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Integral Equations and Operator Theory

Denjoy–Carleman Differentiable Perturbation of Polynomials and Unbounded Operators

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Abstract. Let $t \mapsto A(t)$ for $t \in T$ be a C^M-mapping with values unbounded operators with compact resolvents and common domain of definition which are self-adjoint or normal. Here C^M stands for C^ω (real analytic), a quasianalytic or non-quasianalytic Denjoy–Carleman class, C^∞, or a Hölder continuity class $C^{0,\alpha}$. The parameter domain T is either $\mathbb{R}$ or $\mathbb{R}^n$ or an infinite dimensional convenient vector space. We prove and review results on C^M-dependence on t of the eigenvalues and eigenvectors of $A(t)$.

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Theorem. *Let $t \mapsto A(t)$ for $t \in T$ be a parameterized family of unbounded operators in a Hilbert space H with common domain of definition and with compact resolvent.*

If $t \in T = \mathbb{R}$ and all $A(t)$ are self-adjoint then the following holds:

(A) *If $A(t)$ is real analytic in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized real analytically in t.*
(B) *If $A(t)$ is quasianalytic of class C^Q in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^Q in t.*
(C) *If $A(t)$ is non-quasianalytic of class C^L in $t \in \mathbb{R}$ and if no two different continuously parameterized eigenvalues (e.g., ordered by size) meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^L in t.*
(D) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and if no two different continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^∞ in t.*

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(E) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then the eigenvalues of $A(t)$ can be parameterized twice differentiably in t.*
(F) *If $A(t)$ is $C^{1,\alpha}$ in $t \in \mathbb{R}$ for some $\alpha > 0$, then the eigenvalues of $A(t)$ can be parameterized C^1 in t.*

If $t \in T = \mathbb{R}$ and all $A(t)$ are normal then the following holds:

(G) *If $A(t)$ is real analytic in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue z_0 of $A(t_0)$ there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near z_0 of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized real analytically in s near $s = 0$.*
(H) *If $A(t)$ is quasianalytic of class C^Q in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue z_0 of $A(t_0)$ there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near z_0 of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^Q in s near $s = 0$.*
(I) *If $A(t)$ is non-quasianalytic of class C^L in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue z_0 of $A(t_0)$ at which no two of the different continuously parameterized eigenvalues (which is always possible by [12, II 5.2]) meet of infinite order, there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near z_0 of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^L in s near $s = 0$.*
(J) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue z_0 of $A(t_0)$ at which no two of the different continuously parameterized eigenvalues meet of infinite order, there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near z_0 of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^∞ in s near $s = 0$.*
(K) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue z_0 of $A(t_0)$ at which no two of the different continuously parameterized eigenvalues meet of infinite order, the eigenvalues near z_0 of $A(t)$ and their eigenvectors can be parameterized by absolutely continuous functions in t near $t = t_0$.*

If $t \in T = \mathbb{R}^n$ and all $A(t)$ are normal then the following holds:

(L) *If $A(t)$ is real analytic or quasianalytic of class C^Q in $t \in \mathbb{R}^n$, then for each $t_0 \in \mathbb{R}^n$ and for each eigenvalue z_0 of $A(t_0)$, there exist a neighborhood D of z_0 in $\mathbb{C}$, a neighborhood W of t_0 in $\mathbb{R}^n$, and a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings each of which is either a local blow-up along a real analytic or C^Q submanifold or a local power substitution, such that the eigenvalues of $A(\pi_k(s))$, $s \in U_k$, in D and the corresponding eigenvectors can be parameterized real analytically or C^Q in s. If A is self-adjoint, then we do not need power substitutions.*
(M) *If $A(t)$ is real analytic or quasianalytic of class C^Q in $t \in \mathbb{R}^n$, then for each $t_0 \in \mathbb{R}^n$ and for each eigenvalue z_0 of $A(t_0)$, there exist a neighborhood D of z_0 in $\mathbb{C}$ and a neighborhood W of t_0 in $\mathbb{R}^n$ such that the eigenvalues of $A(t)$, $t \in W$, in D and the corresponding eigenvectors can be parameterized by functions which are special functions of bounded variation (SBV), see [9] or [3], in t.*

If $t \in T \subseteq E$, a c^∞-open subset in a finite or infinite dimensional convenient vector space then the following holds:

(N) *For $0 < \alpha \leq 1$, if $A(t)$ is $C^{0,\alpha}$ (Hölder continuous of exponent α) in $t \in T$ and all $A(t)$ are self-adjoint, then the eigenvalues of $A(t)$ can be parameterized $C^{0,\alpha}$ in t.*

(O) *For $0 < \alpha \leq 1$, if $A(t)$ is $C^{0,\alpha}$ in $t \in T$ and all $A(t)$ are normal, then we have: For each $t_0 \in T$ and each eigenvalue z_0 of $A(t_0)$ consider a simple closed C^1-curve γ in the resolvent set of $A(t_0)$ enclosing only z_0 among all eigenvalues of $A(t_0)$. Then for t near t_0 in the c^∞-topology on T, no eigenvalue of $A(t)$ lies on γ. Let $\lambda(t) = (\lambda_1(t), \dots, \lambda_N(t))$ be the N-tuple of all eigenvalues (repeated according to their multiplicity) of $A(t)$ inside of γ. Then $t \mapsto \lambda(t)$ is $C^{0,\alpha}$ for t near t_0 with respect to the non-separating metric*

$$d(\lambda, \mu) = \min_{\sigma \in \mathcal{S}_N} \max_{1 \leq i \leq N} |\lambda_i - \mu_{\sigma(i)}|$$

on the space of N-tuples.

Part (A) is due to Rellich [22] in 1942, see also [4] and [12, VII 3.9]. Part (D) has been proved in [2, 7.8], see also [13, 50.16], in 1997, which contains also a different proof of (A). (E) and (F) have been proved in [14] in 2003. (G) was proved in [19, 7.1]; it can be proved as (H) with some obvious changes, but it is not a special case since C^ω does not correspond to a sequence which is an $\mathcal{L}$-intersection (see 'definitions and remarks' below and [17]). (J) and (K) were proved in [19, 7.1]. (N) was proved in [15].

The purpose of this paper is to prove the remaining parts (B), (C), (H), (I), (L), (M), and (O).

Definitions and Remarks. Let $M = (M_k)_{k \in \mathbb{N} = \mathbb{N}_{\geq 0}}$ be an increasing sequence ($M_{k+1} \geq M_k$) of positive real numbers with $M_0 = 1$. Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that, for each compact $K \subseteq U$, there exist positive constants C and ρ such that

$$|\partial^\alpha f(x)| \leq C\, \rho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}^n \text{ and } x \in K.$$

The set $C^M(U)$ is a *Denjoy–Carleman class* of functions on U. If $M_k = 1$, for all k, then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U. In general, $C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$.

Throughout this paper $Q = (Q_k)_{k \in \mathbb{N}}$ is a sequence as above which is log-convex (i.e., $Q_k^2 \leq Q_{k-1} Q_{k+1}$ for all k), derivation closed (i.e., $\sup_k (Q_{k+1}/Q_k)^{1/k} < \infty$), quasianalytic (i.e., $\sum_k (k!\, Q_k)^{-1/k} = \infty$), and which is also an $\mathcal{L}$-intersection. We say that Q is an $\mathcal{L}$-intersection if $C^Q = \bigcap \{C^N : N \text{ non-quasianalytic, log-convex, } N \geq Q\}$. Moreover, $L = (L_k)_{k \in \mathbb{N}}$ is a sequence as above which is log-convex, derivation closed, and non-quasi-analytic. Then C^Q and C^L are closed under composition and allow for the implicit function theorem. See [17] or [16] and references therein.

That $A(t)$ is a real analytic, C^M (where M is either Q or L), C^∞, or $C^{k,\alpha}$ family of unbounded operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that $A(t)^* = A(t)$ in the self-adjoint case, or $A(t)$ has closed graph and $A(t)A(t)^* = A(t)^*A(t)$ wherever defined in the normal case. Moreover, we require that $t \mapsto \langle A(t)u, v\rangle$ is of the respective differentiability class for each $u \in V$ and $v \in H$. From now on we treat only $C^M = C^\omega$, C^M for $M = Q$, $M = L$, and $C^M = C^{0,\alpha}$.

This implies that $t \mapsto A(t)u$ is of the same class $C^M(T, H)$ (where T is either $\mathbb{R}$ or $\mathbb{R}^n$) or is in $C^{0,\alpha}(T, H)$ (if T is a convenient vector space) for each $u \in V$ by [13, 2.14.4, 10.3] for C^ω, by [16, 3.1, 3.3, 3.5] for $M = L$, by [17, 1.10, 2.1, 2.3] for $M = Q$, and by [13, 2.3], [11, 2.6.2] or [10, 4.14.4] for $C^{0,\alpha}$ because $C^{0,\alpha}$ can be described by boundedness conditions only and for these the uniform boundedness principle is valid.

A sequence of functions λ_i is said to *parameterize the eigenvalues*, if for each $z \in \mathbb{C}$ the cardinality $|\{i : \lambda_i(t) = z\}|$ equals the multiplicity of z as eigenvalue of $A(t)$.

Let X be a C^ω or C^Q manifold. A *local blow-up* Φ over an open subset U of X means the composition $\Phi = \iota \circ \varphi$ of a blow-up $\varphi : U' \to U$ with center a C^ω or C^Q submanifold and of the inclusion $\iota : U \to X$. A *local power substitution* is a mapping $\Psi : V \to X$ of the form $\Psi = \iota \circ \psi$, where $\iota : W \to X$ is the inclusion of a coordinate chart W of X and $\psi : V \to W$ is given by

$$(y_1, \ldots, y_q) = ((-1)^{\epsilon_1} x_1^{\gamma_1}, \ldots, (-1)^{\epsilon_q} x_q^{\gamma_q}),$$

for some $\gamma = (\gamma_1, \ldots, \gamma_q) \in (\mathbb{N}_{>0})^q$ and all $\epsilon = (\epsilon_1, \ldots, \epsilon_q) \in \{0, 1\}^q$, where $y_1, \ldots, y_q$ denote the coordinates of W (and $q = \dim X$).

This paper became possible only after some of the results of [16] and [17] were proved, in particular the uniform boundedness principles. The wish to prove the results of this paper was the main motivation for us to work on [16] and [17].

Applications. For brevity we confine ourselves to C^Q; the same applies to C^ω. Let X be a compact C^Q manifold and let $t \mapsto g_t$ be a C^Q-curve of C^Q Riemannian metrics on X. Then we get the corresponding C^Q curve $t \mapsto \Delta(g_t)$ of Laplace-Beltrami operators on $L^2(X)$. By theorem (B) the eigenvalues and eigenvectors can be arranged C^Q in t. By [1], the eigenfunctions are also C^Q as functions on X (at least for those C^Q which can be described by a weight function, see [7]). Question: Are the eigenvectors viewed as eigenfunctions then also in $C^Q(X \times \mathbb{R})$?

Let Ω be a bounded region in $\mathbb{R}^n$ with C^Q boundary, and let $H(t) = -\Delta + V(t)$ be a C^Q-curve of Schrödinger operators with varying C^Q potential and Dirichlet boundary conditions. Then the eigenvalues and eigenvectors can be arranged C^Q in t. Question: Are the eigenvectors viewed as eigenfunctions then also in $C^Q(\Omega \times \mathbb{R})$?

Example. This is an elaboration of [2, 7.4] and [14, Example]. Let $S(2)$ be the vector space of all symmetric real (2×2)-matrices. We use the C^L-curve lemma [16, 3.6] or [17, 2.5]: *For each L, there exist sequences $\mu_n \to \infty$, $t_n \to t_\infty$, $s_n > 0$ in $\mathbb{R}$ with the following property: For sequences $A_n, B_n \in S(2)$*

which are μ_n-convergent to 0*, i.e., $\mu_n A_n$ and $\mu_n B_n$ are bounded in $S(2)$, there exists a curve $A \in C^L(\mathbb{R}, S(2))$ such that $A(t_n + t) = A_n + tB_n$ for $|t| \le s_n$.*

Choose a sequence ν_n of reals satisfying $\mu_n \nu_n \to 0$ and $(\nu_n)^n \le s_n$ for all n and use the C^L-curve lemma for

$$A_n := (\nu_n)^{n+1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_n := \nu_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of $A_n + tB_n$ and their derivatives are

$$\lambda_n(t) = \pm \nu_n \sqrt{(\nu_n)^{2n} + t^2}, \quad \lambda_n'(t) = \pm \frac{\nu_n t}{\sqrt{(\nu_n)^{2n} + t^2}}.$$

Then

$$\begin{aligned} \frac{\lambda'(t_n + (\nu_n)^n) - \lambda'(t_n)}{((\nu_n)^n)^\alpha} &= \frac{\lambda_n'((\nu_n)^n) - \lambda_n'(0)}{(\nu_n)^{n\alpha}} = \pm \frac{\nu_n}{(\nu_n)^{n\alpha}\sqrt{2}} \\ &= \pm \frac{(\nu_n)^{1-n\alpha}}{\sqrt{2}} \to \infty \quad \text{for } \alpha > 0. \end{aligned}$$

So the condition (in (C), (D), (I), (J), and (K)) that no two different continuously parameterized eigenvalues meet of infinite order cannot be dropped. By [2, 2.1], we may always find a twice differentiable square root of a non-negative smooth function, so that the eigenvalues λ are functions which are twice differentiable but not $C^{1,\alpha}$ for any $\alpha > 0$.

Note that the normed eigenvectors cannot be chosen continuously in this example (see also example [21, §2]). Namely, we have

$$A(t_n) = (\nu_n)^{n+1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A(t_n + (\nu_n)^n) = (\nu_n)^{n+1} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Resolvent Lemma. *Let C^M be any of C^ω, C^Q, C^L, C^∞, or $C^{0,\alpha}$, and let $A(t)$ be normal. If A is C^M then the resolvent $(t, z) \mapsto (A(t) - z)^{-1} \in L(H, H)$ is C^M on its natural domain, the global resolvent set*

$$\{(t, z) \in T \times \mathbb{C} : (A(t) - z) : V \to H \text{ is invertible}\}$$

which is open (and even connected).

Proof. By definition the function $t \mapsto \langle A(t)v, u\rangle$ is of class C^M for each $v \in V$ and $u \in H$. We may conclude that the mapping $t \mapsto A(t)v$ is of class C^M into H as follows: For $C^M = C^\infty$ we use [13, 2.14.4]. For $C^M = C^\omega$ we use in addition [13, 10.3]. For $C^M = C^Q$ or $C^M = C^L$ we use [17, 2.1] and/or [16, 3.3] where we replace $\mathbb{R}$ by $\mathbb{R}^n$. For $C^M = C^{0,\alpha}$ we use [13, 2.3], [11, 2.6.2], or [10, 4.1.14] because $C^{0,\alpha}$ can be described by boundedness conditions only and for these the uniform boundedness principle is valid.

For each t consider the norm $\|u\|_t^2 := \|u\|^2 + \|A(t)u\|^2$ on V. Since $A(t)$ is closed, $(V, \| \quad \|_t)$ is again a Hilbert space with inner product $\langle u, v\rangle_t := \langle u, v\rangle + \langle A(t)u, A(t)v\rangle$.

Claim. (Cf. [2, in the proof of 7.8], [13, in the proof of 50.16], or [14, Claim 1].) *All these norms $\| \quad \|_t$ on V are equivalent, locally uniformly in t. We then equip V with one of the equivalent Hilbert norms, say $\| \quad \|_0$.*

We reduce this to $C^{0,\alpha}$. Namely, note first that $A(t) : (V, \| \quad \|_s) \to H$ is bounded since the graph of $A(t)$ is closed in $H \times H$, contained in $V \times H$ and thus also closed in $(V, \| \quad \|_s) \times H$. For fixed $u, v \in V$, the function $t \mapsto \langle u, v\rangle_t = \langle u, v\rangle + \langle A(t)u, A(t)v\rangle$ is $C^{0,\alpha}$ since so is $t \mapsto A(t)u$. By the multilinear uniform boundedness principle ([13, 5.18] or [11, 3.7.4]) the mapping $t \mapsto \langle\, , \rangle_t$ is $C^{0,\alpha}$ into the space of bounded sesquilinear forms on $(V, \| \quad \|_s)$ for each fixed s. Thus the inverse image of $\langle\, , \rangle_s + \frac{1}{2}(\text{unit ball})$ in $L(\overline{(V, \| \quad \|_s)} \oplus (V, \| \quad \|_s); \mathbb{C})$ is a c^∞-open neighborhood U of s in T. Thus $\sqrt{1/2}\|u\|_s \le \|u\|_t \le \sqrt{3/2}\|u\|_s$ for all $t \in U$, i.e., all Hilbert norms $\| \quad \|_t$ are locally uniformly equivalent, and the claim follows.

By the linear uniform boundedness theorem we see that $t \mapsto A(t)$ is in $C^M(T, L(V, H))$ as follows (here it suffices to use a set of linear functionals which together recognize bounded sets instead of the whole dual): For $C^M = C^\infty$ we use [13, 1.7, 2.14.3]. For $C^M = C^\omega$ we use in addition [13, 9.4]. For $C^M = C^Q$ or $C^M = C^L$ we use [17, 2.2, 2.3] and/or [16, 3.5] where we replace $\mathbb{R}$ by $\mathbb{R}^n$. For $C^M = C^{0,\alpha}$ see above.

If for some $(t, z) \in T \times \mathbb{C}$ the bounded operator $A(t) - z : V \to H$ is invertible, then this is true locally with respect to the c^∞-topology on the product which is the product topology by [13, 4.16], and $(t, z) \mapsto (A(t)-z)^{-1} : H \to V$ is C^M, by the chain rule, since inversion is real analytic on the Banach space $L(V, H)$.

Note that $(A(t) - z)^{-1} : H \to H$ is a compact operator for some (equivalently any) (t, z) if and only if the inclusion $i : V \to H$ is compact, since $i = (A(t) - z)^{-1} \circ (A(t) - z) : V \to H \to H$.

Polynomial Proposition. *Let P be a curve of polynomials*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + \cdots + (-1)^n a_n(t), \quad t \in \mathbb{R}.$$

- (a) *If P is hyperbolic (i.e., all roots of $P(t)$ are real for each fixed t) and if the coefficient functions a_i are all C^Q then there exist C^Q functions λ_i which parameterize all roots.*
- (b) *If P is hyperbolic, the coefficient functions a_i are C^L, and no two of the different continuously arranged roots (e.g., ordered by size) meet of infinite order, then there exist C^L functions λ_i which parameterize all roots.*
- (c) *If the coefficient functions a_i are C^Q, then for each t_0 there exists $N \in \mathbb{N}_{>0}$ such that the roots of $s \mapsto P(t_0 \pm s^N)$ can be parameterized C^Q in s for s near 0.*
- (d) *If the coefficient functions a_i are C^L and no two of the different continuously arranged roots (by [12, II 5.2]) meet of infinite order, then for each t_0 there exists $N \in \mathbb{N}_{>0}$ such that the roots of $s \mapsto P(t_0 \pm s^N)$ can be parameterized C^L in s for s near 0.*

All C^Q or C^L solutions differ by permutations.

The proof of parts (a) and (b) is exactly as in [2] where the corresponding results were proven for C^∞ instead of C^L, and for C^ω instead of C^Q. For this we need only the following properties of C^Q and C^L:

- They allow for the implicit function theorem (for [2, 3.3]).
- They contain C^ω and are closed under composition (for [2, 3.4]).
- They are derivation closed (for [2, 3.7]).

Part (a) is also in [8, 7.6] which follows [2]. It also follows from the multi-dimensional version [20, 6.10] since blow-ups in dimension 1 are trivial. The proofs of parts (c) and (d) are exactly as in [19, 3.2] where the corresponding result was proven for C^ω instead of C^Q, and for C^∞ instead of C^L, if none of the different roots meet of infinite order. For these we need the properties of C^Q and C^L listed above.

Matrix Proposition. *Let $A(t)$ for $t \in T$ be a family of $(N \times N)$-matrices.*

(e) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^Q-curve of Hermitian matrices, then the eigenvalues and the eigenvectors can be chosen C^Q.*
(f) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^L-curve of Hermitian matrices such that no two different continuously arranged eigenvalues meet of infinite order, then the eigenvalues and the eigenvectors can be chosen C^L.*
(g) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^L-curve of normal matrices such that no two different continuously arranged eigenvalues meet of infinite order, then for each t_0 there exists $N_1 \in \mathbb{N}_{>0}$ such that the eigenvalues and eigenvectors of $s \mapsto A(t_0 \pm s^{N_1})$ can be parameterized C^L in s for s near 0.*
(h) *Let $T \subseteq \mathbb{R}^n$ be open and let $T \ni t \mapsto A(t)$ be a C^ω or C^Q-mapping of normal matrices. Let $K \subseteq T$ be compact. Then there exist a neighborhood W of K, and a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings each of which is either a local blow-up along a C^ω or C^Q submanifold or a local power substitution, such that the eigenvalues and the eigenvectors of $A(\pi_k(s))$ can be chosen C^ω or C^Q in s. Consequently, the eigenvalues and eigenvectors of $A(t)$ are locally special functions of bounded variation (SBV). If A is a family of Hermitian matrices, then we do not need power substitutions.*

The proof of the matrix proposition in case (e) and (f) is exactly as in [2, 7.6], using the polynomial proposition and properties of C^Q and C^L. Item (g) is exactly as in [19, 6.2], using the polynomial proposition and properties of C^L. Item (h) is proved in [20, 9.1, 9.6], see also [18].

Proof of Theorem. We have to prove parts (B), (C), (H), (I), (L), (M), and (O). So let C^M be any of C^ω, C^Q, C^L, or $C^{0,\alpha}$, and let $A(t)$ be normal. Let z be an eigenvalue of $A(t_0)$ of multiplicity N. We choose a simple closed C^1 curve γ in the resolvent set of $A(t_0)$ for fixed t_0 enclosing only z among all eigenvalues of $A(t_0)$. Since the global resolvent set is open, see the resolvent lemma, no eigenvalue of $A(t)$ lies on γ, for t near t_0. By the resolvent lemma, $A : T \to L((V, \| \quad \|_0), H)$ is C^M, thus also

$$t \mapsto -\frac{1}{2\pi i} \int_\gamma (A(t) - z)^{-1} \, dz =: P(t, \gamma) = P(t)$$

is a C^M mapping. Each $P(t)$ is a projection, namely onto the direct sum of all eigenspaces corresponding to eigenvalues of $A(t)$ in the interior of γ, with finite rank. Thus the rank must be constant: It is easy to see that the (finite) rank cannot fall locally, and it cannot increase, since the distance in $L(H,H)$ of $P(t)$ to the subset of operators of rank $\le N = \operatorname{rank}(P(t_0))$ is continuous in t and is either 0 or 1.

So for t in a neighborhood U of t_0 there are equally many eigenvalues in the interior of γ, and we may call them $\lambda_i(t)$ for $1 \le i \le N$ (repeated with multiplicity).

Now we consider the family of N-dimensional complex vector spaces $t \mapsto P(t)H \subseteq H$, for $t \in U$. They form a C^M Hermitian vector subbundle over U of $U \times H \to U$: For given t, choose $v_1, \dots, v_N \in H$ such that the $P(t)v_i$ are linearly independent and thus span $P(t)H$. This remains true locally in t. Now we use the Gram Schmidt orthonormalization procedure (which is C^ω) for the $P(t)v_i$ to obtain a local orthonormal C^M frame of the bundle.

Now $A(t)$ maps $P(t)H$ to itself; in a C^M local frame it is given by a normal $(N \times N)$-matrix parameterized C^M by $t \in U$.

Now all local assertions of the theorem follow:

(B) Use the matrix proposition, part (e).
(C) Use the matrix proposition, part (f).
(H) Use the matrix proposition, part (h), and note that in dimension 1 blow-ups are trivial.
(I) Use the matrix proposition, part (g).
(L,M) Use the matrix proposition, part (h), for $\mathbb{R}^n$.
(O) We use the following

Result. ([6], [5, VII.4.1]) *Let A, B be normal $(N \times N)$-matrices and let $\lambda_i(A)$ and $\lambda_i(B)$ for $i = 1, \dots, N$ denote the respective eigenvalues. Then*

$$\min_{\sigma \in \mathcal{S}_N} \max_j |\lambda_j(A) - \lambda_{\sigma(j)}(B)| \le C\|A - B\|$$

for a universal constant C with $1 < C < 3$. Here $\|\quad\|$ is the operator norm.

Finally, it remains to extend the local choices to global ones for the cases (B) and (C) only. There $t \mapsto A(t)$ is C^Q or C^L, respectively, which imply both C^∞, and no two different eigenvalues meet of infinite order. So we may apply [2, 7.8] (in fact we need only the end of the proof) to conclude that the eigenvalues can be chosen C^∞ on $T = \mathbb{R}$, uniquely up to a global permutation. By the local result above they are then C^Q or C^L. The same proof then gives us, for each eigenvalue $\lambda_i : T \to \mathbb{R}$ with generic multiplicity N, a unique N-dimensional smooth vector subbundle of $\mathbb{R} \times H$ whose fiber over t consists of eigenvectors for the eigenvalue $\lambda_i(t)$. In fact this vector bundle is C^Q or C^L by the local result above, namely the matrix proposition, part (e) or (f), respectively.

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