{t}{2} + 1\right) - \frac{t^2}{4} = -1,$$

a contradiction.

It remains to consider the case when rank $A = 1$. As in the previous case we may, and we will assume that H is an orthogonal direct sum $H = H_1 \oplus H_2$ with $\dim H_1 = 2$, and the matrix representation of A with respect to this direct sum decomposition is

$$A = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let B and C be bounded linear positive finite rank operators on H with the matrix representations

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$B_1 = \begin{bmatrix} \frac{3}{2} & 1 \\ 1 & 1 \end{bmatrix}$$

and C_1 is the 2×2 identity matrix. We obviously have $d(A, B) = d(C, B) = 2$ and $d(A, C) = 1$. Assume that there exists a bounded linear finite-rank positive operator D which is adjacent to A, B, and C. Then, since A and C are adjacent, the operator D belongs to the line joining A and C, and consequently,

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}$$

for some real $t > 0$. From $B \sim D$ we get that $\det(B_1 - D_1) = 0$ which yields $t = -1$, a contradiction. This completes the proof. $\square$

We shall prove Theorem 2.2 in several steps. We will first consider the two-dimensional case. If $\dim H = 2$, then $\mathcal{S}_F(H)^{\geq 0} = \mathcal{S}(H)^{\geq 0}$ can be identified with $\mathcal{H}_2^{\geq 0}$, the set of all 2×2 positive hermitian matrices.

Proposition 2.18. *Let $\phi : \mathcal{H}_2^{\geq 0} \to \mathcal{H}_2^{\geq 0}$ be a bijective map preserving adjacency in both directions. Then there exists an invertible 2×2 complex matrix T such that either*

$$\phi(A) = TAT^*$$

for every $A \in \mathcal{H}_2^{\geq 0}$*; or*

$$\phi(A) = TA^tT^*$$

for every $A \in \mathcal{H}_2^{\geq 0}$*. Here,* A^t *denotes the transpose of the matrix* A*.*

If we add in the above statement the additional assumption that ϕ maps projections to projections, then in particular, $\phi(I) = I$, and consequently, T must be a unitary matrix. It follows that a pair of two orthogonal projections of rank one is mapped into a pair of orthogonal projections. If moreover, we assume that $\phi(E_{11}) = E_{11}$ and $\phi(E_{22}) = E_{22}$ (here and in the sequel the symbol E_{ij} denotes the matrix whose all entries are zero but the (i, j)-entry which is equal to 1), then either

$$\phi\left(\begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix}\right) = \begin{bmatrix} t & w\alpha \\ \overline{w\alpha} & s \end{bmatrix},$$

or

$$\phi\left(\begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix}\right) = \begin{bmatrix} t & w\overline{\alpha} \\ \overline{w}\alpha & s \end{bmatrix}$$

for some complex number w of modulus one.

It is rather easy to see that if we deal with linear positive operators on a two-dimensional Hilbert space H rather than with positive 2×2 matrices, then the above statement can be reformulated in the following way. Let $\dim H = 2$ and assume that $\phi : \mathcal{S}(H)^{\geq 0} \to \mathcal{S}(H)^{\geq 0}$ is a bijective map preserving adjacency in both directions. Then there exists a bijective linear or conjugate-linear map $T : H \to H$ such that $\phi(A) = TAT^*$ for every $A \in \mathcal{S}(H)^{\geq 0}$. Indeed, all we need is to observe that for a hermitian matrix A its transpose A^t is obtained from A by applying complex conjugation entrywise.

Proof of Proposition 2.18*.* We first observe that under our assumptions we have $d(\phi(A), \phi(B)) = d(A, B), A, B \in \mathcal{H}_2^{\geq 0}$. It is now clear that Proposition 2.17 yields $\phi(0) = 0$.

Let $A \in \mathcal{H}_2^{\geq 0}$ be any matrix of rank one. Then the set $\{tA : t \geq 0\}$ will be called a maximal adjacent set of type one. Clearly, a subset $\mathcal{S} \subset \mathcal{H}_2^{\geq 0}$ is a maximal adjacent set of type one if and only if there exists a projection Q of rank one in $\mathcal{H}_2^{\geq 0}$ such that $\mathcal{S}$ consists of all matrices $tQ, t \in [0, \infty)$. If $\mathcal{S}$ is any such set, then obviously any two distinct members of $\mathcal{S}$ are adjacent and $\mathcal{S}$ is a maximal subset of $\mathcal{H}_2^{\geq 0}$ with this property.

Let now A and B be two linearly independent positive 2×2 matrices of rank one. Then the set $\{A + tB : t \geq 0\}$ will be called a maximal adjacent set of type two. Clearly, a subset $\mathcal{S} \subset \mathcal{H}_2^{\geq 0}$ is a maximal adjacent set of type two if and only if there exist a positive real number p and projections $P, Q, P \neq Q$, of rank one in $\mathcal{H}_2^{\geq 0}$ such that $\mathcal{S}$ consists of all matrices $pP + tQ, t \in [0, \infty)$. If $\mathcal{S}$ is any such set, then obviously any two distinct members of $\mathcal{S}$ are adjacent and $\mathcal{S}$ is a maximal subset of $\mathcal{H}_2^{\geq 0}$ with this property.

In general, we will say that $\mathcal{S} \subset \mathcal{H}_2^{\geq 0}$ is a maximal adjacent set if any two distinct members of $\mathcal{S}$ are adjacent, and if $\mathcal{S} \subset \mathcal{M} \subset \mathcal{H}_2^{\geq 0}$ and any two distinct members of $\mathcal{M}$ are adjacent, then $\mathcal{S} = \mathcal{M}$.

We claim that $\mathcal{S} \subset \mathcal{H}_2^{\geq 0}$ is a maximal adjacent set if and only if $\mathcal{S}$ is either a maximal adjacent set of type one, or a maximal adjacent set of type two. One direction is trivial. So, assume that $\mathcal{S}$ is a maximal adjacent set. Then $\mathcal{S}$ contains a nonzero matrix. Thus, we have two possibilities: either $\max\{\operatorname{rank} A \, : \, A \in \mathcal{S}\} = 1$, or $\max\{\operatorname{rank} A \, : \, A \in \mathcal{S}\} = 2$. We start with the first possibility. Let $B \in \mathcal{S}$ be of rank one. Then any $A \in \mathcal{S}$ is either the zero matrix or is a rank one matrix adjacent to B. In the second case we have $A = tB$. By maximality, $\mathcal{S} = \{tB \, : \, t \geq 0\}$, and hence, $\mathcal{S}$ is a maximal adjacent set of type one. It remains to consider the case when there exists $B \in \mathcal{S}$ with $\operatorname{rank} B = 2$. Let $T \in \mathcal{H}_2^{\geq 0}$ be any invertible matrix. Clearly, $T\mathcal{S}T$ is a maximal adjacent set if and only if $\mathcal{S}$ is a maximal adjacent set, and $T\mathcal{S}T$ is a maximal adjacent set of type two if and only if $\mathcal{S}$ is a maximal adjacent set of type two. So, there is no loss of generality in assuming that $B = I$. As $\{A - I \, : \, A \in \mathcal{S}\}$ is a set of pairwise adjacent matrices of rank at most one, we have $\mathcal{S} \subset \{I + tP \, : \, t \geq -1\}$, where P is some projection of rank one. By maximality, $\mathcal{S} = \{(I - P) + tP \, : \, t \geq 0\}$ is a maximal adjacent set of type two.

As $\phi : \mathcal{H}_2^{\geq 0} \to \mathcal{H}_2^{\geq 0}$ is a bijective map preserving adjacency in both directions, it maps each maximal adjacent set onto some maximal adjacent set. Moreover, a maximal adjacent set is of type one if and only if it contains 0. Since $\phi(0) = 0$, the map ϕ maps each maximal adjacent set of type one onto some maximal adjacent set of type one, and the same is true for maximal adjacent sets of type two.

In the next step we will prove that if $P, Q, R \in \mathcal{H}_2^{\geq 0}$ are distinct projections of rank one, then for every positive real number η there exists a positive real number p such that for every nonnegative real number t the following two statements are equivalent:

1. the matrix tR is adjacent to some element of the maximal adjacent set $\{pP + sQ \, : \, s \geq 0\}$,
2. $t < \eta$.

Let $P, Q, R \in \mathcal{H}_2^{\geq 0}$ be pairwise distinct projections of rank one and p a positive real number. It is easy to verify that there exists an invertible 2×2 complex matrix such that $TPT^* = E_{11}$ and $TQT^* = E_{22}$. Indeed, we have $P = xx^*$ and $Q = yy^*$ where x and y are linearly independent 2×1 matrices. We take T to be the invertible 2×2 matrix satisfying $Tx = e_1$ and $Ty = e_2$, where e_1, e_2 are the elements of the standard basis of the space of all 2×1 complex matrices.

Clearly, the matrices tR and $pP + sQ$ (here, t, s are nonnegative real numbers) are adjacent if and only if

$$\operatorname{rank} T(pP + sQ - tR)T^* = 1.$$

Since TRT^* is not a scalar multiple of E_{11} or E_{22}, we have

$$TRT^* = t_0 \begin{bmatrix} a & \alpha \\ \overline{\alpha} & 1 - a \end{bmatrix},$$

where $t_0 > 0, 0 < a < 1$, and $a(1 - a) = |\alpha|^2$ (in other words, TRT^* is a projection of rank one $\neq E_{11}, E_{22}$, multiplied by a positive real number t_0).

Hence, the matrices tR and $pP + sQ$ are adjacent if and only if

$$\text{rank}\left(\begin{bmatrix} p & 0 \\ 0 & s \end{bmatrix} - tt_0 \begin{bmatrix} a & \alpha \\ \overline{\alpha} & 1-a \end{bmatrix}\right) = 1.$$

The above matrix is nonzero no matter how we choose $t, s \geq 0$. A nonzero 2×2 matrix is of rank one if and only if its determinant is zero. Using the fact that $a(1-a) - |\alpha|^2 = 0$ we see that the matrices tR and $pP + sQ$ are adjacent if and only if

$$ps + ptt_0(a-1) - stt_0 a = 0.$$

This holds true if and only if

$$t = \frac{As}{s+B},$$

where $A = \frac{p}{at_0}$ and $B = \frac{p(1-a)}{a}$ are both positive. It follows that tR is adjacent to the matrix $pP + sQ$ for some nonnegative real number s if and only if $t \in [0, A)$.

Thus, if η is any positive real number, we set $p = \eta a t_0$ and notice that then $A = \eta$. Therefore the matrix tR is adjacent to some element of the maximal adjacent set $\{pP + sQ : s \geq 0\}$ if and only if $t < \eta$, as desired.

Let $\mathcal{J} = \{pP + sQ : s \geq 0\}$ be any maximal adjacent set of type two. Here, P and Q are distinct projections and $p > 0$. Note that P, Q and p are uniquely determined. Indeed, there is a unique rank one matrix $T \in \mathcal{J}$. Every such matrix can be writen as a scalar times projection of rank one in a unique way. It follows that p and P are uniquely determined. It is then easy to see that the rank one projection Q is also uniquely determined by the maximal adjacent set $\mathcal{J}$ of type two.

We choose now any rank one projection $R \in \mathcal{H}_2^{\geq 0}$ and we want to know for which nonnegative real numbers t the matrix tR is adjacent to some element of $\mathcal{J}$. Obviously, we have three possibilities. The first one is that R and P as well as R and Q are linearly independent. Then we already know that there exists a positive real number η such that the matrix tR is adjacent to some element of $\mathcal{J}$ if and only if $0 \leq t < \eta$. It remains to consider the cases when either R and P are linearly dependent, or R and Q are linearly dependent. It is trivial to see that two projections of rank one are linearly dependent if and only if they are equal. In the case $R = P$ it is obvious that for every nonnegative real number t the matrix $tR = tP$ is adjacent to some element of $\mathcal{J}$ (indeed, if $t \neq p$, then tP is adjacent to pP, while pP is adjacent to all matrices $pP + sQ$ with $s > 0$), while in the case when $R = Q$ we see that for every nonnegative real number t the matrix tR is adjacent to $pP + tQ \in \mathcal{J}$.

We will next show that for every rank one projection $P \in \mathcal{H}_2^{\geq 0}$ there exist a unique rank one projection $Q \in \mathcal{H}_2^{\geq 0}$ and a bijective monotone increasing continuous function $f_P : [0, \infty) \to [0, \infty)$ such that $\phi(tP) = f_P(t)Q, t \in [0, \infty)$. Indeed, we already know that the maximal adjacent set of type one $\{tP : t \geq 0\}$ is mapped by ϕ bijectively onto some maximal adjacent set $\mathcal{L}$ of type one. There is exactly one rank one projection in $\mathcal{L}$. Denote it by Q. Thus, there exists a bijective function $f_P : [0,\infty) \to [0,\infty)$ with $\phi(tP) = f_P(t)Q, t \in [0,\infty)$. All we need to do is to show that f_P is a monotone increasing function as then the continuity of f_P follows trivially. In order to do this we choose any positive real number η. We know that there exists a maximal adjacent set $\mathcal{K}$ of type two such that tP is adjacent to some element of $\mathcal{K}$ if and only if $t \in [0,\eta)$. Further, $\phi(\mathcal{K}) = \mathcal{M}$ is again a maximal adjacent set of type two. On one hand, we know that $\phi(tP) = f_P(t)Q$ is adjacent to some element of $\mathcal{M}$ if and only if $t \in [0,\eta)$. On the other hand, we know that we have two possibilities: either there exists a positive real number μ such that $f_P(t)Q$ is adjacent to some element of $\mathcal{M}$ if and only if $f_P(t) \in [0,\mu)$, or $f_P(t)Q$ is adjacent to some element of $\mathcal{M}$ for all real numbers $t \in [0,\infty)$. It follows that either $f_p([0,\eta)) = [0,\mu)$, or $f_p([0,\eta)) = [0,\infty)$. The second possibility cannot occur because of the bijectivity of f_P. Thus, we have shown that for every positive real number η there exists a positive real number μ such that $f_p([0,\eta)) = [0,\mu)$. This together with the bijectivity of f_P yields that f_P is an increasing function.

After replacing the map ϕ by $A \mapsto \phi(I)^{-1/2}\phi(A)\phi(I)^{-1/2}$, we may, and we will assume that $\phi(I) = I$. Then $\phi(E_{11})$ is a rank one matrix adjacent to I, and consequently, $\phi(E_{11})$ is a projection of rank one. After composing ϕ with a suitable unitary similarity transformation we asssume with no loss of generality that $\phi(I) = I$ and $\phi(E_{11}) = E_{11}$. There is a unique maximal adjacent set of type two containing I and E_{11}, namely the set $\{E_{11}+sE_{22} : s \geq 0\}$. Thus,

$$\phi(E_{11} + tE_{22}) = E_{11} + f(t)E_{22}, \quad t \in [0,\infty), \tag{4}$$

for some bijective function $f : [0,\infty) \to [0,\infty)$. We know that $\{tE_{11} : t \geq 0\}$ and $\{tE_{22} : t \geq 0\}$ are the only two maximal adjacent sets of type one having the property that each member of this set is adjacent to some element of $\{E_{11} + sE_{22} : s \geq 0\}$. It follows easily that

$$\phi(tE_{11}) = g(t)E_{11} \quad \text{and} \quad \phi(tE_{22}) = f(t)E_{22}, \quad t \in \mathbb{R},$$

where $g : [0,\infty) \to [0,\infty)$ is a monotone increasing bijective function and $f : [0,\infty) \to [0,\infty)$ is the same function as the one appearing in (4). In particular, f is an increasing bijection of $[0,\infty)$.

Let p be any positive real number. We know that ϕ maps $\{pE_{11}+tE_{22} : t \geq 0\}$ onto some maximal adjacent set of type two, say $\{R + sT : s \geq 0\}$. There is only one member of the set $\{pE_{11} + tE_{22} : t \geq 0\}$ that has rank one, that is, pE_{11}. Of, course, ϕ maps it into the unique rank one matrix in $\{R + sT : s \geq 0\}$. Hence, $R = g(p)E_{11}$. And we know that on one hand, $\{tE_{22} : t \geq 0\}$ is the unique maximal adjacent set of type one with the property that each element of this set is adjacent to exactly one element from the set $\{pE_{11} + tE_{22} : t \geq 0\}$, and on the other hand $\{sT : s \geq 0\}$ is the unique maximal adjacent set of type one with the property that each element of this set is adjacent to exactly one element from the set $\{R + sT : s \geq 0\}$. Thus, $\{tE_{22} : t \geq 0\} = \phi(\{tE_{22} : t \geq 0\}) = \{sT : s \geq 0\}$. It follows that for every nonnegative real number t we have $\phi(pE_{11} + tE_{22}) = g(p)E_{11} + qE_{22}$ for some nonnegative q. But $pE_{11} + tE_{22}$ is adjacent to tE_{22} which is mapped by ϕ to $f(t)E_{22}$, and therefore, $q = f(t)$. As $p > 0$ was an arbitrary positive real number, we conclude that

$$\phi(sE_{11} + tE_{22}) = g(s)E_{11} + f(t)E_{22}, \quad t, s \geq 0.$$

Let now

$$\begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix}$$

be any matrix in $\mathcal{H}_2^{\geq 0}$ with $\alpha \neq 0$, and denote

$$\phi\left(\begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix}\right) = \begin{bmatrix} t_1 & \alpha_1 \\ \overline{\alpha_1} & s_1 \end{bmatrix}.$$

For every real number $q, q > s$, we have

$$\begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix} \sim \begin{bmatrix} t + \frac{|\alpha|^2}{q-s} & 0 \\ 0 & q \end{bmatrix}$$

and consequently,

$$\begin{bmatrix} t_1 & \alpha_1 \\ \overline{\alpha_1} & s_1 \end{bmatrix} \sim \begin{bmatrix} g\left(t + \frac{|\alpha|^2}{q-s}\right) & 0 \\ 0 & f(q) \end{bmatrix}.$$

It follows that the determinant of the difference of the above two matrices is equal to zero. Thus, for every $q > s$ we have

$$(f(q) - s_1)\left(g\left(t + \frac{|\alpha|^2}{q-s}\right) - t_1\right) = |\alpha_1|^2.$$

When q tends to infinity, $f(q) - s_1 \to \infty$ as well, and therefore, the second factor on the left-hand side of the above equality tends to zero. Because g is continuous we have $t_1 = g(t)$, and similarly, $s_1 = f(s)$.

Assume next that

$$R = \begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix}$$

is a matrix from $\mathcal{H}_2^{\geq 0}$ of rank one satisfying $\alpha \neq 0$. We know that

$$\phi(R) = \begin{bmatrix} g(t) & \alpha_1 \\ \overline{\alpha_1} & f(s) \end{bmatrix}$$

for some nonzero complex number α_1. Further, for any positive real number p there exists a nonzero complex number α_2 such that

$$\phi(pR) = \begin{bmatrix} g(pt) & \alpha_2 \\ \overline{\alpha_2} & f(ps) \end{bmatrix}.$$

But $\phi(pR)$ and $\phi(R)$ are adjacent hermitian matrices both of rank one, and therefore, $\phi(pR) = c\phi(R)$ for some nonzero real number c. In particular, we have

$$\frac{g(pt)}{g(t)} = \frac{f(ps)}{f(s)}.$$

This equation holds for all $p, s, t > 0$. It follows that the quotient $\frac{g(pt)}{g(t)}$ is independent of t. Hence, there is a function $h : (0, \infty) \to (0, \infty)$ such that

$$g(pt) = h(p)g(t), \quad p, t > 0,$$

and, of course, we have also $f(ps) = h(p)f(s)$ for all real numbers $p, s > 0$. Clearly, $h(1) = 1$. Choosing $t = 1$ in the above equation we see that $g = c_1 h$, where $c_1 = g(1)$. Similarly, $f = c_2 h$ for some positive real number c_2. The above equation now yields that

$$h(pt) = h(p)h(t)$$

for all $p, t > 0$. We define a function $a : \mathbb{R} \to \mathbb{R}$ by $a(t) = log h(e^t)$. It is easy to see that a is additive, that is $a(t + s) = a(t) + a(s)$ for all $t, s \in \mathbb{R}$. Moreover, a is continuous. It is well-known that every continuous additive function $a : \mathbb{R} \to \mathbb{R}$ is of the form $a(t) = kt$ for some real constant k. Hence, we have $h(t) = t^k, t > 0$.

Putting all the facts obtained so far together we see that for every

$$A = \begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix} \in \mathcal{H}_2^{\geq 0}$$

with $t, s > 0$ we have

$$\phi(A) = \begin{bmatrix} c_1 t^k & \alpha_1 \\ \overline{\alpha_1} & c_2 s^k \end{bmatrix}$$

for some $\alpha_1 \in \mathbb{C}$. It follows from $\phi(I) = I$ that $c_1 = c_2 = 1$. Clearly, the rank one projection

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

is adjacent to I, and consequently, its ϕ-image

$$\phi(P) = \begin{bmatrix} \left(\frac{1}{2}\right)^k & * \\ * & \left(\frac{1}{2}\right)^k \end{bmatrix}$$

is adjacent to $\phi(I) = I$. But then $\phi(P)$ is again a projection of rank one. In particular, its trace is equal to 1, that is, $2\left(\frac{1}{2}\right)^k = 1$. This yields $k = 1$.

We have shown that for every

$$A = \begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix} \in \mathcal{H}_2^{\geq 0}$$

there exists $\alpha_1 \in \mathbb{C}$ such that

$$\phi(A) = \begin{bmatrix} t & \alpha_1 \\ \overline{\alpha_1} & s \end{bmatrix}.$$

Of course, we have $\alpha_1 \neq 0$ if and only if $\alpha \neq 0$.

Let $A \in \mathcal{H}_2^{\geq 0}$ be any matrix of rank two, $A \notin (0, \infty)I = \{tI : t > 0\}$. A positive real number t is an eigenvalue of A if and only if A is adjacent to tI, which is true if and only if $\phi(A)$ is adjacent to tI. Hence, $\phi(A)$ and A have the same eigenvalues. As $A \in \mathcal{H}_2^{\geq 0}$ is singular if and only if $\phi(A)$ is singular, we have $\det \phi(A) = \det A$ for every $A \in \mathcal{H}_2^{\geq 0}$. It follows that for every

$$A = \begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix} \in \mathcal{H}_2^{\geq 0}$$

there exists $u \in \mathbb{C}$ with $|u| = 1$ such that

$$\phi(A) = \begin{bmatrix} t & u\alpha \\ \overline{u\alpha} & s \end{bmatrix}.$$

Replacing the map ϕ by the map $A \mapsto U\phi(A)U^*$, where U is an appropriate 2×2 diagonal unitary matrix, we may assume with no loss of generality that

$$\phi\left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

We claim that for every $A \in \mathcal{H}_2^{\geq 0}$ of rank one we have either $\phi(A) = A$, or $\phi(A) = A^t$. It is enough to show this only for the special case when A is a projection of rank one. Indeed, if P is a projection of rank one such that for P we have one of the above two possibilities, say $\phi(P) = P^t$, and if t is any positive real number, then tP is adjacent to tI, and because $\phi(tP) = s\phi(P) = sP^t$ for some positive real number s and $\phi(tP)$ is adjacent to tI as well, we conclude that $\phi(tP) = tP^t$ for every $t \geq 0$.

Hence, all we need to do is to prove that for every projection P of rank one we have either $\phi(P) = P$, or $\phi(P) = P^t$. We already know that this is true when $P = E_{11}$ or $P = E_{22}$. So, assume $P \neq E_{11}, E_{22}$. Then

$$P = \begin{bmatrix} x & \sqrt{x(1-x)}e^{i\varphi} \\ \sqrt{x(1-x)}e^{-i\varphi} & 1-x \end{bmatrix}$$

for some real $x, 0 < x < 1$, and some $\varphi \in [0, 2\pi)$. We already know that

$$\phi(P) = \begin{bmatrix} x & \sqrt{x(1-x)}e^{i\psi} \\ \sqrt{x(1-x)}e^{-i\psi} & 1-x \end{bmatrix}$$

for some some $\psi \in [0, 2\pi)$. There exists a unique positive real number p such that the matrix

$$K = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

is adjacent to pP. Clearly, $K = \phi(K)$ is adjacent to $p\phi(P)$ as well. For this p we have

$$\begin{vmatrix} px - 1 & p\sqrt{x(1-x)}e^{i\varphi} - 1 \\ p\sqrt{x(1-x)}e^{-i\varphi} - 1 & p(1-x) - 2 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} px - 1 & p\sqrt{x(1-x)}e^{i\psi} - 1 \\ p\sqrt{x(1-x)}e^{-i\psi} - 1 & p(1-x) - 2 \end{vmatrix} = 0.$$

Comparing these two equations we arrive at

$$|me^{i\varphi} - 1| = |me^{i\psi} - 1|,$$

where $m = p\sqrt{x(1-x)} > 0$. It follows that either $\psi = \varphi$, or $\psi = -\varphi$, as desired.

Thus, each rank one matrix $A \in \mathcal{H}_2^{\geq 0}$ is mapped by ϕ either into itself, or into the transpose of itself.

After composing ϕ with the transposition, if necessary, we may assume that

$$\phi\left(\begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{bmatrix}.$$

Let $A \in \mathcal{H}_2^{\geq 0}$ be any non-diagonal matrix of rank two. Then

$$\phi(A) = \phi\left(\begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix}\right) = \begin{bmatrix} t & \alpha_1 \\ \overline{\alpha_1} & s \end{bmatrix}.$$

We have $|\alpha| = |\alpha_1| \neq 0$. There exist unique positive real numbers p and q such that

$$\begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix} \quad \text{and} \quad p\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

are adjacent, and

$$\begin{bmatrix} t & \alpha \\ \overline{\alpha} & s \end{bmatrix} \quad \text{and} \quad q\begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

are adjacent. This yields that

$$\left(t - \frac{p}{2}\right)\left(s - \frac{p}{2}\right) - \left|\alpha - \frac{p}{2}\right|^2 = 0$$

and

$$\left(t - \frac{q}{2}\right)\left(s - \frac{q}{2}\right) - \left|\alpha - i\frac{q}{2}\right|^2 = 0.$$

But then the matrices

$$\begin{bmatrix} t & \alpha_1 \\ \overline{\alpha_1} & s \end{bmatrix} \quad \text{and} \quad p\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

are adjacent, and

$$\begin{bmatrix} t & \alpha_1 \\ \overline{\alpha_1} & s \end{bmatrix} \quad \text{and} \quad q\begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

are adjacent, and therefore

$$\left(t - \frac{p}{2}\right)\left(s - \frac{p}{2}\right) - \left|\alpha_1 - \frac{p}{2}\right|^2 = 0$$

and

$$\left(t - \frac{q}{2}\right)\left(s - \frac{q}{2}\right) - \left|\alpha_1 - i\frac{q}{2}\right|^2 = 0.$$

Hence,

$$|\alpha| = |\alpha_1| \quad \text{and} \quad \left|\alpha - \frac{p}{2}\right| = \left|\alpha_1 - \frac{p}{2}\right| \quad \text{and} \quad \left|\alpha - i\frac{q}{2}\right| = \left|\alpha_1 - i\frac{q}{2}\right|,$$

which yields that $\alpha = \alpha_1$. We have shown that $\phi(A) = A$ for every $A \in \mathcal{H}_2^{\geq 0}$ of rank two. We also know that for every $R \in \mathcal{H}_2^{\geq 0}$ of rank one we have either $\phi(R) = R$, or $\phi(R) = R^t$. In order to complete the proof we have to show that the second possibility cannot occur.

Assume on the contrary that there is a rank one matrix $R \in \mathcal{H}_2^{\geq 0}$ such that $\phi(R) = R^t \neq R$. With no loss of generality we may assume that R is a projection of rank one. Clearly, R is adjacent to $R+t(I-R)$ for every positive real number t, and consequently, R^t must be adjacent to $R+t(I-R)$ for every positive real number t as well. Hence, $R = R^t$, a contradiction. This completes the proof. □

The next step is to consider the finite-dimensional case. We denote by $\mathcal{H}_n^{\geq 0}$ the set of all $n \times n$ positive matrices.

Proposition 2.19. *Let $n \geq 2$ be an integer and $\phi : \mathcal{H}_n^{\geq 0} \to \mathcal{H}_n^{\geq 0}$ a bijective map preserving adjacency in both directions. Then there exists an invertible $n \times n$ complex matrix T such that either*

$$\phi(A) = TAT^*$$

for every $A \in \mathcal{H}_n^{\geq 0}$; or

$$\phi(A) = TA^tT^*$$

for every $A \in \mathcal{H}_n^{\geq 0}$.

Proof. As in the previous statement we have $d(\phi(A), \phi(B)) = d(A, B)$, $A, B \in \mathcal{H}_n^{\geq 0}$, and $\phi(0) = 0$. In particular, $\phi(I)$ is a positive invertible matrix, and therefore, after replacing ϕ by a map $A \mapsto \phi(I)^{-1/2}\phi(A)\phi(I)^{-1/2}$, we may, and we will assume that $\phi(I) = I$. We claim that then ϕ maps projections into projections. Moreover, if $P, Q \in \mathcal{H}_n^{\geq 0}$ are two projections, then $Q \leq P \iff \phi(Q) \leq \phi(P)$. Indeed, assume that P, Q are projections with $Q \leq P$. Then $\operatorname{rank} Q = q \leq q + p = \operatorname{rank} P$. Therefore, $\operatorname{rank} \phi(Q) = q, \operatorname{rank} \phi(P) = q+p, d(\phi(Q), \phi(P)) = p, \operatorname{rank} \phi(I) = \operatorname{rank} I = n$, and $d(I, \phi(P)) = n-(q+p)$. Applying Corollary 2.6 two times, we first conclude that $\phi(P)$ is a projection, and then that $\phi(Q)$ is a projection satisfying $\phi(Q)\phi(P) = \phi(Q)$, or equivalently, $\phi(Q) \leq \phi(P)$.

Next, we will prove that for every projection $P \in \mathcal{H}_n^{\geq 0}$ of rank two we have $\phi(P\mathcal{H}_n^{\geq 0}P) = \phi(P)\mathcal{H}_n^{\geq 0}\phi(P)$. As the inverse of ϕ has the same properties as ϕ it is enough to show that $\phi(P\mathcal{H}_n^{\geq 0}P) \subset \phi(P)\mathcal{H}_n^{\geq 0}\phi(P)$. Let $A \in P\mathcal{H}_n^{\geq 0}P$ be a matrix of rank one. Then $A = tQ$ for some positive real number t and a rank one projection Q with $Q \leq P$. It follows that $\phi(Q) \leq \phi(P)$. We are done if $t = 1$. If not, then the rank one matrix $\phi(tQ)$ is adjacent to the rank one matrix $\phi(Q)$, and consequently, $\phi(tQ) = s\phi(Q)$ for some positive real number s. Because $\phi(Q) \in \phi(P)\mathcal{H}_n^{\geq 0}\phi(P)$, we have $\phi(tQ) \in \phi(P)\mathcal{H}_n^{\geq 0}\phi(P)$, as well.

Assume next that $A \in P\mathcal{H}_n^{\geq 0}P$ is of rank two. Then $A = t_1Q_1 + t_2Q_2$, where t_1 and t_2 are positive real numbers and Q_1 and Q_2 are orthogonal projections of rank one. Hence, $\phi(A)$ is adjacent to $\phi(t_1Q_1)$, that is, $\phi(A) = \phi(t_1Q_1)+R$ for some rank one matrix R. By Lemma 2.5 we have $\operatorname{Im} \phi(t_1Q_1) \subset \operatorname{Im} \phi(A)$ (here, we have identified matrices with operators acting on $\mathbb{C}^n$). Similarly, $\operatorname{Im} \phi(t_2Q_2) \subset \operatorname{Im} \phi(A)$. Since $\operatorname{rank} \phi(A) = 2$ and $d(\phi(t_1Q_1), \phi(t_2Q_2)) = 2$, the image of $\phi(A)$ is spanned by the images of $\phi(t_1Q_1)$ and $\phi(t_2Q_2)$. We already know that the image of $\phi(P)$ is spanned by the images of $\phi(t_1Q_1)$ and $\phi(t_2Q_2)$. Hence, $\operatorname{Im} \phi(A) = \operatorname{Im} \phi(P)$, and therefore, $\phi(A) \in \phi(P)\mathcal{H}_n^{\geq 0}\phi(P)$, as desired.

We have proved that for every projection $P \in \mathcal{H}_n^{\geq 0}$ of rank two we have $\phi(P\mathcal{H}_n^{\geq 0}P) = \phi(P)\mathcal{H}_n^{\geq 0}\phi(P)$. And we know that $\phi(P)$ is a projection of rank two as well. So, both sets $P\mathcal{H}_n^{\geq 0}P$ and $\phi(P)\mathcal{H}_n^{\geq 0}\phi(P)$ can be identified in the natural way with $\mathcal{H}_2^{\geq 0}$. And therefore, we can apply Proposition 2.18 for the restriction $\phi_{|P\mathcal{H}_n^{\geq 0}P} : P\mathcal{H}_n^{\geq 0}P \to \phi(P)\mathcal{H}_n^{\geq 0}\phi(P)$. In particular, as this map sends projections to projections, we have $\phi(R_1)\phi(R_2) = 0$ whenever $R_1, R_2 \in P\mathcal{H}_n^{\geq 0}P$ are projections satisfying $R_1R_2 = 0$. Moreover,

$$\phi(ABA) = \phi(A)\phi(B)\phi(A) \tag{5}$$

for all $A, B \in P\mathcal{H}_n^{\geq 0}P$.

Because ϕ maps the set of projections onto itself and preserves order on the set of projections, we may assume, after replacing ϕ by the map $A \mapsto U\phi(A)U^*$ for an appropriate unitary matrix U, that $\phi(E_{11} + \cdots + E_{jj}) = E_{11} + \cdots + E_{jj}, j = 1, \cdots, n$. Considering the restriction of ϕ to $(E_{ii} + E_{jj})\mathcal{H}_n^{\geq 0}(E_{ii} + E_{jj})$ and using the previous paragraph we see that $\phi(E_{ii})\phi(E_{jj}) = 0$ whenever $i \neq j$. Since $\phi(E_{22}) \leq \phi(E_{11} + E_{22}) = E_{11} + E_{22}$ and $0 = \phi(E_{22})\phi(E_{11}) = \phi(E_{22})E_{11}$, we have $\phi(E_{22}) = E_{22}$. By induction we get that $\phi(E_{jj}) = E_{jj}, j = 1, \ldots, n$.

It follows that $\phi((E_{ii} + E_{jj})\mathcal{H}_n^{\geq 0}(E_{ii} + E_{jj})) = (E_{ii} + E_{jj})\mathcal{H}_n^{\geq 0}(E_{ii} + E_{jj}), i \neq j$. Now, using the remark after Proposition 2.18 we conclude that there exists a complex number w_{ij} of modulus one such that either

$$\phi(tE_{ii} + \alpha E_{ij} + \overline{\alpha}E_{ji} + sE_{jj}) = tE_{ii} + w_{ij}\alpha E_{ij} + \overline{w_{ij}\alpha}E_{ji} + sE_{jj}$$

for all $tE_{ii} + \alpha E_{ij} + \overline{\alpha}E_{ji} + sE_{jj} \in (E_{ii} + E_{jj})\mathcal{H}_n^{\geq 0}(E_{ii} + E_{jj})$; or

$$\phi(tE_{ii} + \alpha E_{ij} + \overline{\alpha}E_{ji} + sE_{jj}) = tE_{ii} + w_{ij}\overline{\alpha}E_{ij} + \overline{w_{ij}}\alpha E_{ji} + sE_{jj}$$

for all $tE_{ii} + \alpha E_{ij} + \overline{\alpha}E_{ji} + sE_{jj} \in (E_{ii} + E_{jj})\mathcal{H}_n^{\geq 0}(E_{ii} + E_{jj})$.

Let $A = [a_{ij}] \in \mathcal{H}_n^{\geq 0}$ be any rank one matrix and take any rank one matrix $B \in (E_{ii} + E_{jj})\mathcal{H}_n^{\geq 0}(E_{ii} + E_{jj})$. We can then find a projection P of rank two such that $PAP = A$ and $PBP = B$. Therefore, by (5) we have

$$\phi(BAB) = \phi(B)\phi(A)\phi(B)$$

for every $B = tE_{ii} + \alpha E_{ij} + \overline{\alpha}E_{ji} + sE_{jj}$, where $t, s \geq 0$ and $ts - |\alpha|^2 = 0$. A straightforward computation shows that

$$(E_{ii} + E_{jj})\phi([a_{pq}])(E_{ii} + E_{jj}) = a_{ii}E_{ii} + w_{ij}a_{ij}E_{ij} + \overline{w_{ij}}a_{ji}E_{ji} + a_{jj}E_{jj};$$

or

$$(E_{ii} + E_{jj})\phi([a_{pq}])(E_{ii} + E_{jj}) = a_{ii}E_{ii} + w_{ij}\overline{a_{ij}}E_{ij} + \overline{w_{ij}a_{ji}}E_{ji} + a_{jj}E_{jj}.$$

Hence, for every $[a_{ij}] \in \mathcal{H}_n^{\geq 0}$ of rank one we have $\phi([a_{ij}]) = [w_{ij}f_{ij}(a_{ij})]$, where $w_{11} = \ldots = w_{nn} = 1, w_{ij}$ are complex numbers of modulus one with $w_{ij} = \overline{w_{ji}}, 1 \leq i, j \leq n, i \neq j, f_{11} = \cdots = f_{nn}$ are identity functions on $[0, \infty)$, and every function $f_{ij} = f_{ji} : \mathbb{C} \to \mathbb{C}$ is either the identity, or the complex conjugation, $1 \leq i, j \leq n, i \neq j$. We further know that for every $[a_{ij}] \in \mathcal{H}_n^{\geq 0}$ of rank one the matrix $[w_{ij}f_{ij}(a_{ij})]$ has rank one as well. It follows easily that there exist complex numbers $z_1, \ldots, z_n$ of modulus one such that $w_{ij} = z_i\overline{z_j}$ and either $f_{ij} = id$ for all $i \neq j$, or f_{ij} is the complex conjugation for all $i \neq j$.

We have proved that either for every rank one $A \in \mathcal{H}_n^{\geq 0}$ we have $\phi(A) = UAU^*$, or for every rank one $A \in \mathcal{H}_n^{\geq 0}$ we have $\phi(A) = UA^tU^*$, where U is a diagonal unitary matrix $U = \operatorname{diag}(z_1, \ldots, z_n)$.

After replacing ϕ by either $A \mapsto U^*\phi(A)U$, or $A \mapsto U^*\phi(A^t)U$, we may, and we will assume that $\phi(A) = A$ for every rank one matrix $A \in \mathcal{H}_n^{\geq 0}$. We have to prove that this holds for all $A \in \mathcal{H}_n^{\geq 0}$. We start with matrices A of rank two. For every such A there exists a unique projection P of rank two such $PAP = A$. We know that $\phi(P\mathcal{H}_n^{\geq 0}P) = \phi(P)\mathcal{H}_n^{\geq 0}\phi(P)$. Moreover, $\phi(P)$ is a projection of rank two and $\phi(Q) = Q$ for every projection Q of rank one satisfying $Q \leq P$. It follows that $\phi(P) = P$, and consequently, $\phi(P\mathcal{H}_n^{\geq 0}P) = P\mathcal{H}_n^{\geq 0}P$. Now we apply Proposition 2.18 to the restriction of ϕ on $P\mathcal{H}_n^{\geq 0}P$ together with the fact that this restriction acts like the identity operator on all rank one operators to conclude that this restriction is the identity operator. Hence, $\phi(A) = A$, as desired. It is now straightforward to prove that $\phi(A) = A$ for every $A \in \mathcal{H}_n^{\geq 0}$ by induction on rank A. Namely, the induction step is a direct consequence of Lemma 2.9. □

Now we are ready to prove Theorem 2.2 in full generality.

Proof of Theorem 2.2. We are now in a similar position as in the case of the proof of Theorem 2.1 which was based on the finite-dimensional result formulated as Theorem 2.16. In fact, the present situation is even simpler as we do not need to deal with the degenerate form.

Hence, once again we only need to show that for every projection $P \in \mathcal{S}_F(H)^{\geq 0}$ of rank r there exists a projection $Q \in \mathcal{S}_F(H)^{\geq 0}$ of rank r such that $\phi(P\mathcal{S}_F(H)^{\geq 0}P) = Q\mathcal{S}_F(H)^{\geq 0}Q$. Actually we only need to prove that $\phi(P\mathcal{S}_F(H)^{\geq 0}P) \subset Q\mathcal{S}_F(H)^{\geq 0}Q$ because the inverse ϕ^{-1} has the same properties as ϕ. Assume for a moment that we have already done this. Then the rest of the proof goes in a similar way as the proof of Theorem 2.1 with a difference that now the situation is simpler as we have stronger assumptions (the bijectivity assumption and the assumption of preserving adjacency in both directions) and in our present case the degenerate form cannot occur. Therefore we will leave the details to the readers.

So, let $P \in \mathcal{S}_F(H)^{\geq 0}$ be a projection of rank r. We know that then $\phi(P)$ is a positive operator of rank r. Denote by Q the projection of rank r satisfying $\operatorname{Im} Q = \operatorname{Im} \phi(P)$, that is, $\phi(P) = Q\phi(P)Q$. Let P_1 be any projection of rank one such that $P_1 \leq P$. Then $d(P_1, P) = r - 1$, and consequently, $\phi(P) = \phi(P_1) + S$ where S is a self-adjoint operator of rank $r - 1$. By Lemma 2.5 we have $\operatorname{Im} \phi(P_1) \subset \operatorname{Im} Q$. As $\phi(tP_1)$ is a rank one operator adjacent to $\phi(P_1)$ for every positive real number $t \neq 1$, we conclude that $Q\phi(A)Q = \phi(A)$ for every rank one operator $A \in P\mathcal{S}_F(H)^{\geq 0}P$. We have to show that this holds for every $A \in P\mathcal{S}_F(H)^{\geq 0}P$. We will proceed inductively. Assume that this is true for all $A \in P\mathcal{S}_F(H)^{\geq 0}P$ of rank smaller than $k, k \leq r$, and let $B \in P\mathcal{S}_F(H)^{\geq 0}P$ be of rank k. Then we can write $B = B_1 + B_2$ with $\operatorname{rank} B_1 = k - 1, \operatorname{rank} B_2 = 1, d(B_1, B_2) = k$, and $B_1, B_2 \in P\mathcal{S}_F(H)^{\geq 0}P$. Because $d(B_2, B) = k - 1$ we have $\phi(B) = \phi(B_2) + T$ for some T of rank $k - 1$. Applying Lemma 2.5 again we see that $\operatorname{Im} \phi(B_2) \subset \operatorname{Im} B$. Similarly, $\operatorname{Im} \phi(B_1) \subset \operatorname{Im} B$. Of course, $\dim \operatorname{Im} \phi(B_1) = k - 1$ and $\dim \operatorname{Im} \phi(B_2) = 1$. Clearly, $\operatorname{Im} \phi(B_2) \not\subset \operatorname{Im} \phi(B_1)$, since otherwise we would have $d(\phi(B_1), \phi(B_2)) \leq k - 1$, a contradiction. Thus, the sum of images of $\phi(B_1)$ and $\phi(B_2)$ is a direct sum, and therefore, $\operatorname{Im}\phi(B_1) \oplus \operatorname{Im}\phi(B_2) \subset \operatorname{Im}\phi(B)$. Comparing the dimensions of these subspaces we see that actually $\operatorname{Im}\phi(B_1) \oplus \operatorname{Im}\phi(B_2) = \operatorname{Im}\phi(B)$. By the induction hypothesis we have $\operatorname{Im}\phi(B_1), \operatorname{Im}\phi(B_2) \subset \operatorname{Im} Q$, and consequently, $Q\phi(B)Q = \phi(B)$, as desired. □

Proof of Theorem 2.3. Since $\phi(0) = 0$ and $d(\phi(A), \phi(B)) = d(A, B)$, $A, B \in \mathcal{S}_F(H)^{>-I}$, we have $\operatorname{rank}\phi(A) = \operatorname{rank} A$ for every $A \in \mathcal{S}_F(H)^{>-I}$. Let $A \in \mathcal{S}_F(H)^{>-I}$ be an operator of rank two. Recall that J_A is the set of all rank one operators $R \in \mathcal{S}_F(H)^{>-I}$ with the property that there exists a real number t such that $tR \in \mathcal{S}_F(H)^{>-I}$ and $d(tR, A) = 1$. Let $R \in \mathcal{S}_F(H)^{>-I}$ be a rank one operator. Then $Q \in \mathcal{S}_F(H)^{>-I}$ is of rank one and is adjacent to R if and only if $Q = tR$ for some real number $t \neq 0, 1$ such that $tR \in \mathcal{S}_F(H)^{>-I}$. It is now straightforward to check that

$$\phi(J_A) = J_{\phi(A)} \tag{6}$$

for every $A \in \mathcal{S}_F(H)^{>-I}$ of rank two. This together with Lemma 2.13 yields that the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having one positive and one negative eigenvalue is mapped by ϕ onto itself. It follows that the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having two positive or two negative eigenvalues is mapped by ϕ onto itself as well.

In the next step we will prove that either

- the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having two positive eigenvalues is mapped by ϕ onto itself and the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having two negative eigenvalues is mapped by ϕ onto itself; or
- the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having two positive eigenvalues is mapped by ϕ onto the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having two negative eigenvalues and the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having two negative eigenvalues is mapped by ϕ onto the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having two positive eigenvalues.

Let us first fix some more notation. We will denote by $\mathcal{W}_{2+}, \mathcal{W}_{2-}$, and $\mathcal{W}_{2o}$ the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having two positive eigenvalues, the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having two negative eigenvalues, and the set of all rank two operators from $\mathcal{S}_F(H)^{>-I}$ having one positive and one negative eigenvalue. Set $\mathcal{W}_{2e} = \mathcal{W}_{2+} \cup \mathcal{W}_{2-}$. For an arbitrary pair of operators $A, B \in \mathcal{W}_{2e}$ we write $A \cong B$ whenever there exists a rank one operator $R \in \mathcal{S}_F(H)^{>-I}$ such that $d(A, R) = d(B, R) = 1$, and $A \approx B$ whenever there exists a sequence $A = A_0, A_1, \ldots, A_k = B \in \mathcal{W}_{2e}$ such that $A_{j-1} \cong A_j$, $j = 1, \ldots, k$. We already know that $\phi(\mathcal{W}_{2e}) = \mathcal{W}_{2e}$. It is straightforward to check that for an arbitrary pair $A, B \in \mathcal{W}_{2e}$ we have

$$A \cong B \iff \phi(A) \cong \phi(B)$$

and

$$A \approx B \iff \phi(A) \approx \phi(B).$$

Hence, this step of the proof will be completed once we will show that for all pairs $A, B \in \mathcal{W}_{2e}$ we have $A \approx B$ if and only if either both A, B belong to $\mathcal{W}_{2+}$, or both A, B belong to $\mathcal{W}_{2-}$. In order to verify this statement let us consider an operator $A \in \mathcal{W}_{2-}$. Then $A = tx \otimes x^* + sy \otimes y^*$, where $t, s \in (-1, 0)$ and $x, y \in H$ are unital orthogonal vectors. Clearly, $A \cong -\frac{1}{2}x \otimes x^* + sy \otimes y^*$ and $-\frac{1}{2}x \otimes x^* + sy \otimes y^* \cong -\frac{1}{2}x \otimes x^* - \frac{1}{2}y \otimes y^*$. Thus, for each $A \in \mathcal{W}_{2-}$ there exists a projection P of rank two such that $A \approx -\frac{1}{2}P$. It is easy to see that if P and Q are projections of rank two and $\operatorname{Im} P \cap \operatorname{Im} Q \neq \{0\}$, then $-\frac{1}{2}P \cong -\frac{1}{2}Q$. Indeed, there is nothing to prove when $P = Q$. If $P \neq Q$, then both $-\frac{1}{2}P$ and $-\frac{1}{2}Q$ are adjacent to $-\frac{1}{2}R$, where R is a rank one projection onto $\operatorname{Im} P \cap \operatorname{Im} Q$. We have shown that $A, B \in \mathcal{W}_{2-}$ yields $A \approx B$. Similarly, $A \approx B$ whenever $A, B \in \mathcal{W}_{2+}$. It remains to show that if $A \in \mathcal{W}_{2+}, B \in \mathcal{W}_{2e}$, and $A \approx B$, then $B \in \mathcal{W}_{2+}$, and similarly, that if $A \in \mathcal{W}_{2-}, B \in \mathcal{W}_{2e}$, and $A \approx B$, then $B \in \mathcal{W}_{2-}$. Once again we will prove just one of these two implications, say the first one. Clearly, it is enough to show that if $A \in \mathcal{W}_{2+}, B \in \mathcal{W}_{2e}$, and $A \cong B$, then $B \in \mathcal{W}_{2+}$. Thus, let $A \in \mathcal{W}_{2+}, B \in \mathcal{W}_{2e}$, and $A \cong B$. Then there is a rank one operator $R \in \mathcal{S}_F(H)^{>-I}$ that is adjacent to both A and B. By Corollary 2.7 we have $R \geq 0$, and applying the same corollary once more we conclude that $B \geq 0$ as well.

We have thus proved that either $\phi(\mathcal{W}_{2+}) = \mathcal{W}_{2+}$ and $\phi(\mathcal{W}_{2-}) = \mathcal{W}_{2-}$, or $\phi(\mathcal{W}_{2+}) = \mathcal{W}_{2-}$ and $\phi(\mathcal{W}_{2-}) = \mathcal{W}_{2+}$. In the first case we need to prove that there exists a unitary or an antiunitary operator $U : H \to H$ such that $\phi(A) = UAU^*$ for every $A \in \mathcal{S}_F(H)^{>-I}$.

Assume for a moment that we have already completed the proof in the first case and let us show that then the second case follows from the first one. So, assume that $\phi(\mathcal{W}_{2+}) = \mathcal{W}_{2-}$ and $\phi(\mathcal{W}_{2-}) = \mathcal{W}_{2+}$ and define $\xi : \mathcal{S}_F(H)^{>-I} \to \mathcal{S}(H)$ by

$$\xi(A) = (I + A)^{-1} - I, \quad A \in \mathcal{S}_F(H)^{>-I}.$$

The map is well-defined since $I + A$ is invertible for every $A \in \mathcal{S}_F(H)^{>-I}$. It is straightforward to check that $A, B \in \mathcal{S}_F(H)^{>-I}$ are adjacent if and only if $\xi(A)$ and $\xi(B)$ are adjacent. This together with $\xi(0) = 0$ yields that $\xi(\mathcal{S}_F(H)^{>-I}) \subset \mathcal{S}_F(H)$. Since $(I + A)^{-1}$ is positive for every $A \in \mathcal{S}_F(H)^{>-I}$ we actually have $\xi(\mathcal{S}_F(H)^{>-I}) \subset \mathcal{S}_F(H)^{>-I}$. A direct computation tells that $\xi(\xi(A)) = A$ for every $A \in \mathcal{S}_F(H)^{>-I}$. Hence, ξ is a bijective map from $\mathcal{S}_F(H)^{>-I}$ onto itself preserving adjacency in both directions and satisfying $\xi(0) = 0$. It follows that the map $\psi : \mathcal{S}_F(H)^{>-I} \to \mathcal{S}_F(H)^{>-I}$ defined by $\psi(A) = (I + \phi(A))^{-1} - I$ is a bijective map from $\mathcal{S}_F(H)^{>-I}$ onto itself preserving adjacency in both directions and mapping the zero operator into itself. A direct verification shows that $\psi(\mathcal{W}_{2+}) = \mathcal{W}_{2+}$ and $\psi(\mathcal{W}_{2-}) = \mathcal{W}_{2-}$, and thus, by the first case we have

$$(I + \phi(A))^{-1} - I = UAU^*, \quad A \in \mathcal{S}_F(H)^{>-I},$$

for some unitary or antiunitary operator $U : H \to H$. It follows that $\phi(A) = U(I + A)^{-1}U^* - I, A \in \mathcal{S}_F(H)^{>-I}$, and we are done.

So, we may assume from now on that $\phi(\mathcal{W}_{2+}) = \mathcal{W}_{2+}$ and $\phi(\mathcal{W}_{2-}) = \mathcal{W}_{2-}$. Let us show that for every rank one operator $A \in \mathcal{S}_F(H)^{>-I}$ we have

$A \geq 0$ if and only if $\phi(A) \geq 0$. Assume that $A \in \mathcal{S}_F(H)^{>-I}$ is of rank one and $A \geq 0$. Then one can easily find a rank two operator $B \in \mathcal{S}_F(H)^{>-I}$ such that A and B are adjacent and $B \geq 0$. Thus, $\phi(A)$ and $\phi(B)$ are adjacent, $\phi(A)$ is of rank one, and we already know that $\phi(B) \geq 0$. By Corollary 2.7 we have $\phi(A) \geq 0$, as desired. Since ϕ^{-1} has the same properties as ϕ, we conclude that the set of positive rank one operators is mapped by ϕ onto itself.

We are now ready to show that the restriction of ϕ to $\mathcal{S}_F(H)^{\geq 0}$ is a bijective map of $\mathcal{S}_F(H)^{\geq 0}$ onto itself. Once again it is enough to show that $\phi(\mathcal{S}_F(H)^{\geq 0}) \subset \mathcal{S}_F(H)^{\geq 0}$. So, let $A \in \mathcal{S}_F(H)^{\geq 0}, A \neq 0$, and set $r = \operatorname{rank} A$. By Corollary 2.7 we have $d(A,B) \geq r$ for every rank one operator $B \in \mathcal{S}_F(H)^{>-I}$ satisfying $B \leq 0$. Hence, $\phi(A) \in \mathcal{S}_F(H)^{>-I}$ is an operator of rank r such that $d(\phi(A),B) \geq r$ for every rank one operator $B \in \mathcal{S}_F(H)^{>-I}$ satisfying $B \leq 0$. We have

$$\phi(A) = \sum_{j=1}^{r} t_j x_j \otimes x_j^*,$$

where $x_1,\dots,x_r$ is an orthonormal set of vectors and $t_1 \geq t_2 \geq \cdots \geq t_r$ are nonzero eigenvalues of $\phi(A)$. Clearly, $t_r x_r \otimes x_r^*$ is a rank one operator satisfying $d(\phi(A), t_r x_r \otimes x_r^*) = r-1$, and by the above property of $\phi(A)$, we have $t_r > 0$. Hence, $\phi(A) \geq 0$, as desired.

Thus we can apply Theorem 2.2 to the restriction of ϕ to $\mathcal{S}_F(H)^{\geq 0}$ to conclude that there exists a bijective linear or conjugate-linear map $T : H \to H$ such that

$$\phi\left(\sum_{j=1}^{k} t_j x_j \otimes x_j^*\right) = \sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^*$$

for every $\sum_{j=1}^{k} t_j x_j \otimes x_j^* \in \mathcal{S}_F(H)^{\geq 0}$.

Let $K \subset H$ be any two-dimensional subspace. We denote by $\mathcal{Z}_K \subset \mathcal{S}_F(H)^{>-I}$ the subset of all operators A with the property that $AK \subset K$ and $Az = 0$ for every $z \in K^\perp$. We claim that

$$\phi(\mathcal{Z}_K) \subset \mathcal{Z}_{TK}.$$

We already know that $\phi(A) \subset \mathcal{Z}_{TK}$ for every $A \in \mathcal{Z}_K$ with $A \geq 0$. Assume next that $A \in \mathcal{Z}_K$ is of rank one and $A \leq 0$. Then $\phi(A)$ is of rank one and is adjacent to the rank one operator $\phi(-A)$. Hence, $\phi(A)$ is a scalar multiple of $\phi(-A)$, and therefore, $\phi(A) \subset \mathcal{Z}_{TK}$ as well. And finally, let $A \in \mathcal{Z}_K$ be any operator of rank two. We can find two rank one operators $P, Q \in \mathcal{Z}_K$ such that $\operatorname{Im} P \neq \operatorname{Im} Q$ and A is adjacent to both P and Q. We already know that $\operatorname{Im} \phi(P) = T(\operatorname{Im} P)$ and $\operatorname{Im} \phi(Q) = T(\operatorname{Im} Q)$, and since T is injective, we have $\operatorname{Im} \phi(P) \neq \operatorname{Im} \phi(Q)$. As $\phi(A)$ and $\phi(P)$ are adjacent we have $\phi(A) = \phi(P) + S$ for some rank one operator S. By Lemma 2.5, $\operatorname{Im} \phi(P) \subset \operatorname{Im} \phi(A)$, and similarly, $\operatorname{Im} \phi(Q) \subset \operatorname{Im} \phi(A)$. Thus, $\operatorname{Im} \phi(A)$ is a two-dimensional subspace containing one-dimensional subspaces $\operatorname{Im} \phi(P)$ and $\operatorname{Im} \phi(Q)$ that are not the same and are both continued in TK. It follows that $\operatorname{Im} \phi(A) = TK$, as desired.

Let $x, y \in K$ be any pair of orthonormal vectors, p a positive real number, and s a positive real number smaller than 1. Then $B = px \otimes x^* - sy \otimes y^* \in \mathcal{Z}_K$. By Lemma 2.14 there exists a nonempty open subset $\mathcal{U}$ of the unit sphere of K such that for every $u \in \mathcal{U}$ there exists a positive integer t_u such that B and $t_u u \otimes u^*$ are adjacent. Since $T : K \to TK$ is a bijective linear or conjugate-linear map between the two-dimensional subspaces K and TK, the set $\mathcal{Y} = \{\frac{1}{\|Tu\|} Tu \, : \, u \in \mathcal{U}\}$ is an open subset of the unit sphere of TK. We know that $\operatorname{Im} \phi(B) = TK = \operatorname{span}\{Tx, Ty\}$ and that for every $u \in \mathcal{U}$ the operator $\phi(B)$ is adjacent to

$$t_u (Tu) \otimes (Tu)^* = t_u \|Tu\|^2 w \otimes w^*,$$

where $w = \frac{1}{\|Tu\|} Tu \in \mathcal{Y}$. On the other hand, it is clear that $p(Tx) \otimes (Tx)^* - s(Ty) \otimes (Ty)^*$ is adjacent to $t_u (Tu) \otimes (Tu)^*$ for every $u \in \mathcal{U}$. It follows from Lemma 2.15 that $\phi(px \otimes x^* - sy \otimes y^*) = p(Tx) \otimes (Tx)^* - s(Ty) \otimes (Ty)^*$.

We already know that $\phi(-sy \otimes y^*) = qTy \otimes (Ty)^*$ for some real $q \in (-1, 0)$. Since $px \otimes x^* - sy \otimes y^*$ and $-sy \otimes y^*$ are adjacent, the same must be true for the pair of operators $p(Tx) \otimes (Tx)^* - s(Ty) \otimes (Ty)^*$ and $q(Ty) \otimes (Ty)^*$. This clearly yields that $q = -s$.

We have proved that for every unit vector $y \in H$ and every real number $s > -1$ we have $\phi(sy \otimes y^*) = s(Ty) \otimes (Ty)^*$. Let $u \in H$ be any unit vector. Then the set $\{tu \otimes u^* \, : \, t > -1\}$ is a maximal subset in $\mathcal{S}_F(H)^{>-I}$ of operators of rank at most one with the property that any two different members of this subset are adjacent. As $\phi : \mathcal{S}_F(H)^{>-I} \to \mathcal{S}_F(H)^{>-I}$ is a bijective map preserving adjacency and rank one operators in both directions, such a set is mapped by ϕ onto the set of the same type. In particular,

$$\{s(Ty) \otimes (Ty)^* : s > -1\} = \{tu \otimes u^* : t > -1\}$$

for some unit vector u. It follows easily that $\|Ty\| = 1$. As $y \in H$ was an arbitrary unit vector, we conclude that T is either a unitary, or an antiunitary operator. In particular, the adjoint operator $T^* : H \to H$ exists and then $(Tx) \otimes (Tx)^* = T(x \otimes x^*)T^*$ for every $x \in H$.

Hence, $\phi(A) = TAT^*$ for every $A \in \mathcal{S}_F(H)^{>-I}$ with $A \geq 0$. Replacing the map ϕ by $A \mapsto T^* \phi(A) T$ we may, and we will assume that $\phi(A) = A$ for every $A \in \mathcal{S}_F(H)^{>-I}$ with $A \geq 0$. We know that $\phi(A) = A$ also for all rank one operators $A \in \mathcal{S}_F(H)^{>-I}$. In order to complete the proof we need to show that $\phi(A) = A$ for every $A \in \mathcal{S}_F(H)^{>-I}$. Assume first that $\operatorname{rank} A = 2$. We only need to consider the case when both eigenvalues of A are negative. We have already proved that $\operatorname{Im} \phi(A) = \operatorname{Im} A$. Denote by $\mathcal{P}$ the set of all rank one projections P satisfying $\operatorname{Im} P \subset \operatorname{Im} A$. Clearly, for every $P \in \mathcal{P}$ the operators A and $t_P P$ are adjacent for some $t_P \in (-1, 0)$. It follows that $\phi(A)$ and $t_P P = \phi(t_P P)$ are adjacent. By Lemma 2.15 we have $\phi(A) = A$, as desired.

It is now easy to verify that $\phi(A) = A$ for every $A \in \mathcal{S}_F(H)^{>-I}$. The verification is by induction on $\operatorname{rank} A$. The induction step follows directly from Lemma 2.9. $\square$

3. Symmetries on Bounded Observables

The aim of this section is to apply the main results obtained in the previous section to describe the general form of various symmetries on sets $\mathcal{S}(H), \mathcal{S}(H)^{\geq 0}$, and $\mathcal{S}(H)^{>0}$.

3.1. Comparability Preserving Maps

Let H be a Hilbert space and either $\mathcal{V} = \mathcal{S}(H)$, or $\mathcal{V} = \mathcal{S}(H)^{\geq 0}$. In [11] (see also [13]) Molnár proved that if $\phi : \mathcal{V} \to \mathcal{V}, \dim H \geq 2$, is a bijective map such that for every pair $A, B \in \mathcal{V}$ we have $A \leq B \iff \phi(A) \leq \phi(B)$, then there exist an operator $C \in \mathcal{V}$ and an invertible bounded linear or conjugate-linear operator $T : H \to H$ such that $\phi(A) = TAT^* + C, A \in \mathcal{V}$. In the case when $\mathcal{V} = \mathcal{S}(H)^{\geq 0}$, we have $C = 0$. The proof was based on several deep results including Rothaus's theorem [18] on the automatic linearity of bijective maps between closed convex cones preserving order in both directions, Vigier's theorem [16, Theorem 4.1.1], and the Kadison's well-known structural theorem for bijective linear positive unital maps on C^*-algebras [8, Corollary 5]. In [19] Molnár's result has been improved for maps acting on $\mathcal{S}(H)$. Bijective maps satisfying the weaker condition that comparability is preserved in both directions were characterized. The proof was simpler than the original one given by Molnár.

Here we will reprove the main theorem from [19] and give two new results on comparability preserving maps. All three results will be obtained as rather easy consequences of the theorems from the previous section. Recall that two operators $A, B \in \mathcal{S}(H)$ are comparable if $A \leq B$ or $B \leq A$.

Theorem 3.1. [19] *Let H be a Hilbert space,* $\dim H \geq 2$*, and $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ a bijective map with the property that for every pair $A, B \in \mathcal{S}(H)$ we have*

$$A \text{ and } B \text{ are comparable} \iff \phi(A) \text{ and } \phi(B) \text{ are comparable.}$$

Then there exist $c \in \{-1, 1\}$, an operator $C \in \mathcal{S}(H)$, and an invertible bounded linear or conjugate-linear operator $T : H \to H$ such that

$$\phi(A) = cTAT^* + C$$

for every $A \in \mathcal{S}(H)$.

Theorem 3.2. *Let H be a Hilbert space,* $\dim H \geq 2$*, and $\phi : \mathcal{S}(H)^{\geq 0} \to \mathcal{S}(H)^{\geq 0}$ a bijective map with the property that for every pair $A, B \in \mathcal{S}(H)^{\geq 0}$ we have*

$$A \text{ and } B \text{ are comparable} \iff \phi(A) \text{ and } \phi(B) \text{ are comparable.}$$

Then there exists an invertible bounded linear or conjugate-linear operator $T : H \to H$ such that

$$\phi(A) = TAT^*$$

for every $A \in \mathcal{S}(H)^{\geq 0}$.

Theorem 3.3. *Let H be a Hilbert space,* $\dim H \geq 2$*, and* $\phi : \mathcal{S}(H)^{>0} \to \mathcal{S}(H)^{>0}$ *a bijective map with the property that for every pair* $A, B \in \mathcal{S}(H)^{>0}$ *we have*

$$A \text{ and } B \text{ are comparable} \iff \phi(A) \text{ and } \phi(B) \text{ are comparable.}$$

Then there exists an invertible bounded linear or conjugate-linear operator $T : H \to H$ *such that either*

$$\phi(A) = TAT^*$$

for every $A \in \mathcal{S}(H)^{>0}$*; or*

$$\phi(A) = TA^{-1}T^*$$

for every $A \in \mathcal{S}(H)^{>0}$.

It is interesting to observe the differences between the above results. In the case of bijective maps on $\mathcal{S}(H)^{\geq 0}$ we see that every such map preserving comparability in both directions actually preserves order in both directions. Moreover, every such map is additive. When dealing with bijective maps on $\mathcal{S}(H)$ preserving comparability in both directions we see that they either preserve order in both directions, or satisfy $A \leq B \iff \phi(B) \leq \phi(A), A, B \in \mathcal{S}(H)$. In both cases these maps are real linear up to a translation. Bijective maps on $\mathcal{S}(H)^{>0}$ preserving comparability in both directions are either order automorphisms, or order anti-automorphisms. In the second case they are not additive.

Let $\mathcal{V}$ be any of the sets $\mathcal{S}(H), \mathcal{S}(H)^{\geq 0}$, or $\mathcal{S}(H)^{>0}$. For an arbitrary subset $\mathcal{W} \subset \mathcal{V}$ we define $\mathcal{W}^c$ to be the set of all operators $A \in \mathcal{V}$ such that A and B are comparable for every $B \in \mathcal{W}$. We write shortly $(\mathcal{W}^c)^c = \mathcal{W}^{cc}$ and $\{A\}^c = A^c = \{B \in \mathcal{V} : B \leq A \text{ or } A \leq B\}$. The proof of all three theorems is based on the following lemma which is a modification of [19, Lemma 2.1].

Lemma 3.4. *Let H be a Hilbert space,* $\dim H \geq 2, \mathcal{V}$ *any of the sets* $\mathcal{S}(H), \mathcal{S}(H)^{\geq 0}$*, or* $\mathcal{S}(H)^{>0}$*, and* $A, B \in \mathcal{V}, A \neq B$*. In the case when* $\mathcal{V} = \mathcal{S}(H)^{\geq 0}$ *we further assume that* $A \neq 0, B \neq 0, A$ *is not of rank one, and B is not of rank one. Then the following are equivalent:*

- *A and B are adjacent,*
- *A and B are comparable and* $\{A, B\}^{cc}$ *is an infinite set.*

Proof. Assume first that A and B are adjacent. Then $B = A + tP$ for some rank one projection P and some nonzero real number t. Interchanging A and B, if necessary, we may assume that $t > 0$. Clearly, A and B are comparable. Thus all we need to show is that $\{A, B\}^{cc}$ is an infinte set. It is enough to verify that each operator $A + sP, 0 \leq s \leq t$, belongs to $\{A, B\}^{cc}$. We first observe that since $A \leq B$, the set $\{A, B\}^c$ consists of

- all operators $D \in \mathcal{V}$ satisfying $B \leq D$,
- all operators $D \in \mathcal{V}$ satisfying $D \leq A$, and
- all operators $D \in \mathcal{V}$ satisfying $A \leq D \leq B$.

Let $0 \leq s \leq t$. Then, $A + sP \leq B \leq D$ for every D satisfying the first condition, and similarly, $A + sP \geq D$ for every D with $D \leq A$. Finally, it is clear that $A \le D \le B$ yields that $D = A + qP$ for some $q, 0 \le q \le t$. In this last case $A + sP$ and D are comparable as well.

To prove the other direction assume that A and B are not adjacent. Assume further that they are comparable and that $\{A, B\}^{cc}$ is an infinite set. With no loss of generality we may assume that $A \ge B$. Choose any $C \in \{A, B\}^{cc} \setminus \{0, A, B\}$. Because A and B are comparable, we have $A, B \in \{A, B\}^c$, and therefore A and C are comparable, and B and C are comparable.

Assume first that $C \ge A$. Then $C = A + S$, where $S \ge 0$ is a nonzero operator. We can find a positive operator $D \in \mathcal{S}(H)$ such that S and D are not comparable. For $T = A + D \in \mathcal{V}$ we have $T \ge A \ge B$, but T and C are not comparable, a contradiction.

We next suppose that $B \le C \le A$. Consider nonzero positive operators $A - C$ and $C - B$. If they are both of rank one, then they are linearly independent, since otherwise A and B would be adjacent. Hence, in this case we have $A = C + pP$ and $B = C - qQ$, where p, q are positive real numbers, and P and Q are linearly independent projections of rank one. Set $T = C + \frac{p}{2}P - \frac{q}{2}Q$. Then $T \le A$ and $T \ge B$, and therefore $T \in \{A, B\}^c$. But C and T are not comparable, a contradiction. It follows that one of operators $A - C$ or $C - B$ has at least two-dimensional image. We will consider only the case when $\dim \operatorname{Im}(A - C) \ge 2$. We can find a rank one projection Q and a positive real number q such that $B \le B_1 = C - qQ$. Because $\dim \operatorname{Im}(A - C) \ge 2$ we can find a rank one projection P and a positive real number p such that P and Q are linearly independent and $C + pP = A_1 \le A$. Now we can find in exactly the same way as above an operator T such that $B \le B_1 \le T \le A_1 \le A$, but T and C are not comparable, a contradiction.

It remains to consider the case when $C \le B$. Then $C = B - S \in \mathcal{V}$, where $S \ge 0$ is a nonzero operator. If $\mathcal{V} = \mathcal{S}(H)$ or $\mathcal{V} = \mathcal{S}(H)^{>0}$, we can easily find a positive operator $D \in \mathcal{S}(H)$ such that S and D are not comparable and $T = B - D \in \mathcal{V}$. Then $T \le B \le A$. But T and C are not comparable, a contradiction. In the case when $\mathcal{V} = \mathcal{S}(H)^{\ge 0}$, we know that $\dim \operatorname{Im} B \ge 2$ and $C \ge 0$ and $C \ne 0$. If $B - S$ and S are both of rank one, then they are linearly independent operators, since otherwise B would be of rank one. Hence, we can find linearly independent projections P, Q of rank one and positive real numbers p, q such that

$$(B - S) - pP \ge 0 \quad \text{and} \quad S - qQ \ge 0. \tag{7}$$

Of course, we can find such projections P and Q and real numbers p, q also in the case when at least one of nonzero positive operators $B - S$ and S has image of dimension at least two. It follows now easily that $D = S + pP - qQ \ge 0, D$ and S are not comparable, and $T = B - D \ge 0$, which as before, contradict our assumptions. This completes the proof. □

We have characterized adjacent pairs of operators with the use of comparability relation. However, the characterization is not complete in the case when $\mathcal{V} = \mathcal{S}(H)^{\ge 0}$. In this case we will need two more lemmas.

Lemma 3.5. *Let $A \in \mathcal{S}(H)^{\geq 0}$ be a nonzero operator. Then the following are equivalent:*

- $\operatorname{rank} A = 1$,
- $A^{cc} \neq \{A, 0\}$.

Proof. Assume first that $A = tP$ for some positive real number t and some rank one projection P. Then clearly, $A^c = \{sP : 0 \leq s \leq t\} \cup \{B \in \mathcal{S}(H)^{\geq 0} : B \geq A\}$. It follows directly that $\{sP : 0 \leq s \leq t\} \subset A^{cc}$.

To prove the other direction assume that $\dim \operatorname{Im} A \geq 2$ and $C \in A^{cc}$. Then either $A \leq C$, or $C \leq A$. Exactly the same ideas as in the proof of the previous lemma shows that in the first case we have $C = A$, while in the second case we have $C = A$ or $C = 0$. This completes the proof. □

Lemma 3.6. *Let $A, B \in \mathcal{S}(H)^{\geq 0}$. Assume that* $\operatorname{rank} A = 1$ *and* $\dim \operatorname{Im} B \geq 2$. *Then the following are equivalent:*

- *A and B are adjacent,*
- *A and B are comparable and $\{A, B\}^{cc}$ contains infinitely many operators that are not of rank one.*

Proof. Assume first that A and B are adjacent. Because $\operatorname{rank} A = 1$ and $\dim \operatorname{Im} B \geq 2$ there exist positive real numbers p, q and linearly independent projections P, Q of rank one such that $A = pP$ and $B = pP + qQ$. The set $\{A, B\}^c$ consists of all operators that are either below A, or between A and B, or above B, that is,

$$\{A, B\}^c = \{sP : 0 \leq s \leq p\} \cup \{pP + rQ : 0 \leq r \leq q\} \cup \{D \in \mathcal{S}(H)^{\geq 0} : D \geq B\}.$$

It is now clear that each operator $pP + rQ, 0 < r \leq q$, belongs to $\{A, B\}^{cc}$ and is of rank two.

To prove the other direction assume that $\operatorname{rank} A = 1, \dim \operatorname{Im} B \geq 2, A$ and B are not adjacent, and A and B are comparable. If we had $B \leq A$, then B would be of rank at most one, a contradiction. So, $B \geq A$, and because A and B are not adjacent, we have $B = A + S$, where $S \geq 0$ and $\dim \operatorname{Im} S \geq 2$. Using the same ideas as above we conclude that for any $C \in \{A, B\}^{cc} \setminus \{A, B\}$ the possibilities $C \geq B$ and $A \leq C \leq B$ cannot occur. Hence for each $C \in \{A, B\}^{cc}$ we have either $C = B$, or $C \leq A$. It follows that B and 0 are the only two operators in $\{A, B\}^{cc}$ that are not of rank one. This completes the proof. □

Now we are ready to prove the main results of this subsection. The proofs of the first two theorems have few common steps. So we will prove them simultaneously.

Proof of Theorems 3.1 and 3.2. Let $\mathcal{V}$ denote either $\mathcal{S}(H)$ or $\mathcal{S}(H)^{\geq 0}$. It is clear that if $\phi : \mathcal{V} \to \mathcal{V}$ is a bijective map preserving comparability in both directions and $\mathcal{W} \subset \mathcal{V}$ is any subset, then $\phi(\mathcal{W}^{cc}) = (\phi(\mathcal{W}))^{cc}$.

We start with bijective maps on $\mathcal{S}(H)$ preserving comparability in both directions. Replacing ϕ by $A \mapsto \phi(A) - \phi(0)$ we see that there is no loss of generality in assuming that $\phi(0) = 0$. By Lemma 3.4 we see that ϕ preserves adjacency in both directions. In particular, $\phi(\mathcal{S}_F(H)) \subset \mathcal{S}_F(H)$. Hence, we can apply Theorem 2.1. Moreover, we have stronger assumptions than in Theorem 2.1. Namely, ϕ and its inverse have the same properties, and therefore, ϕ is a bijective map of $\mathcal{S}_F(H)$ onto itself and it preserves adjacency in both directions. It follows that there exist a bijective linear or conjugate-linear map $T : H \to H$ and $c \in \{-1, 1\}$ such that

$$\phi\left(\sum_{j=1}^{k} t_j x_j \otimes x_j^*\right) = c\sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^*$$

for every $\sum_{j=1}^{k} t_j x_j \otimes x_j^* \in \mathcal{S}_F(H)$.

Let $A \in \mathcal{S}(H)$ be positive. Then clearly, $A \geq R$ for every negative rank one operator R. Assume next that $A \in \mathcal{S}(H)$ has the property that A and R are comparable for every negative rank one operator R. We will show that then $A \geq R$ for every negative rank one operator R. Assume on the contrary that $A \leq S$ for some negative S of rank one. If we take a real number t that is large enough we have $A \not\leq tS$. But then $tS \leq A \leq S$. It follows that A is a negative operator of rank one. Consequently, it is easy to find a negative rank one operator R such that A and R are not comparable, a contradiction. Hence, if A and R are comparable for every negative rank one operator R, then actually $A \geq R$ for every negative rank one operator R. But then $A \geq 0$.

It follows that if $c = 1$, then $\phi(\mathcal{S}(H)^{\geq 0}) = \mathcal{S}(H)^{\geq 0}$, and similarly, $\phi(-\mathcal{S}(H)^{\geq 0}) = -\mathcal{S}(H)^{\geq 0}$. We can apply the same arguments when $c = -1$. We have proved that if $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ is a bijective map satisfying $\phi(0) = 0$ and preserving comparability in both directions, then either it maps the set of positive operators onto itself and the set of negative operators onto itself, or it maps the set of positive operators onto the set of negative operators and the set of negative operators onto the set of positive operators. We may, and we will assume in the next few steps that we have the first possibility, since we can multiply ϕ by -1, if necessary. For a while we will consider only the restriction of ϕ to the set of all positive operators.

If ϕ is a bijective map on $\mathcal{S}(H)^{\geq 0}$ preserving comparability in both directions, then we have $\phi(0) = 0$, because 0 is obviously the only operator with the property that it is comparable with all elements of $\mathcal{S}(H)^{\geq 0}$. By Lemma 3.5, $A \in \mathcal{S}(H)^{\geq 0}$ is of rank one if and only if $\phi(A)$ is of rank one. Two rank one operators A and $B, A \neq B$, are adjacent if and only if they are scalar multiples of the same projection of rank one, that is, if and only if they are comparable. All three lemmas now imply that ϕ preserves adjacency in both directions. Hence, we can apply Theorem 2.2.

Thus, in both cases ($\mathcal{V} = \mathcal{S}(H)$ or $\mathcal{V} = \mathcal{S}(H)^{\geq 0}$) we have a bijective map $\phi : \mathcal{S}(H)^{\geq 0} \to \mathcal{S}(H)^{\geq 0}$ which maps the set of finite rank operators onto itself, and preserves adjacency in both directions, and

$$\phi\left(\sum_{j=1}^{k} t_j x_j \otimes x_j^*\right) = \sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^*$$

for every $\sum_{j=1}^{k} t_j x_j \otimes x_j^* \in \mathcal{S}_F(H)^{\geq 0}$. In particular, we have $T(x \otimes x^*) = (Tx) \otimes (Tx)^*$ for every $x \in H$. Let $R, A \in \mathcal{S}(H)^{\geq 0}$ with $\operatorname{rank} R = 1$ and

$\dim \operatorname{Im} A \geq 2$. Then R and A are comparable if and only if $R \leq A$. Indeed, the possibility $R \geq A$ cannot occur. If $x \in H$ is any vector of norm one, then $x \otimes x^* \leq I$, and therefore, $(Tx) \otimes (Tx)^*$ and $\phi(I)$ are comparable. It follows that

$$\|Tx\|^2 \frac{Tx}{\|Tx\|} \otimes \left(\frac{Tx}{\|Tx\|}\right)^* \leq \phi(I) \leq \|\phi(I)\| \cdot I.$$

We conclude that $\|Tx\| \leq \sqrt{\|\phi(I)\|}$. Thus, T is a bounded invertible linear or conjugate-linear operator. Replacing ϕ by the map $A \mapsto T^{-1}\phi(A)\left(T^{-1}\right)^*$, we may, and we will assume that $\phi(A) = A$ for every $A \in \mathcal{S}_F(H)^{\geq 0}$.

Let $A \in \mathcal{S}(H)^{\geq 0}$ be an invertible operator. A positive operator R of rank one satisfies $R \leq A$ if and only if R and A are comparable which is true if and only if $\phi(R) = R$ and $\phi(A)$ are comparable. Thus, $R \leq A$ if and only if $R \leq \phi(A)$. Since A is invertible, there exists a positive real number c such that $A \geq cP$ for every rank one projection P, and consequently, $\phi(A)$ is invertible as well. By [19, Lemma 4.1], we have $\phi(A) = A$.

For an arbitrary $A \in \mathcal{S}(H)^{\geq 0}$ we know that $\phi(A)$ and $\phi(A+\varepsilon I) = A+\varepsilon I$ are comparable for every $\varepsilon > 0$. As we already know that each invertible operator is mapped by ϕ into itself, we conclude that $\phi(A) \leq A + \varepsilon I, \varepsilon > 0$, and therefore, $\phi(A) \leq A$. But of course, the inverse of ϕ has the same properties as ϕ, and thus, $\phi^{-1}(A) \leq A$ for every $A \in \mathcal{S}(H)^{\geq 0}$ as well. It follows that $\phi(A) = A$ for every $A \in \mathcal{S}(H)^{\geq 0}$. We have completed the proof of Theorem 3.2.

So, from now on we assume that $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ is a bijective map preserving comparability in both directions and $\phi(A) = A$ for every $A \geq 0$ and for every A of finite rank. We need to show that $\phi(A) = A$ for every $A \in \mathcal{S}(H)$. Choose $A \in \mathcal{S}(H)$ and find a real number $a \in \mathbb{R}$ such that $A + aI$ is positive and invertible. Define $\psi : \mathcal{S}(H) \to \mathcal{S}(H)$ by $\psi(C) = \phi(C - aI) - \phi(-aI)$. Clearly, ψ is a bijective map preserving comparability in both directions and $\psi(0) = 0$. We then already know that $\psi(C) = cSCS^*, C \in \mathcal{S}(H)^{\geq 0}$, for some bijective bounded linear or conjugate-linear $S : H \to H$ and $c \in \{-1, 1\}$. So, if $C - aI \geq 0$, then

$$cSCS^* = C - aI - \phi(-aI).$$

Replacing C by tC and sending t to infinity we arrive at $cSCS^* = C$ for every $C \geq aI$. It follows easily that $c = 1$ and $S = e^{i\varphi}I$ for some real φ. Thus,

$$\phi(D) = \phi((D + aI) - aI) = \psi(D + aI) + \phi(-aI) = D + aI + \phi(-aI)$$

whenever $D + aI \geq 0$. If we choose D in such a way that both D and $D + aI$ are positive operators we get that $\phi(-aI) = -aI$, and therefore, $\phi(D) = D$ whenever $D + aI \geq 0$. In particular, $\phi(A) = A$, as desired. □

Proof of Theorem 3.3. Replacing the map ϕ by the map $A \mapsto \phi(I)^{-1/2}\phi(A)$ $\phi(I)^{-1/2}, A \in \mathcal{S}(H)^{>0}$, we may assume with no loss of generality that $\phi(I) = I$. It follows from Lemma 3.4 that ϕ preserves adjacency in both directions. Therefore, ϕ maps the set $\{I + A : A \in \mathcal{S}_F(H)^{>-I}\}$ bijectively onto itself. Define a map $\psi : \mathcal{S}_F(H)^{>-I} \to \mathcal{S}_F(H)^{>-I}$ by $\phi(I + A) = I + \psi(A), A \in \mathcal{S}_F(H)^{>-I}$. This is obviously a bijective map preserving adjacency in both directions. By Theorem 2.3, there exists a unitary or antiunitary operator $U : H \to H$ such that either

$$\phi(I + A) = U(I + A)U^*$$

for every $A \in \mathcal{S}_F(H)^{>-I}$, or

$$\phi(I + A) = U(I + A)^{-1}U^*$$

for every $A \in \mathcal{S}_F(H)^{>-I}$. After replacing the map ϕ by either the map $A \mapsto U^*\phi(A)U$, or by the map $A \mapsto U^*\phi(A)^{-1}U$, we may, and we will assume that

$$\phi(I + A) = I + A$$

for every $A \in \mathcal{S}_F(H)^{>-I}$. We need to prove that $\phi(A) = A$ for every $A \in \mathcal{S}(H)^{>0}$. We do this in two steps: first we prove that ϕ acts like the identity on the set of all operators $A \geq I$, and then in the next step we show that $\phi(A) = A$ for all $A \in \mathcal{S}(H)^{>0}$. We can achieve this using almost identical methods as in the proof of the previous two main theorems of this subsection. Therefore we omit the details. □

Let us conclude this section with a remark on possible improvements of the above results. When dealing with comparability preserving maps on the set $\mathcal{S}(H)$ the main tool was Theorem 2.1. One may wonder why Theorems 2.1 and 3.1 differ so much with respect to the assumptions. In Theorem 2.1 we have described maps on $\mathcal{S}_F(H)$ preserving adjacency in one direction only, while maps on $\mathcal{S}(H)$ in Theorem 3.1 are assumed to be bijective and to preserve comparability in both directions. It is natural to ask whether we can relax also the assumptions in the description of comparability preserving maps by removing the bijectivity assumption and/or assuming that comparability is preserved in one direction only. As we shall see the answer is negative even if we replace comparability preserving property with a stronger order preserving property. Before presenting the required counterexamples we need to introduce one more notation. We define a function $\rho : \mathcal{S}(H) \to \mathbb{R}$ by

$$\rho(A) = \sup\{\langle Ax, x\rangle \; : \; x \in H \;\; and \;\; \|x\| = 1\}, \quad A \in \mathcal{S}(H).$$

It is clear that $A \leq B$ yields $\rho(A) \leq \rho(B)$ and $\rho(A + tI) = \rho(A) + t, A, B \in \mathcal{S}(H), t \in \mathbb{R}$.

Example 3.7. Let H be an infinite-dimensional Hilbert space. Then H can be identified with the direct sum of two copies of H. Thus, we may identify $\mathcal{S}(H)$ with $\mathcal{S}(H \oplus H)$. Define a map $\phi : \mathcal{S}(H) \to \mathcal{S}(H \oplus H)$ by

$$\phi(A) = \begin{bmatrix} A & 0 \\ 0 & \rho(A)I \end{bmatrix}, \quad A \in \mathcal{S}(H).$$

It is clear that for any pair of operators $A, B \in \mathcal{S}(H)$ we have $A \leq B \iff \phi(A) \leq \phi(B)$. The map ϕ is injective but not surjective. Moreover, $\phi(0) = 0$. However, ϕ is far from being real-linear.

Example 3.8. Let H be any Hilbert space, $\dim H \geq 2$. Define a map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ by $\phi(A) = A + \rho(A)I, A \in \mathcal{S}(H)$. Obviously, $A \leq B$ yields $\phi(A) \leq \phi(B)$. To show that ϕ is surjective choose any $A \in \mathcal{S}(H)$. Then one can easily check that $\phi\left(A - \frac{\rho(A)}{2} I\right) = A$. Assume that $\phi(A) = \phi(B)$ for some $A, B \in \mathcal{S}(H)$. Then $B = A + sI$ for some real number s. From $A+\rho(A)I = \phi(A) = \phi(A+sI) = A+sI+\rho(A)I+sI$ we conclude that $s = 0$. Hence, ϕ is bijective and preserves order. And clearly, $\phi(0) = 0$. However, ϕ is not real-linear. Of course, it follows then from Theorem 3.1 that ϕ does not preserve order in both directions; to see this directly take a nontrivial projection P and observe that $\phi(I) = 2I \le 2I + 2P = \phi(2P)$, but $I \not\le 2P$.

3.2. Jordan Triple Product Automorphisms

Let $\mathcal{V}$ be any of the sets $\mathcal{S}(H), \mathcal{S}(H)^{\ge 0}$, or $\mathcal{S}(H)^{>0}$. Then $A, B \in \mathcal{V}$ does not imply that $AB \in \mathcal{V}$. However, we have always $ABA \in \mathcal{V}$. This product called Jordan triple product plays an important role in some parts of ring theory as well as in the mathematical foundations of quantum mechanics. Symmetries with respect to Jordan triple product are sometimes called Jordan triple automorphisms. Recently, several papers were devoted to such maps, see [10] and [12] and the references therein.

Theorem 3.9. *Let H be a Hilbert space,* $\dim H \ge 2$*, and $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ a bijective map with the property that for every pair $A, B \in \mathcal{S}(H)$ we have*

$$\phi(ABA) = \phi(A)\phi(B)\phi(A).$$

Then there exist $c \in \{-1, 1\}$ and an either unitary or antiunitary operator $U : H \to H$ such that

$$\phi(A) = cUAU^*$$

for every $A \in \mathcal{S}(H)$.

The above result was proved in [10] under the stronger assumption that $\dim H \ge 3$.

Theorem 3.10. *Let H be a Hilbert space,* $\dim H \ge 2$*, and $\phi : \mathcal{S}(H)^{\ge 0} \to \mathcal{S}(H)^{\ge 0}$ a bijective map with the property that for every pair $A, B \in \mathcal{S}(H)^{\ge 0}$ we have*

$$\phi(ABA) = \phi(A)\phi(B)\phi(A).$$

Then there exists an either unitary or antiunitary operator $U : H \to H$ such that

$$\phi(A) = UAU^*$$

for every $A \in \mathcal{S}(H)^{\ge 0}$.

This is a substantial improvement of one of the main results from [12]. Namely, there the same conclusion was obtained under the additional assumption that ϕ is continuous and as before, under the stronger assumption that $\dim H \ge 3$.

It turns out that our approach based on the reduction to adjacency preserving maps cannot be used to characterize Jordan triple automorphisms of $\mathcal{S}(H)^{>0}$. Namely, if H is finite-dimensional, then the map $\phi : \mathcal{S}(H)^{>0} \to \mathcal{S}(H)^{>0}$ defined by

$$\phi(A) = \det A \cdot A,$$

$A \in \mathcal{S}(H)^{>0}$, is bijective and $\phi(ABA) = \phi(A)\phi(B)\phi(A), A, B \in \mathcal{S}(H)^{>0}$. However, this map is obviously not an adjacency preserving map. It should be mentioned here that this kind of maps were studied in [12] in the presence of the continuity assumption.

We will start with some easy observations. Let $\mathcal{V}$ be either $\mathcal{S}(H)$, or $\mathcal{S}(H)^{\geq 0}$, and $\phi : \mathcal{V} \to \mathcal{V}$ a Jordan triple automorphism. From $\phi(0) = \phi(A \cdot 0 \cdot A) = \phi(A)\phi(0)\phi(A)$ and bijectivity of ϕ we get that $\phi(0) = 0$. Let $A \in \mathcal{V}$ be nonzero. Then the following are equivalent:

- A is a rank one operator,
- if $B \in \mathcal{V}$ and $\{BCB : C \in \mathcal{V}\}$ is a proper subset of $\{ACA : C \in \mathcal{V}\}$, then $B = 0$.

To show this equivalence set $\mathbb{L} = \mathbb{R}$ when $\mathcal{V} = \mathcal{S}(H)$ and $\mathbb{L} = [0, \infty)$ when $\mathcal{V} = \mathcal{S}(H)^{\geq 0}$. It is trivial to check that if A is of rank one then $\{ACA : C \in \mathcal{V}\} = \{tA : t \in \mathbb{L}\}$. If, on the other hand A is nonzero and is not of rank one, then there exists a rank one operator $R \in \mathcal{V}$ such that $ARA = S$ is nonzero. Then S is of rank one. We have $\{tS : t \in \mathbb{L}\} = \{A(tR)A : t \in \mathbb{L}\} \subset \{ACA : C \in \mathcal{V}\}$ and at the same time $A^2 \in \{ACA : C \in \mathcal{V}\}$. Of course, A^2 is a nonzero operator that is not of rank one. The above equivalence can be now easily verified.

Consequently, ϕ maps the set of rank one operators from $\mathcal{V}$ onto itself. Denote by $\mathcal{R}_1 \subset \mathcal{V}$ the subset of all rank one operators from $\mathcal{V}$. For $A, B \in \mathcal{R}_1$ we write $A \perp B$ if $AB = 0$. Clearly, $A \perp B$ if and only if $AB = BA = 0$ if and only if $ABA = 0$. Thus, for every pair $A, B \in \mathcal{R}_1$ we have $A \perp B$ if and only if $\phi(A) \perp \phi(B)$. Let $y \in H$ be a nonzero vector. Denote $\mathcal{R}_1(y) = \{A \in \mathcal{R}_1 : A \perp y \otimes y^*\}$. Then for every nonzero $y \in H$ we can find a nonzero $z \in H$ such that $\phi(\mathcal{R}_1(y)) = \mathcal{R}_1(z)$. We are now ready for the crucial lemma in this subsection.

Lemma 3.11. *Let $A, B \in \mathcal{V}$. Then the following are equivalent:*

- *$A \sim B$,*
- *there exists a nonzero $y \in H$ such that $\{C \in \mathcal{R}_1 : CAC = CBC\} = \mathcal{R}_1(y)$.*

Assume for a moment that we have already proved this lemma. Then we can conclude that the bijective map ϕ preserves adjacency in both directions. Moreover, as $\phi(0) = 0$, the set of all finite rank operators from $\mathcal{V}$ is mapped by ϕ onto itself. Hence, we are in the position where we can apply Theorems 2.1 and 2.2.

Proof of Lemma 3.11. Assume first that A and B are adjacent. Then $B = A \pm y \otimes y^*$ for some nonzero $y \in H$. It is then clear that for $C \in \mathcal{R}_1$ we have $CAC = CBC$ if and only if $C \perp y \otimes y^*$. This completes the proof in one direction.

To prove the other direction assume that there exists a nonzero $y \in H$ such that $\{C \in \mathcal{R}_1 : CAC = CBC\} = \mathcal{R}_1(y)$. It is then clear that $A \neq B$. In other words, we have

$$x \otimes x^* \ (B - A) \ x \otimes x^* = 0$$

for every vector x perpendicular to y. Thus, $\langle (B-A)x, x\rangle = 0$ for every x belonging to the orthogonal complement of y. Standard arguments yield then that $\langle (B-A)x, z\rangle = 0$ for all x, z belonging to the orthogonal complement of y. It follows that

$$B - A = ty \otimes y^* + \lambda y \otimes v^* + \overline{\lambda} v \otimes y^*$$

for some $t \in \mathbb{R}$, some nonzero vector v perpendicular to y and some complex number λ. We may assume with no loss of generality that $\|y\| = \|v\| = 1$. All we need to do to complete the proof is to show that $\lambda = 0$.

Assume on the contrary that $\lambda \neq 0$. Then with respect to the orthogonal direct sum decomposition $H = \text{span}\,\{y, v\} \oplus \{y, v\}^\perp$, the operator $B - A$ has the following matrix representation

$$B - A = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$F = \begin{bmatrix} t & \lambda \\ \overline{\lambda} & 0 \end{bmatrix}.$$

Because $\det F < 0$ the matrix F has one positive and one negative eigenvalue. Hence, we can find linearly independent vectors $x_1, x_2 \in \text{span}\,\{y, v\}$ such that $\langle (B-A)x_j, x_j\rangle = 0, j = 1, 2$. We can choose $x \in \{x_1, x_2\}$ that is not perpendicular to y. We have

$$x \otimes x^* A\; x \otimes \; x^* = x \otimes x^*\; B\; x \otimes x^*$$

and $x \otimes x^* \notin \mathcal{R}_1(y)$, contradicting our assumptions. This shows that $\lambda = 0$, as desired. □

Proof of Theorem 3.9. We already know that we can apply Theorem 2.1 for the restriction of ϕ to the subset of all finite rank operators. As ϕ is bijective and its inverse has the same properties as ϕ we conclude that there exist a bijective linear or conjugate-linear map $T : H \to H$ and $c \in \{-1, 1\}$ such that

$$\phi\left(\sum_{j=1}^{k} t_j x_j \otimes x_j^*\right) = c \sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^*$$

for every $\sum_{j=1}^{k} t_j x_j \otimes x_j^* \in \mathcal{S}_F(H)$. We apply the above formula together with $\phi(A^3) = (\phi(A))^3$ for the special case when $A = x \otimes x^*$ with x being any vector of norm one. We get that $\|Tx\| = 1$. Thus, T is an isometry, and therefore, it is either a unitary, or an antiunitary operator. In particular, $T^* = T^{-1}$. Replacing ϕ by the map $A \mapsto T^*\phi(A)T$ and multiplying it by -1, if necessary, we may, and we will assume that $\phi(A) = A$ for every $A \in \mathcal{S}_F(H)$.

In order to complete the proof we have to show that $\phi(A) = A$ for every $A \in \mathcal{S}(H)$. Let $A \in \mathcal{S}(H)$ be any operator and $P \in \mathcal{S}(H)$ any projection of rank one. Then PAP is of rank at most one, and therefore,

$$PAP = \phi(PAP) = \phi(P)\phi(A)\phi(P) = P\phi(A)P.$$

Since this holds for every projection of rank one we must have $\phi(A) = A$, as desired. □

The proof of Theorem 3.10 goes through in an almost the same way. We leave the details to the reader.

3.3. Jordan Product Automorphisms

While $\mathcal{S}(H)$ is closed under the Jordan product defined by $A \circ B = \frac{1}{2}(AB + BA)$, it is easy to find $A, B \in \mathcal{S}(H)^{>0}$ such that $A \circ B \notin \mathcal{S}(H)^{\geq 0}$. So, when studying the Jordan product automorphisms we have to restrict our attention to maps acting on $\mathcal{S}(H)$.

As another example illustrating the efficiency of our method we will reprove the following result from [1].

Theorem 3.12. *Let H be a Hilbert space, $\dim H \geq 2$, and $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ a bijective map with the property that for every pair $A, B \in \mathcal{S}(H)$ we have*

$$\phi\left(\frac{1}{2}(AB + BA)\right) = \frac{1}{2}(\phi(A)\phi(B) + \phi(B)\phi(A)).$$

Then there exists an either unitary or antiunitary operator $U : H \to H$ such that

$$\phi(A) = UAU^*$$

for every $A \in \mathcal{S}(H)$.

Let $A, B \in \mathcal{S}(H)$ be any pair of operators. We denote $\mathcal{C}(A, B) = \{C \in \mathcal{S}(H) : C \circ C = C \text{ and } A \circ C = B \circ C\}$. Note that $C \circ C = C$ if and only if C is a projection. It is straightforward to check that $\mathcal{C}(\phi(A), \phi(B)) = \phi(\mathcal{C}(A, B))$. Our proof will be based on the following lemma.

Lemma 3.13. *Let $A, B \in \mathcal{S}(H)$ with $A \neq B$. Then the following are equivalent:*

- *$A \sim B$,*
- *if $E, F \in \mathcal{S}(H)$ and $\mathcal{C}(A, B) \subset \mathcal{C}(E, F)$ and $\mathcal{C}(A, B) \neq \mathcal{C}(E, F)$, then $E = F$.*

Proof. We first observe that $\mathcal{C}(A, B)$ is the set of all projections $P \in \mathcal{S}(H)$ satisfying $P \circ (B - A) = 0$. For an arbitrary projection $P \in \mathcal{S}(H)$ we denote by $P^\perp$ the set of all projections $Q \in \mathcal{S}(H)$ with the property $PQ = 0$. We will first prove that if $A, B \in \mathcal{S}(H)$ are adjacent, then $\mathcal{C}(A, B) = Q^\perp$ for some projection Q of rank one, and next, if $A \neq B$ and A and B are not adjacent, then $\mathcal{C}(A, B) \subset Q^\perp$ for some projection of rank at least two. Once we prove these two facts the statement of our lemma follows trivially.

Assume first that $B - A$ is of rank one. Then $B - A = tQ$ for some non-zero real t and some projection Q of rank one. Hence, $\mathcal{C}(A, B)$ is the set of all projections $P \in \mathcal{S}(H)$ such that $P \circ Q = 0$. In other words, $\mathcal{C}(A, B) = Q^\perp$.

We next consider the case when $B - A$ is of rank two with one positive eigenvalue and one negative eigenvalue. Then there exists a two-dimensional subspace $K \subset H$ such that with respect to the direct sum decomposition $H = K \oplus K^\perp$ the operator $B - A$ has the matrix representation

$$B - A = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

with

$$D = \begin{bmatrix} p & 0 \\ 0 & -q \end{bmatrix},$$

where p, q are positive real numbers. Denote the rank two projection onto the subspace K by Q. A straightforward computation shows that if $T \in \mathcal{S}(H)$ satisfies $T \circ (B - A) = 0$, then

$$T = \begin{bmatrix} S & 0 \\ 0 & * \end{bmatrix}$$

with

$$S = \begin{bmatrix} 0 & \lambda \\ \overline{\lambda} & 0 \end{bmatrix},$$

where $*$ stands for any self-adjoint operator on $K^\perp$, and $\lambda = 0$ unless $p = q$. If we further assume that T is a projection, then clearly, λ has to be zero. It follows that $\mathcal{C}(A, B) \subset Q^\perp$.

Assume finally that $B - A$ is neither of rank one, nor of rank two with the nonzero eigenvalues of the opposite signs. Then, by the spectral theorem we can find an orthogonal direct sum decomposition $H = K \otimes K^\perp$ such that $\dim K \geq 2$,

$$B - A = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$$

with S_1 being an invertible either positive, or negative operator. With no loss of generality we may assume that S_1 is positive. As before, we denote by Q the orthogonal projection on K. If

$$T = \begin{bmatrix} T_1 & T_2 \\ T_2^* & T_3 \end{bmatrix} \in \mathcal{S}(H)$$

satisfies $T \circ (B - A) = 0$, then $S_1 T_1 + T_1 S_1 = 0$. The famous Sylvester–Rosenblum theorem states that if the spectra of M and N are disjoint the operator equation $MX - XN = L$ has a unique solution for every L. In our case S_1 is positive invertible, and consequently, $\sigma(S_1) \cap \sigma(-S_1) = \emptyset$. Therefore, $T_1 = 0$. If we further assume that T is a projection, then $T \geq 0$ and since $T_1 = 0$ we must have $T_2 = 0$. We conclude that $\mathcal{C}(A, B) \subset Q^\perp$. □

We are now ready to prove the main theorem of this subsection.

Proof of Theorem 3.12. From the fact that an operator $A \in \mathcal{S}(H)$ is a projection if and only if $A \circ A = A$, it follows that A is a projection if and only if $\phi(A)$ is a projection. In particular, $\phi(0) = Q$ is a projection with the property that

$$Q = \frac{1}{2}(Q\phi(A) + \phi(A)Q)$$

for every $A \in \mathcal{S}(H)$. Multiplying on both sides by Q and applying the bijectivity of ϕ we see that $QCQ = Q$ for every $C \in \mathcal{S}(H)$. It follows that $0 = Q = \phi(0)$.

Lemma 3.13 yields that ϕ is a bijective map preserving adjacency in both directions. As $\phi(0) = 0, \phi$ maps $\mathcal{S}_F(H)$ onto itself. It follows from Theorem 2.1 together with the fact that ϕ preserves projections that there exists a bijective linear or conjugate-linear map $T : H \to H$ such that

$$\phi\left(\sum_{j=1}^{k} t_j x_j \otimes x_j^*\right) = \sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^*$$

for every $\sum_{j=1}^{k} t_j x_j \otimes x_j^* \in \mathcal{S}_F(H)$. Let $x \in H$ be any vector of norm one. Putting $A = B = x \otimes x^*$ into $\phi\left(\frac{1}{2}(AB + BA)\right) = \frac{1}{2}(\phi(A)\phi(B) + \phi(B)\phi(A))$ and applying the above formula we get

$$Tx \otimes (Tx)^* = \langle Tx, Tx \rangle\, Tx \otimes (Tx)^*.$$

Thus, $\|Tx\|^2 = 1$, and consequently, T is either a unitary, or an antiunitary operator. Replacing ϕ by $A \mapsto T^* \phi(A) T$, we may, and we will assume that $\phi(A) = A$ for every finite rank operator $A \in \mathcal{S}(H)$. We need to prove that this holds true for all operators.

So, let $A \in \mathcal{S}(H)$ be any operator. For an arbitrary projection P of rank one we have

$$\frac{1}{2}(PA + AP) = \phi\left(\frac{1}{2}(PA + AP)\right) = \frac{1}{2}\left(P\phi(A) + \phi(A)P\right).$$

Multiplying on both sides by P we arrive at

$$PAP = P\phi(A)P.$$

As this is true for every projection of rank one we must have $\phi(A) = A$. This completes the proof. $\square$

3.4. Maps Preserving the Invertibility of the Difference of Operators

Let H be a finite-dimensional Hilbert space, $\dim H = n$. In this paper we have studied maps on $\mathcal{S}(H)$ preserving adjacency, that is, maps preserving the minimal nonzero arithmetic distance. What about preserving the maximal possible arithmetic distance? As $d(A, B) \leq n$ for all $A, B \in \mathcal{S}(H)$, we are interested in maps preserving pairs of operators A, B with the property that $\operatorname{rank}(A - B) = n$. Equivalently, we are interested in maps having the property that $A - B$ is invertible if and only if $\phi(A) - \phi(B)$ is invertible. With this formulation we do not need to restrict ourselves to the finite-dimensional case.

Such maps defined on the algebra of all bounded linear operators on a Hilbert space H have been already studied in [5]. An interested reader can find there informations on the background of this kind of problems going back to the famous Kaplansky's problem on invertibility preserving maps and Kowalski–Słodkowski's extension of the celebrated Gleason–Kahane–Żelazko theorem.

Here we will characterize such maps defined on the set of all self-adjoint operators, the set of all positive operators, and the set of all positive invertible operators. As far as we know all these results are new. Only the finite-dimensional version of the first result has been known before [4,9].

Theorem 3.14. *Let H be a Hilbert space,* $\dim H \geq 2$*, and* $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ *a bijective map with the property that for every pair* $A, B \in \mathcal{S}(H)$ *we have*

$$A - B \text{ is invertible} \iff \phi(A) - \phi(B) \text{ is invertible.}$$

Then there exist $c \in \{-1, 1\}$*, an operator* $C \in \mathcal{S}(H)$*, and an invertible bounded linear or conjugate-linear operator* $T : H \to H$ *such that*

$$\phi(A) = cTAT^* + C$$

for every $A \in \mathcal{S}(H)$*.*

Theorem 3.15. *Let H be a Hilbert space,* $\dim H \geq 2$*, and* $\phi : \mathcal{S}(H)^{\geq 0} \to \mathcal{S}(H)^{\geq 0}$ *a bijective map with the property that for every pair* $A, B \in \mathcal{S}(H)^{\geq 0}$ *we have*

$$A - B \text{ is invertible} \iff \phi(A) - \phi(B) \text{ is invertible.}$$

Then there exists an invertible bounded linear or conjugate-linear operator $T : H \to H$ *such that*

$$\phi(A) = TAT^*$$

for every $A \in \mathcal{S}(H)^{\geq 0}$*.*

Theorem 3.16. *Let H be a Hilbert space,* $\dim H \geq 2$*, and* $\phi : \mathcal{S}(H)^{>0} \to \mathcal{S}(H)^{>0}$ *a bijective map with the property that for every pair* $A, B \in \mathcal{S}(H)^{>0}$ *we have*

$$A - B \text{ is invertible} \iff \phi(A) - \phi(B) \text{ is invertible.}$$

Then there exists an invertible bounded linear or conjugate-linear operator $T : H \to H$ *such that either*

$$\phi(A) = TAT^*$$

for every $A \in \mathcal{S}(H)^{>0}$*; or*

$$\phi(A) = TA^{-1}T^*$$

for every $A \in \mathcal{S}(H)^{>0}$*.*

For the proofs we need several lemmas. Let H be a Hilbert space with $\dim H \geq 2$. We denote by $\mathcal{P}_2 \subset \mathcal{S}(H)$ the set of all projections of rank two.

Lemma 3.17. *Assume that* $A \in \mathcal{S}(H)$ *satisfy* $\dim \operatorname{Im} A \geq 2$*. Then*

$$\mathcal{A} = \{P \in \mathcal{P}_2 \,:\, \operatorname{rank} PAP = 2\}$$

is an open dense subset of $\mathcal{P}_2$*.*

Proof. The map $P \mapsto PAP$ is a continuous map from $\mathcal{P}_2$ into the set of all bounded self-adjoint operators on H of rank at most two. The set of all bounded self-adjoint operators on H of rank exactly two is an open subset of the set of all bounded self-adjoint operators on H of rank at most two. Hence, $\mathcal{A}$ is open in $\mathcal{P}_2$.

To verify that it is also dense we first choose orthogonal unit vectors $x, y \in H$ such that QAQ is of rank two, where $Q = x \otimes x^* + y \otimes y^*$. Such x and y exist because $\dim \mathrm{Im}\, A \geq 2$. Let $P \in \mathcal{P}_2$ be any projection of rank two. Then $P = u \otimes u^* + v \otimes v^*$ for some orthonormal vectors $u, v \in H$. The operator

$$\begin{aligned} &((u+tx)\otimes(u+tx)^* + (v+ty)\otimes(v+ty)^*)\, A \\ &((u+tx)\otimes(u+tx)^* + (v+ty)\otimes(v+ty)^*) \\ &\quad = A_0 + tA_1 + t^2A_2 + t^3A_3 + t^4A_4, \end{aligned}$$

where $A_4 = QAQ$, is of rank at most two for every real number t. All operators $A_0, A_1, \dots, A_4$ annihilate the orthogonal complement of $\{x, y, u, v\}$ and their images are contained in the linear span of $\{x, y, u, v\}$. So, we can identify them with matrices. We know that there exists a 2×2 submatrix of A_4 which is invertible. It is then clear that there exists a sequence of real numbers (t_n) tending to zero such that the corresponding submatrices of $A_0 + t_nA_1 + t_n^2A_2 + t_n^3A_3 + t_n^4A_4$ are invertible. It follows that operators

$$\begin{aligned} &((u+t_nx)\otimes(u+t_nx)^* + (v+t_ny)\otimes(v+t_ny)^*)\, A \\ &\quad((u+t_nx)\otimes(u+t_nx)^* + (v+t_ny)\otimes(v+t_ny)^*) \\ &= ((u+t_nx)\otimes(u+t_nx)^* + (v+t_ny)\otimes(v+t_ny)^*)\, P_nAP_n \\ &\quad((u+t_nx)\otimes(u+t_nx)^* + (v+t_ny)\otimes(v+t_ny)^*) \end{aligned}$$

are of rank two. Here, P_n is the projection of rank two whose image is the linear span of $\{u+t_nx, v+t_ny\}$. It follows that P_nAP_n is a rank two operator. Clearly, $P_n \to P$ as n tends to infinity, and consequently, $\mathcal{A}$ is dense in $\mathcal{P}_2$, as desired. □

It is even much easier to prove the following statement whose verification is left to the reader.

Lemma 3.18. *Let $A \in \mathcal{S}(H)$ be a nonzero operator. Then*

$$\{P \in \mathcal{P}_2 \,:\, PAP \neq 0\}$$

is an open dense subset of $\mathcal{P}_2$.

Lemma 3.19. *Let H be a Hilbert space with the orthogonal direct sum decomposition $H = H_1 \oplus H_2$. Assume that $A \in \mathcal{S}(H)$ has a corresponding matrix representation*

$$A = \begin{bmatrix} A_1 & B \\ B^* & C \end{bmatrix}$$

with A_1 invertible. Then there exists a positive real number M such that

$$\begin{bmatrix} A_1 & B \\ B^* & tI + C \end{bmatrix}$$

is invertible whenever $t \geq M$.

Proof. The statement follows directly from the equation

$$\begin{bmatrix} A_1 & B \\ B^* & tI + C \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^*A_1^{-1} & I \end{bmatrix} \begin{bmatrix} A_1 & B \\ 0 & tI + (C - B^*A_1^{-1}B) \end{bmatrix}. \qquad \square$$

Lemma 3.20. *Let H be a Hilbert space with the orthogonal direct sum decomposition $H = H_1 \oplus H_2$. Assume that $A \in \mathcal{S}(H)$ has a corresponding matrix representation*

$$A = \begin{bmatrix} A_1 & B \\ B^* & 0 \end{bmatrix}$$

with A_1 being a positive invertible operator. Then there exists a positive real number M such that

$$\begin{bmatrix} A_1 & B \\ B^* & tI \end{bmatrix}$$

is invertible positive operator whenever $t \geq M$.

Proof. By the assumptions there exists a positive real number a such that $\langle A_1x, x\rangle \geq a\|x\|^2$ for every $x \in H_1$. Set

$$M = \frac{\|B\|^2}{a}.$$

Choose and fix $t > M$. Then both the trace and the determinant of the 2×2 hermitian matrix

$$\begin{bmatrix} a & -\|B\| \\ -\|B\| & t \end{bmatrix}$$

are positive, and therefore there exists a positive real number c such that

$$\left\langle \begin{bmatrix} a & -\|B\| \\ -\|B\| & t \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix}, \begin{bmatrix} t \\ s \end{bmatrix} \right\rangle \geq c$$

whenever t, s are real numbers satisfying $t^2 + s^2 = 1$.

Let $z \in H$ be an arbitrary vector with $\|z\| = 1$. Then $z = x + y$ with $x \in H_1, y \in H_2$, and $\|x\|^2 + \|y\|^2 = 1$, and hence,

$$\left\langle \begin{bmatrix} A_1 & B \\ B^* & tI \end{bmatrix} z, z \right\rangle = \left\langle \begin{bmatrix} A_1 & B \\ B^* & tI \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle$$
$$= \langle A_1x, x\rangle + \langle By, x\rangle + \langle B^*x, y\rangle + \langle tIy, y\rangle \geq a\|x\|^2 - 2\|B\|\,\|x\|\,\|y\| + t\|y\|^2$$
$$= \left\langle \begin{bmatrix} a & -\|B\| \\ -\|B\| & t \end{bmatrix} \begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix}, \begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} \right\rangle \geq c.$$

This completes the proof. $\square$

The following lemma has been proved in [9].

Lemma 3.21. *Let $A, B, C \in \mathcal{H}_2$ satisfy $C \neq A, C \neq B$, and* $\operatorname{rank}(B - A) = 2$. *Then there exists $D \in \mathcal{H}_2$ such that $D - C$ is invertible, $D - A$ is singular, and $D - B$ is singular.*

Here we need to prove also a slightly modified version.

Lemma 3.22. *Let $A, B, C \in \mathcal{H}_2$ satisfy $C \neq A, C \neq B$, and* $\operatorname{rank}(B-A) = 2$. *Assume further that $A, B, C \geq 0$ and $A \neq 0$ and $B \neq 0$. Then there exists a positive invertible matrix $D \in \mathcal{H}_2$ such that $D - C$ is invertible, $D - A$ is singular, and $D - B$ is singular.*

Proof. Denote by $\mathcal{P}_1 \subset \mathcal{H}_2$ the set of all projections of rank one. Clearly, one of the operators $A - B$ and $B - A$ has at least one positive eigenvalue. With no loss of generality we will assume that $B - A$ has at least one positive eigenvalue. If $B - A$ has two positive eigenvalues, then for every rank one projection $P \in \mathcal{H}_2$ there exists a positive real number t_P such that $(B-A) - t_P P$ is of rank one. If on the other hand, $B - A$ has one positive and one negative eigenvalue, then by Lemma 2.14, there exists a nonempty open subset $\mathcal{U} \subset \mathcal{P}_1$ such that for every $P \in \mathcal{U}$ there exists a positive real number t_P such that $(B-A) - t_P P$ is singular. If A is invertible, then $A + sP$ is positive invertible for every positive $s \in \mathbb{R}$ and every projection P of rank one. When A is of rank one, the matrix $A + sP$ is positive invertible for every positive $s \in \mathbb{R}$ and every projection P of rank one that is linearly independent of A.

Hence, there exists a nonempty open subset $\mathcal{W} \subset \mathcal{P}_1$ such that for every $P \in \mathcal{W}$ there exists a positive real number t_P such that $(B - A) - t_P P$ is singular and $A + t_P P$ is positive invertible matrix. If for some $P \in \mathcal{W}$ and the corresponding positive real number t_P the matrix $(C-A) - t_P P$ is invertible, then we set $D = A + t_P P$. Clearly, D has all the desired properties.

It remains to prove that the possibility, that for every $P \in \mathcal{W}$ and the corresponding positive real number t_P the matrix $(C - A) - t_P P$ is singular, cannot occur. Assume on the contrary that we have this possibility. If $\operatorname{rank}(C - A) = 1$, then we can find $P \in \mathcal{W}$ linearly independent of $C - A$. It follows that $(C - A) - t_P P$ is of rank two, a contradiction. Thus, we may assume that $C - A$ is invertible. Then by Lemma 2.15, we have $B - A = C - A$, contradicting $B \neq C$. □

Lemma 3.23. *Let $B, C \in \mathcal{S}(H)^{\geq 0}$ satisfy* $\dim \operatorname{Im} B \geq 2, C \neq 0$, *and $C \neq B$. Then there exists a singular operator $D \in \mathcal{S}(H)^{\geq 0}$ such that $D - C$ is invertible and $D - B$ is singular.*

Proof. By Lemma 3.17, the set of all projections P of rank two such that PBP is of rank two is an open dense subset of $\mathcal{P}_2$. It follows from Lemma 3.18 that the set of all projections P of rank two such that $P(C - B)P \neq 0$ is also open and dense in $\mathcal{P}_2$. So is the set of all projections P of rank two such that $PCP \neq 0$. The intersection of finitely many open dense subsets of $\mathcal{P}_2$ is nonempty. Hence, there exists a projection P of rank two such that

$$\operatorname{rank} PBP = 2, \quad PCP \neq PBP, \quad \text{and} \quad PCP \neq 0.$$

Let

$$B = \begin{bmatrix} B_1 & B_2 \\ B_2^* & B_3 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} C_1 & C_2 \\ C_2^* & C_3 \end{bmatrix}$$

be matrix representations of operators B and C with respect to the orthogonal direct sum decomposition $H = \operatorname{Im} P \oplus \operatorname{Ker} P$. Since $B_1 : \operatorname{Im} P \to \operatorname{Im} P$ is positive invertible operator there exists an invertible operator $T : \operatorname{Im} P \to$

$\operatorname{Im} P$ such that TB_1T^* is the identity operator on $\operatorname{Im} P$. Set $W = -B_2^*(B_1^{-1})^*$ and

$$S = \begin{bmatrix} T & 0 \\ W & I \end{bmatrix}.$$

Replacing B and C by SBS^* and SCS^*, respectively, we may, and we will assume that

$$B = \begin{bmatrix} I & 0 \\ 0 & B_3 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} C_1 & C_2 \\ C_2^* & C_3 \end{bmatrix}$$

with $C_1 \neq 0, I$. It follows easily that there exists a rank one projection $Q : \operatorname{Im} P \to \operatorname{Im} P$ such that $C_1 - Q$ is invertible. By Lemma 3.19 there exists a positive real number t such that $D - C$, where

$$D = \begin{bmatrix} Q & 0 \\ 0 & tI \end{bmatrix}$$

is invertible. Clearly, D is a positive singular operator, and $D - B$ is singular as well. This completes the proof. □

Lemma 3.24. *Let $\mathcal{V}$ be any of the sets $\mathcal{S}(H), \mathcal{S}(H)^{\geq 0}$, or $\mathcal{S}(H)^{>0}$. Assume that $A, B \in \mathcal{V}$ with $A \neq B$. Then the following are equivalent:*

- *A and B are adjacent,*
- *there exists $C \in \mathcal{V} \setminus \{A, B\}$ such that for every $D \in \mathcal{V}$ the invertibility of $D - C$ implies that $D - A$ is invertible or $D - B$ is invertible.*

Proof. Assume first that A and B are adjacent. Then they are comparable. With no loss of generality we assume that $A \leq B$. Set $C = A + 2(B - A)$. Clearly, $C \in \mathcal{V}$. In order to verify that C satisfies the second condition of our lemma we assume that D is an operator from $\mathcal{V}$ such that $D - C$ is invertible. Then

$$D - A = (D - C) + 2(B - A) = (D - C)(I + 2R)$$

and

$$D - B = (D - C) + (B - A) = (D - C)(I + R),$$

where $R = (D - C)^{-1}(B - A)$ is of rank one. It follows that at least one of the operators $I + R$ and $I + 2R$ is invertible, and consequently, at least one of $D - A$ and $D - B$ is invertible.

To prove the converse we assume that $\dim \operatorname{Im}(B - A) \geq 2$ and we have to show that for every $C \in \mathcal{V} \setminus \{A, B\}$ there exists $D \in \mathcal{V}$ such that $D - C$ is invertible and both $D - A$ and $D - B$ are singular. So, let $C \in \mathcal{V}$ be any operator, $C \neq A, B$.

As in the proof of the previous lemma we see that there exists a projection P of rank two such that

$$\operatorname{rank} P(B - A)P = 2, \quad PAP \neq PCP, \quad \text{and} \quad PBP \neq PCP,$$

and if $A \neq 0$ and $B \neq 0$, then $PAP \neq 0$ and $PBP \neq 0$. Let

$$A = \begin{bmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_2^* & B_3 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} C_1 & C_2 \\ C_2^* & C_3 \end{bmatrix}$$

be matrix representations of operators A, B, C with respect to the orthogonal direct sum decomposition $H = \operatorname{Im} P \oplus \operatorname{Ker} P$.

We will first consider the case when $\mathcal{V} = \mathcal{S}(H)$. By Lemma 3.21 there exists $D_1 \in \mathcal{H}_2$ such that $D_1 - C_1$ is invertible while both $D_1 - A_1$ and $D_1 - B_1$ are singular. In particular, there exist nonzero vectors $x, y \in \operatorname{Im} P$ such that $D_1 x = A_1 x$ and $D_1 y = B_1 y$. Clearly, x and y are linearly independent, since otherwise we would have $A_1 x = B_1 x$ contradicting the fact that $B_1 - A_1$ is invertible. Then there is a unique linear operator $T : \operatorname{Im} P \to \operatorname{Ker} P$ such that $Tx = A_2^* x$ and $Ty = B_2^* y$. By Lemma 3.19 we can find a positive real number t such that the operator

$$\begin{bmatrix} D_1 - C_1 & T^* - C_2 \\ T - C_2^* & tI - C_3 \end{bmatrix}$$

is invertible. Set

$$D = \begin{bmatrix} D_1 & T^* \\ T & tI \end{bmatrix}.$$

Clearly, $D \in \mathcal{S}(H), D - C$ is invertible, and $(D - A)x = 0$ and $(D - B)y = 0$.

It remains to consider the cases when $\mathcal{V}$ is $\mathcal{S}(H)^{\geq 0}$ or $\mathcal{S}(H)^{>0}$. We will first consider the case when $A, B \neq 0$. Note that this is automatically true when $\mathcal{V} = \mathcal{S}(H)^{>0}$. Then by Lemma 3.22 there exists a positive invertible $D_1 \in \mathcal{H}_2$ such that $D_1 - C_1$ is invertible while both $D_1 - A_1$ and $D_1 - B_1$ are singular. Define $T : \operatorname{Im} P \to \operatorname{Ker} P$ as above. Applying Lemmas 3.19 and 3.20 we can find a positive real t such that

$$\begin{bmatrix} D_1 - C_1 & T^* - C_2 \\ T - C_2^* & tI - C_3 \end{bmatrix}$$

is invertible and

$$D = \begin{bmatrix} D_1 & T^* \\ T & tI \end{bmatrix}$$

is positive invertible. Clearly, the operator $D \in \mathcal{V}$ has all the desired properties.

The only possibility left is that $\mathcal{V} = \mathcal{S}(H)^{\geq 0}$ and that $A = 0$ or $B = 0$. With no loss of generality we may assume that $A = 0$. Then the existence of an operator D with the desired properties is guaranteed by Lemma 3.23. □

Now we are prepared to prove the main results of this subsection.

Proof of Theorem 3.14. After composing ϕ with the translation we may assume with no loss of generality that $\phi(0) = 0$. By Lemma 3.24, ϕ preserves adjacency in both directions. Thus, by Theorem 2.1 there exist $c \in \{-1, 1\}$ and a bijective linear or conjugate-linear map $T : H \to H$ such that

$$\phi\left(\sum_{j=1}^{k} t_j x_j \otimes x_j^*\right) = c \sum_{j=1}^{k} t_j (Tx_j) \otimes (Tx_j)^* \tag{8}$$

for every $\sum_{j=1}^{k} t_j x_j \otimes x_j^* \in \mathcal{S}_F(H)$. With no loss of generality we may assume that $c = 1$. We know that $\phi(I)$ is invertible. Since $I - x \otimes x^*$ is singular for every $x \in H$ of norm one, the operator $\phi(I) - (Tx) \otimes (Tx)^*$ is singular for every $x \in H$ of norm one. It follows that $\phi(I)$ is positive. Thus, after composing ϕ by a congruence transformation, we may, and we will assume that $\phi(I) = I$. We still have (8), with T being now some other bijective linear or conjugate-linear map. We apply once more the fact that $I - x \otimes x^*$ is singular for every $x \in H$ of norm one to conclude that T is a unitary or antiunitary operator. Hence, we may assume, after composing ϕ with a unitary or antiunitary similarity transformation that $\phi(A) = A$ for every $A \in \mathcal{S}_F(H) \cup \{I\}$.

Let t be a nonzero real number. It follows from the fact that for every $x \in H$ of norm one, the operator $tI - sx \otimes x^*$ is singular if and only if $s = t$, that $\phi(tI) = tI, t \in \mathbb{R}$. If $R \in \mathcal{S}_F(H)$ is any operator of rank one, then $tI + R$ is adjacent to tI, and therefore, $\phi(tI + R) = tI + S$ for some $S \in \mathcal{S}_F(H)$ of rank one. If $Q \in \mathcal{S}_F(H)$ is any rank one operator, then $(tI + R) - Q$ is invertible if and only if $\phi(tI + R) - \phi(Q) = tI + S - Q$ is invertible. It follows easily that $R = S$. Hence, $\phi(tI + R) = tI + R$ for every real number t and every operator R of rank one.

Let now $A \in \mathcal{S}(H)$ be any operator and $x \in H$ a vector of norm one. For $t > 2\|A\|, 2\|\phi(A)\|$ we set

$$s(t) = \langle (tI - A)^{-1}x, x\rangle.$$

Since

$$\langle (tI - A)^{-1}x, x\rangle = \frac{1}{t}\left(1 + \frac{1}{t}\langle Ax, x\rangle + \frac{1}{t^2}\langle A^2x, x\rangle + \cdots\right)$$

and

$$\left|\frac{1}{t}\langle Ax, x\rangle + \frac{1}{t^2}\langle A^2x, x\rangle + \cdots\right| < \frac{1}{2} + \frac{1}{2^2} + \cdots = 1$$

we see that $s(t) \neq 0$ for every $t > 2\|A\|$. The operator

$$tI - \frac{1}{s(t)}x \otimes x^* - A = (tI - A)\left(I - \frac{1}{s(t)}(tI - A)^{-1}x \otimes x^*\right)$$

is singular because $\frac{1}{s(t)}(tI - A)^{-1}x \otimes x^*$ is a rank one idempotent. It follows that

$$tI - \frac{1}{s(t)}x \otimes x^* - \phi(A) = (tI - \phi(A))\left(I - \frac{1}{s(t)}(tI - \phi(A))^{-1}x \otimes x^*\right)$$

is singular, which yields that

$$\langle (tI - \phi(A))^{-1}x, x\rangle = s(t) = \langle (tI - A)^{-1}x, x\rangle$$

for all $t > 2\|A\|, 2\|\phi(A)\|$. But then $\langle Ax, x\rangle = \langle \phi(A)x, x\rangle$. As $x \in H$ was any unit vector, we have $\phi(A) = A$, as desired. $\square$

To prove Theorem 3.15 we first observe that ϕ preserves adjacency. Then all we need to do is to check that the set of finite rank operators is mapped by ϕ onto itself. Once we know this we can apply Theorem 2.2 to verify that ϕ restricted to the set of finite rank operators is of the desired form and then we can complete the proof using an almost identical approach as above. We leave the details to the reader.

In order to verify that ϕ maps $\mathcal{S}_F(H)^{\geq 0}$ onto itself we only need to show that $\phi(0) = 0$. This can be achieved by a slight modification of Proposition 2.17. However, a shorter proof is possible under our assumptions. Let $A \in \mathcal{S}(H)^{\geq 0}$ be any operator. Any set of the form $\{A + tP \, : t \in \mathbb{R}$ such that $A + tP \geq 0\}$, where P is a projection of rank one, will be called a line through A. As ϕ preserves adjacency in both directions, each line through A is mapped onto some line through $\phi(A)$. We can show that for an operator $A \in \mathcal{S}(H)^{\geq 0}$ the following are equivalent:

- $A = 0$,
- For every $B \in \mathcal{S}(H)^{\geq 0}$ such that $B - A$ is invertible and every line L through A there exists $C \in L$ such that $B - C$ is singular.

The desired equality $\phi(0) = 0$ follows directly from this equivalence. If $A = 0, B \in \mathcal{S}(H)^{>0}$, and $L = \{tP \, : \, t \in [0, \infty)\}$ is a line through the zero operator passing through a rank one projection $P = x \otimes x^*$, we can easily find a positive real number s such that $B - sP = B(I - sB^{-1}P)$ is singular. All we need to do is to set $s = (\mathrm{tr}\,(B^{-1}P))^{-1} = (\langle B^{-1}x, x\rangle)^{-1} > 0$. If on the other hand, $A \neq 0$, then we can find $B \in \mathcal{S}(H)^{\geq 0}$ such that $B - A$ is invertible and $B - A$ is neither positive, nor negative. It follows that there is a projection P of rank one with $\mathrm{tr}\,((B - A)^{-1}P) = 0$. Consequently, all operators of the form $B - A - tP$ are invertible, as desired.

One can now use a similar approach to prove Theorem 3.16. We leave the details to the reader.

3.5. Maps Preserving the Geometric Mean

Let H be a Hilbert space. For $A, B \in \mathcal{S}(H)^{\geq 0}$ the most natural definition of their geometric mean was given by Ando [2] by the formula

$$A \,\sharp\, B = \max \left\{ X \geq 0 \, : \, \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0 \right\}.$$

Let us list some basic properties of the geometric mean. Clearly, $A \,\sharp\, B = B \,\sharp\, A$. We have $T(A \,\sharp\, B)T^* = (TAT^*) \,\sharp\, (TBT^*)$ for all $A, B \in \mathcal{S}(H)^{\geq 0}$ and every invertible bounded linear or conjugate-linear operator T on H. If $A, B \in \mathcal{S}(H)^{\geq 0}$ with A invertible, then

$$A \,\sharp\, B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}. \tag{9}$$

And finally, if $A \in \mathcal{S}(H)^{\geq 0}$ and P is a rank one projection on H, then

$$A \,\sharp\, P = \sqrt{\sup\{t \in [0, \infty) \, : \, tP \leq A\}}P. \tag{10}$$

The following result was proved in [14].

Theorem 3.25. *Let H be a Hilbert space, $\dim H \geq 2$, and $\phi : \mathcal{S}(H)^{\geq 0} \to \mathcal{S}(H)^{\geq 0}$ a bijective map with the property that for every pair $A, B \in \mathcal{S}(H)^{\geq 0}$ we have*

$$\phi(A \,\sharp\, B) = \phi(A) \,\sharp\, \phi(B).$$

Then there exists an invertible bounded linear or conjugate-linear operator $T : H \to H$ such that

$$\phi(A) = TAT^*$$

for every $A \in \mathcal{S}(H)^{\geq 0}$.

We will see that this result can be reproved using our approach based on adjacency preserving maps. We will give here only the sketch of the proof. But before doing so let us just mention that our approach is not appropriate for studying bijective preservers of geometric mean defined on the set of all positive invertible operators. Namely, in the finite-dimensional case we can consider the map $A \mapsto \det(A)\, A$ defined on the set of all positive invertible $n \times n$ hermitian matrices. Clearly, such a map is bijective. Because of (9), it is an automorphism with respect to the geometric mean. But obviously, it does not preserve adjacency.

In order to prove the above theorem one first observes that $A \in \mathcal{S}(H)^{\geq 0}$ is invertible if and only if $A \,\sharp\, \mathcal{S}(H)^{\geq 0} = \{A \,\sharp\, B \,:\, B \in \mathcal{S}(H)^{\geq 0}\} = \mathcal{S}(H)^{\geq 0}$. Hence, ϕ maps the set of positive invertible operators onto itself. After composing ϕ with the appropriate congruence transformation we may assume with no loss of generality that $\phi(I) = I$. It is easy to verify that $A = 0$ if and only if the set $A \,\sharp \mathcal{S}(H)^{\geq 0} = \{0\}$ is minimal among the sets of the form $B \,\sharp\, \mathcal{S}(H)^{\geq 0}, B \in \mathcal{S}(H)^{\geq 0}$. Among all such sets the non-trivial minimal sets are $R \,\sharp\, \mathcal{S}(H)^{\geq 0}$ with $R \in \mathcal{S}(H)^{\geq 0}$ being of rank one. Hence, ϕ maps the zero operator into itself and the set of all rank one operators onto itself. We can then characterize adjacent pairs $A, B \in \mathcal{S}(H)^{\geq 0}$ as pairs of two different operators for which the set of all rank one operators R with the property that $A \,\sharp\, R = B \,\sharp\, R$ is maximal. Thus, the restriction of ϕ to the set of all finite rank operators is of the form described in Theorem 2.2. Because of $\phi(I) = I$ and (10) the map T appearing in the conclusion of Theorem 2.2 is a unitary or antiunitary operator. After composing ϕ with yet another congruence transformation we may assume with no loss of generality that $\phi(A) = A$ for every A of finite rank. It then follows from (10) that $\phi(A) = A$ for all $A \in \mathcal{S}(H)^{\geq 0}$.

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Peter Šemrl (✉)
Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 19
1000 Ljubljana
Slovenia
e-mail: peter.semrl@fmf.uni-lj.si

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