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Quasianalytic Wave Front Sets for Solutions of Linear Partial Differential Operators

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Abstract. In the present paper, we introduce and study Beurling and Roumieu quasianalytic (and nonquasianalytic) wave front sets, WF_*, of classical distributions. In particular, we have the following inclusion

$$WF_*(u) \subset WF_*(Pu) \cup \Sigma, \quad u \in \mathcal{D}'(\Omega),$$

where Ω is an open subset of $\mathbb{R}^n$, P is a linear partial differential operator with coefficients in a suitable ultradifferentiable class, and Σ is the characteristic set of P. Some applications are also investigated.

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1. Introduction

Classes of ultradifferentiable functions have been investigated intensively since the 20ies of the last century. According to the theorem of Denjoy–Carleman they split in two groups: the quasianalytic and the nonquasianalytic classes. Several authors have introduced the classes in different ways (e.g., Beurling [1]; Komatsu [16]; Hörmander [14]; Braun, Meise, and Taylor [9]). In general, the classes defined in one way cannot be defined in another way (see [8]). On the other hand, independently of the definition there are several known methods to construct functions with prescribed properties in nonquasianalytic classes, while this is not the case for quasianalytic classes, for which other techniques are needed, for example, complex variable methods.

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The notion of wave front set was introduced by Hörmander in 1970 to simplify the study of the propagation of singularities of distributional solutions of linear partial differential operators. In 1971, Hörmander [14] established the following microlocal regularity result:

$$WF_L(u) \subset \Sigma \cup WF_L(Pu), \quad u \in \mathcal{D}'(\Omega), \tag{1}$$

where P is a linear partial differential operator with analytic coefficients, and Σ is the characteristic set of P. This type of result represents a fundamental tool in the study of propagation of singularities. This inclusion has been also applied, modified and adapted several times since the seventies of the last century to study the problem of iterates from a microlocal point of view (we refer to [2, 3, 5, 24], among others; see also [4] for a survey on this topic). In the inclusion (1) the wave front set $WF_L(u)$ is defined with respect to the ultradifferentiable class C^L of Roumieu type and u is always a classical distribution on an open set Ω of $\mathbb{R}^n$. The function spaces C^L were introduced in [14] and cover the classical spaces of Gevrey functions and of analytic functions. In the case of the analytic wave front set, WF_A, such an inclusion was proved by Kato [15] for hyperfunctions, but for a very different definition of WF_A. We also mention that a version of inclusion (1) for wave front sets with respect to the C^∞-class and an operator P with C^∞-coefficients can be found in [13, Chapter VIII].

In this paper we present a version of inclusion (1) for wave front sets defined with respect to the ultradifferentiable classes as introduced by Braun, Meise, and Taylor [9] and classical distributions. By modifying the arguments in [14] in a suitable way, we cover both quasianalytic and non-quasianalytic cases at the same time. These classes have the advantage that can be studied using the decay properties of their Fourier transform and the decay behaviour of their derivatives. We give two versions of the result. First, we prove the Beurling case in Theorem 4.1. In this case, we require the assumption $\omega(t) = o(t)$, as t tends to infinity, on the weight function. The Roumieu version is obtained as a consequence of the Beurling case and the description of Roumieu wave front set of a classical distribution as the closure of the union of all the Beurling wave front sets contained in it. Such a description is proved for arbitrary weight functions, quasianalytic or not, in Proposition 4.5 (see [11] for a version for non-quasianalytic weight functions). To show our inclusion results, Theorem 4.1 and Theorem 4.8, we need to assume that the weight functions are equivalent to subadditive weight functions (see [18, 19]). In particular, our results apply to the most relevant cases considered by Komatsu [16] (see [8]). Finally, our results are applied to study the propagation of singularities of the operator $P = \partial_{x_n}$ in $\mathbb{R}^n$.

2. Notation and preliminaries

In this preliminary section we fix the notation and provide a number of basic results that will be used in the subsequent sections. Throughout this article

$|\cdot|$ denotes the Euclidian norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ and $B_r(a)$ denotes the open ball of radius r and center a.

Definition 2.1. *A weight function is a continuous increasing function* $\omega : [0,\infty[\to [0,\infty[$ *with the following properties:*

(α) *there exists* $L \geq 0$ *such that* $\omega(2t) \leq L(\omega(t)+1)$ *for all* $t \geq 0$,
(β) $\omega(t) = O(t)$, *as* t *tends to* ∞,
(γ) $\log(t) = o(\omega(t))$ *as* t *tends to* ∞,
(δ) $\varphi : t \to \omega(e^t)$ *is convex.*

A weight function ω is called *quasianalytic* if

$$\int_1^\infty \frac{\omega(t)}{t^2} dt = \infty.$$

If this integral is finite, then ω is called *nonquasianalytic.*

A weight function ω is equivalent to a sub-additive weight if, and only if, it has the property

$$(\alpha_0) \quad \exists D > 0 \ \ \exists t_0 > 0 \ \ \forall \lambda \geq 1 \ \ \forall t \geq t_0 : \quad \omega(\lambda t) \leq \lambda D \omega(t).$$

The condition above should be compared with [19, p.19] and [18, Lemma 1].

The *Young conjugate* $\varphi^* : [0,\infty[\to \mathbb{R}$ of φ is given by

$$\varphi^*(s) := \sup\{st - \varphi(t), t \geq 0\}.$$

There is no loss of generality to assume that ω vanishes on $[0,1]$. Then φ^* has only non-negative values, it is convex and $\varphi^*(t)/t$ is increasing and tends to ∞ as $t \to \infty$ and $\varphi^{**} = \varphi$. For more details on properties of ω and φ^* we refer to [9, 12].

Example 2.2. The following are examples of weight functions (eventually after a change on the interval $[0,\delta]$ for a suitable $\delta > 0$):

(1) $\omega(t) = t^\alpha$, $0 < \alpha < 1$;
(2) $\omega(t) = (\log(1+t))^\beta$, $\beta > 1$;
(3) $\omega(t) = t\,(\log(e+t))^{-\beta}$, $\beta > 0$;
(4) $\omega(t) = t$.

The weight function in (3) is quasianalytic for $\beta \in]0,1]$ and nonquasianalytic for $\beta > 1$. The weight function in (4) is also quasianalytic. Moreover, all the weight functions above satisfy property (α_0). For further examples of quasianalytic weight functions we refer to [8].

Definition 2.3. *Let* ω *be a weight function. For an open set* $\Omega \subset \mathbb{R}^n$ *we set*

$$\mathcal{E}_{(\omega)}(\Omega) := \{f \in C^\infty(\Omega) : \|f\|_{K,\lambda} < \infty, \textit{ for every } K \subset\subset \Omega \textit{ and every } \lambda > 0\},$$

and

$$\mathcal{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega) : \textit{for every } K \subset\subset \Omega, \textit{ there exists } \lambda > 0 \textit{ such that } \|f\|_{K,\lambda} < \infty\},$$

where

$$\|f\|_{K,\lambda} := \sup_{x\in K}\sup_{\alpha\in\mathbf{N}_0^N} |f^{(\alpha)}(x)| \exp\Big(-\lambda\varphi^*\big(\frac{|\alpha|}{\lambda}\big)\Big).$$

The spaces $\mathcal{E}_{(\omega)}(\Omega)$ and $\mathcal{E}_{\{\omega\}}(\Omega)$ are endowed with their natural topologies. Then $\mathcal{E}_{(\omega)}(\Omega)$ is a nuclear Fréchet space, while $\mathcal{E}_{\{\omega\}}(\Omega)$ is a countable projective limit of (DFN)-spaces, which is reflexive and complete. If ω is nonquasianalytic, the space $\mathcal{E}_{\{\omega\}}(\Omega)$ is ultrabornological, see [9, Proposition 4.9]. If ω is quasianalytic, the space $\mathcal{E}_{\{\omega\}}(\Omega)$ is surely ultrabornological under the assumption of convexity of Ω and, this follows from [21, Satz 3.25], together with [22, Theorem 3.4], and [23, Theorem 3.5].

The elements in the space $\mathcal{E}_{(\omega)}(\Omega)$ (respectively, in the space $\mathcal{E}_{\{\omega\}}(\Omega)$) are called ω-ultradifferentiable functions of Beurling type (respectively, of Roumieu type) in Ω. By $\mathcal{E}'_{(\omega)}(\Omega)$ and $\mathcal{E}'_{\{\omega\}}(\Omega)$ we denote the duals of $\mathcal{E}_{(\omega)}(\Omega)$ and $\mathcal{E}_{\{\omega\}}(\Omega)$. When ω is quasianalytic the elements of $\mathcal{E}'_{(\omega)}(\Omega)$ (respectively, $\mathcal{E}'_{\{\omega\}}(\Omega)$) are called quasianalytic functionals of Beurling (respectively, Roumieu) type.

We observe that in the case $\omega(t) = t^\alpha$, $0 < \alpha \le 1$, the corresponding Roumieu class is the Gevrey class with exponent $s = 1/\alpha$. In particular, $\mathcal{E}_{\{\omega\}}(\Omega)$ coincides with the space $\mathcal{A}(\Omega)$ of all real analytic functions on Ω.

We will write $*$ to denote (ω) or $\{\omega\}$ when it is not necessary to distinguish between both cases.

If ω is quasianalytic, the elements with compact support in $\mathcal{E}_{\{\omega\}}(\Omega)$ or in $\mathcal{E}_{(\omega)}(\Omega)$ are trivial. While, if ω is nonquasianalytic, the space $\mathcal{D}_*(K) := \mathcal{E}_*(\Omega) \cap \mathcal{D}(K) \neq \{0\}$, being $K \subset \Omega$ a compact set. Then $\mathcal{D}_*(\Omega) := \mathrm{ind}_n \mathcal{D}_*(K_n)$, where (K_n) is any compact exhaustion of Ω. The elements of $\mathcal{D}'_{(\omega)}(\Omega)$ (respectively, $\mathcal{D}'_{\{\omega\}}(\Omega)$) are called ω-ultradistributions of Beurling (respectively, Roumieu) type.

The $*$-singular support of a classical distribution $u \in \mathcal{D}'(\Omega)$, denoted by $\mathrm{sing}_*\,\mathrm{supp}\, u$, is the complementary in Ω of the biggest open set $U \subset \Omega$ satisfying $u|_U \in \mathcal{E}_*(U)$.

Remark 2.4. We also recall the following properties (see, for example, [12, Remark 2.8]):

(a) If $\sigma(t) = o(\omega(t))$ as t tends to infinity, then

$$\mathcal{E}_{\{\omega\}}(\Omega) \subset \mathcal{E}_{(\sigma)}(\Omega)$$

with continuous inclusion.

(b) If $\omega(t) = o(t)$ as t tends to infinity, then for each constant $l \in \mathbb{N}$, there is a constant $C_l > 0$ such that

$$y\log y \le y + l\varphi^*\big(\frac{y}{l}\big) + C_l, \quad y > 0.$$

3. Quasianalytic wave front sets and properties

In this section, we introduce and study Beurling and Roumieu quasianalytic and non quasianalytic wave front sets of a classical distribution. To provide this, we begin with two lemmas which clarify the behaviour of the Fourier transform with respect to the weight function in Roumieu and Beurling settings. The first one treats the Roumieu case.

Lemma 3.1. *Let f be a continuous function defined in a cone $\Gamma \subset \mathbb{R}^n \setminus \{0\}$ taking values in $[0, +\infty[$. The following statements are equivalent:*

(*i*) *There exist constants $C, \varepsilon > 0$ such that*

$$f(\xi) \leq C \exp(-\varepsilon\omega(\xi)), \quad \xi \in \Gamma,$$

(*ii*) *There exists a constant $C > 0$ such that*

$$f(\xi) \leq C^{N+1} N! \left(\frac{1}{\omega(\xi)}\right)^N, \quad N = 0, 1, 2, \ldots, \quad \xi \in \Gamma,$$

(*iii*) *There exists a constant $C > 0$ such that*

$$f(\xi) \leq C \left(\frac{CN}{\omega(\xi)}\right)^N, \quad N = 0, 1, 2, \ldots, \quad \xi \in \Gamma,$$

(*iv*) *There exist constants $C > 0$ and $k \in \mathbb{N}$ satisfying*

$$|\xi|^N f(\xi) \leq C e^{\frac{1}{k}\varphi^*(Nk)}, \quad N = 0, 1, 2, \ldots, \quad \xi \in \Gamma.$$

(*v*) *There exist constants $C > 0$ and $k \in \mathbb{N}$ such that*

$$|\xi|^N f(\xi) \leq C^{N+1} e^{\frac{1}{k}\varphi^*(Nk)}, \quad N = 0, 1, 2, \ldots, \quad \xi \in \Gamma.$$

Proof. $(i) \iff (ii) \iff (iii)$ can be proved as in [20, Lemma 1.6.2]. We first show $(i) \iff (iv)$. To prove this, it is sufficient to check, for $t > 1$, that:

$$e^{-\frac{1}{k}\omega(t)} \leq \inf_{N \in \mathbb{N}_0} t^{-N} e^{\frac{1}{k}\varphi^*(Nk)} \leq e^{-\frac{1}{k}\omega(t) + \log t}. \tag{2}$$

Indeed, since $\log(t) = o(\omega(t))$ as $t \to \infty$, for $0 < \varepsilon < \frac{1}{k}$ there is $t_0 > 0$ so that $-\frac{1}{k}\omega(t) + \log t \leq -\varepsilon\omega(t)$ for all $t \geq t_0$.

The first inequality in (2) follows by observing that $(\varphi^*)^* = \varphi$ and hence, we have

$$\begin{aligned}
\omega(t) &= \sup_{s>0}\{s \log t - \varphi^*(s)\} \\
&\geq \sup_{N \in \mathbb{N}_0}\{Nk \log t - \varphi^*(Nk)\} \\
&= k \sup_{N \in \mathbb{N}_0}\{N \log t - \frac{1}{k}\varphi^*(Nk)\}.
\end{aligned}$$

The second inequality in (2) follows by observing that φ^* is increasing and hence, we have

$$\begin{aligned}\omega(t) &= \sup_{s>0}\{s\log t - \varphi^*(s)\} \\ &= \sup_{N\in\mathbb{N}_0}\left(\sup_{Nk\le s\le (N+1)k}\{s\log t - \varphi^*(s)\}\right) \\ &\le \sup_{N\in\mathbb{N}_0}\{(N+1)k\log t - \varphi^*(Nk)\} \\ &= k\log t + \sup_{N\in\mathbb{N}_0}\{Nk\log t - \varphi^*(Nk)\} \\ &= k\left(\log t + \sup_{N\in\mathbb{N}_0}\{N\log t - \frac{1}{k}\varphi^*(Nk)\}\right).\end{aligned}$$

It is obvious that (iv) implies (v). To finish the proof, it remains to show that (v) implies (iv). We proceed as in [10, p. 404]. We take $s\in\mathbb{N}$ to be the smallest natural number such that $C\le e^s$, being C the constant that appears in (iv). Let m the smallest natural number bigger than kL^s, where k is the constant of (iv) and $L>1$ the one that appears in property (α) of the weight function ω. We have

$$\frac{1}{k}\varphi^*(Nk) + Ns \le \frac{1}{m}\varphi^*(Nm) + \frac{1}{m}\sum_{j=1}^{s} L^j,$$

and hence,

$$C^{N+1}e^{\frac{1}{k}\varphi^*(Nk)} \le Ce^{Ns+\frac{1}{k}\varphi^*(Nk)} \le Ce^{\frac{1}{m}\sum_{j=1}^{s}L^j}e^{\frac{1}{m}\varphi^*(Nm)}. \qquad \square$$

The following lemma treats the Beurling case.

Lemma 3.2. *Let f be a continuous function defined in a cone $\Gamma\subset\mathbb{R}^n\setminus\{0\}$ taking values in $[0,+\infty[$. The following statements are equivalent:*

(i) *For every $k\in\mathbb{N}$ there exists a constant $C_k>0$ such that*

$$f(\xi)\le C_k\exp(-k\omega(\xi)),\quad \xi\in\Gamma,$$

(ii) *For every $k\in\mathbb{N}$ there exists a constant $C_k>0$ such that*

$$f(\xi)\le C_k N!\left(\frac{1}{k\omega(\xi)}\right)^N,\quad N=0,1,2,\dots,\quad \xi\in\Gamma,$$

(iii) *For every $k\in\mathbb{N}$ there exists a constant $C_k>0$ such that*

$$f(\xi)\le C_k\left(\frac{N}{k\omega(\xi)}\right)^N,\quad N=0,1,2,\dots,\quad \xi\in\Gamma,$$

(iv) *For every $k\in\mathbb{N}$ there exists a constant $C_k>0$ such that*

$$|\xi|^N f(\xi)\le C_k e^{k\varphi^*(N/k)},\quad N=0,1,2,\dots,\quad \xi\in\Gamma.$$

(v) *There exists a constant $C>0$ such that for every $k\in\mathbb{N}$ there exists a constant $C_k>0$ for which*

$$|\xi|^N f(\xi)\le C_k C^N e^{k\varphi^*(N/k)},\quad N=0,1,2,\dots,\quad \xi\in\Gamma.$$

Proof. It follows easily that (i) implies (ii) by using the Taylor expansion of the exponential function. It is clear that (ii) implies (iii). On the other hand, (iii) implies (ii) since $N^N \leq e^N N!$ and therefore, if f satisfies (iii) for some sequence $\{C_k\}$, then for every $k \in \mathbb{N}$,

$$f(\xi) \leq C_{3k} N! \, [k\,\omega(\xi)]^{-N}, \quad N = 0, 1, 2, \ldots, \ \xi \in \Gamma.$$

(ii) implies that, for each $k \in \mathbb{N}$, $f(\xi)k^N \omega(\xi)^N / N! \leq C_{2k}(1/2)^N$ for all $N = 0, 1, 2, \ldots$. So, using the expansion of the exponential function, we obtain

$$f(\xi) \exp(k\,\omega(\xi)) \leq 2C_{2k},$$

and hence, (i) is satisfied.

$(i) \iff (iv)$ is proved in a similar way to Lemma 3.1. It remains to prove that (v) implies (iv). As in the Roumieu setting, we take $s \in \mathbb{N}$ such that $C \leq e^s$, being C the constant that appears in (v). By (v) for each $k \in \mathbb{N}$ there is $C_k > 0$ such that, for $A_k = k(L^s + \cdots + L)$,

$$f(\xi) \leq C_k C^N e^{kL^s \varphi^*(N/(kL^s))} \leq C_k e^{A_k} e^{k\varphi^*(N/k)}. \qquad \square$$

The following proposition describes the $*$-singular support of a classical distribution. The proof follows the lines of [14] (see also [12, Lemma 4.7]).

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $u \in \mathcal{D}'(\Omega)$ and $x_0 \in \Omega$.*

(a) *Then u is a $\mathcal{E}_{\{\omega\}}$-function in some neighborhood of x_0 if and only if for some neighborhood U of x_0 there exists a bounded sequence $u_N \in \mathcal{E}'(\Omega)$ which is equal to u in U and satisfies, for some $C > 0$ and $k \in \mathbb{N}$, the estimates*

$$|\xi|^N \, |\widehat{u}_N(\xi)| \leq C e^{\frac{1}{k}\varphi^*(Nk)}, \quad N = 1, 2, \ldots, \quad \xi \in \mathbb{R}^n. \tag{3}$$

(b) *Then u is a $\mathcal{E}_{(\omega)}$-function in some neighborhood of x_0 if and only if for some neighborhood U of x_0 there exists a bounded sequence $u_N \in \mathcal{E}'(\Omega)$ which is equal to u in U and such that for every $k \in \mathbb{N}$ there exists a constant $C_k > 0$ satisfying*

$$|\xi|^N \, |\widehat{u}_N(\xi)| \leq C_k e^{k\varphi^*(N/k)}, \quad N = 1, 2, \ldots, \quad \xi \in \mathbb{R}^n. \tag{4}$$

Proof. (a) *Necessity.* Let $u \in \mathcal{E}_{\{\omega\}}(U)$ with $U = B_{3r}(x_0)$ and choose $\chi_N \in \mathcal{D}(\Omega)$ so that $\chi_N = 1$ in $B_r(x_0)$ and $\chi_N = 0$ on $(B_{2r}(x_0))^c$ in such a way that

$$|D^\alpha \chi_N| \leq (C_1 N)^{|\alpha|}, \quad |\alpha| \leq N, \tag{5}$$

where C_1 does not depend on N (we refer to [14] for a proof of the existence of sequences $\{\chi_N\}$ satisfying (5)). Now, we put $u_N = \chi_N u$. Then, for $|\alpha| \leq N$,

$$|D^\alpha(\chi_N u)| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (C_1 N)^{|\beta|} D_r e^{\frac{1}{m}\varphi^*(m|\alpha - \beta|)}, \tag{6}$$

being D_r the value of the seminorm $\|u\|_{\overline{B_{2r}(x_0)}, 1/m}$. Since φ^* is a convex function, we have $\frac{1}{m}\varphi^*(m|\alpha - \beta|) + \frac{1}{m}\varphi^*(m|\beta|) \leq \frac{1}{m}\varphi^*(m|\alpha|)$ and therefore, the right hand side of (6) is less than or equal to

$$\begin{aligned}
&D_r e^{\frac{1}{m}\varphi^*(m|\alpha|)} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} e^{|\beta|\log(C_1N)-\frac{1}{m}\varphi^*(m|\beta|)}\\
&\le 2^N D_r e^{\frac{1}{m}\varphi^*(mN)} \sup_{|\beta|\le N} e^{|\beta|\log(C_1N)-\frac{1}{m}\varphi^*(m|\beta|)}\\
&\le 2^N D_r e^{\frac{1}{m}\varphi^*(mN)} e^{\frac{1}{m}\sup_{|\beta|\le N}\{m|\beta|\log(C_1N)-\varphi^*(m|\beta|)\}}\\
&\le 2^N D_r e^{\frac{1}{m}\varphi^*(mN)} e^{\frac{1}{m}\varphi^{**}(\log(C_1N))}\\
&\le D_r e^{N+\frac{1}{m}\omega(C_1N)+\frac{1}{m}\varphi^*(mN)},
\end{aligned}$$

where we have used the fact that $\varphi^{**}=\varphi$, the definition of φ and $|\alpha|\le N$.

Then, we have, for $|\alpha|=N$,

$$|\xi^\alpha \widehat{u}_N(\xi)| = \left|\int e^{-i\langle x,\xi\rangle} D^\alpha u_N(x)dx\right| \le C D_r e^{N+\frac{1}{m}\omega(C_1N)+\frac{1}{m}\varphi^*(mN)},$$

where the constant C depends on the Lebesgue measure of $B_{2r}(x_0)$. Now, we select $i=1,\dots,n$ such that $|\xi_i|=\max_{1\le j\le n}|\xi_j|$ and set $\alpha=Ne_i$, where e_i is the $i-$th vector of the canonical basis of $\mathbb{R}^n$. Then,

$$|\xi|^N \le n^{N/2}\max_{1\le j\le n}|\xi_j|^N = n^{N/2}|\xi_i|^N = n^{N/2}|\xi^\alpha|.$$

Consequently,

$$\begin{aligned}
|\xi|^N|\widehat{u}_N(\xi)| &\le n^{N/2}|\xi^\alpha\widehat{u}_N(\xi)|\\
&\le C D_r n^{N/2} e^{N+\frac{1}{m}\omega(C_1N)+\frac{1}{m}\varphi^*(Nm)}\\
&= C D_r e^{\frac{N}{2}\log n+N+\frac{1}{m}\omega(C_1N)+\frac{1}{m}\varphi^*(Nm)}.
\end{aligned}$$

Since $\omega(t)=O(t)$ as t tends to infinity, we can find a positive constant C_2 such that $\omega(t)\le C_2t+C_2$ for all $t>0$. Now, proceeding as in the proof Lemma 3.2, $(v)\Rightarrow(iv)$, we take $s\in\mathbb{N}$ greater than $1+\frac{\log n}{2}+\frac{C_1C_2}{m}$. Then, if k is the smallest natural number bigger than mL^s, where $L>1$ is the constant that appears in property (α) of the weight function ω, we have

$$\frac{1}{m}\varphi^*(Nm)+Ns \le \frac{1}{k}\varphi^*(Nk)+\frac{1}{k}\sum_{j=1}^{s}L^j.$$

Hence, we obtain

$$\begin{aligned}
|\xi|^N|\widehat{u}_N(\xi)| &\le C D_r e^{C_2+N(\frac{\log n}{2}+1+\frac{1}{m}C_2C_1)+\frac{1}{m}\varphi^*(Nm)}\\
&\le C D_r e^{C_2} e^{Ns+\frac{1}{m}\varphi^*(Nm)} \le C_m e^{\frac{1}{k}\varphi^*(Nk)},
\end{aligned}$$

where the constant $C_m = C D_r e^{C_2+\frac{1}{k}\sum_{j=1}^{s}L^j}$ depends on m, on the weight ω, on r, and the selection of χ_N.

Sufficiency. For $x\in U$, we have

$$D^\alpha u(x) = (2\pi)^{-n}\int \xi^\alpha\widehat{u}_N(\xi)e^{i\langle x,\xi\rangle}d\xi,$$

when $N \geq |\alpha| + n + 1$, for $\xi^\alpha \widehat{u}_N(\xi)$ is then integrable since (3) is satisfied by hypothesis. Now, as $\{u_N\}$ is a bounded sequence in $\mathcal{E}'(\Omega)$, an application of the Banach-Steinhaus Theorem gives, for all $N \in \mathbb{N}$,

$$|\widehat{u}_N(\xi)| \leq C_1(1 + |\xi|)^M,$$

for some constants $C_1 > 0$ and $M \in \mathbb{N}$ that do not depend on N.

Hence, for $x \in U$,

$$|D^\alpha u(x)| \leq (2\pi)^{-n} \int |\xi^\alpha||\widehat{u}_N(\xi)|d\xi = I_1 + I_2,$$

where I_1 denotes the integral when $|\xi| \leq \exp(\frac{1}{kN}\varphi^*(kN))$, and I_2 denotes the integral when $|\xi| \geq \exp(\frac{1}{kN}\varphi^*(kN))$. Since $|\xi^\alpha| \leq |\xi|^{|\alpha|}$,

$$I_1 \leq m_n C_1 e^{\frac{|\alpha|}{kN}\varphi^*(Nk)} \left(1 + e^{\frac{1}{kN}\varphi^*(kN)}\right)^M,$$

being m_n the Lebesgue measure of the set $\{\xi \ : \ |\xi| \leq \exp(\frac{1}{kN}\varphi^*(kN))\}$ in $\mathbb{R}^n$, that is less than or equal to the Lebesgue measure of the hypercube $\{\xi \ : \ \|\xi\|_\infty \leq \exp(\frac{1}{kN}\varphi^*(kN))\}$, where $\|\cdot\|_\infty$ denotes the maximum norm in $\mathbb{R}^n$, which is equal to $2^n \exp\left(\frac{n}{kN}\varphi^*(kN)\right)$. Summing up, we have

$$I_1 \leq 2^{n+M} C_1 \left(e^{\frac{1}{kN}\varphi^*(kN)}\right)^{n+|\alpha|+M} \leq 2^{n+M} C_1 e^{\frac{1}{k}\varphi^*(k(n+|\alpha|+M))},$$

when $N = n + |\alpha| + M$. On the other hand, by (3),

$$\begin{aligned} I_2 &\leq \int_{|\xi| \geq e^{\frac{1}{Nk}\varphi^*(Nk)}} |\xi|^{N-n-1}|\widehat{u}_N(\xi)|d\xi \\ &\leq C e^{\frac{1}{k}\varphi^*(Nk)} \int_{|\xi| \geq e^{\frac{1}{kN}\varphi^*(kN)}} \frac{1}{|\xi|^{n+1}} d\xi \\ &\leq C e^{\frac{1}{k}\varphi^*(Nk)} \int_{|\xi| \geq 1} \frac{1}{|\xi|^{n+1}} d\xi. \end{aligned}$$

Therefore, we deduce that there is a constant $C' > 0$, that depends only on n and M, such that

$$|D^\alpha u| \leq C' e^{\frac{1}{k}\varphi^*(k(|\alpha|+n+M))},$$

on U. Now, from the convexity of φ^* we obtain

$$\frac{1}{k}\varphi^*(k(|\alpha| + n + M)) \leq \frac{1}{2k}\varphi^*(2k|\alpha|) + \frac{1}{2k}\varphi^*(2k(n + M)),$$

and hence, $C' e^{\frac{1}{k}\varphi^*(k(|\alpha|+n+M))} \leq C_k e^{\frac{1}{m}\varphi^*(m|\alpha|)}$, with $m = 2k$, which concludes the proof of (a).

(b) The proof is similar to the one of (a), and we only indicate the main changes. *Necessity.* As in (a), we put $u_N = \chi_N u$. Then, for $|\alpha| \leq N$ and $k \in \mathbb{N}$ there is $C_k > 0$ such that

$$|D^\alpha(\chi_N u)| \leq C_k \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (C_1 N)^{|\beta|} e^{k\varphi^*(|\alpha-\beta|/k)}. \tag{7}$$

Proceeding as in the Roumieu case, the right hand side of (7) is less than or equal to

$$C_k e^{N+k\omega(C_1N)+k\varphi^*(N/k)},$$

and therefore,

$$|\xi|^N|\widehat{u}_N(\xi)| \le DC_k e^{\frac{N}{2}\log n+N+k\omega(C_1N)+k\varphi^*(N/k)},$$

where the constant D depends on the Lebesgue measure of $B_{2r}(x_0)$.

Now, since $\omega(t)=o(t)$ as t tends to infinity, we can find, for each $k\in\mathbb{N}$, a positive constant A_k such that $k\omega(t)\le t+A_k$ for $t>0$. Hence, if we put $\overline{C}_k = DC_ke^{A_k}$ and $B=e^{\frac{\log n}{2}+1+C_1}$, we obtain

$$|\xi|^N|\widehat{u}_N(\xi)| \le \overline{C}_k B^N e^{k\varphi^*(N/k)}.$$

Proceeding as in Lemma 3.2, $(v)\Rightarrow(iv)$, we obtain an estimate like (4).

Sufficiency. As in (a), for $x\in U$, and a fixed $k\in\mathbb{N}$,

$$|D^\alpha u(x)| \le (2\pi)^{-n}\int |\xi^\alpha||\widehat{u}_N(\xi)|d\xi = I_1+I_2,$$

where I_1 denotes the last integral when $|\xi|\le \exp\left(\frac{2k}{N}\varphi^*\left(\frac{N}{2k}\right)\right)$, and I_2 denotes the integral when $|\xi|\ge \exp\left(\frac{2k}{N}\varphi^*\left(\frac{N}{2k}\right)\right)$. Since $|\xi^\alpha|\le|\xi|^{|\alpha|}$,

$$I_1 \le m_nC_1e^{\frac{2k|\alpha|}{N}\varphi^*\left(\frac{N}{2k}\right)}\left(1+e^{\frac{2k}{N}\varphi^*\left(\frac{N}{2k}\right)}\right)^M,$$

being m_n the Lebesgue measure of the set $\{\xi\ :\ |\xi|\le\exp\left(\frac{2k}{N}\varphi^*\left(\frac{N}{2k}\right)\right)\}$ in $\mathbb{R}^n$. We obtain, as in (a),

$$I_1 \le 2^{n+M}C_1\left(e^{\frac{2k}{N}\varphi^*\left(\frac{N}{2k}\right)}\right)^{n+|\alpha|+M} \le 2^{n+M}C_1e^{2k\varphi^*\left(\frac{n+|\alpha|+M}{2k}\right)},$$

when $N=n+|\alpha|+M$, and by (4),

$$\begin{aligned} I_2 &\le \int_{|\xi|\ge e^{\frac{2k}{N}\varphi^*\left(\frac{N}{2k}\right)}} |\xi|^{N-n-1}|\widehat{u}_N(\xi)|d\xi \\ &\le Ce^{2k\varphi^*\left(\frac{N}{2k}\right)}\int_{|\xi|\ge1}\frac{1}{|\xi|^{n+1}}d\xi. \end{aligned}$$

Therefore, we deduce that there is a constant $C'>0$, that depends only on n and M, such that

$$|D^\alpha u| \le C'e^{2k\varphi^*\left(\frac{|\alpha|+n+M}{2k}\right)},$$

on U. Now, from the convexity of φ^* we obtain

$$2k\varphi^*\left(\frac{|\alpha|+n+M}{2k}\right) \le k\varphi^*(|\alpha|/k)+k\varphi^*((n+M)/k). \qquad \square$$

We can now give the definition of quasianalitic (and non quasianlytic) wave front set of a classical distribution in Roumieu and Beurling settings.

Definition 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in \mathcal{D}'(\Omega)$. Let ω be a weight function. The $\{\omega\}-$wave (respectively $(\omega)-$wave) front set $WF_{\{\omega\}}(u)$ (respectively $WF_{(\omega)}(u)$) of u is the complement in $\Omega \times (\mathbb{R}^n \setminus 0)$ of the set of points (x_0, ξ_0) such that there exist an open neighborhood U of x_0 in Ω, a conic neighborhood Γ of ξ_0 and a bounded sequence $u_N \in \mathcal{E}'(\Omega)$ equal to u in U satisfying* (3) (*respectively satisfying* (4)) *in* Γ.

Lemma 3.5. *Let $u \in \mathcal{D}'(\Omega)$ and let K be a compact subset of Ω with non-empty interior, F a closed cone in $\mathbb{R}^n$. Let ω be a weight function. Suppose that $\{\chi_N\} \subset \mathcal{D}(K)$ and, for every $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}$, that*

$$|D^{\alpha+\beta}\chi_N| \leq C_\alpha \left(C_\alpha N\right)^{|\beta|}, \quad |\beta| \leq N. \tag{8}$$

Then $\{\chi_N u\}$ is a bounded sequence in ${\mathcal{E}'}^M$ if u is of order M in a neighborhood of K. Moreover:

(a) *If $WF_{\{\omega\}}(u) \cap (K \times F) = \emptyset$, there exist a constant $C > 0$ and $k \in \mathbb{N}$ such that*

$$|\xi|^N \left|\widehat{\chi_N u}(\xi)\right| \leq C^{N+1} e^{\frac{1}{k}\varphi^*(kN)}, \quad N = 1, 2, \dots, \quad \xi \in F. \tag{9}$$

(b) *If $WF_{(\omega)}(u) \cap (K \times F) = \emptyset$ and $\omega(t) = o(t)$ as t tends to infinity, there is a constant $C > 0$ such that for every $k \in \mathbb{N}$ there is $C_k > 0$ for which*

$$|\xi|^N \left|\widehat{\chi_N u}(\xi)\right| \leq C_k C^N e^{k\varphi^*(\frac{N}{k})}, \quad N = 1, 2, \dots, \quad \xi \in F. \tag{10}$$

Proof. The condition (8) with $\beta = 0$ implies that the sequence χ_N is bounded in $\mathcal{D}(K)$ and hence $\chi_N u$ is bounded in ${\mathcal{E}'}^M$ if u is of order M in a neighborhood of K.

Let $x_0 \in K$, $\xi_0 \in F \setminus \{0\}$ and choose U, Γ and u_N according to Definition 3.4. If the support of χ_N is in U, we have $\chi_N u = \chi_N u_N$.

We first prove (a). By hypothesis u_N satisfies in Γ

$$|\xi|^N |\widehat{u}_N(\xi)| \leq D_1 e^{\frac{1}{k}\varphi^*(Nk)}, \quad N = 1, 2, \dots, \tag{11}$$

for some constant $D_1 > 0$ that does not depend on N. On the other hand, $|\widehat{u}_N(\xi)| \leq D_2(1 + |\xi|)^M$ for some constants $D_2, M > 0$ and all $\xi \in \mathbb{R}^n$, as the sequence u_N is bounded in ${\mathcal{E}'}^M$.

Since the function $\varphi^*(x)/x$ is increasing and $k \geq 1$, it follows from (8) that

$$|D^{\alpha+\beta}\chi_N| \leq C_\alpha \left(C_\alpha e^{\frac{1}{kN}\varphi^*(Nk)}\right)^{|\beta|}, \quad |\beta| \leq N,$$

where C_α denotes again a suitable positive constant depending only on α. Therefore, we obtain

$$|\widehat{\chi}_N(\eta)| \leq C^{N+1} \frac{\exp\left(\frac{1}{kN}\varphi^*(Nk)\right)^N}{\left(|\eta| + \exp\left(\frac{1}{kN}\varphi^*(Nk)\right)\right)^N} (1+|\eta|)^{-n-1-M}, \quad \eta \in \mathbb{R}^n. \tag{12}$$

The properties of the Fourier transform give

$$\widehat{\chi_N u}(\xi) = (2\pi)^{-n} \int \widehat{\chi}_N(\eta)\widehat{u}_N(\xi - \eta)d\eta.$$

Let $0 < c < 1$. We split the integral into two parts:

$$\widehat{\chi_N u}(\xi) = (2\pi)^{-n} \left(\int_{|\eta| \le c|\xi|} \widehat{\chi}_N(\eta) \widehat{u}_N(\xi - \eta) d\eta + \int_{|\eta| \ge c|\xi|} \widehat{\chi}_N(\eta) \widehat{u}_N(\xi - \eta) d\eta \right). \tag{13}$$

In the second integral we have $|\xi - \eta| \le (1 + c^{-1})|\eta|$, and from the boundedness of $\{u_N\}$, it can be estimated by

$$D_2(1 + c^{-1})^M \int_{|\eta| \ge c|\xi|} |\widehat{\chi}_N(\eta)|(1 + |\eta|)^M d\eta. \tag{14}$$

Now, we estimate the first integral in (13) by

$$\|\widehat{\chi}_N\|_{L_1} \sup_{|\eta| \le c|\xi|} |\widehat{u}_N(\xi - \eta)|. \tag{15}$$

On the other hand, for a conic closed neighborhood Γ_1 of ξ_0 contained in $\Gamma \setminus \{0\}$ we can choose the constant c so that $\eta \in \Gamma$ if $\xi \in \Gamma_1$ and $|\xi - \eta| \le c|\xi|$. We observe that $|\eta| \ge (1 - c)|\xi|$ and that the last supremum can be written as $\sup_{|\xi - \eta| \le c|\xi|} |\widehat{u}_N(\eta)|$. Therefore, we conclude, from formulas (11), (12), (14) and (15), and for $N \ge 0$,

$$\begin{aligned} \sup_{\Gamma_1} |\xi|^N \, |\widehat{\chi_N u}(\xi)| &\le (1 - c)^{-N} \|\widehat{\chi}_N\|_{L_1} \sup_{\Gamma} |\widehat{u}_N(\eta)| \cdot |\eta|^N \\ &+ D_2(1 + c^{-1})^{N+M} \int (1 + |\eta|)^M |\eta|^N |\widehat{\chi}_N(\eta)| d\eta \\ &\le (1 - c)^{-N} C_1 C^N D_1 e^{\frac{1}{k}\varphi^*(Nk)} + D_2(1 + c^{-1})^{N+M} C_2 e^{\frac{1}{k}\varphi^*(Nk)}, \end{aligned}$$

where the constants C_1, C_2, D_1, D_2 do not depend on N. Now, F can be covered by a finite number of neighborhoods like Γ_1 so that (9) is valid if $\operatorname{supp} \chi_N \subset U$ for a sufficiently small neighborhood of x_0. We can cover K by such neighborhoods U_j, $j = 1, \ldots, J$, and choose $\chi_{N,j} \in \mathcal{D}(U_j)$ so that $\sum \chi_{N,j} = 1$ in K and $\chi_{N,j}$ satisfies (8) for $j = 1, \ldots, J$. But, if $\chi_N \in \mathcal{D}(K)$ satisfies (8), the same is valid also for the product $\chi_{N,j}\chi_N$ with some other constants. Hence (9) is valid with χ_N replaced by $\chi_{N,j}\chi_N$. The proof of (a) is complete since $\sum \chi_{N,j}\chi_N = \chi_N$.

We now prove (b). By hypothesis, u_N satisfies in Γ

$$|\xi|^N |\widehat{u}_N(\xi)| \le C_k e^{k\varphi^*(N/k)}, \quad N = 1, 2, \ldots,$$

for some sequence $\{C_k\}$ of positive constants that does not depend on N. On the other hand, $|\widehat{u}_N(\xi)| \le D(1 + |\xi|)^M$ for some constants $D, M > 0$ and all $\xi \in \mathbb{R}^n$, as the sequence u_N is bounded in $\mathcal{E}'^M$.

It follows from (8) that

$$|\widehat{\chi}_N(\eta)| \le C^{N+1} \frac{N^N}{(|\eta| + N)^N} (1 + |\eta|)^{-n-1-M}, \quad \eta \in \mathbb{R}^n.$$

Now we can proceed as in the Roumieu case (a) taking into account that $N^N \le e^N N!$ and the following property: as $\omega(t) = o(t)$ as t tends to infinity, from Remark 2.4(b) and Lemma 3.2, it follows that for every $B > 0$ and $\lambda > 0$ there is a constant $C_{B,\lambda} > 0$ such that

$$B^n n! \leq C_{B,\lambda} e^{\lambda\varphi^*(\frac{n}{\lambda})}, \quad n \in \mathbb{N}. \qquad \square$$

At this point, we can give some properties and consequences.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set and ω be a weight function. Then the projection of $WF_*(u)$ in Ω is equal to* $\mathrm{sing}_*\mathrm{supp}\, u$ *if $u \in \mathcal{D}'(\Omega)$.*

Proof. We give the proof only for the Roumieu case. The Beurling case is similar. If u is a $\mathcal{E}_{\{\omega\}}$-function in a neighborhood of x_0 it follows from Proposition 3.3 that $(x_0, \xi_0) \notin WF_{\{\omega\}}(u)$ for each $\xi_0 \in \mathbb{R}^n \setminus \{0\}$. Assume that $(x_0, \xi_0) \notin WF_{\{\omega\}}(u)$ for every $\xi_0 \in \mathbb{R}^n \setminus \{0\}$. Then we can choose a compact neighborhood K of x_0 so that $WF_{\{\omega\}}(u) \cap (K \times \mathbb{R}^n) = \emptyset$. By Lemma 3.5 there is a sequence $\chi_N \in \mathcal{D}(K)$ which is equal to 1 in a neighborhood of x_0 such that $\chi_N u$ is bounded in $\mathcal{E}'$ and satisfies (3). Therefore, $x_0 \notin \mathrm{sing}_{\{\omega\}}\mathrm{supp}\, u$ by Proposition 3.3. $\square$

The condition (8) is satisfied by any fixed function in $\mathcal{E}_*(\Omega)$ with support in K, where $* = \{\omega\}$ or (ω). Therefore, if ω is a non-quasianalytic weight function, an equivalent definition of wave front set is given by the following proposition.

Proposition 3.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in \mathcal{D}'(\Omega)$. Let ω be a nonquasianalytic weight function. Then $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus 0)$ is not in the wave front set $WF_{\{\omega\}}(u)$ (resp. $WF_{(\omega)}(u)$) of u if and only if there is a neighborhood $U \subset \Omega$ of x_0, a conic neighborhood Γ of ξ_0 and $v \in \mathcal{E}'_{\{\omega\}}(\Omega)$ (resp. $v \in \mathcal{E}'_{(\omega)}(\Omega)$) which is equal to u in U and has a Fourier transform satisfying* (3) *(resp.* (4)*) in Γ.*

Combining Proposition 3.7 with Proposition 3.3 and Lemmas 3.1 and 3.2, we recover the definition of wave front set in the Gevrey setting, [20, p. 36], and in the nonquasianalytic Beurling setting, [11].

The properties of the Young conjugate φ^*_ω of φ_ω for a given weight function ω lead to the following property:

Proposition 3.8. *Let $\Omega \subset \mathbb{R}^n$ be an open set. If ω and σ are two weight functions such that $\omega = O(\sigma)$, then $WF_{\{\omega\}}(u) \subset WF_{\{\sigma\}}(u)$, and $WF_{(\omega)}(u) \subset WF_{(\sigma)}(u)$, for each $u \in \mathcal{D}'(\Omega)$.*

Since $\log t = o(\omega)$, and $\omega = O(\omega_1)$, where $\omega_1(t) = \max(t-1, 0)$ for every weight function ω, we observe that

$$WF(u) \subset WF_{\{\omega\}}(u) \subset WF_A(u), \quad u \in \mathcal{D}'(\Omega),$$

being $WF(u)$ the classical wave front set, and $WF_A(u)$ the Roumieu wave front set with respect to ω_1. Moreover, if $\omega(t) = o(t)$ as t tends to infinity, we have

$$WF(u) \subset WF_{\{\omega\}}(u) \subset WF_{(\omega)}(u) \subset WF_A(u), \quad u \in \mathcal{D}'(\Omega).$$

Proposition 3.9. *If $\Omega \subset \mathbb{R}^n$ is an open set and S is a closed cone in $\Omega\times(\mathbb{R}^n\backslash 0)$, then there is $u \in \mathcal{D}'(\Omega)$ with*

$$WF(u) = WF_{\{\omega\}}(u) = S \quad (\textit{respectively, } WF(u) = WF_{(\omega)}(u) = S)$$

for every weight function ω (respectively, for every weight function ω satisfying $\omega(t) = o(t)$ as t tends to infinity).

Proof. By [13, Theorem 8.4.14], there is $u \in \mathcal{D}'(\Omega)$ such that

$$WF(u) = WF_A(u) = S.$$

Since

$$WF(u) \subset WF_{\{\omega\}}(u) \subset WF_A(u)$$

(respectively,

$$WF(u) \subset WF_{(\omega)}(u) \subset WF_A(u))$$

for every weight function ω (respectively, for every weight function ω satisfying $\omega(t) = o(t)$ as t tends to infinity), the result follows. □

The conditions in (8) remain valid if we multiply all χ_N by the same function in $\mathcal{E}_*(\Omega)$. Therefore, we obtain

Theorem 3.10. *$WF_*(au) \subset WF_*(u)$ if $a \in \mathcal{E}_*(\Omega)$ and $u \in \mathcal{D}'(\Omega)$.*

On the other hand, it is clear that

$$WF_{\{\omega\}}(\partial u/\partial x_j) \subset WF_{\{\omega\}}(u).$$

In fact,

$$\begin{aligned}\left|\widehat{\frac{\partial u_{N+1}}{\partial x_j}}(\xi)\right| &= |\xi_j \widehat{u}_{N+1}(\xi)| \leq C|\xi| \left(e^{\frac{1}{(N+1)k}\varphi^*((N+1)k)}/|\xi|\right)^{N+1} \\ &= Ce^{\frac{1}{k}\varphi^*((N+1)k)}/|\xi|^N \leq C|\xi|^{-N} e^{\frac{1}{2k}\varphi^*(2Nk)} e^{\frac{1}{2k}\varphi^*(2k)}.\end{aligned}$$

If we combine this property with Theorem 3.10, we obtain

Theorem 3.11. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $P(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ be a linear partial differential operator with coefficients in $\mathcal{E}_*(\Omega)$. Then*

$$WF_{\{\omega\}}(P(x, D)u) \subset WF_{\{\omega\}}(u), \quad u \in \mathcal{D}'(\Omega),$$

(respectively,

$$WF_{(\omega)}(P(x, D)u) \subset WF_{(\omega)}(u), \quad u \in \mathcal{D}'(\Omega))$$

for every weight function ω (respectively, for every weight function ω satisfying $\omega(t) = o(t)$ as t tends to infinity).

4. Propagation of singularities

In this section we prove a converse of Theorem 3.11 related to the propagation of singularities of solutions of linear partial differential operators. We first study the Beurling case. The corresponding Roumieu version is then obtained as a consequence of a description of Roumieu wave front sets via a suitable union of Beurling wave front sets.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let σ be a weight function satisfying property (α_0) and ω be a weight function such that $\omega(t) = o(\sigma(t))$ as t tends to infinity. If $P := P(x, D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ is a linear partial differential operator with coefficients in the Roumieu class $\mathcal{E}_{\{\sigma\}}(\Omega)$, then*

$$WF_{(\omega)}(u) \subset WF_{(\omega)}(Pu) \cup \Sigma, \quad u \in \mathcal{D}'(\Omega), \tag{16}$$

where $P_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is the principal symbol of P and Σ the characteristic set of P which is defined by

$$\Sigma = \{(x, \xi) \in \Omega \times (\mathbb{R}^n \setminus 0) \,:\, P_m(x, \xi) = 0\}\,.$$

Proof. We must prove that if (x_0, ξ_0) does not belong to the right hand side of (16) and $\xi_0 \neq 0$, then $(x_0, \xi_0) \notin WF_{(\omega)}(u)$. If we assume this hypothesis, we can choose a compact neighborhood K of x_0 and a closed conic neighborhood Γ of ξ_0 in $\mathbb{R}^n \setminus 0$ such that

$$P_m(x, \xi) \neq 0 \;\text{ in } K \times \Gamma, \tag{17}$$

$$(K \times \Gamma) \cap WF_{(\omega)}(Pu) = \emptyset. \tag{18}$$

Now, we take a sequence $\chi_N \in \mathcal{D}(K)$ equal to 1 in a fixed neighborhood U of x_0 satisfying property (8) for every α. Then, the sequence $u_N = \chi_{2N} u$ is bounded in $\mathcal{E}'$ and equal to u in U. The theorem will be proved if we show that (4) is valid for the weight ω when $\xi \in \Gamma$ and $|\xi| \ge N$, since (4) is true for $|\xi| \le N$. In fact, when $|\xi| \le N$ we can argue in the following way. As in Proposition 3.3, $|\widehat{u_N}(\xi)| \le C_1(1 + |\xi|)^M \le C_1(1 + N)^M \le C^N$ for some positive constants C_1, C and a natural number $M \in \mathbb{N}$. On the other hand, since $\omega(t) = o(t)$ as t tends to infinity, we have that, for each $k \in \mathbb{N}$, there exists $C_k > 0$ satisfying $N^N \le C_k e^{k\varphi^*_\omega(N/k)}$. Then, we obtain

$$|\xi|^N |\widehat{u_N}(\xi)| \le C^N N^N \le C_k C^N e^{k\varphi^*_\omega(N/k)}.$$

To estimate $\widehat{u}_N(\xi)$ in Γ we will solve in an approximate way the equation

$$P^t v(x) = \chi_{2N}(x) e^{-i\langle x, \xi\rangle}. \tag{19}$$

Following [14], we put $v = e^{-i\langle x,\xi\rangle} w / P_m(x, \xi)$ and observe that the principal symbol of P^t is $P_m(x, -\xi)$. Hence, we obtain instead of (19) an equation of the form

$$w - Rw = \chi_{2N}, \quad R = R_1 + R_2 + \cdots + R_m \tag{20}$$

where $R_j |\xi|^j$ is a differential operator of order less than or equal to j with $\mathcal{E}_{\{\sigma\}}$-coefficients which is homogeneous of degree 0 with respect to ξ for $\xi \in \Gamma$ and $x \in K$.

We now write

$$w_N = \sum_{j_1+\cdots+j_k\le N-m} R_{j_1}\cdots R_{j_k}\chi_{2N}. \tag{21}$$

From this formula, we obtain

$$\begin{aligned} w_N - Rw_N &= \chi_{2N} - \sum_{j_1+\cdots+j_k>N-m\ge j_2+\cdots+j_k} R_{j_1}\cdots R_{j_k}\chi_{2N} \\ &= \chi_{2N} - e_N, \end{aligned} \tag{22}$$

which means that

$$P^t(x,D)(e^{-i\langle x,\xi\rangle}w_N(x,\xi)/P_m(x,\xi)) = e^{-i\langle x,\xi\rangle}(\chi_{2N}(x) - e_N(x,\xi)).$$

With integrals denoting action of distributions we obtain, with $f = P(x,D)u$,

$$\begin{aligned} \int u(x)\chi_{2N}(x)e^{-i\langle x,\xi\rangle}dx &= \int u(x)e_N(x,\xi)e^{-i\langle x,\xi\rangle}dx \\ &\quad + \int f(x)e^{-i\langle x,\xi\rangle}w_N(x,\xi)/P_m(x,\xi)dx. \end{aligned} \tag{23}$$

To estimate the right-hand side of (23) we need the following lemma:

Lemma 4.2. *There exists a constant $C' > 0$ such that for $j = j_1+\cdots+j_k$ and $j+|\beta|\le 2N$,*

$$|D^\beta R_{j_1}\cdots R_{j_k}\chi_{2N}| \le C'^{N+1}\left(e^{\frac{1}{2hN}\varphi^*_\sigma(2hN)}\right)^{j+|\beta|}|\xi|^{-j}, \quad \xi\in\Gamma. \tag{24}$$

Proof of Lemma 4.2. By homogeneity it suffices to prove the lemma when $|\xi| = 1$ $(x\in K)$. But then, this lemma is a consequence of the next one. □

Lemma 4.3. *Let $K\subset\Omega$ be a compact set and $\chi_N\in\mathcal{D}(K)$ be a sequence satisfying property* (8). *If $a_1,\ldots,a_{j-1}$ are functions in $\mathcal{E}_{\{\sigma\}}(\Omega)$ such that, for some constant $C>0$ and some $h\in\mathbb{N}$, that*

$$\sup_K |D^\alpha a_s| \le Ce^{\frac{1}{h}\varphi^*_\sigma(h|\alpha|)} \tag{25}$$

for every $\alpha\in\mathbb{N}_0^n$ and $s = 1,\ldots,j-1$, we have, for some constant $C_1>0$ that depends only on C and the sequence $\{\chi_N\}$, and $j\le N$,

$$\sup_K |D_{i_1}a_1D_{i_2}\cdots a_{j-1}D_{i_j}\chi_N| \le C_1^{j+1}\left(e^{\frac{1}{hN}\varphi^*_\sigma(hN)}\right)^j \tag{26}$$

Proof of Lemma 4.3. Since $\sigma(t) = O(t)$ as t tends to infinity, $N^N \le C^Ne^{\varphi^*_\sigma(N)}$ for a sufficiently large constant $C>0$. Therefore, from (8), for $\alpha = 0$, we obtain

$$|D^\beta\chi_N| \le C_0\left(C_0e^{\frac{1}{N}\varphi^*_\sigma(N)}\right)^{|\beta|}, \quad |\beta|\le N. \tag{27}$$

It is clear that $D_{i_1}a_1D_{i_2}\cdots a_{j-1}D_{i_j}\chi_N$ is a sum of terms of the form

$$(D^{\alpha_1}a_1)\cdots(D^{\alpha_{j-1}}a_{j-1})D^{\alpha_j}\chi_j$$

with $|\alpha_1|+\cdots+|\alpha_j| = j$.

If there are $C_{k_1,\ldots,k_j}$ terms with $|\alpha_1| = k_1,\ldots,|\alpha_j| = k_j$, we have

$$
\begin{aligned}
&|D_{i_1} a_1 D_{i_2} \cdots a_{j-1} D_{i_j} \chi_N| \\
\leq& \sum C_{k_1,\ldots,k_j} C^{j-1} e^{\frac{1}{h}\varphi^*_\sigma(hk_1)} \cdots e^{\frac{1}{h}\varphi^*_\sigma(hk_{j-1})} C_0 C_0^{k_j} \left(e^{\frac{1}{N}\varphi^*_\sigma(N)}\right)^{k_j} \qquad (28)\\
\leq& \sum C_{k_1,\ldots,k_j} k_1! \cdots k_{j-1}! C^{j-1} \frac{e^{\frac{1}{h}\varphi^*_\sigma(hk_1)}}{k_1!} \cdots \frac{e^{\frac{1}{h}\varphi^*_\sigma(hk_{j-1})}}{k_{j-1}!} C_0 C_0^{k_j} \left(e^{\frac{1}{N}\varphi^*_\sigma(N)}\right)^{k_j}.
\end{aligned}
$$

As in [10], since σ satisfies property (α_0), we can suppose that σ is equivalent to a sub-additive weight and then, we have

$$\frac{e^{\frac{1}{h}\varphi^*_\sigma(hk_1)}}{k_1!} \cdots \frac{e^{\frac{1}{h}\varphi^*_\sigma(hk_{j-1})}}{k_{j-1}!} \leq \frac{e^{\frac{1}{h}\varphi^*_\sigma(h(j-k_j))}}{(j-k_j)!}.$$

We also observe that

$$\frac{k_1! \cdots k_{j-1}!}{(j-k_j)!} = \frac{k_1! \cdots k_j! j!}{(j-k_j)! k_j! j!} \leq 2^j \frac{k_1! \cdots k_j!}{j!}$$

and that $\sum C_{k_1,\ldots,k_j} k_1! \cdots k_j! = (2j-1)!!$. We can assume that $C, C_0 > 1$ and hence, if we put $C_1 = CC_0$, we have $C^{j-1} C_0 C_0^{k_j} \leq C_1^{j+1}$. Since $\varphi^*_\sigma(x)/x$ is increasing, from (28), we obtain

$$
\begin{aligned}
&|D_{i_1} a_1 D_{i_2} \cdots a_{j-1} D_{i_j} \chi_N| \\
\leq& C_1^{j+1} 2^j / j! \sum C_{k_1,\ldots,k_j} k_1! \cdots k_j! e^{\frac{1}{h}\varphi^*_\sigma(h(j-k_j))} \left(e^{\frac{1}{hN}\varphi^*_\sigma(hN)}\right)^{k_j} \\
\leq& (2C_1)^{j+1} / j! \sum C_{k_1,\ldots,k_j} k_1! \cdots k_j! \left(e^{\frac{1}{h(j-k_j)}\varphi^*_\sigma(h(j-k_j))}\right)^{j-k_j} \left(e^{\frac{1}{hN}\varphi^*_\sigma(hN)}\right)^{k_j} \\
\leq& (4C_1)^{j+1} \left(e^{\frac{1}{hN}\varphi^*_\sigma(hN)}\right)^j \frac{(2j-1)!!}{j! 2^j},
\end{aligned}
$$

which concludes the proof of the lemma since $\frac{(2j-1)!!}{j!2^j} \leq 1$. □

We now finish the proof of Theorem 4.1. If M is the order of u in a neighborhood of K, we can estimate the first term on the right-hand side of (23) for large N and $|\xi| > N$, with $\xi \in \Gamma$, by

$$C \sum_{|\alpha| \leq M} (1 + |\xi|)^{M-|\alpha|} \sup_x |D^\alpha e_N(x, \xi)|.$$

The number of terms in e_N cannot exceed 2^N, and each term can be estimated by means of (24), where $j_1 + \ldots + j_k = j > N - m$ by (22), which gives the bound

$$|D_x^\alpha e_N(x, \xi)| \leq C'^{N+1} 2^N \left(e^{\frac{1}{2hN}\varphi^*_\sigma(2hN)}\right)^{N+|\alpha|} |\xi|^{-N+m}.$$

Therefore, the first term on the right-hand side of (23) can be estimated by

$$C'^{N+1} 2^{N+M} \left(e^{\frac{1}{2hN}\varphi^*_\sigma(2hN)}\right)^{N+M} |\xi|^{M-N+m}. \qquad (29)$$

Now, as φ_σ^* is convex and $\varphi_\sigma^*(x)/x$ is increasing, we have

$$\begin{aligned}\left(e^{\frac{1}{2hN}\varphi_\sigma^*(2hN)}\right)^{N+M} &\le \left(e^{\frac{1}{2h(N+M)}\varphi_\sigma^*(2h(N+M))}\right)^{N+M} \\ &\le e^{\frac{1}{2h}\varphi_\sigma^*(2h(N+M))} \le e^{\frac{1}{4h}\varphi_\sigma^*(4hM)}e^{\frac{1}{4h}\varphi_\sigma^*(4hN)} \\ &= C_{h,M}e^{\frac{1}{4h}\varphi_\sigma^*(4hN)}.\end{aligned} \tag{30}$$

Since $\omega(t) = o(\sigma(t))$ as t tends to ∞, if N is replaced by $N + m + M$, the bound (29) and (30) imply an estimate of the form (11) for the first integral on the right of (23). Indeed, from $\omega(t) = o(\sigma(t))$ as t tends to ∞ we deduce that for every $k \in \mathbb{N}$ there exists $d_k > 0$ such that

$$\frac{1}{4h}\varphi_\sigma^*(4hN) \le d_k + k\varphi_\omega^*\left(\frac{N}{k}\right), \quad N \in \mathbb{N}. \tag{31}$$

Now, combining (29) togheter (31) and (30) with N replaced by $N+m+M$, we easily obtain an estimate of the form (11) for the first integral on the right of (23).

To estimate the last term in (23) we observe that (24) gives

$$|D^\beta w_N| \le C_1^{N+1}\left(e^{\frac{1}{2hN}\varphi_\sigma^*(2hN)}\right)^{|\beta|}, \quad |\beta| \le N,\ \xi \in \Gamma,\ |\xi| > N. \tag{32}$$

We have a similar bound for $w_N|\xi|^m/P_m(x,\xi)$. The proof is completed by the following lemma:

Lemma 4.4. *Let $f \in \mathcal{D}'(\Omega)$. Let $K \subset \Omega$ be a compact set and Γ a closed cone $\subset \mathbb{R}^n \setminus 0$ such that*

$$WF_{(\omega)}(f) \cap (K \times \Gamma) = \emptyset.$$

If $w_N \in \mathcal{D}(K)$ and (32) is fulfilled, then there exists $C_2 > 0$ such that for every $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$|\widehat{w_N f}(\xi)| \le C_k C_2^N \left(e^{\frac{k}{N-M-n}\varphi_\omega^*\left(\frac{N-M-n}{k}\right)}/|\xi|\right)^{N-M-n} \tag{33}$$

if $\xi \in \Gamma$, $|\xi| > N$, and $N > M+n$. Here M is the order of f in a neighborhood of K.

Proof. By Lemma 3.5 we can find a sequence f_N which is bounded in $\mathcal{E}'^M$ and equal to f in a neighborhood of K so that there exists $D > 0$ such that for every $k \in \mathbb{N}$ there exists $D_k > 0$ for which

$$|\widehat{f}_N(\eta)| \le D_k D^N \left(e^{\frac{k}{N}\varphi_\omega^*(N/k)}/|\eta|\right)^N, \quad N = 0,1,2,\ldots, \quad \eta \in \Gamma',$$

where Γ' is a conic neighborhood of Γ. Then $w_N f = w_N f_{N'}$, $N' = N-M-n$. On the other hand, using (32) and proceeding as in Lemma 3.5 to show (12), we obtain that there exists $E > 0$ such that

$$|\widehat{w}_N(\eta)| \le E^{N+1}(e^{\frac{1}{2hN}\varphi_\sigma^*(2hN)}/(e^{\frac{1}{2hN}\varphi_\sigma^*(2hN)} + |\eta|))^N, \quad \eta \in \mathbb{R}^n.$$

Proceeding again as in the proof of Lemma 3.5, it follows (the integrals that appear there are convergent from the selection of N'), that there exists $D' > 0$ such that for every $k \in \mathbb{N}$ there exists $D'_k > 0$ for which

$$\sup_{\xi\in\Gamma} |\xi|^{N'} |\widehat{w_N f}(\xi)| \leq D'_k D'^N \left(e^{k\varphi^*_\omega(N'/k)} + e^{\frac{1}{2h}\varphi^*_\sigma(2hN)} \right), \quad \xi \in F,\ |\xi| > N. \tag{34}$$

We now observe that the convexity of φ^*_σ implies that

$$\frac{1}{2h}\varphi^*_\sigma(2hN) \leq \frac{1}{4h}\left[\varphi^*_\sigma(4h(N-M-n)) + \varphi^*_\sigma(4h(M+n))\right], \quad N > M+n. \tag{35}$$

On the other hand, $\omega(t) = o(\sigma(t))$ as t tends to ∞ and hence, for every $k \in \mathbb{N}$ there exists $D''_k > 0$ such that

$$\frac{1}{4h}\varphi^*_\sigma(4hN) \leq D''_k + k\varphi^*_\omega\left(\frac{N}{k}\right), \quad N \in \mathbb{N}. \tag{36}$$

Combining (34) with (35) and (36), (33) follows. Thus the lemma is proved. □

The last lemma gives the key to estimate the second integral on the right-hand side of (23), and finishes the proof of Theorem 4.1. □

An examination of the proofs above shows that we can repeat the same arguments to prove the analogous result of Theorem 4.1 in the Roumieu setting, even assuming the coefficients of $P(x, D)$ in $\mathcal{E}_{\{\omega\}}(\Omega)$ with ω a weight function satisfying property (α_0). We point out that it is necessary only to state and prove Lemma 4.4 appropriately. But, we also present another proof of the Roumieu version based on an application of Theorem 4.1 and of the following proposition, which is an extension to the quasianalytic case of [11, Proposition 2] and could be of independent interest.

Given two weight functions σ_0 and ω such that $\sigma_0(t) = o(\omega(t))$ as t tends to infinity, we set

$$S := \{\sigma \text{ weight function}: \ \sigma_0 \leq \sigma = o(\omega)\}.$$

Proposition 4.5. *Let ω be weight function. If σ_0 and σ are as above, we have*

$$WF_{\{\omega\}}(u) = \overline{\bigcup_{\sigma\in S} WF_{(\sigma)}(u)}, \quad u \in \mathcal{D}'(\Omega).$$

Proof. The inclusion $\overline{\cup_{\sigma\in S} WF_{(\sigma)}(u)} \subset WF_{\{\omega\}}(u)$ follows easily from the definition of S, and the facts that $\mathcal{E}_{\{\omega\}}(\Omega) \subset \mathcal{E}_{(\sigma)}(\Omega)$ if $\sigma = o(\omega)$ (see Remark 2.4) and that the wave front set is closed.

Conversely, suppose that $(x_0, \xi_0) \notin \Gamma := \overline{\cup_{\sigma\in S} WF_{(\sigma)}(u)}$. We can then find a compact neighborhood K of x_0 and a closed conic neighborhood F of ξ_0 in $\mathbb{R}^n \setminus 0$ such that

$$(K \times F) \cap \overline{\cup_{\sigma\in S} WF_{(\sigma)}(u)} = \emptyset.$$

For each $N \in \mathbb{N}$ let $\chi_N \in \mathcal{D}(K)$ be equal to 1 in a fixed neighborhood U of x_0 such that, for every $\alpha \in \mathbb{N}_0^n$,

$$|D^{\alpha+\beta}\chi_N| \le C_\alpha (C_\alpha N)^{|\beta|}, \quad |\beta| \le N.$$

Then, by Lemma 3.5. (b), $\chi_N u$ is a bounded sequence in $\mathcal{E}'^M$ if M is the order of u in a neighborhood of K and, for every $\sigma \in S$ and $k \in \mathbb{N}$, there is $C_k^\sigma > 0$ so that

$$|\xi|^N |\widehat{\chi_N u}(\xi)| \le C_k^\sigma e^{k\varphi_\sigma^*(N/k)}, \quad \xi \in F, \; N = 0, 1, 2, \dots. \tag{37}$$

We will deduce from this fact that $(x_0, \xi_0) \notin WF_{\{\omega\}}(u)$, after showing that there exist $C > 0$ and $h \in \mathbb{N}$ for which

$$|\xi|^N |\widehat{\chi_N u}(\xi)| \le C^{N+1} e^{\frac{1}{h}\varphi_\omega^*(hN)}, \quad \xi \in F, \; N = 0, 1, 2, \dots. \tag{38}$$

In order to prove such an inequality, we will proceed as follows. By (37) we obtain that, for every $\sigma \in S$, $k \in \mathbb{N}$, and $r > 0$,

$$g_N(r) := r^N \sup_{|\xi| = r, \xi \in F} |\widehat{\chi_N u}(\xi)| \le C_k^\sigma e^{k\varphi_\sigma^*(N/k)}, \qquad N = 0, 1, 2, \dots$$

and hence

$$\sup_{r>0} g_N(r) \le C_k^\sigma e^{k\varphi_\sigma^*(N/k)}, \qquad N = 0, 1, 2, \dots.$$

This implies that, for every $\sigma \in S$ and $k \in \mathbb{N}$,

$$a_N := \log \sup_{r>0} g_N(r) \le k\varphi_\sigma^*\left(\frac{N}{k}\right) + \log C_k^\sigma, \qquad N = 0, 1, 2, \dots. \tag{39}$$

We claim that (39) implies that there exist $h \in \mathbb{N}$ and $C > 0$ so that

$$a_N \le \frac{1}{h}\varphi_\omega^*(Nh) + C, \quad N = 0, 1, 2, \dots. \tag{40}$$

Proceeding by contradiction we can construct an increasing sequence $(N(h))_h$ of positive integers ($N(1) := 0$) such that

$$a_{N(h)} > \frac{1}{h}\varphi_\omega^*(N(h)h) + C_h, \tag{41}$$

for every $h \in \mathbb{N}$ ($C_h := h$).

We will show that inequalities (41) are in contradiction with inequalities (39) by constructing a weight function $\sigma \in S$ for which the inequality

$$\frac{1}{h}\varphi_\omega^*(N(h)h) + C_h < \varphi_\sigma^*(N(h)) + \log C_1^\sigma \tag{42}$$

does not hold for infinitely many indices $h \in \mathbb{N}$.

Without loss of generality, we can suppose that $\omega_{|[R,+\infty[}$ is a C^1 function for some $R \ge 0$ (see [9, Lemma 1.7]). Then the function $\varphi := \varphi_\omega$ is a C^1 function too on $[R, +\infty[$ and φ' is a nondecreasing continuous function on $[R, +\infty[$. In particular, $\lim_{x\to+\infty} \varphi'(x) = +\infty$ because $\log(1+t) = o(\omega(t))$ as t tends to infinity. Then, $\varphi'([0, +\infty[) \supset [\varphi'(R), +\infty[$.

Consequently, we can find an increasing sequence $(x_h)_h \subset [0, +\infty[$ ($x_1 := 0$) satisfying $x_h \to \infty$ and $\varphi'(x_h) = N(h)h$ for all $h \in \mathbb{N}$. Set $x_1 = y_1 = z_1 = 0$. As $x_h \to \infty$, we can inductively define an increasing sequence $(h(n))_n$ of positive integers $(h(0) = h(1) = 1)$ and the sequences $(y_n)_n$ and $(z_n)_n$ with $x_{h(2)} > R$, $y_2 = z_2 = x_{h(2)}$ and for all $n \geq 3$

$$N(h(n)) > \frac{N(h(n-1))h(n-1)}{h(n-3)}, \tag{43}$$

$$x_{h(n)} > y_{n-1} + n, \tag{44}$$

$$\sigma_0(e^x) \leq \frac{\varphi(x)}{h^2(n-1)} \text{ for all } x \geq x_{h(n)}, \tag{45}$$

$$\varphi(x_{h(n)}) \geq h(n-1) \sum_{i=1}^{n-1} \varphi(z_i), \tag{46}$$

$$\varphi'(y_n) = \frac{h(n-1)}{h(n-2)} \varphi'(x_{h(n)}), \tag{47}$$

and

$$\begin{aligned} &[h(n-1) - h(n-2)]\varphi(z_n) \\ &= h(n-1)\varphi(x_{h(n)}) - h(n-2)\varphi(y_n) + h(n-1)(y_n - x_{h(n)})\varphi'(x_{h(n)}). \end{aligned} \tag{48}$$

We have that

$$x_{h(n)} \leq z_n \leq y_n \tag{49}$$

for all $n \in \mathbb{N}$.

From (47) we get that $y_n \geq x_{h(n)}$ as φ' is a nondecreasing function. Hence, by (48) we get

$$\begin{aligned} &\frac{\varphi(z_n) - \varphi(x_{h(n)})}{y_n - x_{h(n)}} \\ &= \frac{1}{h(n-1) - h(n-2)} \\ &\quad \times \frac{h(n-1)\varphi(x_{h(n-1)}) - h(n-2)\varphi(y_n) - (h(n-1) - h(n-2))\varphi(x_{h(n)})}{y_n - x_{h(n)}} \\ &\quad + \frac{h(n-1)}{h(n-1) - h(n-2)} \varphi'(x_{h(n)}) \\ &= \frac{h(n-2)}{h(n-1) - h(n-2)} \frac{\varphi(x_{h(n)}) - \varphi(y_n)}{y_n - x_{h(n)}} + \frac{h(n-1)}{h(n-1) - h(n-2)} \varphi'(x_{h(n)}) \\ &\geq \frac{h(n-2)}{h(n-1) - h(n-2)} \left[\varphi'(x_{h(n)}) - \frac{\varphi(y_n) - \varphi(x_{h(n)})}{y_n - x_{h(n)}} \right] \geq 0, \end{aligned}$$

because φ' is a nondecreasing function. On the other hand, (48) also implies that

$$\begin{aligned} [h(n-1) - h(n-2)]\varphi(z_n) &= h(n-1)[\varphi(x_{h(n)}) + (y_n - x_{h(n)})\varphi'(x_{h(n)})] \\ &\quad - h(n-2)\varphi(y_n) \leq [h(n-1) - h(n-2)]\varphi(y_n) \end{aligned}$$

because φ is a convex function.

We define a function ψ on $[0,+\infty[$ by setting

$$\psi(x)=\begin{cases}\frac{1}{h(n-2)}\varphi(x_{h(n)})+\sum_{i=1}^{n-2}\frac{h(i)-h(i-1)}{h(i-1)h(i)}\varphi(z_{i+1}) & \text{if } x_{h(n)}\le x<y_n,\\ +\frac{x-x_{h(n)}}{h(n-2)}\varphi'(x_{h(n)}) & \\ \frac{1}{h(n-1)}\varphi(x)+\sum_{i=1}^{n-1}\frac{h(i)-h(i-1)}{h(i-1)h(i)}\varphi(z_{i+1}) & \text{if } y_n\le x\le x_{h(n+1)}.\end{cases} \tag{50}$$

From (47) and (48) it follows that ψ is a C^1 function. Moreover, it is convex because it consists of linear parts and of dilated and shifted parts of φ. We define σ by $\sigma(t)=\psi(\max(\log t,0))$, which is again a C^1 function.

Proceeding as in the proof of [9, Lemma 1.7], it is easily seen that

$$\psi(x)\ge\frac{1}{h(n-1)}\varphi(x)\quad\text{for all } x\in[x_{h(n)},x_{h(n+1)}],\ n\ge 3;$$

hence by (45) we deduce

$$\sigma_0(t)\le\frac{1}{h^2(n-1)}\varphi(x)\le\frac{1}{h(n-1)}\psi(x)\quad\text{for all } x\in[x_{h(n)},x_{h(n+1)}],\ n\ge 3.$$

Therefore $\sigma_0(t)=o(\sigma(t))$ as $t\to\infty$. We also have

$$\frac{\psi(x)}{\varphi(x)}\le\frac{2}{h(n-2)}\quad\text{for all } x\in[x_{h(n)},x_{h(n+1)}],\ n\ge 2,$$

and hence, $\sigma(t)=o(\omega(t))$ as $t\to\infty$.

Now, we have, for every $n\in\mathbb{N}$, that

$$\varphi^*(N(h(n))h(n))=N(h(n))h(n)x_{h(n)}-\varphi(x_{h(n)}). \tag{51}$$

Indeed,

$$\varphi'(x_{h(n)})=N(h(n))h(n),$$

and, since φ' is nondecreasing, $N(h(n))h(n)-\varphi'(s)\ge 0$ for all $0\le s\le x_{h(n)}$. Moreover, $N(h(n))h(n)s-\varphi(s)\to-\infty$ as $s\to+\infty$ and $(N(h(n))h(n)s-\varphi(s))(0)=0$.

On the other hand, by (50) and proceeding as above we deduce, for every $n\in\mathbb{N}$, that

$$\psi^*(N(h(n)))=N(h(n))\zeta_n-\psi(\zeta_n), \tag{52}$$

where $\psi'(\zeta_n)=N(h(n))$ (indeed, the function $\delta_n(s)=N(h(n))s-\psi(s)$ is C^1 with derivative $\delta_n'(s)=N(h(n))-\psi'(s)$ and hence $\delta_n'(s)\ge 0$ if and only if $\psi'(s)\le N(h(n))$). In particular, $\zeta_n\in[y_{n-1},x_{h(n)}]$. In fact, if $x\in[y_{n-1},x_{h(n)}]$ then, by (50),

$$\frac{\varphi'(y_{n-1})}{h(n-2)}\le\psi'(x)\le\frac{\varphi'(x_{h(n)})}{h(n-2)}.$$

Moreover, by (43) and (47),

$$\frac{\varphi'(y_{n-1})}{h(n-2)}=\frac{1}{h(n-3)}\varphi'(x_{h(n-1)})=\frac{N(h(n-1))h(n-1)}{h(n-3)}<N(h(n)),$$

while

$$\frac{\varphi'(x_{h(n)})}{h(n-2)}=\frac{N(h(n))h(n)}{h(n-2)}>N(h(n)).$$

Consequently, we have $N(h(n)) \in \psi'([y_{n-1}, x_{h(n)}])$ and so we can conclude from the fact that ψ' is continuous.

Therefore, we have constructed a C^1–function $\sigma\colon [0,+\infty[\to [0,+\infty[$ such that $\sigma_0(t) = o(\sigma(t))$ and $\sigma(t) = o(\omega(t))$ as $t \to +\infty$. Then, by [9, Lemma 1.7] we can find a weight function v such that $\sigma(t) = o(v(t))$ and $v(t) = o(\omega(t))$ as $t \to +\infty$ (hence, $v \in S$), and

$$\psi(t) \le \varphi_v(t) \quad \text{for all } t \ge 0.$$

This implies that

$$\varphi_v^*(t) \le \psi^*(t) \quad \text{for all } t \ge 0. \tag{53}$$

As $v \in S$, by (39) and (41) we then obtain

$$\frac{1}{h(n)}\varphi^*(N(h(n))h(n)) + h(n) < \varphi_v^*(N(h(n))) + \log C_1^v,$$

for all $n \in \mathbb{N}$. Thus, by (53) we get

$$\frac{1}{h(n)}\varphi^*(N(h(n))h(n)) + h(n) < \psi^*(N(h(n)) + \log C_1^v,$$

for all $n \in \mathbb{N}$. By (51) and (52) we also have

$$N(h(n))x_{h(n)} - \frac{\varphi(x_{h(n)})}{h(n)} + h(n) < N(h(n))\zeta_n - \psi(\zeta_n) + \log C_1^v.$$

Then, by (50), since φ' is a nondecreasing function and since $\varphi'(x_{h(n)}) = N(h(n))h(n)$, we obtain

$$\begin{aligned}
&N(h(n))(x_{h(n)} - \zeta_n) + h(n)\\
&< \frac{\varphi(x_{h(n)})}{h(n)} - \frac{\varphi(\zeta_n)}{h(n-2)} - \sum_{i=1}^{n-2} \frac{h(i)-h(i-1)}{h(i-1)h(i)}\varphi(z_{i+1}) + \log C_1^v\\
&< \frac{\varphi(x_{h(n)})}{h(n)} - \frac{\varphi(\zeta_n)}{h(n)} + \frac{\varphi(\zeta_n)}{h(n)} - \frac{\varphi(\zeta_n)}{h(n-2)} + \log C_1^v\\
&\le \int_{\zeta_n}^{x_{h(n)}} \frac{\varphi'(s)}{h(n)} ds + \log C_1^v\\
&\le \int_{\zeta_n}^{x_{h(n)}} \frac{\varphi'(x_{h(n)})}{h(n)} ds + \log C_1^v\\
&= N(h(n))(x_{h(n)} - \zeta_n) + \log C_1^v,
\end{aligned}$$

which is a contradiction.

Clearly, (41) implies (38) and the proof is complete. □

An immediate consequence of Proposition 4.5 is the following result.

Corollary 4.6. *Let ω be a weight function and Ω be an open set in $\mathbb{R}^n$. Then*

$$\mathcal{E}_{\{\omega\}}(\Omega) = \bigcap_{\sigma \in S} \mathcal{E}_{(\sigma)}(\Omega).$$

Proof. Let $g \in \mathcal{D}'(\Omega)$. By Theorem 3.6, we have that $g \in \mathcal{E}_{\{\omega\}}(\Omega)$ if, and only if, $WF_{\{\omega\}}(g)$ is empty. Then, $WF_{(\sigma)}(g)$ is empty for all $\sigma \in S$, and the conclusion follows. □

There exists an unpublished version of Corollary 4.6, with a different proof, for open convex sets, due to J. Bonet and R. Meise, and is an extension for quasianalytic classes of [6, Proposition 3.5]. Another consequence of Proposition 4.5 is the following:

Corollary 4.7. *Let ω be a weight function and Ω be an open set in $\mathbb{R}^n$. Then*

$$\operatorname{sing}_{\{\omega\}} \operatorname{supp}(u) = \overline{\bigcup_{\sigma \in S} \operatorname{sing}_{(\omega)} \operatorname{supp}(u)}, \quad u \in \mathcal{D}'(\Omega).$$

Proof. Fix $u \in \mathcal{D}'(\Omega)$. Then, combining Proposition 4.5 and Theorem 3.6, we easily obtain

$$\operatorname{sing}_{\{\omega\}} \operatorname{supp}(u) \subset \overline{\bigcup_{\sigma \in S} \operatorname{sing}_{(\omega)} \operatorname{supp}(u)}.$$

The opposite inclusion follows from Remark 2.4(a) and the fact that the singular support is always a closed set. □

We now state and prove the Roumieu version of Theorem 4.1. We do not need any change of weight function, but the weight function must satisfy property (α_0), as in the Beurling version (Theorem 4.1).

Theorem 4.8. *Let ω be a weight function satisfying property (α_0) and $\Omega \subset \mathbb{R}^n$ be an open set. If $P(x, D)$ is a linear partial differential operator whose coefficients belong to $\mathcal{E}_{\{\omega\}}(\Omega)$, then*

$$WF_{\{\omega\}}(u) \subset WF_{\{\omega\}}(Pu) \cup \Sigma, \quad u \in \mathcal{D}'(\Omega),$$

where Σ is the characteristic set of P.

Proof. Let S be the set of weight functions σ such that $\sigma = o(\omega)$. Then, for each $\sigma \in S$ and $u \in \mathcal{D}'(\Omega)$, by Theorem 4.1 we have

$$WF_{(\sigma)}u \subset WF_{(\sigma)}(Pu) \cup \Sigma, \quad u \in \mathcal{D}'(\Omega),$$

as the coefficients of $P(x, D)$ belong to $\mathcal{E}_{\{\omega\}}(\Omega)$. Now, we apply Proposition 4.5 to conclude. □

Finally, we point out that from Theorems 4.1 and 4.8 we immediately get (see, for example, [20, p.105]):

Corollary 4.9. *Let $P = P(x, D)$ be an elliptic linear partial differential operator defined on an open set Ω of $\mathbb{R}^n$ (elliptic means that $\Sigma = \emptyset$). Then:*

(a) *If ω is a weight function satisfying property (α_0) and the coefficients of P belong to $\mathcal{E}_{\{\omega\}}(\Omega)$, we have*

$$WF_{\{\omega\}}(u) = WF_{\{\omega\}}(Pu), \quad u \in \mathcal{D}'(\Omega),$$

and hence,

$$\operatorname{sing}_{\{\omega\}} \operatorname{supp}(u) = \operatorname{sing}_{\{\omega\}} \operatorname{supp}(Pu), \quad u \in \mathcal{D}'(\Omega).$$

(b) *If ω and σ are two weight functions such that ω satisfies property (α_0) and $\omega = o(\sigma)$, and the coefficients of P belong to $\mathcal{E}_{\{\sigma\}}(\Omega)$, then*

$$WF_{(\omega)}(u) = WF_{(\omega)}(Pu), \quad u \in \mathcal{D}'(\Omega),$$

and hence,

$$\text{sing}_{(\omega)} \text{supp}\,(u) = \text{sing}_{(\omega)} \text{supp}\,(Pu), \quad u \in \mathcal{D}'(\Omega).$$

We conclude the paper by studying the wave front set of the solutions of the following (non hypoelliptic) partial differential operator of principal type in $\mathbb{R}^n$:

$$P = \frac{\partial}{\partial x_n}.$$

Observe that the characteristic set of P is

$$\Sigma = \{(x, \xi) \in \mathbb{R}^{2n} : \xi_n = 0,\ \xi \neq 0\}.$$

Moreover, we point out that $u \in \mathcal{D}'(\mathbb{R}^n)$ is a solution of $Pu = 0$ if, and only if, $u = v \otimes \mathbf{1}$ for some $v \in \mathcal{D}'(\mathbb{R}^{n-1})$, being $\mathbf{1}$ the function identically 1 in the x_n-variable. Indeed, if u is of the form $v \otimes \mathbf{1}$, then $\frac{\partial}{\partial x_n}(v \otimes \mathbf{1}) = 0$. On the other hand, if $Pu = 0$, then u satisfies $\tau_h u = u$ for every $h = (0, \ldots, 0, h_n)$, where $\tau_h u$ denotes the h-translation of the distribution u (see, for example, [20]). From this fact, by an approximation procedure, it is easy to conclude that u must be of the form $v \otimes \mathbf{1}$ for some distribution $v \in \mathcal{D}'(\mathbb{R}^{n-1})$.

We can now state the following result:

Proposition 4.10. *Let ω be a weight function satisfying property (α_0), and write, as usual, $*$ for $\{\omega\}$ or (ω). Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be a solution of the equation $Pu = 0$. If $(x_0, \xi_0) \in WF_*(u)$, then $(x_0, \xi_0) \in \Sigma$, and splitting $\mathbb{R}^n \ni x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n)$, we have that the straight line*

$$L = \{(x_0', x_n, \xi_0),\ x_n \in \mathbb{R}\}$$

is contained in $WF_*(u)$. Moreover, if ω is non-quasianalytic, for every $(x_0, \xi_0) \in \Sigma$ there exists a solution $u \in \mathcal{D}_*'(\mathbb{R}^n)$ of $Pu = 0$ whose $*$-wave front set is given by the set

$$\{(x_0', x_n, \lambda\xi_0),\ x_n \in \mathbb{R},\ \lambda > 0\}.$$

Proof. Since $u \in \mathcal{D}'(\mathbb{R}^n)$ is a solution of $Pu = 0$, by Theorems 4.1 and 4.8 we have

$$WF_*(u) \subset \Sigma$$

and $u = v \otimes \mathbf{1}$ for some suitable $v \in \mathcal{D}'(\mathbb{R}^{n-1})$. We claim that

$$WF_*(u) = \{(x, \xi) \in \Sigma : (x', \xi') \in WF_* v\}. \tag{54}$$

First, we show property (54) in Beurling case proceeding as follows. Let $(\overline{x}, \overline{\xi}) \in \Sigma$. If $(\overline{x}', \overline{\xi}') \notin WF_{(\omega)}(v)$ then, by Definition 3.4, there exist an open neighborhood U' of $\overline{x}'$, a conic neighborhood Γ' of $\overline{\xi}'$ and a bounded sequence $v_N \in \mathcal{E}'(\mathbb{R}^{n-1})$, $v_N = v$ in U', such that for every $k \in \mathbb{N}$ there exists a positive constant C_k satisfying

$$|\xi'|^N |\widehat{v}_N(\xi')| \leq C_k e^{k\varphi^*(N/k)} \tag{55}$$

in Γ', for $N = 1, 2, \ldots$. Let $\chi \in \mathcal{D}(\mathbb{R})$, be a function equal to 1 in a neighborhood I of $\overline{x}_n$. Then, we have that $u_N := v_N \otimes \chi$ is a bounded sequence in $\mathcal{E}'(\mathbb{R}^n)$, $u_N = u$ in $U := U' \times I$. Let Γ be a conic neighborhood of $(\overline{\xi}', 0)$ (since $(\overline{x}, \overline{\xi}) \in \Sigma$ we have $\overline{\xi}_n = 0$) satisfying $\Gamma \cap \{\xi_n = 0\} \subset \Gamma'$. Then there exists a positive constant c_1 such that

$$|\xi_n| \leq c_1 |\xi'| \quad \text{for} \quad \xi = (\xi', \xi_n) \in \Gamma. \tag{56}$$

We also observe that

$$\hat{u}_N(\xi) = \widehat{v_N \otimes \chi}(\xi) = \widehat{v}_N(\xi')\widehat{\chi}(\xi_n).$$

Now, from (55) and (56), it follows that for every $k \in \mathbb{N}$,

$$\begin{aligned} |\xi|^N |\widehat{u}_N(\xi)| &\leq \big(|\xi'| + |\xi_n|\big)^N |\widehat{v}_N(\xi')|\,|\widehat{\chi}(\xi_n)| \\ &\leq (1 + c_1)^N |\xi'|^N |\widehat{v}_N(\xi')|\,|\widehat{\chi}(\xi_n)| \\ &\leq c_2\, C_k\, (1 + c_1)^N e^{k\varphi^*(N/k)}, \end{aligned}$$

for each $N \in \mathbb{N}$ and $\xi \in \Gamma$. In view of Definition 3.4 and Lemma 3.2, this inequality implies that $(\overline{x}, \overline{\xi}) \notin WF_{(\omega)}(u)$. Then, we deduce that

$$WF_{(\omega)}(u) \subset \{(x, \xi) \in \Sigma \,:\, (x', \xi') \in WF_{(\omega)}(v)\}.$$

Conversely, let $(\overline{x}, \overline{\xi}) \in \Sigma$ such that $(\overline{x}, \overline{\xi}) \notin WF_{(\omega)}(u)$. By Definition 3.4 there exist an open neighborhood U of $\overline{x}$, a conic neighborhood Γ of $\overline{\xi}$ and a bounded sequence $u_N \in \mathcal{E}'(\mathbb{R}^n)$, $u_N = u$ in U, such that (4) is satisfied in Γ. Since $u = v \otimes \mathbf{1}$, without loss of generality we can assume that $u_N = v_N \otimes \chi_N$, eventually multiplying u_N by a tensor product test function equal to 1 in a neighborhood V of $\overline{x}$ with $V \subset U$ (see Lemmas 3.5 and 3.2 and remarks before Proposition 3.7). Therefore, there exists a sequence C_k of positive constants such that

$$|\xi|^N |\widehat{u}_N(\xi)| = |\xi|^N |\widehat{v}_N(\xi')|\,|\widehat{\chi}_N(\xi_n)| \leq C_k e^{k\varphi^*(N/k)} \tag{57}$$

for $N \in \mathbb{N}$ and $\xi \in \Gamma$. We may also suppose that $\chi_N =: \chi$ is independent of N and satisfies $\widehat{\chi}(0) \neq 0$. So, from (57) it follows that

$$|\xi'|^N |\widehat{v}_N(\xi')| \leq \frac{C_k}{|\widehat{\chi}(0)|} e^{k\varphi^*(N/k)}$$

for $N = 1, 2, \ldots$ and $\xi' \in \Gamma' = \Gamma \cap \{\xi_n = 0\}$. Then $(\overline{x}', \overline{\xi}') \notin WF_{(\omega)}(v)$, and so we have shown that

$$\{(x, \xi) \in \Sigma \,:\, (x', \xi') \in WF_{(\omega)}(v)\} \subset WF_{(\omega)}(u).$$

This concludes the proof of (54). In the Roumieu case the proof is similar with minor changes.

Now, it follows immediately that if $(\overline{x}, \overline{\xi}) \in WF_*(u)$, being u a solution of $Pu = 0$, then every point of the kind (x_0', x_n, ξ_0), with $x_n \in \mathbb{R}$, belongs to $WF_*(u)$. Thereby, the straight line L is contained in $WF_*(u)$.

In the following, let ω be a non-quasianalytic weight function. Now, we show that there exists an $*$-ultradistribution $u \in \mathcal{D}'_*(\mathbb{R}^n)$ solution of $Pu = 0$ with the prescribed wave front set

$$WF_*(u) = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} : x = (x'_0, x_n),\ \xi = \lambda\xi_0,\ x_n \in \mathbb{R},\ \lambda > 0\}. \tag{58}$$

Fix $(x_0, \xi_0) \in \Sigma$. Proceeding in a similar way as in Example 1 in [11], we can construct $\tilde{v} \in \mathcal{E}'_*(\mathbb{R}^{n-1})$ satisfying

$$\begin{aligned} &WF_*(\tilde{v}) \\ &= \{(x', \xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \setminus \{0\} : x' = 0,\ \xi' = (0, \dots, 0, \xi_{n-1}),\ \xi_{n-1} > 0\}. \end{aligned}$$

By a linear change of variable and a translation, we then find v satisfying

$$WF_*(v) = \{(x', \xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \setminus \{0\} : x' = x'_0,\ \xi' = \lambda\xi'_0,\ \lambda > 0\}.$$

Set $u = v \otimes \mathbf{1} \in \mathcal{D}'_*(\mathbb{R}^n)$. Then, we have that $Pu = 0$ and that, by (54), equality (58) is satisfied. □

Observe that an analogous result holds for the equation $Pu = f$, with $f \in C^\infty(\mathbb{R}^n)$. Indeed, every solution $u \in \mathcal{D}'(\mathbb{R}^n)$ of $Pu = f$ can be written as

$$u(x) = u_0(x) + \int_0^{x_n} f(x', t)\, dt, \quad x = (x', x_n) \in \mathbb{R}^n,$$

where u_0 is a solution of $Pu = 0$. If $(x_0, \xi_0) \in \Sigma$ and $(x_0, \xi_0) \notin WF_*(f)$, then $(x_0, \xi_0) \in WF_*(u)$ implies that $(x'_0, x_n, \xi_0) \in WF_*(u)$ for x_n in a suitable interval I containing $x_{n,0}$. In fact, if $(x_0, \xi_0) \notin WF_*(f)$ then there exists a neighborhood U of (x_0, ξ_0) with empty intersection with $WF_*(f)$. So, in a neighborhood of x_0 the wave front set of u coincides with the wave front set of u_0.

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