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Iterates and Hypoellipticity of Partial Differential Operators on Non-Quasianalytic Classes

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Abstract. Let P be a linear partial differential operator with constant coefficients. For a weight function ω and an open subset Ω of $\mathbb{R}^N$, the class $\mathcal{E}_{P,\{\omega\}}(\Omega)$ of Roumieu type involving the successive iterates of the operator P is considered. The completeness of this space is characterized in terms of the hypoellipticity of P. Results of Komatsu and Newberger-Zielezny are extended. Moreover, for weights ω satisfying a certain growth condition, this class coincides with a class of ultradifferentiable functions if and only if P is elliptic. These results remain true in the Beurling case $\mathcal{E}_{P,(\omega)}(\Omega)$.

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1. Introduction

In 1960, Komatsu [9], using tools introduced by Hörmander [7], characterized when a smooth function $f \in \mathcal{C}^\infty(\Omega)$ in an open subset $\Omega \subset \mathbb{R}^N$ is real analytic in terms of the successive iterates of a elliptic partial differential operator $P(D)$. In particular, given a elliptic differential operator $P(D)$ of order m, a function $f \in \mathcal{C}^\infty(\Omega)$ is real analytic if and only if for each compact subset $K \subset\subset \Omega$ there exists a constant $C > 0$ such that for each $j \in \mathbb{N}_0$

$$\|P^j(D)f\|_{2,K} \leq C^{j+1}(j!)^m,$$

where $P^j(D)$ is the jth iterate of $P(D)$, i.e., $P^j(D) = P(D)\underbrace{\circ\cdots\circ}_{j}P(D)$.

See also the theorem by Kotake–Narasimhan [11, Theorem 1].

In 1973, Newberger and Zielezny [19] treated this problem in the setting of the Gevrey classes: let $\mathcal{G}^d(\Omega)$ be the Gevrey class of exponent $d > 1$ and

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let $\mathcal{G}_P^d(\Omega)$ be the class of smooth functions in Ω such that for each $K \subset\subset \Omega$ there exists a constant $C > 0$ such that $\forall j \in \mathbb{N}_0$,

$$\|P^j(D)f\|_{2,K} \leq C^{j+1}(j!)^d,$$

then

$$\mathcal{G}^d(\Omega) = \mathcal{G}_P^{md}(\Omega)$$

whenever P is an elliptic polynomial with degree m.

Moreover, in case P is a hypoelliptic polynomial and Q is an arbitrary polynomial, it is proved the equivalence between the inequality $|Q(\xi)|^2 \leq C(1 + |P(\xi)|^2)^h$, $\forall \xi \in \mathbb{R}^N$ and the inclusion $\mathcal{G}_P^d(\Omega) \subset \mathcal{G}_Q^{dh}(\Omega)$.

This research is continued by several authors like Bolley et al. [1], Zanghirati [22–24] and Bouzar and Chaili [3]. Langenbruch utilized generalized Gevrey classes in connection with different problems, like boundary values of zero solutions of hypoelliptic differential operators [12,13], diametral dimension of solution spaces [14] and isomorphic classification [15].

The problem of the iterates consists in giving conditions on P in order to guarantee the equality $\mathcal{G}^d(\Omega) = \mathcal{G}_P^{md}(\Omega)$. In this paper we extend the results of Komatsu [9] and Newberger and Zielezny [19] to the setting of non-quasianalytic classes in the sense of Braun–Meise–Taylor [4]. The precise definition of the spaces will be given in Sect. 2, in which we recall the definition of the non-quasianalytic classes of ultradifferentiable functions $\mathcal{E}_{(\omega)}(\Omega)$ (Beurling type) and $\mathcal{E}_{\{\omega\}}(\Omega)$ (Roumieu type) in the sense of Braun et al. [4] and the classes $\mathcal{E}_{P,(\omega)}(\Omega)$ and $\mathcal{E}_{P,\{\omega\}}(\Omega)$ of ultradifferentiable functions with respect to the iterates of P.

In Sect. 3 we prove in Theorem 3.3 that $\mathcal{E}_{P,(\omega)}(\Omega)$ and $\mathcal{E}_{P,\{\omega\}}(\Omega)$, endowed with their natural topologies, are complete if and only if P is hypoelliptic. In case P is not hypoelliptic, a finer topology on $\mathcal{E}_{P,*}(\Omega)$ can be defined so that the space becomes complete. In spite of the importance of the completeness when dealing with functional analytic tools, as far as we know this is the first time that the completeness of these spaces is discussed. In Sect. 4 we extend the results of Komatsu [9] and Newberger–Zielezny [19] for weight functions ω verifying a growth condition considered by Bonet et al. [2]. For this type of weight functions, we characterize in Theorem 4.12 when $\mathcal{E}_{P,\{\omega\}}(\Omega)$ (respectively $\mathcal{E}_{P,(\omega)}(\Omega)$) coincides with the class of ultradifferentiable functions $\mathcal{E}_{\{\sigma\}}(\Omega)$ (respectively $\mathcal{E}_{(\sigma)}(\Omega)$), for $\sigma(t) = \omega(t^{\frac{1}{m}})$, where m denotes the degree of the polynomial P.

2. Notation and Preliminaries

2.1. Non-Quasianalytic Classes in the Sense of Braun–Meise–Taylor

We follow the point of view of Braun–Meise–Taylor (see [4]). A non-quasi-analytic *weight* function is an increasing continuous function $\omega : [0, \infty[\to [0, \infty[$ with the following properties:

(α) there exists $L \geq 0$ with $\omega(2t) \leq L(\omega(t) + 1)$ for all $t \geq 0$,

(β) $\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty$,

(γ) $\log(t) = o(\omega(t))$ as t tends to ∞,
(δ) $\varphi : t \to \omega(e^t)$ is convex.

For a weight function ω we define $\overline{\omega} : \mathbb{C}^N \to [0, +\infty[$ by $\overline{\omega}(z) := \omega(|z|)$ and again we denote this function by ω.

The condition (β) is called non-quasianalytic condition and it implies $\omega(t) = o(t)$. Moreover, this condition implies the existence of functions with compact support in the class of ultradifferentiable functions.

The *Young conjugate* $\varphi^* : [0, \infty[\to \mathbb{R}$ of φ is given by

$$\varphi^*(s) := \sup\{st - \varphi(t), t \geq 0\}.$$

There is no loss of generality to assume that ω vanishes on $[0, 1]$. Then φ^* has only non-negative values, it is convex and $\varphi^*(t)/t$ is increasing and tends to ∞ as $t \to \infty$ and $\varphi^{**} = \varphi$.

Lemma 2.1. *Given φ as above, we suppose that there exists $L \geq 0$ such that*

$$\varphi(x + 1) \leq L(1 + \varphi(x))$$

for all $x \in [0, \infty[$. Then, there exists $y_0 > 0$ such that

$$\varphi^*(y) - y \geq L\varphi^*\left(\frac{y}{L}\right) - L$$

for all $y \geq y_0$.

Lemma 2.2. *Given $\lambda > 0$ there exists a constant $C > 0$ (depending on λ) such that*

$$\exp\left(2\lambda\varphi^*\left(\frac{x+1}{2\lambda}\right)\right) \leq C \exp\left(\lambda\varphi^*\left(\frac{x}{\lambda}\right)\right) \quad \forall x > 0.$$

Proof. From the convexity of φ^* we obtain

$$\varphi^*\left(\frac{x}{2\lambda} + \frac{1}{2\lambda}\right) \leq \frac{1}{2}\varphi^*\left(\frac{x}{\lambda}\right) + \frac{1}{2}\varphi^*\left(\frac{1}{\lambda}\right).$$

The conclusion follows with $C = \exp\left(\lambda\varphi^*\left(\frac{1}{\lambda}\right)\right)$. □

Examples. The following functions are, after a change in some interval $[0, M]$, examples of weight functions:

(i) $\omega(t) = t^d$ for $0 < d < 1$.
(ii) $\omega(t) = (\log(1 + t))^s$, $s > 1$.
(iii) $\omega(t) = t(\log(e + t))^{-\beta}$, $\beta > 1$.
(iv) $\omega(t) = \exp(\beta(\log(1 + t))^\alpha)$, $0 < \alpha < 1$.

For a non-quasianalytic weight function ω, the spaces of ω-ultradifferentiable functions of Beurling and Roumieu case are defined as follows.

$$\mathcal{E}_{(\omega)}(\Omega) := \{f \in C^\infty(\Omega) : p_{K,\lambda}(f) < \infty, \text{ for every } K \subset\subset \Omega \text{ and every } \lambda > 0\},$$

and

$$\mathcal{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega) : \text{for every } K \subset\subset \Omega \text{ there exists } \lambda > 0 \text{ such that } p_{K,\lambda}(f) < \infty\},$$

where

$$p_{K,\lambda}(f) := \sup_{x\in K}\sup_{\alpha\in\mathbb{N}_0^N} |f^{(\alpha)}(x)|\exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right).$$

$\mathcal{E}_{(\omega)}(\Omega)$ is a Fréchet space, that is, a complete and metrizable locally convex space, while $\mathcal{E}_{\{\omega\}}(\Omega)$ is a PLS-space, that is, a projective limit of a sequence E_n, where each E_n is an inductive limit of Banach spaces with compact linking maps. In the case that $\omega(t) := t^d$ $(0 < d < 1)$, the corresponding Roumieu class is the Gevrey class with exponent $1/d$. In the limit case $d = 1$, not included in our setting, the corresponding Roumieu class $\mathcal{E}_{\{\omega\}}(\Omega)$ is the space of real analytic functions on Ω whereas the Beurling class $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ gives the entire functions. The elements of $\mathcal{E}_{(\omega)}(\Omega)$ (resp. $\mathcal{E}_{\{\omega\}}(\Omega)$) are called ultradifferentiable functions of Beurling type (resp. Roumieu type) in Ω.

If a statement holds in the Beurling and the Roumieu case then we will use the notation $\mathcal{E}_*(\Omega)$. It means that in all cases * can be replaced either by (ω) or $\{\omega\}$.

The corresponding classes of test functions are defined as follows: for a compact set K in $\mathbb{R}^N$, define

$$\mathcal{D}_*(K) := \{f \in \mathcal{E}_*(\mathbb{R}^N) : \mathrm{supp} f \subset K\},$$

endowed with the induced topology. In [4, Remark 3.2 (1) and Corollary 3.6 (1)] it is shown that $\mathcal{D}_*(K) \neq \{0\}$ is the strong dual of a nuclear Fréchet space (i.e., it is a (DFN)-space). For an open set Ω in $\mathbb{R}^N$, define

$$\mathcal{D}_*(\Omega) := \underset{K\subset\subset\Omega}{\underrightarrow{\mathrm{ind}}}\ \mathcal{D}_*(K).$$

According to [4](Proposition 4.7), the following inclusion

$$\mathcal{D}_*(\Omega) \hookrightarrow \mathcal{E}_*(\Omega)$$

is continuous with dense range. The following lemma is well-known, but it is not easy to find a precise reference.

Lemma 2.3. *The spaces $\mathcal{E}_*(\Omega)$ and $\mathcal{D}_*(\Omega)$ can be described with the $\mathcal{L}^2$-norm, i.e., we can replace $p_{K,\lambda}$ by the seminorms $q_{K,\lambda}(f) := \sup_{\alpha\in\mathbb{N}_0^N} \left\|f^{(\alpha)}\right\|_{2,K}$ $\exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right)$, where*

$$\|f\|_{2,K} = \left(\int_K |f|^2\right)^{\frac{1}{2}}.$$

Proof. We only need to prove that for each compact subset $K \subset\subset \Omega$ and $\lambda > 0$, there is other compact subset $L \subset\subset \Omega$, $\mu > 0$ and a constant $D > 0$ (depending only on K and λ) such that for all $f \in \mathcal{E}_*(\Omega)$,

$$p_{K,\lambda}(f) \leq D q_{L,\mu}(f).$$

We fix $K \subset\subset \Omega$ and $\lambda > 0$. We take $L \subset\subset \Omega$ such that $K \subset \overset{\circ}{L} \subset \Omega$. By the Sobolev Lemma, there exists a constant $C > 0$ such that

$$\sup_{x\in K} |f(x)| \leq C \sup_{|\beta|\leq N+1} \int_L |f^{(\beta)}| \ \forall f \in \mathcal{C}^\infty(\Omega).$$

Then,

$$\begin{aligned}
\sup_{x\in K} |f^{(\alpha)}(x)| &\leq C \sup_{|\beta|\leq N+1} \int_L |f^{(\alpha+\beta)}| \\
&= C \sup_{|\beta|\leq N+1} \int_L |f^{(\alpha+\beta)}| \exp\left(-\lambda\varphi^*\left(\frac{|\alpha|+|\beta|}{\lambda}\right)\right) \exp\left(\lambda\varphi^*\left(\frac{|\alpha|+|\beta|}{\lambda}\right)\right) \\
&\leq C_2 q_{L,\lambda}(f) \exp\left(\lambda\varphi^*\left(\frac{|\alpha|+N+1}{\lambda}\right)\right) \\
&\leq C_3 q_{L,\lambda}(f) \exp\left(\frac{\lambda}{2^{N+1}}\varphi^*\left(\frac{2^{N+1}}{\lambda}|\alpha|\right)\right).
\end{aligned}$$

where we have applied that φ^* is increasing and also Lemma 2.2 and Hölder's inequality.

As a consequence,

$$p_{K,\frac{\lambda}{2^{N+1}}}(f) \leq C_3 q_{L,\lambda}(f).$$

So, given $\lambda > 0$ we take $\mu = 2^{N+1}\lambda$. □

2.2. The Classes $\mathcal{E}_{P,\omega}(\Omega)$

We now consider smooth functions in an open set Ω verifying for each $j \in \mathbb{N}_0$

$$\|P^j(D)f\|_{2,K} \leq C \exp\left(-\lambda\varphi^*\left(\frac{j}{\lambda}\right)\right),$$

where K is a compact subset in Ω and $P^j(D)$ is the jth iterate of the partial differential operator $P(D)$, i.e, $P^j(D) = P(D)\underbrace{\circ\cdots\circ}_{j}P(D)$. If $j = 0$, then $P^0(D)f = f$.

Given a polynomial $P \in \mathbb{C}[z_1, \ldots, z_N]$ with degree m,

$$P(z) = \sum_{|\alpha|\leq m} a_\alpha z^\alpha,$$

the partial differential operator $P(D)$ is the following:

$$P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha, \text{ where } D^\alpha = \frac{1}{i}\partial^\alpha.$$

Let ω be a weight function. Given a polynomial P, an open set Ω of $\mathbb{R}^N$, a compact subset $K \subset\subset \Omega$ and $\lambda > 0$, we define

$$\mathcal{E}^{\lambda}_{P,\omega}(K) = \{f \in \mathcal{C}^{\infty}(K) : \|f\|_{K,\lambda} := \sup_{j\in\mathbb{N}_0} \|P^j(D)f\|_{2,K} \exp\left(-\lambda\varphi^*\left(\frac{j}{\lambda}\right)\right) < +\infty\}.$$

The spaces of ultradifferentiable functions with respect to the successive iterates of P are defined as follows:

Beurling case:

$$\mathcal{E}_{P,(\omega)}(\Omega) = \{f \in \mathcal{C}^{\infty}(\Omega) : \ \|f\|_{K,\lambda} < +\infty \text{ for each } K \subset\subset \Omega \text{ and } \lambda > 0\}.$$

It is endowed with the topology given by

$$\mathcal{E}_{P,(\omega)}(\Omega) := \operatorname*{proj}_{\overleftarrow{K\subset\subset\Omega}} \operatorname*{proj}_{\overleftarrow{\lambda>0}} \mathcal{E}^{\lambda}_{P,\omega}(K).$$

If $\{K_n\}_{n\in\mathbb{N}}$ is a compact exhaustion of Ω we have

$$\mathcal{E}_{P,(\omega)}(\Omega) = \operatorname*{proj}_{\overleftarrow{n\in\mathbb{N}}} \operatorname*{proj}_{\overleftarrow{k\in\mathbb{N}}} \mathcal{E}^{k}_{P,\omega}(K_n) = \operatorname*{proj}_{\overleftarrow{n\in\mathbb{N}}} \mathcal{E}^{n}_{P,\omega}(K_n).$$

This metrizable locally convex topology is defined by the fundamental system of seminorms $\{\|\cdot\|_{K_n,n}\}_{n\in\mathbb{N}}$.

Roumieu case:

$$\mathcal{E}_{P,\{\omega\}}(\Omega) = \{f \in \mathcal{C}^{\infty}(\Omega) : \text{ for each } K \subset\subset \Omega \text{ there is } \lambda > 0 \text{ such that } \|f\|_{K,\lambda} < +\infty\}.$$

Its topology is defined by

$$\mathcal{E}_{P,\{\omega\}}(\Omega) := \operatorname*{proj}_{\overleftarrow{K\subset\subset\Omega}} \operatorname*{ind}_{\overrightarrow{\lambda>0}} \mathcal{E}^{\lambda}_{P,\omega}(K).$$

3. Completeness of the Spaces $\mathcal{E}_{P,*}(\Omega)$

In order to extend the results of [19] we need to apply the Closed Graph Theorem and the Grothendieck's Factorization Theorem. So, it is important to know whether the spaces $\mathcal{E}_{P,*}(\Omega)$ are complete or not. In this section we show that the spaces $\mathcal{E}_{P,*}(\Omega)$ are not necessarily complete spaces. In fact, completeness is characterized in terms of the hypoellipticity of the polynomial P. Moreover, in case completeness fails, a finer topology on $\mathcal{E}_{P,*}(\Omega)$ is introduced so that the space becomes complete. This topology will be considered in Theorem 4.5.

Proposition 3.1. *Let Ω be an open subset of $\mathbb{R}^N$. If the space $\mathcal{E}_{P,*}(\Omega)$ is complete, then P is hypoelliptic.*

Proof. Proceeding by contradiction we assume that P is not hypoelliptic. We first analyze the case that $\Omega = \mathbb{R}^N$. Since P is not hypoelliptic, theorems [8, 11.1.5 and 10.1.25] imply the existence of a continuous function

$$u \in C(\mathbb{R}^N)\backslash C^{\infty}(\mathbb{R}^N) \ \text{ such that } P(D)u = 0.$$

Beurling case. We take a regularizing sequence $\{\rho_n\}$ with $\text{supp}\rho_n = \overline{B(0, \frac{1}{n})}$ and we show that $\{u * \rho_n\}$ is a Cauchy sequence in $\mathcal{E}_{P,(\omega)}(\mathbb{R}^N)$ which is not convergent.

It is clear that $u*\rho_n \in \mathcal{C}^\infty(\mathbb{R}^N)$ for all $n \in \mathbb{N}$. Moreover, $P(D)(u*\rho_n) = P(D)u * \rho_n = 0$. As a consequence, $P^j(D)(u * \rho_n) = 0$ if $j \neq 0$. Therefore,

$$\|u * \rho_n\|_{K,\lambda} \leq (m(K))^{\frac{1}{2}} \sup_{x\in K} |u * \rho_n(x)| < +\infty,$$

for all $K \subset\subset \mathbb{R}^N$ and for all $\lambda > 0$, i.e, $u * \rho_n \in \mathcal{E}_{P,(\omega)}(\mathbb{R}^N)$ for each $n \in \mathbb{N}$. In a similar way,

$$\|u * \rho_n - u * \rho_l\|_{K,\lambda} \leq (m(K))^{\frac{1}{2}} \sup_{x\in K} |u * (\rho_n - \rho_l)(x)|$$

which implies that $\{u * \rho_n\}$ is a Cauchy sequence in $\mathcal{E}_{P,(\omega)}(\mathbb{R}^N)$. If $\{u * \rho_n\}$ converges to $f \in \mathcal{E}_{P,(\omega)}(\mathbb{R}^N)$, then $\{u * \rho_n\}$ converges to f uniformly on the compact sets, hence $f = u$. This is a contradiction since u is not infinitely differentiable.

Roumieu case. The sequence $\{u * \rho_n\}$ constructed in the Beurling case is a Cauchy sequence in $\mathcal{E}_{P,\{\omega\}}(\mathbb{R}^N)$ since the inclusion map

$$\mathcal{E}_{P,(\omega)}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_{P,\{\omega\}}(\mathbb{R}^N)$$

is continuous. We see that $\{u*\rho_n\}$ does not have limit in $\mathcal{E}_{P,\{\omega\}}(\mathbb{R}^N)$. We call $\mathcal{L}^2_{loc}(\mathbb{R}^N) = \underset{\overleftarrow{K\subset\subset\mathbb{R}^N}}{\text{proj}} \{f \text{ mesurable} : \|f\|_{2,K} < \infty\}$, then the inclusion map

$$\mathcal{E}_{P,\{\omega\}}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^2_{loc}(\mathbb{R}^N)$$

is continuous. If $\{u*\rho_n\}$ converges to f in $\mathcal{E}_{P,\{\omega\}}(\mathbb{R}^N)$, then $\{u*\rho_n\}$ converges to f in $\mathcal{L}^2_{loc}(\mathbb{R}^N)$. However, for each $K \subset\subset \mathbb{R}^N$

$$\|u * \rho_n - u\|_{2,K} \leq (m(K))^{\frac{1}{2}} \sup_{x\in K} |u * \rho_n(x) - u(x)| \to 0 \text{ as } n \to +\infty.$$

Then $f = u$, which is a contradiction since u is not $\mathcal{C}^\infty$.

In the case that Ω is an arbitrary open subset of $\mathbb{R}^N$, we can assume (after a suitable translation if necessary) that

$$\exists u \in C(\Omega + B(0,1))\backslash C^\infty(\Omega) \ \text{ such that } P(D)u = 0.$$

Then the convolutions $u * \rho_n$ are defined on Ω and we can proceed as above. □

In order to prove the converse of Proposition 3.1, we introduce the following spaces. Let ω be a weight function. Given a polynomial P, an open set Ω of $\mathbb{R}^N$, a compact subset $K \subset\subset \Omega$ and $\lambda > 0$. Define

$$r_{K,\lambda}(f) := \sup_{j\in\mathbb{N}_0} \|P^j(D)f\|_{2,K} \exp\left(-\lambda\varphi^*\left(\frac{j}{\lambda}\right)\right)$$

and

$$\mathcal{L}^\lambda_{P,\omega}(K) := \{f \in \mathcal{L}^2(K) : \ P^j(D)f \in \mathcal{L}^2(K) \ \forall j \in \mathbb{N}_0, \ r_{K,\lambda}(f) < +\infty\}.$$

Beurling case:

$$\mathcal{L}_{P,(\omega)}(\Omega) = \{f \in \mathcal{L}^2_{loc}(\Omega) : \ f \in \mathcal{L}^{\lambda}_{P,\omega}(K) \text{ for each } K \subset\subset \Omega \text{ and } \lambda > 0\}.$$

If $\{K_n\}_{n\in\mathbb{N}}$ is a compact exhaustion of Ω this space is endowed with the topology given by

$$\mathcal{L}_{P,(\omega)}(\Omega) = \operatorname*{proj}_{\overleftarrow{n\in\mathbb{N}}} \operatorname*{proj}_{\overleftarrow{k\in\mathbb{N}}} \mathcal{L}^k_{P,\omega}(K_n) = \operatorname*{proj}_{\overleftarrow{n\in\mathbb{N}}} \mathcal{L}^n_{P,\omega}(K_n).$$

This metrizable locally convex topology is defined by the fundamental system of seminorms $\{r_{K_n,n}(\cdot)\}_{n\in\mathbb{N}}$. Using standard arguments it follows that $\mathcal{L}_{P,(\omega)}(\Omega)$ is a Fréchet space.

Roumieu case:

$$\mathcal{L}_{P,\{\omega\}}(\Omega) = \{f \in \mathcal{L}^2_{loc}(\Omega) : \text{ for each } K \subset\subset \Omega \text{ there is } \lambda > 0 \text{ such that } \ f \in \mathcal{L}^{\lambda}_{P,\omega}(K)\}.$$

This space is endowed with the following locally convex topology:

$$\mathcal{L}_{P,\{\omega\}}(\Omega) := \operatorname*{proj}_{\overleftarrow{K\subset\subset\Omega}} \operatorname*{ind}_{\overrightarrow{\lambda>0}} \mathcal{L}^{\lambda}_{P,\omega}(K).$$

Proposition 3.2. *The space of Roumieu type $\mathcal{L}_{P,\{\omega\}}(\Omega)$ is complete.*

Proof. We fix a compact subset K of Ω. If suffices to prove that the countable inductive limit of Banach spaces

$$X = \operatorname*{ind}_{\overrightarrow{n\in\mathbb{N}}} \mathcal{L}^n_{P,\omega}(K)$$

is complete. According to a Theorem of Mujica [18] (see also [20, Corollary 8.5.22 (ii)]), we only need to check that there is a Hausdorff locally convex topology s on X such that the unit ball of each $\mathcal{L}^n_{P,\omega}(K)$ is compact in (X, s). To do this it is enough to consider $E := \prod_{j\in\mathbb{N}_0}(\mathcal{L}^2(K_n), t_n)$, where t_n denotes the weak topology, and define s as the topology induced on X by the injective map

$$X \to E, \ f \mapsto \{P^j(D)f\}_{j\in\mathbb{N}_0}.$$

□

Theorem 3.3. *The space $\mathcal{E}_{P,*}(\Omega)$ is complete if and only if P is a hypoelliptic polynomial.*

Proof. By Theorem 3.2 it suffices to prove that the spaces $\mathcal{E}_{P,*}(\Omega)$ and $\mathcal{L}_{P,*}(\Omega)$ coincide algebraically and topologically. Let m denote the degree of P, let Ω' be an open subset of Ω and let $f \in \mathcal{L}_{P,*}(\Omega')$ be given. We use condition H5 of [6, Theorem 6.36]. Since $P(D)f \in \mathcal{L}^2_{loc}(\Omega') = \mathcal{H}^{loc}_0(\Omega')$, there exists $\delta > 0$ such that $f \in \mathcal{H}^{loc}_{m\delta}(\Omega')$. Analogously, $P^j(D)f \in \mathcal{L}^2_{loc}(\Omega')$ implies $f \in \mathcal{H}^{loc}_{jm\delta}(\Omega')$. As a consequence, $f \in \mathcal{L}_{P,*}(\Omega')$ implies $f \in \bigcap_{s>0} \mathcal{H}^{loc}_s(\Omega') = \mathcal{C}^\infty(\Omega')$ as a consequence of theorems [6, Theorems 6.7 and 6.13]. Denote by $(K_j)_j$ a fundamental sequence of compact sets in Ω such that each K_j is contained in the interior of K_{j+1}. Our argument above shows that the restriction maps $\mathcal{L}^{\lambda}_{P,\omega}(K_{j+1})$ into $\mathcal{E}^{\lambda}_{P,\omega}(K_j)$ continuously, from where the conclusion follows. □

In order to construct a complete finer locally convex topology on $\mathcal{E}_{P,*}(\Omega)$ in the general case, we fix a compact exhaustion $\{K_n\}$ of Ω and we consider the following system of seminorms on $\mathcal{E}_{P,(\omega)}(\Omega)$:

$$\{\|\cdot\|_n\}_{n\in\mathbb{N}} \cup \{p_n\}_{n\in\mathbb{N}}$$

where

$$\|f\|_n := \|f\|_{K_n,n} = \sup_{j\in\mathbb{N}_0} \|P^j(D)f\|_{2,K_n} \exp\left(-n\varphi^*\left(\frac{j}{n}\right)\right),$$

and $\{p_n\}_{n\in\mathbb{N}}$ is a fundamental system of seminorms of $\mathcal{E}(\Omega)$, i.e,

$$p_n(f) := \sup_{|\alpha|\leq n} \sup_{x\in K_n} |f^{(\alpha)}(x)|.$$

Then, $\{\max(\|\cdot\|_n, p_n)\}_{n\in\mathbb{N}}$ is a fundamental system of seminorms of a locally convex topology $\tau_{(\omega),\infty}$ on $\mathcal{E}_{P,(\omega)}(\Omega)$. The proof of the following result is standard.

Proposition 3.4. *Let Ω be an open subset of $\mathbb{R}^N$. For a weight function ω and a polynomial P, the space $(\mathcal{E}_{P,(\omega)}(\Omega), \tau_{(\omega),\infty})$ is a Fréchet space.*

In the Roumieu case, we consider the following topology: for $n \in \mathbb{N}$ and $K \subset\subset \Omega$, we endow $\mathcal{E}_{P,\omega}^{\frac{1}{n}}(K)$ with the fundamental system of seminorms

$$\left\{\max\left(\|\cdot\|_{K,\frac{1}{n}}, p_m\right)\right\}_{m\in\mathbb{N}}.$$

It is easy to see that $\mathcal{E}_{P,\omega}^{\frac{1}{n}}(K)$ is a Fréchet space. The topology $\tau_{\{\omega\},\infty}$ on $\mathcal{E}_{P,\{\omega\}}(\Omega)$ is defined by

$$(\mathcal{E}_{P,\{\omega\}}(\Omega), \tau_{\{\omega\},\infty}) = \underset{K\subset\subset\Omega}{\underleftarrow{\text{proj}}}\ \underset{n\in\mathbb{N}}{\underrightarrow{\text{ind}}}\ \mathcal{E}_{P,\omega}^{\frac{1}{n}}(K).$$

The space $\underset{n\in\mathbb{N}}{\underrightarrow{\text{ind}}}\ \mathcal{E}_{P,\omega}^{\frac{1}{n}}(K)$ is an (LF)-space, i.e, a countable inductive limit of Fréchet spaces.

Proposition 3.5. *Let Ω be an open subset of $\mathbb{R}^N$. For a weight function ω and a polynomial P, the space $(\mathcal{E}_{P,\{\omega\}}(\Omega), \tau_{\{\omega\},\infty})$ is complete.*

The proof requires the following result for (LF)-spaces.

Definition 3.6. Let $X = \underset{n\in\mathbb{N}}{\underrightarrow{\text{ind}}}\ X_n$ be an (LF)-space. X is called boundedly stable if on each set which is bounded in some X_n all but finitely many of the step topologies coincide.

Next theorem due to Wengenroth follows from Theorem 6.4 (page 112) and Corollary 6.4 of [21, p. 113]:

Theorem 3.7. *Let* $X = \underset{n\in\mathbb{N}}{\underrightarrow{\operatorname{ind}}} X_n$ *be an (LF)-space and* $\{\|\cdot\|_{n,m}\}_{m\in\mathbb{N}}$ *a fundamental system of seminorms of* X_n*. If* X *is boundedly stable and satisfies the condition (P3*), i.e,*

$$\forall n\ \exists l \geq n\ \forall m \geq l\ \exists N \in \mathbb{N}\ \forall M \in \mathbb{N}\ \exists K \in \mathbb{N}, S > 0\ \forall x \in X_n$$
$$\|x\|_{l,M} \leq S(\|x\|_{m,K} + \|x\|_{n,N}),$$

then X *is complete.*

Proof of Proposition 3.5. It is enough to show that for each $K \subset\subset \Omega$, the space $X = \underset{n\in\mathbb{N}}{\underrightarrow{\operatorname{ind}}} \mathcal{E}^{\frac{1}{n}}_{P,\omega}(K)$ is complete. We denote $X_n = \mathcal{E}^{\frac{1}{n}}_{P,\omega}(K)$ and $\{\|\cdot\|_{n,m}\}_{m\in\mathbb{N}} = \{\max(\|\cdot\|_{\frac{1}{n}}, p_m)\}_{m\in\mathbb{N}}$. To see that X is complete, we show that X verifies the hypothesis of Theorem 3.7, i.e, X is boundedly stable and satisfies the condition (P3*). We apply Lemma 2.1 and the condition (α) of weight function to get a natural number L and $y_0 > 0$ such that

$$\varphi^*(y) - y \geq L\varphi^*\left(\frac{y}{L}\right) - L \quad \text{for each } y \geq y_0.$$

That implies

$$\frac{\exp\left(-\frac{1}{nL}\varphi^*(jnL)\right)}{\exp\left(-\frac{1}{n}\varphi^*(jn)\right)} \longrightarrow 0 \text{ as } j \to +\infty. \tag{1}$$

To see that X is boundedly stable it suffices to prove that any bounded set $\mathcal{B}$ in X_n is relatively compact in X_{nL}. Since $\mathcal{E}(\Omega)$ is a Montel space, we only need to show that if $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in X_n and converges to 0 in $\mathcal{E}(\Omega)$, then

$$\{f_k\}_{k\in\mathbb{N}} \text{ converges to 0 in } X_{nL}, \text{ i.e, } \|f_k\|_{K,\frac{1}{nL}} \longrightarrow 0 \text{ si } k \to +\infty.$$

Since $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in X_n, there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$, $\|f_k\|_{K,\frac{1}{n}} \leq C$. Let $\eta > 0$, in view of (1) there exists $j_0 \in \mathbb{N}_0$ such that

$$\|P^j(D)(f_k)\|_{2,K} \exp\left(-\frac{1}{Ln}\varphi^*(jnL)\right)$$
$$\leq \eta\left(\sup_{i\in\mathbb{N}_0} \|P^i(D)(f_k)\|_{2,K} \exp\left(-\frac{1}{n}\varphi^*(in)\right)\right) \leq \eta\, C \quad \text{if } j > j_0.$$

Therefore,

$$\|f_k\|_{K,\frac{1}{Ln}} \leq \max_{i=0,1,\ldots,j_0}\left(\eta C, \|P^i(D)(f_k)\|_{2,K} e^{-\frac{1}{Ln}\varphi^*(iLn)}\right).$$

Since $\{f_k\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{E}(\Omega)$ we have $\|f_k\|_{K,\frac{1}{Ln}} \leq \eta C$ if k is enough large. The property (P3*) is satisfied. Indeed, take $l = n$, $N = 1$, $K = M$ and $S = 1$, then

$$\forall n\ \forall m \geq n\ \forall M \in \mathbb{N}\ \forall x \in X_n$$
$$\|x\|_{n,M} \leq \|x\|_{m,M} + \|x\|_{n,1},$$

since

$$\|x\|_{n,M} = \max\left(\|x\|_{K,\frac{1}{n}}, p_M(x)\right) = \begin{cases} \|x\|_{K,\frac{1}{n}} \le \|x\|_{n,1} \le \|x\|_{m,M} + \|x\|_{n,1} \\ p_M(x) \le \|x\|_{m,M} \le \|x\|_{m,M} + \|x\|_{n,1} \end{cases}$$

□

4. Hypoelliptic and Elliptic Polynomials and the Growth of $\mathcal{E}_{P,*}(\Omega)$

The following result asserts that the class of ultradifferentiable functions $\mathcal{E}_\omega(\Omega)$ is always contained in $\mathcal{E}_{P,\sigma}(\Omega)$ where σ depends on ω and the degree of P. Let ω be a weight function and $m \ge 1$ the degree of P, it is easy to prove that $\sigma(t) := \omega(t^{\frac{1}{m}})$ is also a weight function. Moreover, $\varphi^*_\sigma(x) = \varphi^*_\omega(mx)$. We will denote φ^*_ω simply by φ^*.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be an open subset. For a weight function ω and a polynomial P with degree m, the inclusion $\mathcal{E}_{*(t)}(\Omega) \subseteq \mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega)$ holds and the inclusion map is continuous.*

Notation. $\mathcal{E}_{*(t)}(\Omega) \subseteq \mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega)$ means that both inclusions $\mathcal{E}_{(\omega)}(\Omega) \subseteq \mathcal{E}_{P,(\sigma)}(\Omega)$ and $\mathcal{E}_{\{\omega\}}(\Omega) \subseteq \mathcal{E}_{P,\{\sigma\}}(\Omega)$ hold.

We need the following technical lemma.

Lemma 4.2. *Given a constant $C \ge 1$ and a weight function ω, there exist two constants A and B (depending on C and ω) such that for all $j \in \mathbb{N}$ and $\lambda > 0$,*

$$C^j \exp\left(\lambda\varphi^*\left(\frac{j}{\lambda}\right)\right) \le \exp\left(\lambda\frac{A}{B}\right)\exp\left(\frac{\lambda}{B}\varphi^*\left(\frac{B}{\lambda}j\right)\right).$$

Proof. From de definition of Young conjugate we get

$$\exp\left(\lambda\varphi^*(\frac{j}{\lambda})\right) = \sup_{s\ge 1} s^j \exp\left(-\lambda\omega(s)\right). \tag{2}$$

Choose $l \in \mathbb{N}$ such that $2^l \ge C$. By condition (α), there exist A and B such that

$$\omega(t) = \omega\left(C\frac{t}{C}\right) \le \omega\left(2^l\frac{t}{C}\right) \le A + B\omega\left(\frac{t}{C}\right). \tag{3}$$

Then,

$$\begin{aligned} C^j \exp\left(\lambda\varphi^*\left(\frac{j}{\lambda}\right)\right) &\overset{(2)}{=} \sup_{s\ge 1}(sC)^j \exp\left(-\lambda\omega(s)\right) \\ &= \sup_{t\ge C} t^j \exp\left(-\lambda\omega\left(\frac{t}{C}\right)\right) \\ &\overset{(3)}{\le} \sup_{t\ge 1} t^j \exp\left(\lambda\left(\frac{A}{B} - \frac{\omega(t)}{B}\right)\right) \end{aligned}$$

$$= \exp\left(\lambda\frac{A}{B}\right)\sup_{t\geq 1} t^j \exp\left(-\lambda\frac{1}{B}\omega(t)\right)$$
$$\overset{(2)}{=} \exp\left(\lambda\frac{A}{B}\right)\exp\left(\frac{\lambda}{B}\varphi^*\left(\frac{B}{\lambda}j\right)\right).$$

□

Proof of Theorem 4.1. We set $P(z) = \sum_{|\alpha|\leq m} a_\alpha z^\alpha$, $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ and $M := \max\{|a_\alpha| : |\alpha| \leq m\}$. Choose F large enough such that for each $j \in \mathbb{N}$, $M^j(mj)^N \leq F^j$ and take the constants A and B of Lema 4.2 such that for each $j \in \mathbb{N}$,

$$F^j \exp\left(\lambda\varphi^*\left(\frac{j}{\lambda}\right)\right) \leq \exp\left(\lambda\frac{A}{B}\right)\exp\left(\frac{\lambda}{B}\varphi^*\left(\frac{B}{\lambda}j\right)\right). \tag{4}$$

To see $\mathcal{E}_{(\omega(t))}(\Omega) \subseteq \mathcal{E}_{P,\left(\omega\left(t^{\frac{1}{m}}\right)\right)}(\Omega)$, take $f \in \mathcal{E}_{(\omega(t))}(\Omega)$ and we fix $K \subset\subset \Omega$, $\lambda > 0$ and $j \geq 1$. We observe that the polynomial of the operator $P^j(D)$ is $P^j = P\underbrace{\cdots}_{j}P$ has degree mj. Moreover, there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^N$,

$$\|f^{(\alpha)}\|_{2,K} \leq C\exp\left(B\lambda\varphi^*\left(\frac{|\alpha|}{B\lambda}\right)\right). \tag{5}$$

We can choose $C = p_{K,B\lambda}(f)$. Hence,

$$\begin{aligned}\|P^j(D)f\|_{2,K} &\leq M^j p_{K,B\lambda}(f)\sum_{|\gamma|\leq mj}\exp\left(B\lambda\varphi^*\left(\frac{|\gamma|}{B\lambda}\right)\right)\\ &\leq p_{K,B\lambda}(f)M^j(mj)^N\exp\left(B\lambda\varphi^*\left(\frac{mj}{B\lambda}\right)\right)\\ &\leq p_{K,B\lambda}(f)F^j\exp\left(B\lambda\varphi^*\left(\frac{mj}{B\lambda}\right)\right)\\ &\overset{(4)}{\leq} p_{K,B\lambda}(f)\exp(\lambda A)\exp\left(\lambda\varphi^*\left(\frac{mj}{\lambda}\right)\right).\end{aligned}$$

Therefore,

$$\|P^j(D)f\|_{2,K}\exp\left(-\lambda\varphi^*\left(\frac{mj}{\lambda}\right)\right) \leq p_{K,B\lambda}(f)\exp(\lambda A). \tag{6}$$

It has been proved $\mathcal{E}_{(\omega(t))}(\Omega) \subseteq \mathcal{E}_{P,(\omega(t^{\frac{1}{m}}))}(\Omega)$ and this inclusion is continuous. This settles the Beurling case.

In the Roumieu case, let $f \in \mathcal{E}_{\{\omega(t)\}}(\Omega)$ be given. For each $K \subset\subset \Omega$ there exists $\lambda > 0$ such that $f \in \mathcal{E}^\lambda_{\omega(t)}(K)$. Proceeding as above $f \in \mathcal{E}^{\frac{\lambda}{B}}_{P,\omega\left(t^{\frac{1}{m}}\right)}(K) \subset \mathcal{E}_{P,\left\{\omega\left(t^{\frac{1}{m}}\right)\right\}}(\Omega)$. Now, from (6) we get the continuity of the inclusion

$$\mathcal{E}^\lambda_{\omega(t)}(K) \hookrightarrow \mathcal{E}^{\frac{\lambda}{B}}_{P,\omega\left(t^{\frac{1}{m}}\right)}(K)$$

and the theorem follows. □

Proposition 4.3. *$\mathcal{E}_*(\Omega)$ is a dense subspace of $\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega)$.*

Proof. First, we suppose $\Omega = \mathbb{R}^N$.

Beurling case. Applying Lema 2.1 there exist $L \geq 1$ and $y_0 > 0$ such that

$$\varphi^*(y) - y \geq L\varphi^*\left(\frac{y}{L}\right) - L \text{ for each } y \geq y_0.$$

This implies

$$\frac{\exp\left(-\lambda\varphi^*\left(\frac{mj}{\lambda}\right)\right)}{\exp\left(-\lambda L\varphi^*\left(\frac{mj}{\lambda L}\right)\right)} \longrightarrow 0 \quad \text{if } j \to +\infty. \tag{7}$$

Using Theorem 4.1

$$\mathcal{D}_{(\omega)}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_{P,\left(\omega\left(t^{\frac{1}{m}}\right)\right)}(\mathbb{R}^N).$$

Let $f \in \mathcal{E}_{P,(\omega)}(\mathbb{R}^N)$ be given. Proceeding as in [4, Lema 3.8] we take a regularizing sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ and prove that $f * \rho_n \in \mathcal{E}_{(\omega)}(\mathbb{R}^N)$. To get this aim we fix a compact subset K of $\mathbb{R}^N$ and $\lambda > 0$, then

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{N}_0^N} \left\|(f * \rho_n)^{(\alpha)}\right\|_{2,K} \exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right)\\
&\quad= \sup_{\alpha\in\mathbb{N}_0^N} \left\|f * \rho_n^{(\alpha)}\right\|_{2,K} \exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right)\\
&\quad= \sup_{\alpha\in\mathbb{N}_0^N} \left\|\int_{\mathrm{supp}\rho_n} f(\cdot - y)\rho_n^{(\alpha)}(y)dy\right\|_{2,K} \exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right)\\
&\quad\leq \sup_{\alpha\in\mathbb{N}_0^N} \sup_{y\in\mathrm{supp}\rho_n} \left|\rho_n^{(\alpha)}(y)\right| \exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right) \left\|\int_{\mathrm{supp}\rho_n} |f(\cdot - y)|dy\right\|_{2,K}\\
&\quad< \infty.
\end{aligned}$$

Now, it is enough to show that for each compact subset K of $\mathbb{R}^N$, $\lambda > 0$ and $p \in \mathbb{N}$

$$\|f - f * \rho_n\|_{K,\lambda} \longrightarrow 0 \quad \text{if } n \to \infty.$$

Given $\eta > 0$, using (7) and (6) there exists $j_0 \in \mathbb{N}_0$ such that

$$\begin{aligned}
&\|P^j(D)(f - f * \rho_n)\|_{2,K} \exp\left(-\lambda\varphi^*\left(\frac{mj}{\lambda}\right)\right)\\
&\quad\leq \left(\sup_{i\in\mathbb{N}_0} \|P^i(D)(f - f * \rho_n)\|_{2,K} \exp\left(-L\lambda\varphi^*\left(\frac{mi}{L\lambda}\right)\right)\right) \frac{e^{-\lambda\varphi^*\left(\frac{mj}{\lambda}\right)}}{e^{-L\lambda\varphi^*\left(\frac{mj}{L\lambda}\right)}}\\
&\quad\leq (\|f\|_{K,L\lambda} + p_{K,L\lambda B}(f * \rho_n)) \frac{e^{-\lambda\varphi^*\left(\frac{mj}{\lambda}\right)}}{e^{-L\lambda\varphi^*\left(\frac{mj}{L\lambda}\right)}} < \eta \quad \text{if } j > j_0.
\end{aligned}$$

As a consequence,

$$\|f - f * \rho_n\|_{K,\lambda} \leq \max_{i=0,1,\ldots,j_0} \left(\eta, \|P^i(D)(f - f * \rho_n)\|_{2,K} e^{-\lambda\varphi^*\left(\frac{mi}{\lambda}\right)}\right).$$

Hence

$$\|f - f * \rho_n\|_{K,\lambda} \leq \eta \quad \text{if } n \text{ is large enough.}$$

Roumieu case. Let $f \in \mathcal{E}_{P,\left\{\omega\left(t^{\frac{1}{m}}\right)\right\}}(\mathbb{R}^N)$. We take $\{\rho_n\}_{n\in\mathbb{N}} \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$ as above and fix a compact subset K of $\mathbb{R}^N$. Since $f \in \mathcal{E}_{P,\left\{\omega\left(\frac{1}{m}\right)\right\}}(\mathbb{R}^N)$, there exists $\lambda > 0$ such that $f \in \mathcal{E}^{\lambda}_{P,\omega}(K)$. Proceeding as above $f * \rho_n \in \mathcal{E}^{\lambda}_{P,\omega}(K)$ and $f * \rho_n$ converges to f in $\mathcal{E}^{\frac{\lambda}{L}}_{P,\omega}(K)$. As a consequence, $f * \rho_n$ converges to f in $\mathcal{E}_{P,\left\{\omega\left(\frac{1}{m}\right)\right\}}(\mathbb{R}^N)$.

To finish, we suppose that Ω is an arbitrary open subset of $\mathbb{R}^N$. We fix a compact subset K and $\lambda > 0$ and we choose a compact subset L such that $K \subset \mathring{L} \subset L \subset \Omega$. Let $f \in \mathcal{E}_{P,\left(\omega\left(t^{\frac{1}{m}}\right)\right)}(\Omega)$. We define $\widetilde{f} = f$ on L and $\widetilde{f} = 0$ in other case. Then $\widetilde{f} \in \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ and according to [4, Proposition 6.4] $\rho_n * \widetilde{f}$ is an ultradifferentiable function which coincides in $f * \rho_n$ on K if n is large enough. Then, given $\varepsilon > 0$ there exists n_0 such that $\|f - \rho_{n_0} * \widetilde{f}\|_{K,\lambda} < \varepsilon$.From this the conclusion follows. □

As a consequence of the former results we get:

1. $\mathcal{D}_*(\Omega)$ is a dense subset of $\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega)$.
2. The functions in $\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega)$ with compact support are a dense subset of $\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega)$.

According to a well known result of Hörmander (see [7, Theorem 3.2]), if P is an hypoelliptic polynomial and Q any polynomial, there are constants $h > 0$ and $C > 0$ such that

$$|Q(\xi)|^2 \leq C(1 + |P(\xi)|^2)^h, \quad \forall \xi \in \mathbb{R}^N.$$

Moreover, the smaller h is a rational number.

For an hypoelliptic polynomial P and an arbitrary polynomial Q we show the equivalence between the inequality $|Q(\xi)|^2 \leq C(1 + |P(\xi)|^2)^h$ and the inclusion $\mathcal{E}_{P,*(t)}(\Omega) \subseteq \mathcal{E}_{Q,*(t^{\frac{1}{h}})}(\Omega)$ for weight functions satisfying the following growth *condition B-M-M:*

There exits a constant $H \geq 1$ such that for all $t \geq 0$

$$2\omega(t) \leq \omega(Ht) + H. \tag{8}$$

This condition is considered by Bonet et al. [2] in order to characterize those weight functions ω such that there exists a sequence $\{M_p\}$ with the property that the class of ultradifferentiable functions in the sense of Braun–Meise–Taylor associated to the weight ω coincides with the non-quasianalytic class in the sense of Komatsu (see [10]) defined by the sequence $\{M_p\}$. Gevrey weights verify this condition.

We will prove the following theorems.

Theorem 4.4. *Let P and Q be hypoelliptic polynomials with $h > 0$ and $C > 0$ such that $|Q(\xi)|^2 \leq C(1+|P(\xi)|^2)^h$, $\forall \xi \in \mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be an open subset and ω a weight function, then there exits m_0 such that if $m \geq m_0$*

$$\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega) \subseteq \mathcal{E}_{Q,*\left(t^{\frac{1}{mh}}\right)}(\Omega),$$

and the inclusion map is continuous.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, ω a weight function satisfying the condition B-M-M, cf. (8). Let P be an hypoelliptic polynomial and let Q be an arbitrary polynomial such that $\mathcal{E}_{P,*(t)}(\Omega) \subseteq \mathcal{E}_{Q,*\left(t^{\frac{1}{h}}\right)}(\Omega)$ for some $h \geq 1$. Then*

$$|Q(\xi)|^2 \leq C(1+|P(\xi)|^2)^h, \quad \forall \xi \in \mathbb{R}^N.$$

In the proofs we need two technical lemmata.

Lemma 4.6. *Let ω be a weight function, $m \geq 1$ and $\gamma,\mu > 0$ such that $\gamma \leq \mu m$. Then for each $k \in \mathbb{N}$, $i = 0, 1, \ldots, k$ and $\lambda > 0$,*

$$k^{i\gamma} \exp\left(\lambda\varphi^*\left(\frac{(k-i)\mu m}{\lambda}\right)\right) \leq \exp\left(\lambda\omega(k)\right) \exp\left(\lambda\varphi^*\left(\frac{k\mu m}{\lambda}\right)\right).$$

As a consequence, since $\omega(t) = o(t)$ there exists $C > 0$ (depending on ω) such that

$$k^{i\gamma} \exp\left(\lambda\varphi^*\left(\frac{(k-i)\mu m}{\lambda}\right)\right) \leq C\exp(\lambda Ck) \exp\left(\lambda\varphi^*\left(\frac{k\mu m}{\lambda}\right)\right).$$

Proof.

$$\begin{aligned} k^{i\gamma} \exp\left(\lambda\varphi^*\left(\frac{(k-i)\mu m}{\lambda}\right)\right) \ \exp\left(-\lambda\varphi^*\left(\frac{k\mu m}{\lambda}\right)\right) \\ = k^{i\gamma} \exp\left(-\lambda\left(\varphi^*\left(\frac{k\mu m}{\lambda}\right) - \varphi^*\left(\frac{(k-i)\mu m}{\lambda}\right)\right)\right). \end{aligned}$$

Since φ^* is a convex function, we have $\varphi^*(A) - \varphi^*(B) \geq \varphi^*(A-B)$ if $0 \leq B < A$. Therefore,

$$k^{i\gamma} \exp\left(\lambda\varphi^*\left(\frac{(k-i)\mu m}{\lambda}\right)\right) \ \exp\left(-\lambda\varphi^*\left(\frac{k\mu m}{\lambda}\right)\right)$$

is less than or equal to

$$\begin{aligned} k^{i\mu m} \exp\left(-\lambda\varphi^*\left(\frac{i\mu m}{\lambda}\right)\right) &= \exp\left(\lambda\left(\frac{i\mu m\ln(k)}{\lambda} - \sup_{s\geq 0}\left\{\frac{i\mu m s}{\lambda} - \omega(e^s)\right\}\right)\right) \\ &\leq \exp\left(\lambda\omega(k)\right) \end{aligned}$$

taking $s = \ln(k)$ in the last inequality. □

The next lemma is stated in [5, 1.4] without a proof. We include a proof here for the sake of completeness.

Lemma 4.7. *Let $h > 0$,$\lambda > 0$ be positive constants, then for all $t \geq 1$*

$$(1)\ \sup_{j\in\mathbb{N}_0} t^j \exp\left(-\lambda\varphi^*\left(\frac{hj}{\lambda}\right)\right) \leq \exp\left(\lambda\omega\left(t^{\frac{1}{h}}\right)\right)$$

and

$$(2)\ \sup_{j\in\mathbb{N}_0} t^j \exp\left(-\lambda\varphi^*\left(\frac{hj}{\lambda}\right)\right) \geq \frac{1}{t}\exp\left(\lambda\omega\left(t^{\frac{1}{h}}\right)\right).$$

Proof. Proof of (1):

$$\sup_{j\in\mathbb{N}_0} t^j \exp\left(-\lambda\varphi^*\left(\frac{hj}{\lambda}\right)\right) = \sup_{j\in\mathbb{N}_0} t^j \exp\left(-\lambda \sup_{s\geq 0}\left\{\frac{hjs}{\lambda} - \omega(e^s)\right\}\right)$$

We take $s = \frac{\ln(t)}{h} \geq 0$, therefore

$$\sup_{j\in\mathbb{N}_0} t^j \exp\left(-\lambda\varphi^*\left(\frac{hj}{\lambda}\right)\right) \leq \sup_{j\in\mathbb{N}_0} t^j t^{-j} \exp\left(\lambda\omega\left(t^{\frac{1}{h}}\right)\right) = \exp\left(\lambda\omega\left(t^{\frac{1}{h}}\right)\right).$$

In order to show (2), having in mind $j \leq s < j+1$:

$$\begin{aligned}\sup_{j\in\mathbb{N}_0} t^j \exp\left(-\lambda\varphi^*\left(\frac{hj}{\lambda}\right)\right) &= \sup_{j\in\mathbb{N}_0} \exp\left(\lambda\left(\frac{j}{\lambda}\ln(t) - \varphi^*\left(\frac{hj}{\lambda}\right)\right)\right)\\ &\geq \sup_{s\geq 0} \exp\left(\lambda\left(\frac{s}{\lambda}\ln(t) - \varphi^*\left(\frac{hs}{\lambda}\right)\right) - \ln(t)\right)\\ &= \frac{1}{t}\exp\left(\lambda\varphi^{**}\left(\frac{\ln(t)}{h}\right)\right)\\ &= \frac{1}{t}\exp\left(\lambda\omega\left(t^{\frac{1}{h}}\right)\right)\bigg).\end{aligned}$$

□

Proof of Theorem 4.4. Using [7, Theorem 3.2] we can suppose $h = \frac{\mu}{\nu}$, where $\mu, \nu \in \mathbb{N}$. Then, for some constant $C > 0$,

$$|Q^\nu(\xi)|^2 \leq C(1 + |P^\mu(\xi)|^2) \quad \xi \in \mathbb{R}^N. \tag{9}$$

Let Ω' be an open subset relatively compact in Ω. Given $\delta > 0$, we set $\Omega'_\delta := \{x \in \Omega' : d(x, \partial\Omega') > \delta\}$ where $\partial\Omega'$ is the boundary of Ω'. The condition (9) and [7, Theorem 4.2] imply that there exist $\gamma > 0$ and C (which depends of P,Q and the diameter of Ω') such that for each $s \geq 0$ and $t > 0$,

$$\sup_{0<\tau\leq t} \tau^\gamma \|Q^\nu(D)f\|_{2,\Omega'_{s+\tau}} \leq C\left\{\sup_{0<\tau\leq t} \tau^\gamma \|P^\mu(D)f\|_{2,\Omega'_{s+\tau}} + \|f\|_{2,\Omega'_s}\right\},$$
$$f \in \mathcal{C}^\infty(\Omega).$$

Moreover, $\gamma = \frac{\mu}{b}$ where $0 < b \leq 1$. Hence,

$$\|Q^\nu(D)f\|_{2,\Omega'_{s+t}} \leq C\left\{\|P^\mu(D)f\|_{2,\Omega'_s} + t^{-\gamma}\|f\|_{2,\Omega'_s}\right\}, \quad f \in \mathcal{C}^\infty(\Omega).$$

Let $k \in \mathbb{N}$, $k \geq 1$, $\delta > 0$. Applying repeatedly the last inequality to $s = \left(1 - \frac{i}{k}\right)\delta$, $t = \frac{\delta}{k}$ and $Q^{(k-i)\nu}f$ for $i = 0, 1, \ldots, k$ we obtain (see [19, Theorem 1])

$$\|Q^{k\nu}(D)f\|_{2,\Omega'_{2\delta}} \leq C^k \sum_{i=0}^k \binom{k}{i}\left(\frac{k}{\delta}\right)^{i\gamma} \|P^{(k-i)\mu}(D)f\|_{2,\Omega'}\ f \in \mathcal{C}^\infty(\Omega). \tag{10}$$

For $k = 0$ this inequality remains true.

On the other hand, for $i = 0, 1, \ldots, \nu - 1$ we have

$$|Q^i(\xi)|^2 \le 1 + |Q^\nu(\xi)|^2, \quad \xi \in \mathbb{R}^N.$$

Again, [7, Theorem 4.2] implies $\forall i = 0, 1, \ldots, \nu - 1$,

$$\|Q^i(D)f\|_{2,\Omega'_{2\delta}} \le C' \left\{ \|Q^\nu(D)f\|_{2,\Omega'_\delta} + \|f\|_{2,\Omega'_\delta} \right\}, \quad f \in \mathcal{C}^\infty(\Omega). \tag{11}$$

where C' depends of δ.

For $j \in \mathbb{N}_0$, we put $j = k\nu + l$, k,$l \in \mathbb{N}_0$, $l \le \nu - 1$. Applying (11) to $i = l$ and $Q^{k\nu} f$,

$$\|Q^j(D)f\|_{2,\Omega'_{2\delta}} \le C' \left\{ \|Q^{(k+1)\nu}(D)f\|_{2,\Omega'_\delta} + \|Q^{k\nu}(D)f\|_{2,\Omega'_\delta} \right\}, \quad f \in \mathcal{C}^\infty(\Omega). \tag{12}$$

As a consequence, using (10) we obtain

$$\begin{aligned} \|Q^j(D)f\|_{2,\Omega'_{2\delta}} \le\ & C' C^{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i} \left(\frac{k+1}{\delta}\right)^{i\gamma} \|P^{(k+1-i)\mu}(D)f\|_{2,\Omega'} \\ & + C' C^k \sum_{i=0}^{k} \binom{k}{i} \left(\frac{k}{\delta}\right)^{i\gamma} \|P^{(k-i)\mu}(D)f\|_{2,\Omega'}, \quad f \in \mathcal{C}^\infty(\Omega). \end{aligned} \tag{13}$$

We set $m_0 := \frac{1}{b} \ge 1$. Let Ω be an open subset of $\mathbb{R}^N$ and let $m \ge m_0$, $f \in \mathcal{E}_{P,\left(\omega\left(t^{\frac{1}{m}}\right)\right)}(\Omega)$. We fix $K \subset\subset \Omega$ and $\lambda > 0$. There exist Ω' and $\delta > 0$ such that $K \subset \Omega'_{2\delta} \subset \overline{\Omega'} \subset\subset \Omega$. We call E the constant of the Lemma 4.6 such that for each $k \in \mathbb{N}$, $i = 0, 1, \ldots, k$ and $\lambda > 0$,

$$k^{i\gamma} \exp\left(\lambda\varphi^*\left(\frac{(k-i)\mu m}{\lambda}\right)\right) \le \exp(\lambda E k) \exp\left(\lambda\varphi^*\left(\frac{k\mu m}{\lambda}\right)\right).$$

Denote $F := 2C\left(\frac{1}{\delta}\right)^\gamma e^{\lambda E}$. We can suppose $F \ge 1$ and $\frac{1}{\delta} > 1$. We take A and B the constants of Lemma 4.2 such that $\forall k \in \mathbb{N}$ and $\forall \lambda > 0$,

$$F^k \exp\left(\lambda\varphi^*\left(\frac{k}{\lambda}\right)\right) \le \exp\left(\lambda\frac{A}{B}\right) \exp\left(\frac{\lambda}{B}\varphi^*\left(\frac{B}{\lambda}k\right)\right).$$

We apply the inequality (13) and $f \in \mathcal{E}_{P,\left(\omega\left(t^{\frac{1}{m}}\right)\right)}(\Omega)$ to get

$$\begin{aligned} \|Q^j(D)f\|_{2,K} & \le \|Q^j(D)\|_{2,\Omega'_{2\delta}} \\ & \le C'' C^{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i} \left(\frac{k+1}{\delta}\right)^{i\gamma} \exp\left(2^p B\lambda\varphi^*\left(\frac{(k+1-i)\mu m}{2^p B\lambda}\right)\right) \\ & \quad + C'' C^k \sum_{i=0}^{k} \binom{k}{i} \left(\frac{k}{\delta}\right)^{i\gamma} \exp\left(B\lambda\varphi^*\left(\frac{(k-i)\mu m}{B\lambda}\right)\right) \end{aligned}$$

when p is the first entire after μm. Therefore,

$$\|Q^j(D)f\|_{2,K}$$

is less than or equal to

$$C^{''}C^{k+1}\sum_{i=0}^{k+1}\binom{k+1}{i}\left(\frac{k+1}{\delta}\right)^{i\gamma}\exp\left(2^pB\lambda\varphi^*\left(\frac{(k-i)\mu m+p}{2^pB\lambda}\right)\right)$$
$$+C^{''}C^{k}\sum_{i=0}^{k}\binom{k}{i}\left(\frac{k}{\delta}\right)^{i\gamma}\exp\left(B\lambda\varphi^*\left(\frac{(k-i)\mu m}{B\lambda}\right)\right)$$

since φ^* is increasing. Now, using Lemmas 2.2 and 4.6 we have

$$\begin{aligned}\|Q^j(D)f\|_{2,K} &\leq D\left(2C\left(\frac{1}{\delta}\right)^{\gamma}e^{B\lambda E}\right)^k\exp\left(B\lambda\varphi^*\left(\frac{k\mu m}{B\lambda}\right)\right)\\ &= DF^k\exp\left(B\lambda\varphi^*\left(\frac{k\mu m}{B\lambda}\right)\right).\end{aligned}$$

In view of Lemma 4.2,

$$\begin{aligned}\|Q^j(D)f\|_{2,K} &\leq D\exp\left(B\lambda\frac{A}{B}\right)\exp\left(\lambda\varphi^*\left(\frac{k\mu m}{\lambda}\right)\right)\\ &= D\exp(A\lambda)\exp\left(\lambda\varphi^*\left(\frac{kh\nu m}{\lambda}\right)\right)\\ &\leq D\exp(A\lambda)\exp\left(\lambda\varphi^*\left(\frac{jhm}{\lambda}\right)\right).\end{aligned}$$

As a consequence for each $j\in\mathbb{N}$,

$$\|Q^j(D)f\|_{2,K}\exp\left(-\lambda\varphi^*\left(\frac{jhm}{\lambda}\right)\right)\leq D\exp(A\lambda).$$

Then,

$$f\in\mathcal{E}_{Q,\left(\omega\left(t^{\frac{1}{mh}}\right)\right)}(\Omega).$$

In order to see the inclusion map

$$\mathcal{E}_{P,\left(\omega\left(t^{\frac{1}{m}}\right)\right)}(\Omega)\hookrightarrow\mathcal{E}_{Q,\left(\omega\left(t^{\frac{1}{mh}}\right)\right)}(\Omega)$$

is continuous, let $f\in\mathcal{E}_{P,\left(\omega\left(t^{\frac{1}{m}}\right)\right)}(\Omega)$ and we fix $K\subset\subset\Omega$, $\lambda>0$. Proceeding as above:

$$\begin{aligned}&\|Q^j(D)f\|_{2,K}\exp\left(-\lambda\varphi^*\left(\frac{jhm}{\lambda}\right)\right)\\ &\leq C^{'}C^{k+1}\sum_{i=0}^{k+1}\binom{k+1}{i}\left(\frac{k+1}{\delta}\right)^{i\gamma}\exp\left(-\lambda\varphi^*\left(\frac{jhm}{\lambda}\right)\right)\|P^{(k+1-i)\mu}(D)f\|_{2,\Omega'}\\ &\quad+C^{'}C^{k}\sum_{i=0}^{k}\binom{k}{i}\left(\frac{k}{\delta}\right)^{i\gamma}\exp\left(-\lambda\varphi^*\left(\frac{jhm}{\lambda}\right)\right)\|P^{(k-i)\mu}(D)f\|_{2,\Omega'}.\end{aligned}$$

Note by Lemma 4.6, for each $k\in\mathbb{N}$, $i=0,1,\ldots,k$ and $\lambda>0$,

$$k^{i\gamma}\exp\left(-\lambda\varphi^*\left(\frac{jhm}{\lambda}\right)\right)\leq\exp(\lambda Ek)\exp\left(-\lambda\varphi^*\left(\frac{(k-i)\mu m}{\lambda}\right)\right).$$

So, for each $j \in \mathbb{N}$,

$$\|Q^j(D)f\|_{2,K} \exp\left(-\lambda\varphi^*\left(\frac{jhm}{\lambda}\right)\right) \leq DF^k \sup_{l\in\mathbb{N}_0} \|P^l(D)f\|_{2,\overline{\Omega'}} \exp\left(-\lambda\varphi^*\left(\frac{lm}{\lambda}\right)\right).$$

Hence,

$$\|Q^j(D)f\|_{2,K} \exp\left(-\frac{\lambda}{B}\varphi^*\left(\frac{jhmB}{\lambda}\right)\right) \leq D\exp\left(\lambda\frac{A}{B}\right) \sup_{l\in\mathbb{N}_0} \|P^l(D)f\|_{2,\overline{\Omega'}} \exp\left(-\lambda\varphi^*\left(\frac{lm}{\lambda}\right)\right).$$

Proceeding as in Theorem 4.1, the Roumieu case is analogous. □

Proof of Theorem 4.5. Roumieu Case. We fix a compact subset $K_0 \subset\subset \Omega$. The following inclusions hold:

$$\mathcal{E}_{P,(\omega(t))}(\Omega) \subseteq \mathcal{E}_{P,\{\omega(t)\}}(\Omega) \subseteq \mathcal{E}_{Q,\left\{\omega\left(t^{\frac{1}{h}}\right)\right\}}(\Omega) \subseteq \operatorname*{ind}_{\overrightarrow{n\in\mathbb{N}}} \mathcal{E}^{\frac{1}{n}}_{Q,\omega\left(t^{\frac{1}{h}}\right)}(K_0).$$

From Theorem 3.3 we get that $\mathcal{E}_{P,(\omega(t))}(\Omega)$ is a Fréchet space. Now we consider on $\mathcal{E}^{\frac{1}{n}}_{Q,\omega\left(t^{\frac{1}{h}}\right)}(K_0)$ the topology of Theorem 3.5, so that $\operatorname*{ind}_{\overrightarrow{n\in\mathbb{N}}}\mathcal{E}^{\frac{1}{n}}_{Q,\omega\left(t^{\frac{1}{h}}\right)}(K_0)$ is an (LF)-space. By Closed Graph Theorem and Grothendieck's Factorization Theorem (see [16, Theorems 24.31 and 24.33]), there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{E}_{P,(\omega(t))}(\Omega) \subseteq \mathcal{E}^{\frac{1}{n_0}}_{Q,\omega\left(t^{\frac{1}{h}}\right)}(K_0)$$

with continuous inclusion. So, given any seminorm $\max\left(p_m, \|\cdot\|_{Q,K_0,\frac{1}{n_0}}\right)$, of $\mathcal{E}^{\frac{1}{n_0}}_{Q,\omega\left(t^{\frac{1}{h}}\right)}(K_0)$, there exist $C > 0$, a compact $K \subset\subset \Omega$, $p \in \mathbb{N}_0$ and $\lambda > 0$ such that for all $f \in \mathcal{E}_{P,(\omega(t))}(\Omega)$,

$$\sup_{j\in\mathbb{N}_0} \|Q^j(D)f\|_{2,K_0} \exp\left(-\frac{1}{n_0}\varphi^*(hjn_0)\right) \leq \max\left(p_m, \|f\|_{Q,K_0,\frac{1}{n_0}}\right) \leq C\sup_{j\in\mathbb{N}_0} \|P^j(D)f\|_{2,K} \exp\left(-\lambda\varphi^*\left(\frac{j}{\lambda}\right)\right). \tag{14}$$

If $\xi \in \mathbb{R}^N$, we denote $f_\xi(x) = e^{i<x,\xi>}$ and observe that

$$Q^j(D)e^{i<x,\xi>} = Q(\xi)^j e^{i<x,\xi>} \quad \text{and} \quad f_\xi^\alpha(x) = \xi^\alpha i^\alpha e^{i<x,\xi>}.$$

Moreover, $f_\xi \in \mathcal{E}_{P,(\omega)}(\Omega)$ since for a compact subset $K \subset\subset \Omega$ and $\lambda > 0$

$$\|P^j(D)f_\xi\|_{2,K} = m(K)|P(\xi)|^j \leq C\exp\left(\lambda\varphi^*\left(\frac{j}{\lambda}\right)\right)$$

by Lemma 4.2.

Applying inequality (14) to f_ξ we get

$$\sup_{j\in\mathbb{N}_0} |Q(\xi)|^j \exp\left(-\frac{1}{n_0}\varphi^*(hjn_0)\right) \le C_2 \sup_{j\in\mathbb{N}_0} |P(\xi)|^j \exp\left(-\lambda\varphi^*\left(\frac{j}{\lambda}\right)\right). \quad (15)$$

For $|Q(\xi)|$ and $|P(\xi)|$ greater or equal than 1 we obtain from (15) and Lemma 4.7,

$$\exp\left(\frac{1}{2n_0}\omega\left(|Q(\xi)|^{\frac{1}{h}}\right)\right) \le C_2 \exp\left(\lambda\omega(|P(\xi)|)\right).$$

Hence,

$$\omega\left(|Q(\xi)|^{\frac{1}{h}}\right) \le C_3 + C_4\omega(|P(\xi)|) \le C_5\omega(|P(\xi)|)$$

whenever $|P(\xi)|$, $|Q(\xi)| \ge 1$. On the other hand, condition B-M-M implies that for each $k \in \mathbb{N}$ there exists H_k such that $2^{k-1}\omega(t) \le \omega(H_k t)$ whenever $t \ge 1$. Then,

$$\omega\left(|Q(\xi)|^{\frac{1}{h}}\right) \le \omega(C_6|P(\xi)|) \quad \text{whenever } |P(\xi)|,\ |Q(\xi)| \ge 1.$$

Having in mind ω vanishes on $[0,1]$ and it is increasing and $|P|$ tends to $+\infty$ if $|\xi|$ tends to $+\infty$ we finally conclude that there is $C_7 > 0$ such that,

$$|Q(\xi)|^{\frac{1}{h}} \le C_7|P(\xi)| \quad \text{for every } \xi \in \mathbb{R}^N.$$

The Beurling case is easier because $\mathcal{E}_{P,(\omega)}(\Omega)$ is a metrizable space. □

Next we show that for any elliptic polynomial P of degree m, the classes $\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega)$ and $\mathcal{E}_{,*(t)}(\Omega)$ are the same as sets and as topological vector spaces. This is an extension of a result of Komatsu (see [9]). The converse in the Gevrey setting is due to Métivier [17]. We need the following lemma due to Komatsu (see [9, Lemma 3]).

Lemma 4.8. *Let Ω be an open subset of $\mathbb{R}^N$. Suppose that P is an elliptic operator of order m and let $\rho_0 > 0$ be given. Then, there exists a constant $C > 0$, which only depends on N, ρ_0 and P, such that for each $f \in \mathcal{C}^\infty(\Omega)$ and for each $\alpha \in {\mathbb{N}_0}^N$ verifying $|\alpha| \le m$,*

$$\|f^{(\alpha)}\|_{2,\Omega_{\rho+\sigma}} \le C\|P(D)f\|_{2,\Omega_\sigma}{}^{\frac{|\alpha|}{m}}\|f\|_{2,\Omega_\sigma}{}^{1-\frac{|\alpha|}{m}} + C\frac{1}{\rho^{|\alpha|}}\|f\|_{2,\Omega_\sigma}$$

for every $0 < \rho \le \rho_0$ and $\sigma > 0$.

Theorem 4.9. *Let ω be a weight function and let Ω be an open subset of $\mathbb{R}^N$. For any elliptic polynomial P of degree m we have*

$$\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega) \subseteq \mathcal{E}_{*(t)}(\Omega)$$

and the inclusion map is continuous.

Proof. Let Ω' be an open subset relatively compact in Ω. We first estimate the derivatives $f^{(\alpha)}$ of order $|\alpha| = km$, $k \in \mathbb{N}_0$. We write $\alpha = p + q$ with

$|p| = m$ and $|q| = (k-1)m$. Using Lemma 4.8 with $\rho = \frac{\delta}{k}$ and $\sigma = \delta\left(1 - \frac{1}{k}\right)$, δ small enough, we get a constant $C > 0$ such that $\forall f \in \mathcal{C}^\infty(\Omega')$,

$$\begin{aligned} \|f^{(\alpha)}\|_{2,\Omega'_\delta} &= \| \left(f^{(q)}\right)^{(p)} \|_{2,\Omega'_\delta} \\ &\leq C \left\{ \|P(D) f^{(q)}\|_{2,\Omega'_{\delta\left(1-\frac{1}{k}\right)}} + \left(\frac{k}{\delta}\right)^m \|f^{(q)}\|_{2,\Omega'_{\delta\left(1-\frac{1}{k}\right)}} \right\}. \end{aligned}$$

Applying this lemma k times as is (10) we obtain

$$\|f^{(\alpha)}\|_{2,\Omega'_\delta} \leq C^k \sum_{i=0}^{k} \binom{k}{i} \left(\frac{k}{\delta}\right)^{im} \|P^{(k-i)}(D) f\|_{2,\Omega'} \tag{16}$$

whenever $|\alpha| = km$. In the case that $\alpha \in {\mathbb{N}_0}^N$ is an arbitrary multi-index we write $\alpha = \beta + \gamma$, where $|\beta| = km$ and $|\gamma| < m$.

We observe that $x^a y^{1-a} \leq x + y$ holds if $x, y \geq 0$ and $0 < a < 1$. Therefore,

$$\|P(D)f\|_{2,\Omega'_\delta}^{\frac{|\gamma|}{m}} \|f\|_{2,\Omega'_\delta}^{1-\frac{|\gamma|}{m}} \leq \|P(D)f\|_{2,\Omega'_\delta} + \|f\|_{2,\Omega'_\delta}.$$

Hence it follows from Lemma 4.8 that there is a constant C' such that

$$\|f^{(\gamma)}\|_{2,\Omega'_{2\delta}} \leq C'(\|P(D)f\|_{2,\Omega'_\delta} + \|f\|_{2,\Omega'_\delta}) \quad \forall f \in \mathcal{C}^\infty(\Omega'_\delta) \text{ y } \forall |\gamma| < m. \tag{17}$$

C' depends of δ. We apply (17) to $f^{(\beta)}$ to obtain

$$\|f^{(\alpha)}\|_{2,\Omega'_{2\delta}} \leq C'(\|P(D)f^{(\beta)}\|_{2,\Omega'_\delta} + \|f^{(\beta)}\|_{2,\Omega'_\delta}). \tag{18}$$

Now the inequality (16) implies

$$\begin{aligned} \|P(D)f^{(\beta)}\|_{2,\Omega'_\delta} &= \|(P(D)f)^{(\beta)}\|_{2,\Omega'_\delta} \\ &\leq C^k \sum_{i=0}^{k} \binom{k}{i} \left(\frac{k}{\delta}\right)^{im} \|P^{(k+1-i)}(D) f\|_{2,\Omega'} \\ &\leq C^{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i} \left(\frac{k+1}{\delta}\right)^{im} \|P^{(k+1-i)}(D) f\|_{2,\Omega'}. \end{aligned}$$

From this inequality, (18) and (16) we conclude

$$\begin{aligned} \|f^{(\alpha)}\|_{2,\Omega'_{2\delta}} \leq\ & C' C^{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i} \left(\frac{k+1}{\delta}\right)^{im} \|P^{(k+1-i)}(D) f\|_{2,\Omega'} \\ & + C' C^k \sum_{i=0}^{k} \binom{k}{i} \left(\frac{k}{\delta}\right)^{im} \|P^{(k-i)}(D) f\|_{2,\Omega'}. \end{aligned}$$

As $\gamma \leq \mu m$, we can use Lemma 4.7, proceeding as in Theorem 4.4 with $h = \frac{\mu}{\nu} = \frac{1}{m}$ and $\gamma = \frac{\mu}{b} = m$, to conclude that

$$\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega) \subseteq \mathcal{E}_{*(t)}(\Omega)$$

with continuous inclusion. □

Corollary 4.10. *Let ω be a weight function and let Ω be an open subset of $\mathbb{R}^N$ and P an elliptic polynomial. Then the equality $\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega) = \mathcal{E}_{*(t)}(\Omega)$ holds.*

Proof. It is a consequence of Theorems 4.1 and 4.9. □

Finally, we characterize when $\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega)$ and $\mathcal{E}_{*(t)}(\Omega)$ coincide for weight functions verifying the growth condition B-M-M.

Lemma 4.11. *If $\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega) = \mathcal{E}_{*(t)}(\Omega)$ algebraically, then P is hypoelliptic and the previous equality also holds in the topological sense.*

Proof. We will present the proof in the Beurling case and for $\Omega = \mathbb{R}^N$. We put $\sigma(t) = \omega\left(t^{\frac{1}{m}}\right)$. Since

$$\{f \in \mathcal{C}^\infty(\mathbb{R}^N) : P(D)f = 0\} \subset \mathcal{E}_{P,(\sigma)}(\mathbb{R}^N),$$

our hypothesis implies that

$$\{f \in \mathcal{C}^\infty(\mathbb{R}^N) : P(D)f = 0\} = \{f \in \mathcal{E}_{(\omega)}(\mathbb{R}^N) : P(D)f = 0\}.$$

We fix a compact subset $K \subset \mathbb{R}^N$ and $\lambda > 0$. From the Open Mapping Theorem we deduce that there are a constant $C > 0, m \in \mathbb{N}$ and a compact set Q such that

$$p_{K,\lambda}(f) \leq C \sup_{|\alpha|\leq m} \sup_{x\in Q} |f^{(\alpha)}(x)| \tag{19}$$

whenever $f \in \mathcal{C}^\infty(\mathbb{R}^N)$ and $P(D)f = 0$. We now assume that P is not hypoelliptic. Then we can apply theorems [8, 11.1.5 and 10.1.25] to find a function $f \in \mathcal{C}^m(\mathbb{R}^N)\backslash\mathcal{C}^\infty(\mathbb{R}^N)$ such that $P(D) = 0$. We take $\{\rho_n\}$ a regularizing sequence. Then $\{f * \rho_n\}$ is a Cauchy sequence in $\mathcal{C}^m(\mathbb{R}^N)$ and from inequality (19) we conclude that also $\{f * \rho_n\}$ is a Cauchy sequence in $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$. This is a contradiction. □

Theorem 4.12. *Let ω be a weight function verifying condition B-M-M. Suppose P is polynomial which degree is m. Then $\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega) = \mathcal{E}_{*(t)}(\Omega)$ holds algebraically if and only if P is elliptic. In this case the equality $\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega) = \mathcal{E}_{*(t)}(\Omega)$ is also topological.*

Proof. If P is elliptic then $\mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega) = \mathcal{E}_{*(t)}(\Omega)$ by Theorem 4.9. In order to show the converse, we apply once again Theorem 4.9 to the elliptic polynomial $Q(\xi) = \xi_1^2 + \cdots \xi_N^2$ to get

$$\mathcal{E}_{P,*\left(t^{\frac{1}{m}}\right)}(\Omega) = \mathcal{E}_{*(t)}(\Omega) = \mathcal{E}_{\xi_1^2+\cdots+\xi_N^2,*\left(t^{\frac{1}{2}}\right)}(\Omega).$$

Moreover, according to Lemma 4.11, P is hypoelliptic and we can proceed as in Theorem 4.5 to deduce

$$(\xi_1^2 + \cdots + \xi_N^2)^{\frac{1}{2}} \leq C|P(\xi)|^{\frac{1}{m}} \quad \text{if } |\xi| \text{ is large enough.}$$

That is, for some constant $C > 0$,

$$(\xi_1^2 + \cdots + \xi_N^2)^m \leq C(1 + |P(\xi)|^2).$$

□

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