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Integral Equations and Operator Theory

Operational Properties of Two Integral Transforms of Fourier Type and their Convolutions

Bui Thi Giang, Nguyen Van Mau and Nguyen Minh Tuan

Abstract. In this paper we present the operational properties of two integral transforms of Fourier type, provide the formulation of convolutions, and obtain eight new convolutions for those transforms. Moreover, we consider applications such as the construction of normed ring structures on $L_1(\mathbb{R})$, further applications to linear partial differential equations and an integral equation with a mixed Toeplitz-Hankel kernel.

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1. Introduction

The Fourier-cosine and Fourier-sine integral transforms are defined as follows

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \cos xy f(y) dy := g_c(x), \tag{1.1}$$

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \sin xy f(y) dy := g_s(x) \tag{1.2}$$

(see Sneddon [15], Titchmarsh [18]). These transforms and the Fourier integral transform have been studied for a long time, and applied to many fields of mathematics (see Hörmander [9], Rudin [13], or [18]). We mention interesting properties of the transforms F_c, F_s (see [1, 15, 18]):

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- For $f \in L_1[0, +\infty)$, the functions $g_c(x), g_s(x)$ exist for every $x \in [0, +\infty)$.
- If $f, g_c \in L_1[0, +\infty)$, then the inversion formula of F_c holds

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \cos xy g_c(y) dy.$$

- If $f, g_s \in L_1[0, +\infty)$, then the inversion formula of F_s holds

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \sin xy g_s(y) dy.$$

- For an arbitrary function $f \in L_2[0, +\infty)$, the functions g_c, g_s are determined for almost every $x \in \mathbb{R}$, and g_c, g_s belong to $L_2[0, +\infty)$ according to the Plancherel theorem for the Fourier transform. Moreover, F_c, F_s are isometric operators in $L_2[0, +\infty)$ satisfying the identities: $F_c^2 = I,\ F_s^2 = I$ (see [2, 18]).

If F_c, F_s were defined as

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \cos xy f(y) dy, \tag{1.3}$$

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \sin xy f(y) dy, \tag{1.4}$$

then $(F_c f)(x)$, $(F_s f)(x)$ would exist for any $f \in L_1(-\infty, \infty)$ and for every $x \in \mathbb{R}$, but there would be no inversion formula due to the fact that $(F_c f)(x) = 0$, or $(F_s f)(x) = 0$ if f were an odd or even function. Furthermore, for $f \in L_2(-\infty, \infty)$ one can give definitions so that the integrals on the right-side of (1.3), (1.4) are determined for almost every $x \in \mathbb{R}$. But in this case, F_c, F_s are non-isometric, non-injective linear operators in $L_2(-\infty, \infty)$.

We also consider the following transforms

$$(T_1 f)(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cos\left(xy + \frac{\pi}{4}\right) f(y) dy,$$

$$(T_2 f)(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sin\left(xy + \frac{\pi}{4}\right) f(y) dy,$$

where f is a real-valued or complex-valued function defined on $(-\infty, \infty)$. The main difference between T_1, T_2 and F_c, F_s is the fact that the kernel functions $\cos xy, \sin xy$ of the integrals (1.1), (1.2) changed to $\cos\left(xy + \frac{\pi}{4}\right), \sin\left(xy + \frac{\pi}{4}\right)$ respectively, and the lower limits zero changed to $-\infty$.

This paper is devoted to the investigation of operational properties of T_1, T_2, to the construction of new convolutions and to applications.

The paper is divided into four sections and organized as follows.

In Section 2, there are several interpretations so that T_1, T_2 become bounded linear operators in $L_2(-\infty, \infty)$. In fact, the definitions of T_1, T_2 in $L_2(-\infty, \infty)$ may be dropped if we accept the Plancherel's theorem for the Fourier integral transform and use the formulae

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

(see [2, 13, 18]). However, Section 2 remains necessary as there are stated operational properties of T_1, T_2 which are different from those of the Fourier transform. Namely, T_1, T_2 are unitary operators in $L_2(-\infty, \infty)$, and they fulfill the identities $T_1^2 = I,\ T_2^2 = I$. Some properties of T_1, T_2 related to the Hermite functions and to differential operators are also proved in this section.

In Section 3, we give some general definitions of convolutions for linear operators maping from a linear space U to a commutative algebra V, and construct eight new convolutions with and without weight for T_1, T_2. We will see that there exist different convolutions for the same integral transform.

The applications for constructing normed ring structures of $L_1(-\infty, \infty)$, for solving some partial differential equations and integral equations are considered in Section 4. In particular, explicit solutions of some classical partial differential equations, of an integral equation of convolution type, and of the integral equation with a mixed Toeplitz-Hankel kernel are obtained.

2. Operational properties

Through the paper we write $\mathbb{N} := \{0, 1, 2, \dots\}$. Let $\mathcal{S}$ denote the set of all $\mathbb{K}$-valued functions f on $\mathbb{R}$ which are infinitely differentiable such that

$$P_m(f) := \sup_{n \le m} \sup_{x \in \mathbb{R}} (1 + |x|^2)^m |(D_n f)(x)| < \infty \tag{2.1}$$

for $m \in \mathbb{N}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $D_n f = f^{(n)}$ for $n \in \mathbb{N}$. $\mathcal{S}$ is a vector space which becomes a Frechet space by the countable collection of semi-norms (2.1) (see [13]).

We start with some facts related to the Hermite functions.

2.1. Transforms of the Hermite functions

The Hermite polynomial of degree n is defined by

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2},$$

and the corresponding Hermite function ϕ_n by

$$\phi_n(x) = (-1)^n e^{\frac{1}{2}x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} \quad \text{(see [18])}.$$

Theorem 2.1. *Let $n = 4m + k, \quad k = 0, 1, 2, 3$. Then*

$$T_1 \phi_n = \begin{cases} \phi_n, & \text{if } k = 0, 3 \\ -\phi_n, & \text{if } k = 1, 2, \end{cases} \tag{2.2}$$

and

$$T_2 \phi_n = \begin{cases} \phi_n, & \text{if } k = 0, 1 \\ -\phi_n, & \text{if } k = 2, 3. \end{cases} \tag{2.3}$$

Proof. Obviously, $\phi_n \in \mathcal{S}$. Using the formulae

$$\cos\left(xy+\frac{\pi}{4}\right) = \frac{e^{i\left(xy+\frac{\pi}{4}\right)} + e^{-i\left(xy+\frac{\pi}{4}\right)}}{2},$$

$$\frac{d^n}{dx^n} e^{\frac{1}{2}(x\pm iy)^2} = (\mp i)^n \frac{d^n}{dy^n} e^{\frac{1}{2}(x\pm iy)^2},$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\pm ixy - \frac{1}{2}x^2} dx = e^{-\frac{1}{2}y^2},$$

and integrating by parts n times yields the relationship

$$(T_1\phi_n)(y) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \phi_n(x) \cos\left(xy+\frac{\pi}{4}\right) dx = \sqrt{2}\cos\left(\frac{\pi}{4}+\frac{n\pi}{2}\right)\phi_n(y).$$

Since

$$\sqrt{2}\cos\left(\frac{\pi}{4}+\frac{n\pi}{2}\right) = \begin{cases} 1, & \text{if} \quad k=0,3, \\ -1, & \text{if} \quad k=1,2 \end{cases}$$

for $m \in \mathbb{N}$, we have proved the assertion (2.2). The proof of (2.3) is similar and left to the reader. □

2.2. Definition of $\mathbf{T_1}, \mathbf{T_2}$ in the spaces $\mathcal{S}, \mathbf{L_1}(\mathbb{R}), \mathbf{L_2}(\mathbb{R})$

Let $C_0(\mathbb{R})$ denote the supremum-normed Banach space of all continuous functions on $\mathbb{R}$ that vanish at infinity.

Proposition 2.2. *If $f \in L_1(\mathbb{R})$, then $T_1 f, T_2 f \in C_0(\mathbb{R})$ and $\|T_1 f\|_\infty \le \|f\|_1$, $\|T_2 f\|_\infty \le \|f\|_1$, where $\|\cdot\|_1$ is the L_1-norm.*

Proof. Using the Riemann-Lebesgue Lemma (see [18, Theorem 1]), we have $T_1 f$, $T_2 f \in C_0(\mathbb{R})$. Since $\left|\cos\left(xy+\frac{\pi}{4}\right)\right| \le 1$, $\left|\sin\left(xy+\frac{\pi}{4}\right)\right| \le 1$, we obtain

$$|T_1 f(x)| \le \frac{1}{\pi}\|f\|_1, \quad |T_2 f(x)| \le \frac{1}{\pi}\|f\|_1, \text{ for all } x \in \mathbb{R}, \tag{2.4}$$

□

For $f \in \mathcal{S}$ define $g_m(x) = x^m f(x)$, $x \in \mathbb{R}$, $m \in \mathbb{N}$. The function $D_n g_m$ belongs to $\mathcal{S}$ for all $n, m \in \mathbb{N}$. We prove the following statement.

Theorem 2.3. *Let $f \in \mathcal{S}$. For all $m, n \in \mathbb{N}$ and all $x \in \mathbb{R}$ we have*

$$x^m D_n(T_1 f)(x) = \begin{cases} T_1 D_m g_n(x), & \text{if } n+m=0 \mod (4) \\ -T_2 D_m g_n(x), & \text{if } n+m=1 \mod (4) \\ -T_1 D_m g_n(x), & \text{if } n+m=2 \mod (4) \\ T_2 D_m g_n(x), & \text{if } n+m=3 \mod (4) \end{cases} \tag{2.5}$$

and

$$x^m D_n(T_2 f)(x) = \begin{cases} T_2 D_m g_n(x), & \text{if } n+m=0 \mod (4) \\ T_1 D_m g_n(x), & \text{if } n+m=1 \mod (4) \\ -T_2 D_m g_n(x), & \text{if } n+m=2 \mod (4) \\ -T_1 D_m g_n(x), & \text{if } n+m=3 \mod (4). \end{cases} \tag{2.6}$$

Proof. Obviously $\dfrac{\partial^k}{\partial x^k}\cos\left(xy+\frac{\pi}{4}\right) = y^k\cos(xy+\frac{\pi}{4}+\frac{k\pi}{2})$ for $k\in\mathbb{N}$. We infer that

$$\begin{aligned}D_n(T_1 f)(x) &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\cos\left(xy+\frac{\pi}{4}+\frac{n\pi}{2}\right)y^n f(y)dy\\ &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\cos\left(xy+\frac{\pi}{4}+\frac{n\pi}{2}\right)g_n(y)dy\end{aligned}$$

for $x\in\mathbb{R}$. Integrating by parts m times yields

$$\begin{aligned}x^m D_n(T_1 f)(x) &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} x^m\cos\left(xy+\frac{\pi}{4}+\frac{n\pi}{2}\right)g_n(y)dy\\ &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\frac{\partial^m}{\partial y^m}\cos\left(xy+\frac{\pi}{4}+\frac{(n-m)\pi}{2}\right)g_n(y)dy\\ &= \frac{(-1)^m}{\sqrt{\pi}}\int_{\mathbb{R}}\cos\left(xy+\frac{\pi}{4}+\frac{(n-m)\pi}{2}\right)D_m g_n(y)dy\\ &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\cos\left(xy+\frac{\pi}{4}+\frac{(n+m)\pi}{2}\right)D_m g_n(y)dy\end{aligned}$$

for all $m,n\in\mathbb{N}$ and all $x\in\mathbb{R}$ whence the formula (2.5) is proved. The proof of the relation (2.6) is left to the reader. □

Theorem 2.4. *The operators T_1 and T_2 are continuous linear maps of the Frechet space $\mathcal{S}$ into itself.*

Proof. Let $f\in\mathcal{S}$. Obviously, $T_1 f$ is an infinitely differentiable function on $\mathbb{R}$. By Proposition 2.2 and formula (2.5), we obtain

$$|x^m D_n(T_1 f)(x)| \le \frac{1}{\pi}\|D_m g_n\|_1 < \infty$$

which proves that $T_1 f$ belongs to $\mathcal{S}$. We shall show that T_1 is a closed operator in $\mathcal{S}$. Let f and g be in $\mathcal{S}$, $\{f_i\}_{i=0}^{\infty}$ a sequence in $\mathcal{S}$ such that $f_i\to f$ and $T_1 f_i\to g$ in $\mathcal{S}$ for $i\to\infty$. We have to show that $T_1 f = g$. Since convergence in $\mathcal{S}$ implies convergence in $L_1(\mathbb{R})$, we conclude from (2.4) that

$$|T_1(f_i - f)(x)| \le \|f_i - f\|_1 \to 0\ (i\to\infty).$$

Hence $T_1 f_i$ converges uniformly on $\mathbb{R}$ to $T_1 f$ as well as to g, whence $T_1 f = g$. By the closed graph theorem for Frechet spaces [13], T_1 is a continuous linear operator on $\mathcal{S}$.
The proof for T_2 is analogous. □

The following lemma is useful for the proof of Theorem 2.6.

Lemma 2.5 ([18, Theorem 3])**.** *Let f belong to $L_1(\mathbb{R})$. If f is a function of bounded variation on an interval including the point x, then*

$$\frac{1}{2}\{f(x+0)+f(x-0)\} = \frac{1}{\pi}\int_0^{\infty}du\int_{-\infty}^{\infty}f(t)\cos u(x-t)dt.$$

If f is continuous and of bounded variation in an interval (a, b), then

$$f(x) = \frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(t)\cos u(x-t)dt,$$

the integral converging uniformly in any interval interior to (a, b).

Theorem 2.6 (Inversion theorem). 1) *If $g \in \mathcal{S}$, then*

$$g(x) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} (T_1 g)(y)\cos\left(xy + \frac{\pi}{4}\right) dy, \tag{2.7}$$

and

$$g(x) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} (T_2 g)(y)\sin\left(xy + \frac{\pi}{4}\right) dy. \tag{2.8}$$

2) *T_1, T_2 are continuous linear one-to-one maps of $\mathcal{S}$ onto itself, $T_1^2 = I = T_2^2$, i.e., $T_1^{-1} = T_1,\ T_2^{-1} = T_2$.*
3) *If $f, T_1 f \in L_1(\mathbb{R})$ (or if $f, T_2 f \in L_1(\mathbb{R})$), and if*

$$f_0(x) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} (T_1 f)(y)\cos\left(xy + \frac{\pi}{4}\right) dy,$$

$$(\textit{or if}\quad f_0(x) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} (T_2 f)(y)\sin\left(xy + \frac{\pi}{4}\right) dy),$$

then $f(x) = f_0(x)$ for almost every $x \in \mathbb{R}$.

Proof. 1) By Theorem 2.4, the inner function on the right-side of (2.7) belongs to $\mathcal{S}$. Using Fubini's theorem and Lemma 2.5, we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} (T_1 g)(y)\cos\left(xy + \frac{\pi}{4}\right) dy \\
&= \lim_{\lambda\to\infty} \frac{1}{\sqrt{\pi}}\int_{-\lambda}^{\lambda} \cos\left(xy + \frac{\pi}{4}\right)(T_1 g)(y)dy \\
&= \lim_{\lambda\to\infty} \frac{1}{\pi}\int_{\mathbb{R}} g(t)dt \int_{-\lambda}^{\lambda} \cos\left(xy + \frac{\pi}{4}\right)\cos(yt + \frac{\pi}{4})dy \\
&= \frac{1}{2\pi}\lim_{\lambda\to\infty}\int_{\mathbb{R}} g(t)\frac{2\sin\lambda(x-t)}{x-t}dt = g(x),
\end{aligned}$$

which proves (2.7). Identity (2.8) is proved similarly.
2) The inversion formulae (2.7), (2.8) show that the operators T_1 and T_2 are one-to-one onto $\mathcal{S}$, and $T_1^2 = I$, $T_2^2 = I$.
3) By assumption $f, T_1 f \in L_1(\mathbb{R})$. Let $g \in \mathcal{S}$. We apply Fubini's Theorem to the double integral

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y)\cos\left(xy + \frac{\pi}{4}\right) dxdy$$

and get the identity

$$\int_{\mathbb{R}} f(x)(T_1 g)(x)dx = \int_{\mathbb{R}} g(y)(T_1 f)(y)dy. \tag{2.9}$$

Since $T_1 f \in L_1(\mathbb{R})$ and $g \in \mathcal{S}$, we can use the inversion formula (2.7) into the right-side of (2.9) and again Fubini's theorem, we obtain

$$\int_{\mathbb{R}} f(x)(T_1 g)(x)dx = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(T_1 g)(x)\cos\left(xy+\frac{\pi}{4}\right)dx\right)(T_1 f)(y)dy$$
$$= \int_{\mathbb{R}}(T_1 g)(x)\left(\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}(T_1 f)(y)\cos\left(xy+\frac{\pi}{4}\right)dy\right)dx = \int_{\mathbb{R}} f_0(x)(T_1 g)(x)dx.$$

Let $\mathcal{D}(\mathbb{R})$ denote the vector space of all infinitely differentiable functions on $\mathbb{R}$ with compact supports. Using Theorem 2.4 and $\mathcal{D}(\mathbb{R}) \subset \mathcal{S}$, we conclude that

$$\int_{\mathbb{R}}(f_0(x) - f(x))\Phi(x)dx = 0,$$

for every $\Phi \in \mathcal{D}(\mathbb{R})$. Thus $f_0(x) - f(x) = 0$ for almost every $x \in \mathbb{R}$ (see [13]). The fact related to T_2 is proved similarly. □

Corollary 2.7 (Uniqueness theorems for T_1, T_2)**.** 1) *If $f \in L_1(\mathbb{R})$, and if $T_1 f = 0$ in $L_1(\mathbb{R})$, then $f = 0$ in $L_1(\mathbb{R})$.*

2) *If $f \in L_1(\mathbb{R})$, and if $T_2 f = 0$ in $L_1(\mathbb{R})$, then $f = 0$ in $L_1(\mathbb{R})$.*

Remark 2.8. a) Recall that the Fourier transform F of $\phi_n(x)$ is $i^n\phi_n(x)$ (see [18, Theorem 57]). So, the Hermite functions are the eigenfunctions of T_1, T_2 and F with the eigenvalues $\{-1, 1\}$ and $\{-1, -i, 1, i\}$, respectively.

b) It is well-known that the functions $\{\phi_n\}$ form a complete orthogonal system in $L_2(\mathbb{R})$, and $\mathcal{S}$ is dense in it. These facts and Theorem 2.4 suggest us to prove $T_1^2 = I$, $T_2^2 = I$ in $L_2(\mathbb{R})$.

Theorem 2.9. (Plancherel's Theorem) *There is a linear isometric operator $\overline{T}_1$ ($\overline{T}_2$) of $L_2(\mathbb{R})$ into itself which is uniquely determined by the requirement that*

$$\overline{T}_1 f = T_1 f \;\; (\overline{T}_2 f = T_2 f), \qquad \text{for every } f \in \mathcal{S}.$$

Moreover, the extension operators fulfill the identities: $\overline{T}_1^2 = I$, $\overline{T}_2^2 = I$, where I is the identity operator in $L_2(\mathbb{R})$.

Proof. It suffices to prove the conclusion of T_1. If $f, g \in \mathcal{S}$, the inversion theorem yields

$$\int_{\mathbb{R}} f(x)\overline{g}(x)dx = \int_{\mathbb{R}}\overline{g}(x)dx\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}(T_1 f)(t)\cos\left(xt+\frac{\pi}{4}\right)dt$$
$$= \int_{\mathbb{R}}(T_1 f)(t)dt\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\overline{g}(x)\cos\left(xt+\frac{\pi}{4}\right)dx.$$

We thus get the Parseval Formula

$$\int_{\mathbb{R}} f(x)\overline{g}(x)dx = \int_{\mathbb{R}}(T_1 f)(t)\overline{T_1 g}(t)dt, \qquad f, g \in \mathcal{S}.$$

If $g = f$, then

$$\|f\|_2 = \|T_1 f\|_2, \qquad f \in \mathcal{S}. \tag{2.10}$$

Note that $\mathcal{S}$ is dense in $L_2(\mathbb{R})$, for the same reason that $\mathcal{S}$ is dense in $L_1(\mathbb{R})$. By (2.10), the map $f \to T_1 f$ is an isometry (relative to the L_2-metric) of the dense subspace $\mathcal{S}$ of $L_2(\mathbb{R})$ onto $\mathcal{S}$. It follows that $f \to T_1 f$ has a unique continuous extension $\overline{T}_1 : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ and that this operator $\overline{T}_1$ is a linear isometry onto $L_2(\mathbb{R})$ (see [2, Theorems 47, 48], [13, Ex. 19 in Chapter 1, or Ex. 16 in Chapter 7]). $\square$

The Parseval formula gives the following corollary.

Corollary 2.10. *$\overline{T}_1$, and $\overline{T}_2$ are unitary operators in the Hilbert space $L_2(\mathbb{R})$.*

Thanks to the uniqueness of the extension, the Plancherel theorems for T_1, T_2 might be stated in some clearer ways as follows.

Theorem 2.11 (Plancherel's Theorem for T_1**).** *Let f be a function (real or complex) in $L_2(\mathbb{R})$, and let*

$$T_1(x,k) = \frac{1}{\sqrt{\pi}} \int_{-k}^{k} \cos\left(xy + \frac{\pi}{4}\right) f(y)dy.$$

Then, as $k \to +\infty$, $T_1(x,k)$ converges in mean over $\mathbb{R}$ to a function in $L_2(\mathbb{R})$, say $(\overline{T}_1 f)$, and reciprocally

$$f(x,k) = \frac{1}{\sqrt{\pi}} \int_{-k}^{k} \cos\left(xy + \frac{\pi}{4}\right) (\overline{T}_1 f)(y)dy$$

converges in mean to f. Moreover, the functions $(\overline{T}_1 f)$ and f are connected by the formulae

$$(\overline{T}_1 f)(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_{\mathbb{R}} f(y) \frac{2\sin\left(xy + \frac{\pi}{4}\right) - \sqrt{2}}{2y} dy,$$
$$f(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_{\mathbb{R}} (\overline{T}_1 f)(y) \frac{2\sin\left(xy + \frac{\pi}{4}\right) - \sqrt{2}}{2y} dy,$$

for almost every $x \in \mathbb{R}$.

Proof. Let $f \in L_2(\mathbb{R})$. There exists a sequence of functions $\{f_n\} \in \mathcal{S}$ such that $\|f_n - f\|_2 \to 0$. By (2.10) $\|T_1 f_m - T_1 f_n\|_2 = \|T_1(f_m - f_n)\|_2 = \|f_m - f_n\|_2$ for $m, n \in \mathbb{N}$. It implies that $\{T_1 f_n\}$ is a Cauchy sequence converging to a function in $L_2(\mathbb{R})$, say $(\overline{T}_1 f)(x)$. Since $\{f_n\} \in \mathcal{S}$, we have

$$\begin{aligned}\int_0^{\xi} (T_1 f_n)(x)dx &= \frac{1}{\sqrt{\pi}} \int_0^{\xi} dx \int_{\mathbb{R}} f_n(y) \cos\left(xy + \frac{\pi}{4}\right) dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f_n(y) \frac{2\sin(\xi y + \frac{\pi}{4}) - \sqrt{2}}{2y} dy. \qquad (2.11)\end{aligned}$$

As $\frac{2\sin(\xi y + \frac{\pi}{4}) - \sqrt{2}}{2y} \in L_2(\mathbb{R})$ and $f_n \in \mathcal{S}$, the dominated convergence theorem can be applied to the integrals in (2.11). Letting $n \to \infty$ we obtain

$$\int_0^{\xi} (\overline{T}_1 f)(x)dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(y) \frac{2\sin(\xi y + \frac{\pi}{4}) - \sqrt{2}}{2y} dy.$$

For almost every $x \in \mathbb{R}$ we thus have

$$(\overline{T}_1 f)(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_{\mathbb{R}} f(y) \frac{2\sin\left(xy + \frac{\pi}{4}\right) - \sqrt{2}}{2y} dy. \tag{2.12}$$

Changing f_n to $T_1 f_n$ into (2.11), using Theorem 2.6 with the same argument, we obtain

$$f(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_{\mathbb{R}} (\overline{T}_1 f)(y) \frac{2\sin\left(xy + \frac{\pi}{4}\right) - \sqrt{2}}{2y} dy, \tag{2.13}$$

for almost every $x \in \mathbb{R}$. In summary, for any $f \in L_2(\mathbb{R})$, there is a unique function $\overline{T}_1 f \in L_2(\mathbb{R})$ (apart from sets of measure zero) such that (2.12), (2.13) hold. This extension operator of $L_2(\mathbb{R})$ into itself actually coincides with the operator $\overline{T}_1$ in Theorem 2.9. Now we set $f_k(x) = f(x)$ if $|x| \le k$, zero if $|x| > k$. Then, $f_k \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, and $\|f_k - f\|_2 \to 0$ as $k \to \infty$. By (2.12) we get

$$\begin{aligned}(\overline{T}_1 f_k)(x) &= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_{-k}^{k} f(y) \frac{2\sin\left(xy + \frac{\pi}{4}\right) - \sqrt{2}}{2y} dy \\ &= \frac{1}{\sqrt{\pi}} \int_{k}^{k} f(y) \cos\left(xy + \frac{\pi}{4}\right) dy = T_1(x,k).\end{aligned}$$

By Theorem 2.9 and Corollary 2.10, $\|\overline{T}_1 f_m - \overline{T}_1 f_n\|_2 = \|f_m - f_n\|_2 \to 0$ as $m, n \to \infty$. Thus, $T_1(x,k)$ converges in $L_2(\mathbb{R})$ to $(\overline{T}_1 f)(x)$ as $k \to +\infty$.. $\square$

Theorem 2.12 below can be proved similarly.

Theorem 2.12 (Plancherel's Theorem for T_2)**.** *Let f be a function (real or complex) in $L_2(\mathbb{R})$, and let*

$$T_2(x,k) = \frac{1}{\sqrt{\pi}} \int_{-k}^{k} \sin\left(xy + \frac{\pi}{4}\right) f(y) dy.$$

Then, as $k \to +\infty$, $T_2(x,k)$ converges in mean over $\mathbb{R}$ to a function in $L_2(\mathbb{R})$, say $(\overline{T}_2 f)$, and reciprocally

$$f(x,k) = \frac{1}{\sqrt{\pi}} \int_{-k}^{k} \sin\left(xy + \frac{\pi}{4}\right) (\overline{T}_2 f)(y) dy$$

converges in mean to f. Moreover, the functions $(\overline{T}_2 f)$ and f are connected by the formulae

$$(\overline{T}_2 f)(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_{\mathbb{R}} f(y) \frac{-2\cos\left(xy + \frac{\pi}{4}\right) + \sqrt{2}}{2y} dy,$$

$$f(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_{\mathbb{R}} (\overline{T}_2 f)(y) \frac{-2\cos\left(xy + \frac{\pi}{4}\right) + \sqrt{2}}{2y} dy,$$

for almost every $x \in \mathbb{R}$.

In the following, we denote by $\operatorname{l.i.m}$ the limit in mean, i.e. the limit in theL_2-norm.

Corollary 2.13. *Let $f \in L_2(\mathbb{R})$. Then the transforms $\overline{T}_1, \overline{T}_2$ defined by*

$$\overline{T}_1 f(x) = \underset{n\to\infty}{\mathrm{l.i.m}} \frac{1}{\sqrt{\pi}} \int_{-n}^{n} \cos\left(xy + \frac{\pi}{4}\right) f(x)dx := \mathcal{T}_1(y),$$

and

$$\overline{T}_2 f(x) = \underset{n\to\infty}{\mathrm{l.i.m}} \frac{1}{\sqrt{\pi}} \int_{-n}^{n} \sin\left(xy + \frac{\pi}{4}\right) f(x)dx := \mathcal{T}_2(y),$$

are unitary operators of $L_2(\mathbb{R})$ onto itself. Moreover, whenever the relation

$$\mathcal{T}_1(y) = \underset{n\to\infty}{\mathrm{l.i.m}} \frac{1}{\sqrt{\pi}} \int_{-n}^{n} \cos\left(xy + \frac{\pi}{4}\right) f(x)dx,$$

or

$$\mathcal{T}_2(y) = \underset{n\to\infty}{\mathrm{l.i.m}} \frac{1}{\sqrt{\pi}} \int_{-n}^{n} \sin\left(xy + \frac{\pi}{4}\right) f(x)dx$$

holds, then so does the other one

$$f(x) = \underset{n\to\infty}{\mathrm{l.i.m}} \frac{1}{\sqrt{\pi}} \int_{-n}^{n} \cos\left(xy + \frac{\pi}{4}\right) \mathcal{T}_1(y)dy,$$

or

$$f(x) = \underset{n\to\infty}{\mathrm{l.i.m}} \frac{1}{\sqrt{\pi}} \int_{-n}^{n} \sin\left(xy + \frac{\pi}{4}\right) \mathcal{T}_2(y)dy$$

respectively.

3. Convolutions

Convolutions were introduced early in 20th century and, since then, they have been studied and developed vigorously. One reason for this is that they have many applications in pure and applied mathematics (see Gohberg-Feldman [7]), Vladimirov [23] and references therein). Each convolution is a new transform which can be an object of study (see [4, 5, 6, 10, 20, 21, 22]). Moreover, convolution is a mathematical way of combining two signals to form a third signal, which is a very important technique in digital signal processing (see Smith [14]). In our view, integral transforms of Fourier type deserve interest.

3.1. General definitions of convolutions

Let U be a linear space and let V be a commutative algebra on the field $\mathcal{K}$. Let $T \in L(U, V)$ be a linear operator from U to V.

Definition 3.1. A bilinear map $* \ U \times U :\longrightarrow U$ is called a convolution for T, if $T(*(f, g)) = T(f)T(g)$ for any $f, g \in U$. We denote this proerty of the bilinear form $*(f, g)$ with respect to T by $f \underset{T}{*} g$.

Let δ be the element in algebra V.

Definition 3.2. A bilinear map $*$ $U \times U :\longrightarrow U$ is called the convolution with the weight-element δ for T, if $T(*(f,g)) = \delta T(f)T(g)$ for any $f, g \in U$. For short we denote this proerty of the bilinear form $*(f,g)$ with respect to T by $f \overset{\delta}{\underset{T}{*}} g$.

Each of the identities in Definitions 3.1, 3.2 is called factorization identity (see Britvina [3] and references therein). Let U_1, U_2, U_3 be linear spaces over $\mathcal{K}$. Suppose that $K_1 \in L(U_1, V)$, $K_2 \in L(U_2, V)$, $K_3 \in L(U_3, V)$ are linear operators from U_1, U_2, U_3 to V respectively.

Definition 3.3. A bilinear map $*$ $U_1 \times U_2 :\longrightarrow U_3$ is called a convolution with the weight-element δ for K_3, K_1, K_2 (in that order) if $K_3(*(f,g)) = \delta K_1(f)K_2(g)$ for any $f \in U_1, g \in U_2$. We denote this proerty of the bilinear form $*(f,g)$ briefly by $f \overset{\delta}{\underset{K_3,K_1,K_2}{*}} g$. If δ is the unit of V, we speak of convolutions for K_3, K_1, K_2.

Remark 3.4. If K_3 is injective, then the convolution $f \overset{\delta}{\underset{K_3,K_1,K_2}{*}} g$ is formally determined uniquely as $f \overset{\delta}{\underset{K_3,K_1,K_2}{*}} g = K_3^{-1}(\delta K_1(f)K_2(g))$ for any $f \in U_1, g \in U_2$.

Throughout the paper, we consider $U_k = L_1(\mathbb{R})$ $(k = 1,2,3)$ with the Lebesgue integral, and V the algebra of all (real-valued or complex-valued) measurable functions defined on $\mathbb{R}$.

3.2. Convolutions of $\mathbf{T_1}$

In this subsection we provide four convolutions for T_1.

Theorem 3.5. *If $f, g \in L_1(\mathbb{R})$, then*

$$(f \underset{T_1}{*} g)(x) := \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \Big[f(x-y) + f(x+y) + f(-x+y) - f(-x-y)\Big] g(y)dy \quad (3.1)$$

defines a convolution for T_1.

Proof. Let us first prove that $f \underset{T_1}{*} g \in L_1(\mathbb{R})$. We have

$$\begin{aligned}\int_{\mathbb{R}} |(f \underset{T_1}{*} g)(x)|dx &\leq \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} |g(y)|dy \Big[\int_{\mathbb{R}} |f(x-y)|dx + \int_{\mathbb{R}} |f(x+y)|dx \\ &\quad + \int_{\mathbb{R}} |f(-x+y)|dx + \int_{\mathbb{R}} |f(-x-y)|dx\Big] \\ &\leq \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} |g(y)|dy \int_{\mathbb{R}} |f(x)|dx < +\infty.\end{aligned}$$

We now prove the factorization identity. Since

$$\begin{aligned}\cos\Big(xu + \frac{\pi}{4}\Big) \cos(xv + \frac{\pi}{4}) &= \cos x(u-v) - \sin x(u+v) \\ &= \cos\Big[x(u+v) + \frac{\pi}{4}\Big] + \cos\Big[x(u-v) + \frac{\pi}{4}\Big]\end{aligned}$$

we obtain, by simple substitution,

$$(T_1f)(x)(T_1g)(x) = \frac{1}{2\sqrt{2}\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\cos\left(xt+\frac{\pi}{4}\right)\Big[f(t-y)+f(t+y)+f(-t+y) - f(-t-y)\Big]g(y)dydt = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\cos\left(xt+\frac{\pi}{4}\right)(f \underset{T_1}{*} g)(t)dt = T_1(f \underset{T_1}{*} g)(x). \qquad \square$$

Write $(f \underset{F}{*} g)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x-y)g(y)dy$ for the Fourier convolution. The following corollary shows the relationship between the convolution (3.1) and the Fourier convolution.

Corollary 3.6. *If $f, g \in L_1(\mathbb{R})$, then*

(i) $(f \underset{T_1}{*} g)(x) = \frac{1}{2}\Big[(f \underset{F}{*} g)(x) + (f(-y) \underset{F}{*} g)(x) + (f(-y) \underset{F}{*} g)(-x) - (f \underset{F}{*} g)(-x)\Big]$.

(ii) $(f \underset{F}{*} g)(x) = \frac{1}{2}\Big[-(f \underset{T_1}{*} g)(x) - (f(-y) \underset{T_1}{*} g)(x) + (f(-y) \underset{T_1}{*} g)(-x) + (f \underset{T_1}{*} g)(-x)\Big]$.

Theorem 3.7. *Put $\gamma_1(x) = \cos(x - \frac{\pi}{4})$. If $f, g \in L_1(\mathbb{R})$, then*

$$(f \overset{\gamma_1}{\underset{T_1}{*}} g)(x) := \frac{1}{4\sqrt{\pi}}\int_{\mathbb{R}} g(u)\big[f(x+u+1) - f(-x-u+1) + f(x-u-1) + f(-x+u-1)\big]du \tag{3.2}$$

defines a convolution with the weight-function γ_1 for T_1; the corresponding factorization identity is

$$T_1(f \overset{\gamma_1}{\underset{T_1}{*}} g)(x) = \gamma_1(x)(T_1f)(x)(T_1g)(x).$$

Proof. The fact that $f \overset{\gamma_1}{\underset{T_1}{*}} g \in L_1(\mathbb{R})$ is proved in the same way as in the proof of Theorem 3.5. We prove the factorization identity. By definition

$$\gamma_1(x)(T_1f)(x)(T_1g)(x) = \frac{\cos(x-\frac{\pi}{4})}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\cos\left(xu+\frac{\pi}{4}\right)\cos(xv+\frac{\pi}{4})f(v)g(u)dudv.$$

Since

$$\cos\left(x-\frac{\pi}{4}\right)\cos\left(xu+\frac{\pi}{4}\right)\cos(xv+\frac{\pi}{4}) = \cos\big[x(u+v+1)+\frac{\pi}{4}\big] + \cos\big[x(u-v+1)-\frac{\pi}{4}\big] - \cos\big[x(u+v-1)-\frac{\pi}{4}\big] + \cos\big[x(u-v-1)+\frac{\pi}{4}\big],$$

we obtain by simple integral substitution

$$\gamma_1(x)(T_1f)(x)(T_1g)(x) = \frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \cos\left(xt+\frac{\pi}{4}\right)\Big[f(t-y-1)+f(t+y+1) - f(-t-y+1)+f(-t+y-1)\Big]g(y)dydt = T_1(f \overset{\gamma_1}{\underset{T_1}{*}} g)(x).$$

□

The following corollary shows the relation between the convolution (3.2) and the Fourier convolution.

Corollary 3.8. *If $f,g\in L_1(\mathbb{R})$, then*

$$(f \overset{\gamma_1}{\underset{T_1}{*}} g)(x) = \frac{1}{2\sqrt{2}}\Big[(f(-u)\underset{F}{*}g)(x+1)-(f\underset{F}{*}g)(-x+1) + (f\underset{F}{*}g)(x-1)+(f(-u)\underset{F}{*}g)(-x-1)\Big].$$

Theorem 3.9. *Put $\gamma_2(x)=e^{-\frac{1}{2}x^2}$. If $f,g\in L_1(\mathbb{R})$, then*

$$(f \overset{\gamma_2}{\underset{T_1}{*}} g)(x) = \frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\Big[-e^{\frac{-(x+u+v)^2}{2}} + e^{\frac{-(x+u-v)^2}{2}}+e^{\frac{-(x-u+v)^2}{2}}+e^{\frac{-(x-u-v)^2}{2}}\Big]dudv \tag{3.3}$$

defines a convolution with the weight-function γ_2 for T_1.

Proof. By $\int_{\mathbb{R}} e^{-\frac{1}{2}w^2}dw=\sqrt{2\pi}$, we obtain

$$\int_{\mathbb{R}}|(f \overset{\gamma_2}{\underset{T_1}{*}} g)|(x)dx \le \frac{\sqrt{2}}{\sqrt{\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}|f(u)||g(v)|dudv<+\infty.$$

Hence, $f \overset{\gamma_2}{\underset{T_1}{*}} g\in L_1(\mathbb{R})$. We prove the factorization identity. We have

$$\begin{aligned}\gamma_2(x)(T_1f)(x)(T_1g)(x) &= \frac{e^{-\frac{1}{2}x^2}}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\cos\left(xu+\frac{\pi}{4}\right)\cos(xv+\frac{\pi}{4})dudv\\ &= -\frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u+v)+\sin x(u+v)]dudv\\ &\quad+\frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u-v)+\sin x(u-v)]dudv\\ &\quad+\frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u-v)-\sin x(u-v)]dudv\\ &\quad+\frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u+v)-\sin x(u+v)]dudv.\end{aligned}$$

By using Theorem 2.1 for the Hermite function $\phi_0(x) = e^{-\frac{1}{2}x^2}$, we obtain

$$
\begin{aligned}
&-\frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u+v) + \sin x(u+v)]dudv \\
&= -\frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\Big[\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} \cos(x(y+u+v)+\frac{\pi}{4})e^{\frac{-(y+u+v)^2}{2}}\cos x(u+v)dy \\
&\quad + \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} \sin(x(y+u+v)+\frac{\pi}{4})e^{\frac{-(y+u+v)^2}{2}}\sin x(u+v)dy\Big]dudv \\
&= -\frac{1}{4\pi\sqrt{\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\int_{\mathbb{R}}\cos\left(xy+\frac{\pi}{4}\right)e^{\frac{-(y+u+v)^2}{2}}dydudv. \qquad (3.4)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u-v) + \sin x(u-v)]dudv \\
&= \frac{1}{4\pi\sqrt{\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\int_{\mathbb{R}}\cos\left(xy+\frac{\pi}{4}\right)e^{\frac{-(y+u-v)^2}{2}}dydudv, \qquad (3.5) \\
&\frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u-v) - \sin x(u-v)]dudv \\
&= \frac{1}{4\pi\sqrt{\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\int_{\mathbb{R}}\cos\left(xy+\frac{\pi}{4}\right)e^{\frac{-(y-u+v)^2}{2}}dydudv, \qquad (3.6) \\
&\frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u+v) - \sin x(u+v)]dudv \\
&= \frac{1}{4\pi\sqrt{\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\int_{\mathbb{R}}\cos\left(xy+\frac{\pi}{4}\right)e^{\frac{-(y-u-v)^2}{2}}dydudv. \qquad (3.7)
\end{aligned}
$$

Adding these four formulae we obtain

$$\gamma_2(x)(T_1f)(x)(T_1g)(x) = T_1(f \overset{\gamma_2}{\underset{T_1}{*}} g)(x). \qquad \square$$

Remark 3.10. Perhaps we should indicate the non-triviality of the convolutions (3.1), (3.2), (3.3). By Theorem 2.6, if $f, g \in \mathcal{S}\setminus\{0\}$, then $T_1fT_1g, \gamma_1T_1fT_1g$, $\gamma_2T_1fT_1g \in \mathcal{S}\setminus\{0\}$. By the factorization identities and Theorem 2.6, we get $f \underset{T_1}{*} g$, $f \overset{\gamma_1}{\underset{T_1}{*}} g$, $f \overset{\gamma_2}{\underset{T_1}{*}} g \in \mathcal{S}\setminus\{0\}$. Hence, the three last functions are non-zero functions in $\mathcal{S}$, so they are in $L_1(\mathbb{R})$.

The following corollary shows the relation between the convolution (3.3) and the Fourier convolution.

Corollary 3.11. *If $f, g \in L_1(\mathbb{R})$, then*

$$
\begin{aligned}
(f \overset{\gamma_2}{\underset{T_1}{*}} g)(x) = \frac{1}{2}\Big[&-[f \underset{F}{*} (e^{-\frac{1}{2}v^2} \underset{F}{*} g(v))](-x) + [f_1 \underset{F}{*} (e^{-\frac{1}{2}v^2} \underset{F}{*} g(v))](x) \\
&+ [f_1 \underset{F}{*} (e^{-\frac{1}{2}v^2} \underset{F}{*} g(v))](-x) + f \underset{F}{*} (e^{-\frac{1}{2}v^2} \underset{F}{*} g(v))](x)\Big],
\end{aligned}
$$

where $f_1(x) = f(-x)$.

Theorem 3.12. *If* $f, g \in L_1(\mathbb{R})$, *then*

$$(f \underset{T_1,T_1,F}{\overset{\gamma_2}{*}} g)(x) = \frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\big[-ie^{\frac{-(x+u+v)^2}{2}} + e^{\frac{-(x-u+v)^2}{2}} + e^{\frac{-(x-u-v)^2}{2}} + ie^{\frac{-(x+u-v)^2}{2}}\big]dudv \tag{3.8}$$

defines a convolution with weight-function γ_2 *for* T_1, T_1, F*; the factorization identity is*

$$T_1(f \underset{T_1,T_1,F}{\overset{\gamma_2}{*}} g)(x) = \gamma_2(x)(T_1 f)(x)(Fg)(x).$$

Proof. The proof that $f \underset{T_1,T_1,F}{\overset{\gamma_2}{*}} g \in L_1(\mathbb{R})$ is similar to that of Theorem 3.9. We prove the factorization identity. We have

$$\begin{aligned}
\gamma_2(x)(T_1 f)(x)(Fg)(x) &= e^{-\frac{1}{2}x^2}\frac{1}{\sqrt{2}\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\cos\Big(xu+\frac{\pi}{4}\Big)e^{-ixv}dudv \\
&= -\frac{ie^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u+v)+\sin x(u+v)]dudv \\
&\quad + \frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u-v)-\sin x(u-v)]dudv \\
&\quad + \frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u+v)-\sin x(u+v)]dudv \\
&\quad + \frac{ie^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u-v)+\sin x(u-v)]dudv.
\end{aligned}$$

Using the formulae (3.4), (3.5), (3.6), (3.7) we obtain

$$\begin{aligned}
&-\frac{ie^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u+v)+\sin x(u+v)]dudv \\
&= -\frac{i}{4\pi\sqrt{\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\int_{\mathbb{R}}\cos\Big(xy+\frac{\pi}{4}\Big)e^{-\frac{(y+u+v)^2}{2}}dydudv, \\
&\frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u-v)-\sin x(u-v)]dudv \\
&= \frac{1}{4\pi\sqrt{\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\int_{\mathbb{R}}\cos\Big(xy+\frac{\pi}{4}\Big)e^{-\frac{(y-u+v)^2}{2}}dydudv, \\
&\frac{e^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u+v)-\sin x(u+v)]dudv \\
&= \frac{1}{4\pi\sqrt{\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\int_{\mathbb{R}}\cos\Big(xy+\frac{\pi}{4}\Big)e^{-\frac{(y-u-v)^2}{2}}dydudv, \\
&\frac{ie^{-\frac{1}{2}x^2}}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)[\cos x(u-v)+\sin x(u-v)]dudv
\end{aligned}$$

$$= \frac{i}{4\pi\sqrt{\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v) \int_{\mathbb{R}} \cos\left(xy + \frac{\pi}{4}\right) e^{-\frac{(y+u-v)^2}{2}} dydudv.$$

Adding these four formulae, we obtain

$$\gamma_2(x)(T_1 f)(x)(Fg)(x) = T_1(f \overset{\gamma_2}{\underset{T_1,T_1,F}{*}} g)(x). \qquad \square$$

Remark 3.13. We state the non-triviality of the convolution (3.8). Indeed, choose $f, g \in \mathcal{S} \setminus \{0\}$. By Theorem 2.6 and Theorem 7.7 in [13], $\gamma_2 T_1 f F g \in \mathcal{S} \setminus \{0\}$. Using the factorization identity and Theorem 2.6, we infer $f \overset{\gamma_2}{\underset{T_1,T_1,F}{*}} g \not\equiv 0$.

3.3. Convolutions of $\mathbf{T_2}$

In this subsection we provide four convolutions for T_2. The proof of the following theorems are analogous to the corresponding proofs of the theorems 3.5, 3.7, 3.9, and 3.12 for T_1 and therefore left to the reader.

Theorem 3.14. *If $f, g \in L_1(\mathbb{R})$, then*

$$(f \underset{T_2}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \Big[f(x-y) + f(x+y) + f(-x+y) - f(-x-y)\Big] g(y)dy \quad (3.9)$$

defines a convolution for T_2; the factorization identity is

$$T_2(f \underset{T_2}{*} g)(x) = (T_2 f)(x)(T_2 g)(x).$$

Corollary 3.15. *If $f, g \in L_1(\mathbb{R})$, then*

(i) $(f \underset{T_2}{*} g)(x) = \frac{1}{2}\Big[(f \underset{F}{*} g)(x) + (f(-y) \underset{F}{*} g)(x) + (f(-y) \underset{F}{*} g)(-x) - (f \underset{F}{*} g)(-x)\Big].$

(ii) $(f \underset{F}{*} g)(x) = \frac{1}{2}\Big[-(f \underset{T_2}{*} g)(x) - (f(-y) \underset{T_2}{*} g)(x) + (f(-y) \underset{T_2}{*} g)(-x) + (f \underset{T_2}{*} g)(-x)\Big].$

Theorem 3.16. *Put $\beta_1(x) = \sin(x + \frac{\pi}{4})$. If $f, g \in L_1(\mathbb{R})$, then*

$$(f \overset{\beta_1}{\underset{T_2}{*}} g)(x) = \frac{1}{4\sqrt{\pi}} \int_{\mathbb{R}} g(u)\big[-f(-x-u-1) + f(x+u-1) \quad (3.10)$$

$$+ f(x-u+1) + f(-x+u+1)\big]du \quad (3.11)$$

defines a convolution with weight-function β_1 for T_2; the factorization identity is

$$T_2(f \overset{\beta_1}{\underset{T_2}{*}} g)(x) = \beta_1(x)(T_2 f)(x)(T_2 g)(x).$$

Corollary 3.17. *If $f, g \in L_1(\mathbb{R})$, then*

$$(f \overset{\beta_1}{\underset{T_2}{*}} g)(x) = \frac{1}{2\sqrt{2}}\Big[(f(-u) \underset{F}{*} g)(x-1) - (f \underset{F}{*} g)(-x-1) + (f \underset{F}{*} g)(x+1) + (f(-u) \underset{F}{*} g)(-x+1)\Big].$$

Theorem 3.18. *If $f, g \in L_1(\mathbb{R})$, then*

$$(f \overset{\gamma_2}{\underset{T_2}{*}} g)(x) = \frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\big[-e^{\frac{-(x+u+v)^2}{2}} + e^{\frac{-(x+u-v)^2}{2}} \tag{3.12}$$

$$+ e^{\frac{-(x-u+v)^2}{2}} + e^{\frac{-(x-u-v)^2}{2}}\big]dudv \tag{3.13}$$

defines a convolution with weight-function γ_2 for T_2; the factorization identity is

$$T_2(f \overset{\gamma_2}{\underset{T_2}{*}} g)(x) = \gamma_2(x)(T_2 f)(x)(T_2 g)(x).$$

Corollary 3.19. *If $f, g \in L_1(\mathbb{R})$, then*

$$(f \overset{\gamma_2}{\underset{T_2}{*}} g)(x) = \frac{1}{2}\Big[-(f \underset{F}{*} (e^{-\frac{1}{2}v^2} \underset{F}{*} g(v)))(-x) + (f_1 \underset{F}{*} (e^{-\frac{1}{2}v^2} \underset{F}{*} g(v)))(x)$$
$$+ (f_1 \underset{F}{*} (e^{-\frac{1}{2}v^2} \underset{F}{*} g(v)))(-x) + (f \underset{F}{*} (e^{-\frac{1}{2}v^2} \underset{F}{*} g(v)))(x)\Big],$$

where $f_1(x) = f(-x)$.

Theorem 3.20. *If $f, g \in L_1(\mathbb{R})$, then*

$$(f \overset{\gamma_2}{\underset{T_2,T_2,F}{*}} g)(x) = \frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)\big[ie^{\frac{-(x+u+v)^2}{2}} + e^{\frac{-(x-u+v)^2}{2}}$$
$$+ e^{\frac{-(x-u-v)^2}{2}} - ie^{\frac{-(x+u-v)^2}{2}}\big]dudv \tag{3.14}$$

defines a convolution with weight-function γ_2 for T_2, T_2, F; the factorization identity is

$$T_2(f \overset{\gamma_2}{\underset{T_2,T_2,F}{*}} g)(x) = \gamma_2(x)(T_2 f)(x)(Fg)(x).$$

Remark 3.21. The non-triviality of the convolutions in this subsection can be proved in the same way as in the proofs in Subsection 3.2.

4. Some applications

4.1. Normed ring structures on $\mathbf{L_1}(\mathbb{R})$

Definition 4.1. (Naimark [12]) A vector space V with a ring structure and a vector norm is called a normed ring if $\|vw\| \le \|v\|\|w\|$, for all $v, w \in V$.
If V has a multiplicative unit element e, it is also required that $\|e\| = 1$.

Let X denote the linear space $L_1(\mathbb{R})$. For each of the convolutions (3.1), (3.3), (3.8), (3.9), (3.12), and (3.14), the norm of f is chosen as

$$\|f\| = \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}} |f(x)|dx,$$

and for each of the convolutions (3.2), (3.10), the norm is

$$\|f\| = \sqrt{\frac{1}{\pi}}\int_{\mathbb{R}} |f(x)|dx.$$

Theorem 4.2. *X, equipped with each of the above-mentioned convolution multiplications, becomes a normed ring having no unit. Moreover,*

1) *For the convolutions* (3.1), (3.2), (3.3), (3.9), (3.10), *or* (3.12), *X is commutative.*

2) *For the convolutions* (3.8) *or* (3.14), *X is non-commutative.*

Proof. The proof of the first statement is divided into two steps.

Step 1. X has a normed ring structure. It is clear that X, equipped with each of those convolution multiplications, has a ring structure. We have to prove the multiplicative inequality. It is sufficient to prove that for the convolution (3.12) as the others can be proved in the same way. By using the formula

$$\int_{\mathbb{R}} e^{-\frac{1}{2}x^2}dx = \sqrt{2\pi},$$

we obtain

$$\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}} |f \overset{\gamma_2}{\underset{T_2}{*}} g|(x)dx \leq \frac{2}{\pi}\Big(\int_{\mathbb{R}} |f(u)|du\Big)\Big(\int_{\mathbb{R}} |g(v)|dv\Big) = \|f\|.\|g\|.$$

Therefore, $\|f \overset{\gamma_2}{\underset{T_2}{*}} g\| \leq \|f\|.\|g\|$.

Step 2. X has no unit. Suppose that there exists an element $e \in X$ such that $f * e = e * f = f$ for any $f \in X$. For short let us use the common symbol $*$ for the above-mentioned convolutions.

i) <u>The convolutions (3.8), (3.14).</u> By the factorization identities of these convolutions, $T_k f(\gamma_2 Fe - 1) = 0,\ k = 1, 2$. Choosing $f = \phi_0$ and using Theorem 2.1, we get $(T_k f)(x) = e^{-\frac{1}{2}x^2} \neq 0$ for $x \in \mathbb{R}$. Hence, $\gamma_2(x)(Fe)(x) = 1$ for every $x \in \mathbb{R}$ which is impossible as $\sup_{x\in\mathbb{R}} |\gamma_2(x)| = 1$ and $\lim_{x\to\infty}(Fe)(x) = 0$ (see [18, Theorem 1]).

ii) <u>The other convolutions.</u> By the factorization identities of those convolutions, $T_k f(\gamma_0 T_k e - 1) = 0$ $(k = 1, 2)$, where $\gamma_0 = 1$ if the convolution is one of (3.1) and (3.9), $\gamma_0 = \gamma_1$ if it is of (3.2), $\gamma_0 = \beta_1$ if it is of (3.10), and $\gamma_0 = \gamma_2$ if it is one of the others. Choosing $f = \phi_0$ and using Theorem 2.1, $\gamma_0(x)(T_k e)(x) = 1$ for every $x \in \mathbb{R}$, which is impossible as $\sup_{x\in\mathbb{R}} |\gamma_0(x)| = 1$ and $\lim_{x\to\pm\infty}(T_k e)(x) = 0$. Thus, X has no unit. We now prove the last conclusions of Theorem 4.2.

1) It is easily seen that X, equipped with each of the convolutions (3.1), (3.2), (3.3), (3.9), (3.10), and (3.12), is commutative.

2) Choose $f = \phi_1,\ g = \phi_0$. Using Theorem 2.1, Theorem 57 in [18] and the factorization identities of the convolutions, we obtain

$$T_k(\phi_1 * \phi_0) = \gamma_2(-\phi_1)\phi_0 = -\gamma_2\phi_0\phi_1,$$
$$T_k(\phi_0 * \phi_1) = \gamma_2\phi_0(i\phi_1) = i\gamma_2\phi_0\phi_1.$$

This implies that $T_k(\phi_1 * \phi_0) \not\equiv T_k(\phi_0 * \phi_1)$ in $L_1(\mathbb{R})$. Due to Corollary 2.7, we get $\phi_1 * \phi_0 \not\equiv \phi_0 * \phi_1$. Therefore, X is non-commutative. □

4.2. Partial differential equations and integral equations of convolution type

It is possible to use T_1, T_2 and the above defined convolutions for solving linear partial differential equations and integral equations of convolution type in a similar way as the Fourier, Fourier-cosine, or Fourier-sine transforms. In Examples 4.1, 4.2, 4.3 we consider formal solutions of three typical types of classical partial differential equations, and in Examples 4.4, 4.5 we obtain explicit solutions in $L_1(\mathbb{R})$ of two integral equations of convolution type.

Example 4.1. (see [18, 10.6]) Find the solution $u(x,t)$ of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{such that } u(x,0) = f(x) \ (-\infty < x < \infty,\ t > 0).$$

Let

$$U(\xi,t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} u(x,t) \cos\Big(x\xi + \frac{\pi}{4}\Big)\, dx.$$

Integrating by parts twice, and assuming that the terms at $+\infty$ and $-\infty$ vanish, we obtain

$$\begin{aligned}\frac{\partial U}{\partial t} &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{\partial u}{\partial t} \cos\Big(x\xi + \frac{\pi}{4}\Big)\, dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{\partial^2 u}{\partial x^2} \cos\Big(x\xi + \frac{\pi}{4}\Big)\, dx \\ &= -\frac{\xi^2}{\sqrt{\pi}} \int_{\mathbb{R}} u \cos\Big(x\xi + \frac{\pi}{4}\Big)\, dx = -\xi^2 U.\end{aligned}$$

This implies that $U(\xi,t) = A(\xi)e^{-\xi^2 t}$. Putting $t = 0$, we obtain

$$A(\xi) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x) \cos\Big(x\xi + \frac{\pi}{4}\Big)\, dx = (T_1 f)(\xi).$$

Hence $U(\xi,t) = (T_1 f)(\xi)e^{-\xi^2 t}$. Thus, the solution is

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} (T_1 f)(\xi) e^{-\xi^2 t} \cos\Big(x\xi + \frac{\pi}{4}\Big)\, d\xi.$$

Example 4.2. (see [18, 10.11]) Find the solution $v(x,y)$ of the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \ (-\infty < x < \infty, 0 < y < b)$$

such that $v(x,0) = f(x),\ v(x,b) = 0.$

Formally, let

$$V(\xi,y) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} v(x,y) \cos\Big(x\xi + \frac{\pi}{4}\Big)\, dx.$$

By assuming that the terms at $+\infty$ and $-\infty$ vanish, we get

$$\frac{\partial^2 V}{\partial y^2} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{\partial^2 v}{\partial y^2} \cos(\xi x + \frac{\pi}{4}) dx = -\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{\partial^2 v}{\partial x^2} \cos(\xi x + \frac{\pi}{4}) dx = \xi^2 V.$$

Hence,

$$V(\xi,y) = A(\xi)\cosh \xi y + B(\xi)\sinh \xi y. \tag{4.1}$$

Letting $y \to 0$, we obtain

$$A(\xi) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x) \cos\left(x\xi + \frac{\pi}{4}\right) dx = (T_1 f)(\xi).$$

Inserting $y = b$ into the identity (4.1), we obtain $A(\xi)\cosh \xi b + B(\xi) \sinh \xi b = 0$. Then $B(\xi) = -\coth \xi b (T_1 f)(\xi)$. Hence

$$V(\xi, y) = (T_1 f)(\xi)(\cosh \xi y - \sinh \xi y \coth \xi b) = (T_1 f)(\xi) \frac{\sinh \xi(b-y)}{\sinh \xi b}.$$

We thus have a solution

$$v(x, y) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} (T_1 f)(\xi) \frac{\sinh \xi(b-y)}{\sinh \xi b} \cos\left(x\xi + \frac{\pi}{4}\right) d\xi.$$

Example 4.3. (see [18, 10.12])

Obtain the solution of the equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} \qquad (-\infty < x < \infty, t > 0)$$

such that $w(x, 0) = f(x), \ \ w_t(x, 0) = g(x)$.

For a formal solution, let

$$W(\xi, t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} w(x, t) \cos\left(x\xi + \frac{\pi}{4}\right) dx.$$

Integrating by parts twice, we get

$$\frac{\partial^2 W}{\partial t^2} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{\partial^2 w}{\partial t^2} \cos\left(x\xi + \frac{\pi}{4}\right) dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{\partial^2 w}{\partial x^2} \cos\left(x\xi + \frac{\pi}{4}\right) dx = -\xi^2 W,$$

Hence $W = A(\xi) \cos \xi t + B(\xi) \sin \xi t$. Inserting $t = 0$ into the last identity and it derivative, we get $A(\xi) = (T_1 f)(\xi), \ \ \xi B(\xi) = (T_1 g)(\xi)$. Hence, the solution is of the following form

$$w(x,t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} (T_1 f)(\xi) \cos \xi t \cos\left(x\xi + \frac{\pi}{4}\right) d\xi + \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{(T_1 g)(\xi)}{\xi} \sin \xi t \cos\left(x\xi + \frac{\pi}{4}\right) d\xi.$$

Remark 4.3. a) In fact, calculating the integrals we can reduce the solutions $u(x, t)$, $v(x, y), w(x, t)$ obtained in Examples 4.1, 4.2, 4.3 to the following forms

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} f(u) e^{-\frac{(x-u)^2}{4t}} du,$$

$$v(x, y) = \frac{1}{2b} \sin \frac{\pi y}{b} \int_{\mathbb{R}} f(u) \left\{ \frac{1}{\cos(b-y)\pi/b + \cosh(x-u)\pi/b} - \frac{1}{\cos(b-y)\pi/b + \cosh(x+u)\pi/b} \right\} du,$$

$$w(x, t) = \frac{1}{2}\{f(x+t) + f(x-t)\} + \frac{1}{2} \int_{x-t}^{x+t} g(u) du,$$

as given by Titchmarsh in [18, 10.6, 10.11, 10.12].

b) For rigorous solutions in Examples 4.1, 4.2, 4.3 one has to add some necessary assumptions underlying the initial conditions f, g, and predetermine the solution to be in a specific class of functions (for instance, f, g and solutions are assumed in $\mathcal{S}$).

Example 4.4. Consider the following integral equation

$$\lambda f(x) + \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} [k_1(x+y) + k_2(x-y) + k_3(-x+y) + k_4(-x-y)] f(y)dy = g(x), \tag{4.2}$$

where the functions g, k_p $(p = 1, 2, 3, 4)$ are given, $\lambda \in \mathbb{C}$ is predetermined, and f is the unknown function. Equation (4.2) is a generalization of the integral equation of convolution type with a mixed Toeplitz-Hankel kernel (see Tsitsiklis-Levy [19]).

In the case of $k_1 = k_2 = k_3 = -k_4 = k$, the equation (4.2) is

$$\lambda f(x) + \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} [k(x-y) + k(x+y) + k(-x+y) - k(-x-y)] f(y)dy = g(x). \tag{4.3}$$

We shall deal with the solvability of (4.2) in $L_1(\mathbb{R})$, i.e., $k, g \in L_1(\mathbb{R})$ are given, and f is to be determined. In what follows, the functional identity $f(x) = g(x)$ means that it is valid for almost every $x \in \mathbb{R}$. However, if both functions f, g are continuous, then of course the above identity is true for every $x \in \mathbb{R}$. In Theorems 4.4 and 4.5 below, we obtain explicit solutions of two integral equations of convolution type.

Theorem 4.4. *Assume that* $\lambda + (T_1 k)(x) \neq 0$ *for every* $x \in \mathbb{R}$. *Then the equation* (4.3) *has a solution in* $L_1(\mathbb{R})$ *if and only if* $T_1 \left(\frac{T_1 g}{\lambda + T_1 k} \right) \in L_1(\mathbb{R})$. *If this is the case, then the solution is given by*

$$f(x) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{(T_1 g)(u)}{\lambda + (T_1 k)(u)} \cos\left(xu + \frac{\pi}{4} \right) du. \tag{4.4}$$

Proof. Equation (4.3) is rewritten in the following form

$$\lambda f(x) + (f \underset{T_1}{*} k)(x) = g(x). \tag{4.5}$$

Necessity. Suppose that $f \in L_1(\mathbb{R})$ is a solution of (4.5). Applying T_1 to both sides of (4.5) and using the factorization identity of the convolution (3.1), we get

$$(T_1 f)(x) = \frac{(T_1 g)(x)}{\lambda + (T_1 k)(x)}. \tag{4.6}$$

By the assumption and Theorem 2.6, we get $f(x)$ as in (4.4). Since $f \in L_1(\mathbb{R})$, the function on the right side of (4.4) belongs to $L_1(\mathbb{R})$.

Sufficiency. Put

$$f(x) := T_1 \left(\frac{T_1 g}{\lambda + T_1 k} \right)(x).$$

By the assumption, $f \in L_1(\mathbb{R})$. We apply Theorem 2.6 to obtain

$$(T_1 f)(x) = \frac{(T_1 g)(x)}{\lambda + (T_1 k)(x)}.$$

Equivalently, $T_1\Big(\lambda f + (f \underset{T_1}{*} k) - g\Big)(x) = 0$. By Corollary 2.7, we conclude that $\lambda f(x) + (f \underset{T_1}{*} k)(x) = g(x)$ for almost every $x \in \mathbb{R}$. Therefore, $f(x)$ fulfills the equation (4.5). □

Example 4.5. Consider the equation

$$\lambda f(x) + \frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} h(v)\big[- ie^{\frac{-(x+y+v)^2}{2}} + e^{\frac{-(x-y+v)^2}{2}} \tag{4.7}$$
$$+ e^{\frac{-(x-y-v)^2}{2}} + ie^{\frac{-(x+y-v)^2}{2}}\big] f(y)dydv = g(x),$$

where h, g are given in $L_1(\mathbb{R})$, and f is to be determined. The kernel of this equation is

$$k(x,y) = \int_{\mathbb{R}} h(v)\big[- ie^{\frac{-(x+y+v)^2}{2}} + e^{\frac{-(x-y+v)^2}{2}} \tag{4.8}$$
$$+ e^{\frac{-(-x+y+v)^2}{2}} + ie^{\frac{-(-x-y+v)^2}{2}}\big] dv. \tag{4.9}$$

According to Theorem 3.12, the equation (4.7) can be rewritten as follows

$$\lambda f(x) + (f \overset{\gamma_2}{\underset{T_1,T_1,F}{*}} h)(x) = g(x).$$

In the same way as in the proof of Theorem 4.4, we can prove the following theorem.

Theorem 4.5. *Assume that* $\lambda + \gamma_2(x)(Fh)(x) \neq 0$ *for every* $x \in \mathbb{R}$. *Then the equation* (4.7) *has a solution in* $L_1(\mathbb{R})$ *if and only if* $T_1\left(\dfrac{T_1 g}{\lambda + \gamma_2 Fh}\right) \in L_1(\mathbb{R})$. *If this is the case, then the solution is given by*

$$f = T_1\left(\frac{(T_1 g}{\lambda + \gamma_2 Fh}\right).$$

In the general theory of integral equations, the assumptions that $\lambda + (T_1 k)(x) \neq 0$, and $\lambda + \gamma_2(x)(Fh)(x) \neq 0$ for every $x \in \mathbb{R}$ as in Theorems 4.4, 4.5 are the conditions of normal solvability of the equations.
The equations (4.2), (4.7) are Fredholm integral equations of first the kind if $\lambda = 0$, and that of the second kind if $\lambda \neq 0$. In the case of the second kind, the following proposition serves as an illustration of the assumptions in Theorems 4.4, 4.5.

Proposition 4.6. *Let* $\lambda \neq 0$. *Then each of the two functions* $\lambda + (T_1 k)(x)$ *and* $\lambda + \gamma_2(x)(Fh)(x)$ *does not vanish outside a finite interval. If* $|\lambda|$ *is sufficiently large, then the equations* (4.3) *and* (4.7) *are solvable in* $L_1(\mathbb{R})$.

Proof. By Proposition 2.2, we have $\lim_{x\to\pm\infty}(T_1k)(x) = 0$. As the function $(T_1k)(x)$ is continuous on $\mathbb{R}$ and $|\lambda| > 0$, there exists a number $R > 0$ such that $|(T_1k)(x)| < |\lambda|$ for every $|x| > R$. We thus have $\lambda + (T_1k)(x) \neq 0$ for every $|x| > R$.
Similarly, by using the Riemann-Lebesgue lemma of the Fourier transform F it is possible to prove that $\lambda + \gamma_2(x)(Fh)(x) \neq 0$ for every x outside a finite interval.

By Proposition 2.2 and the Riemann-Lebesgue Lemma, the two functions $(T_1k), \gamma_2(Fh)$ are continuous on $\mathbb{R}$ and vanish at infinity. It follows that there exist $x_1, x_2 \in \mathbb{R}$ such that $M_0 := |(T_1k)(x_1)| \geq |(T_1k)(x)|$, $N_0 := |\gamma_2(x_2)(Fh)(x_2)| \geq |\gamma_2(x)(Fh)(x)|$ for evey $x \in \mathbb{R}$. Hence, if $|\lambda| > \max\{M_0, N_0\}$, then we have $\lambda + \gamma_2(x)(Fh)(x) \neq 0$ and $\lambda + (T_1k)(x) \neq 0$ for every $x \in \mathbb{R}$. □

Comparison. a) By the use of T_1, T_2 and their inverse transforms as presented in Examples 4.1, 4.2, and 4.3, we have a new approach (apart from the use of the Fourier transform) for the solution of linear partial differential equations.

b) There is an approach to integral equations of convolution type by using an appropriate convolution and the Wiener-Lèvy theorem as e.g. in [8, 11, 16, 17]. However, that approach usually includes only sufficient conditions (no necessary conditions) for the solvability of the equations and obtains the solutions in implicit (not explicit) form. By means of the normally solvable conditions we are able to state necessary and sufficient conditions of the equations (4.3), (4.7), and give the solutions in explicit form by using the convolutions (3.1) and (3.8).

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Bui Thi Giang
Department of Basic Science, Institute of Cryptography Science
141 Chien Thang Str., Thanh Tri Dist.
Hanoi, Vietnam

Nguyen Van Mau and Nguyen Minh Tuan
Department of Mathematical Analysis
University of Hanoi, 334 Nguyen Trai Str.
Hanoi, Vietnam
e-mail: `maunv@vnu.edu.vn`
`tuannm@hus.edu.vn`

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