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Integral Equations and Operator Theory

Eigenvalues of Integral Operators Defined by Smooth Positive Definite Kernels

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Abstract. We consider integral operators defined by positive definite kernels $K : X \times X \to \mathbb{C}$, where X is a metric space endowed with a strictly-positive measure. We update upon connections between two concepts of positive definiteness and upgrade on results related to Mercer like kernels. Under smoothness assumptions on K, we present decay rates for the eigenvalues of the integral operator, employing adapted to our purposes multidimensional versions of known techniques used to analyze similar problems in the case where X is an interval. The results cover the case when X is a subset of $\mathbb{R}^m$ endowed with the induced Lebesgue measure and the case when X is a subset of the sphere S^m endowed with the induced surface Lebesgue measure.

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1. Introduction

This paper is concerned with the analysis of decay rates for eigenvalues of integral operators construct from positive definite kernels on subsets of metric spaces. Such operators appear quite naturally in approximation theory, integral equations and operator theory, playing an important role in many problems.

We will consider two different notions of positive definiteness as explained below. If X is a nonempty set, a kernel $K : X \times X \to \mathbb{C}$ is *positive definite* when the inequality

$$\sum_{i,j=1}^{n} \overline{c_i}\, c_j K(x_i, x_j) \geq 0,$$

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holds for all $n \geq 1$, $x_1, x_2, \ldots, x_n \in X$ and scalars $c_1, c_2, \ldots, c_n$. If X is endowed with a measure ν and K belongs to $L^2(X \times X, \nu \times \nu)$, we say that K is L^2-*positive definite* when the corresponding integral operator

$$\mathcal{K}(f)(x) := \int_X K(x,y) f(y)\, d\nu(y), \quad f \in L^2(X,\nu), \quad x \in X, \tag{1.1}$$

is positive, that is, when the following condition holds

$$\int_X \left(\int_X K(x,y) f(y)\, d\nu(y) \right) \overline{f(x)}\, d\nu(x) \geq 0, \quad f \in L^2(X,\nu).$$

We will write $PD(X)$ and $L^2PD(X,\nu)$ to denote these two classes of kernels. Needless to say that the expression on the left-hand side of the above inequality is just $\langle \mathcal{K}(f), f \rangle_2$, in which $\langle \cdot, \cdot \rangle_2$ is the inner product of $L^2(X,\nu)$.

Keeping the context as general as possible, we will investigate possible connections between these two concepts of positive definiteness. Assuming positive definiteness of the kernel and reasonable additional smoothness assumptions on K, we will update on the corresponding Mercer's theory and analyze decay rates for the eigenvalues of the integral operator (1.1). A quite general formulation for the classical result of Mercer is as follows (see [11, 13]).

Theorem 1.1 (Mercer's Theorem). *Let X be a topological Hausdorff space equipped with a finite Borel measure ν. Then for every continuous positive definite kernel $K : X \times X \to \mathbb{C}$ there exist a scalar sequence $\{\lambda_n\} \in l_1$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ and an orthonormal system $\{\phi_n\}$ in $L^2(X,\nu)$ consisting of continuous functions only, such that the expansion*

$$K(x,y) = \sum_{n=1}^{\infty} \lambda_n(\mathcal{K}) \phi_n(x) \overline{\phi_n(y)}, \quad x, y \in \operatorname{supp}(\nu),$$

converges uniformly.

If the integral operator has countably many eigenvalues $\lambda_1(\mathcal{K}) \geq \lambda_2(\mathcal{K}) \geq \cdots \geq 0$, the basic decay rate given by Mercer's theory is $\lambda_n(\mathcal{K}) = o(n^{-1})$, as $n \to \infty$. The following example ([16]) shows that this rate can not be improved, unless additional assumptions are added. Indeed, consider $X = [-1,1]$ endowed with the Lebesgue measure μ. The kernel

$$K(x,y) = \sum_{n=1}^{\infty} \frac{1}{n^{p+1+\epsilon}} \cos(n\pi x) \cos(n\pi y), \quad x, y \in [-1,1],$$

where p is a nonnegative integer and $\epsilon > 0$, is an element of $L^2PD([-1,1],\mu)$. If

$$\phi_n(x) := \cos(n\pi x), \quad x \in [-1,1],$$

then the sequence $\{\phi_n\}$ is $L^2([-1,1],\mu)$-orthonormal and $\lambda_n(\mathcal{K}) = n^{-1-p-\epsilon}$, $n = 1, 2, \ldots$. As so, $\lambda_n(\mathcal{K}) = O(n^{-1-p-\epsilon}) = o(n^{-1-p})$, as $n \to \infty$, but

$\lambda_n(\mathcal{K}) \neq o(n^{-1-p-\epsilon})$, as $n \to \infty$. The series

$$\sum_{n=1}^{\infty} \frac{n^q}{n^{p+1+\epsilon}}, \quad 0 \leq q \leq p,$$

being convergent, it is not hard to see that the partial derivatives of order at most p of K are continuous. As so, K is of class C^p.

Improvements on the basic estimate exhibit above can be found in many contexts under different sets of hypotheses. For instance, reference [15] considers the case when X is a compact interval while references [12, 13] analyze the case when X is a compact metric space or even a differentiable manifold, endowed with a finite measure. In [2], now dropping the compactness assumption on X, Buescu and Paixão investigated generalizations when X is a closed interval and K satisfies certain smoothness hypotheses. A similar analysis can be found in [3, 4].

In this paper these questions will be take up again, just assuming that X is a metric space endowed with a strictly-positive measure. In Section 2, we investigate possible connections between the notions of positive definiteness and recover Mercer's Theorem in this new setting. In Section 3, we analyze important properties of the square root of an integral operator, under the light of the assumptions adopted on X and on the kernel. In particular we show that the square root is also an integral operator and deduce a recovery formula via the kernel defining the original operator. In Section 4, using the results in Section 3, we discuss basic finite approximations for the integral operator based upon special decompositions of X. The last result of the section describes an estimate for the sum of the eigenvalues of the operator deduced from such finite approximations. Section 5 begins with the concept of (q, t)-compactness, a very special decomposition for metric spaces endowed with a measure. Later in the section, we deduce the main results of the paper. They describe decay rates for the eigenvalues of the integral operator when X is (q, t)-compact and the generating kernel satisfies a smoothness condition of Lipschitz type. It is important to emphasize that the approach we take here is based upon arguments found in [2] and references therein.

2. Positive definiteness and Mercer's theory revisited

The results in this section indicate possible contexts in which the classes $PD(X)$ and $L^2PD(X, \nu)$ coincide. Henceforth, if X is a metric space, we will write $C(X)$ to denote the set of continuous functions on X and $C_B(X)$ to denote the subset of $C(X)$ formed by bounded functions vanishing outside a bounded subset of X. The letter ν will be used to denote a measure over X. If X is a subset of $\mathbb{R}^m$ and the measure is the restriction of the usual Lebesgue measure of $\mathbb{R}^m$ to X, the letter μ will be used instead.

The first result describes a setting where the inclusion $PD(X) \subset L^2PD(X, \nu)$ holds.

Theorem 2.1. *If X is a measurable subset of $\mathbb{R}^m$ endowed with the usual Lebesgue measure μ then*

$$PD(X) \cap C(X \times X) \cap L^2(X \times X, \mu \times \mu) \subset L^2PD(X, \mu).$$

Proof. Let K be in $PD(X) \cap C(X \times X) \cap L^2(X \times X, \mu \times \mu)$. Since $C_B(X)$ is dense in $L^2(X, \mu)$ ([8, p.217]), to show that $\langle \mathcal{K}(f), f\rangle_2 \geq 0$, $f \in L^2(X, \mu)$, it suffices to verify that $\langle \mathcal{K}(f), f\rangle_2 \geq 0$, $f \in C_B(X)$. Let $f \in C_B(X)$ and write X_f to denote a bounded subset of X for which $f(x) = 0$, $x \in X \setminus X_f$. There exists a sequence $\{A_n\}$ of compact subsets of X_f such that $A_n \subset A_{n+1}$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} \mu(X_f \setminus A_n) = 0$. In particular, the kernel K_f given by $K_f(x, y) = K(x, y)\overline{f(x)}f(y)$, $x, y \in X$, is uniformly continuous in $A_n \times A_n$. The Monotone Convergence Theorem shows that $\{K_f \chi_{A_n \times A_n}\}$ converges to K_f in $L^1(X \times X, \mu \times \mu)$. Next, for each n, we can find $r = r(n) > 0$ so that $A_n \subset [-r/2, r/2]^m$. Writing $[-r/2, r/2]^m = \cup_{j=1}^{k^m} C_j^k$, in which $C_1^k, C_2^k, \ldots, C_{k^m}^k$, are m-dimensional cubes having sides of length r/n, parallel to the coordinate axes, we may decompose A_n in the form

$$A_n = \cup_{j=1}^{k^m} A_j^k, \quad A_j^k \subset C_j^k, \quad A_j^k \cap A_l^k = \emptyset, \quad l \neq k.$$

Choosing $x_j^k \in A_j^k$, $j = 1, 2, \ldots, k^m$ and defining

$$g_k^n = \sum_{i,j=1}^{k^m} K(x_i^k, x_j^k)\overline{f(x_i^k)}f(x_j^k)\chi_{A_i^k \times A_j^k},$$

it is easily seen that $\{g_k^n\}$ converges uniformly to $K_f \chi_{A_n \times A_n}$ in $A_n \times A_n$, when $k \to \infty$. Also, since $K \in PD(X)$, it follows that $g_k^n(x, y) \geq 0$, $x, y \in A_n$. Taking into account that $K_f \chi_{A_n \times A_n}$ is bounded and the fact that $\mu(A_n) < \infty$, we can use the Dominated Convergence Theorem to deduce that

$$\begin{aligned}\int_{X \times X} K_f(x, y)\, d\mu(x)\, d\mu(y) &= \int_{X_f \times X_f} K_f(x, y)\, d\mu(x)\, d\mu(y) \\ &= \lim_{n\to\infty} \int_{A_n \times A_n} K_f(x, y)\, d\mu(x)\, d\mu(y) \\ &= \lim_{n\to\infty} \left(\lim_{k\to\infty} \int_{A_n \times A_n} g_k^n(x, y)\, d\mu(x)\, d\mu(y) \right) \geq 0.\end{aligned}$$

Thus, $K \in L^2PD(X, \mu)$. □

In a similar manner, the following extension can be proved.

Corollary 2.2. *If X is a locally compact Hausdorff space endowed with a Radon measure ν that is finite on compact subsets then*

$$PD(X) \cap C(X \times X) \cap L^2(X \times X, \nu \times \nu) \subset L^2PD(X, \nu).$$

The other inclusion can be guaranteed in a quite general context. If X is a topological space, we say that a measure ν on X is *strictly-positive* when it is a Borel measure fulfilling the following requirements: every open nonempty subset of X has positive measure and every $x \in X$ belongs to an open subset of X having finite measure.

Theorem 2.3. *If a topological space X is endowed with a strictly-positive measure ν, then*

$$L^2PD(X,\nu) \cap C(X \times X) \subset PD(X).$$

Proof. Let $K \in L^2PD(X) \cap C(X \times X)$, $x_1, x_2, \ldots, x_n \in X$ and $c_1, c_2, \ldots, c_n \in \mathbb{C}$. Due to the continuity of K, for each $\epsilon > 0$ and $j \in \{1, 2, \ldots, n\}$ there exist open sets X_j^ϵ so that $x_j \in X_j^\epsilon$ and

$$|K(x,y) - K(x_i, x_j)| < \epsilon, \quad x \in X_i^\epsilon, \quad y \in X_j^\epsilon, \quad i,j = 1, 2, \ldots, n.$$

Since ν is strictly-positive, we can assume that $0 < \mu(X_j^\epsilon) < \infty$, $j = 1, 2 \ldots, n$. As so, integration implies that

$$\frac{1}{\nu(X_i^\epsilon)\nu(X_j^\epsilon)} \int_{X_i^\epsilon} \int_{X_j^\epsilon} |K(x,y) - K(x_i, x_j)| \, d\nu(x) \, d\nu(y) < \epsilon.$$

In particular,

$$\lim_{\epsilon \to 0^+} \frac{1}{\nu(X_i^\epsilon)} \frac{1}{\nu(X_j^\epsilon)} \int_{X_i^\epsilon} \int_{X_j^\epsilon} K(x,y) \, d\nu(x) \, d\nu(y) = K(x_i, x_j).$$

Since the functions

$$f_\epsilon := \sum_{j=1}^n \frac{c_j}{\mu(X_j^\epsilon)} \chi_{X_j^\epsilon}, \quad \epsilon > 0,$$

belong to $L^2(X,\nu)$, the inequality

$$0 \leq \langle \mathcal{K}(f_\epsilon), f_\epsilon \rangle_2 = \sum_{i,j=1}^n \overline{c_i} c_j \frac{1}{\nu(X_i^\epsilon)\nu(X_j^\epsilon)} \int_{X_i^\epsilon} \int_{X_j^\epsilon} K(x,y) \, d\nu(x) \, d\nu(y)$$

leads to

$$0 \leq \sum_{i,j=1}^n \overline{c_i} c_j K(x_i, x_j),$$

that is, $K \in PD(X)$. □

Next, we introduce a general formulation for Mercer's Theorem. We will need some additional notation attached to a measure space (X,ν). Precisely, we will write $\mathcal{A}(X,\nu)$ to denote the subset of $C(X \times X) \cap L^2PD(X,\nu)$ formed by all kernels $K : X \times X \to \mathbb{C}$ for which $x \in X \to K(x,x)$ is an element of $L^1(X,\nu)$.

Theorem 2.4 below includes in its statement a generalization of the classical Mercer's Theorem. The proof we include here uses adaptations of methods introduced in [1, 6, 12, 14, 18]. Recall that an operator T on a Hilbert space $\mathcal{H}$ is *trace-class* ([9]), when its square root $|T| := (T^*T)^{1/2}$ satisfies the condition

$$\sum_{f \in \mathfrak{B}} \langle |T|(f), f \rangle_{\mathcal{H}} < \infty, \tag{2.1}$$

for every orthonormal basis $\mathfrak{B}$ of $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$. The trace acts linearly over the vector space of all trace-class operators over $\mathcal{H}$. If T is trace-class, the sum in (2.1) does not depend upon the basis and it is called the *trace* of T, here denoted by $\mathrm{tr}(T)$. If T is a compact operator, then $|T|$ is compact, positive and self-adjoint. As so, denoting by $\{s_n(T)\}$ the sequence of eigenvalues of $|T|$, each repeated as often as its multiplicity, then T is trace-class if and only if $\sum_{n=1}^{\infty} s_n(T) < \infty$. As a matter of fact, the sum coincides with the trace of $|T|$ when T is trace-class. If T is also self-adjoint, then $s_n(T) = |\lambda_n(T)|$, $n = 1, 2, \ldots$, so that

$$\mathrm{tr}\,(T) = \sum_{n=1}^{\infty} \lambda_n(T).$$

An important family of trace-class operators is that encompassing all finite rank operators on $\mathcal{H}$. If $\mathcal{H}$ is a separable Hilbert space, the set of all trace-class operators on $\mathcal{H}$ is a vector space and the formula

$$\|T\|_{tr} := \sum_{n=1}^{\infty} s_n(T)$$

defines a norm on it, the so-called *trace norm.* From now on, all general Hilbert spaces mentioned in the paper are assumed to be separable.

Theorem 2.4. *Let X be a metric space endowed with a strictly-positive measure ν. If $K \in \mathcal{A}(X, \nu)$ then the following assertions hold:*

(*i*) *The range of $\mathcal{K}$ is a subset of $C(X) \cap L^2(X, \nu)$;*

(*ii*) *The operator $\mathcal{K}$ is compact and selfadjoint, having an $L^2(X, \nu)$-convergent series representation in the form*

$$\mathcal{K}(f) = \sum_{n=1}^{\infty} \lambda_n(\mathcal{K}) \langle f, \phi_n \rangle_2 \phi_n, \quad f \in L^2(X, \nu).$$

The series is absolutely and uniformly convergent on compact subsets of X;

(*iii*) *K has a $L^2(X \times X, \nu \times \nu)$-convergent series representation in the form*

$$K(x, y) = \sum_{n=1}^{\infty} \lambda_n(\mathcal{K}) \phi_n(x) \overline{\phi_n(y)}, \quad x, y \in X,$$

$\{\lambda_n(\mathcal{K})\}$ decreases to 0 and $\{\phi_n\}$ is $L^2(X, \nu)$-orthonormal. The convergence of the series is absolute and uniform on compact subsets of $X \times X$. If $\lambda_n(\mathcal{K}) > 0$ then ϕ_n is an eigenfunction of $\mathcal{K}$ associated with the eigenvalue $\lambda_n(\mathcal{K})$, taking into account multiplicities;

(*iv*) *The operator $\mathcal{K}$ is trace-class and*

$$tr(\mathcal{K}) = \int_X K(x, x)\, d\nu(x).$$

Proof. Assume $K \in \mathcal{A}(X, \nu)$. Since ν is strictly-positive, Theorem 2.3 shows that $K \in PD(X)$. In particular,

$$|K(x, y)|^2 \leq K(x, x) K(y, y), \quad x, y \in X. \tag{2.2}$$

If $y \in X \to K(y,y)$ is integrable, it is quite clear that every function $y \in X \to K(x,y)$, $x \in X$, belongs to $L^2(X,\nu)$. Next, we show that the mapping $x \in X \to K(x,\cdot) \in L^2(X,\nu)$ is continuous. Let $\{x_n\}$ be a sequence in X converging to $x_0 \in X$. Since K is continuous, the sequence $\{K(x_n,y)\}$ converges to $K(x_0,y)$, for every $y \in X$ fixed. Using (2.2), we deduce that

$$\begin{aligned}|K(x_n,y)-K(x_0,y)|^2 &\le |K(x_n,y)|^2+2|K(x_n,y)||K(x_0,y)|+|K(x_0,y)|^2\\ &\le K(y,y)\left(K(x_n,x_n)+K(x_0,x_0)\right)\\ &\qquad +2K(y,y)K(x_n,x_n)^{1/2}K(x_0,x_0)^{1/2}\\ &\le 4\sup_{m\in\mathbb{Z}_+}\{K(x_m,x_m)\}\,K(y,y),\quad y\in X.\end{aligned}$$

The Dominated Convergence Theorem leads to

$$\lim_{n\to\infty}\int_X |K(x_n,y)-K(x_0,y)|^2\,d\nu(x)=0.$$

The continuity of $x \in X \mapsto K(x,\cdot) \in L^2(X,\nu)$ follows. Since

$$\mathcal{K}(f)(x)=\langle f,\overline{K(x,\cdot)}\rangle_2,\quad f\in L^2(X,\nu),\quad x\in X,$$

assertion (i) follows.
Since (X,ν) is a measure space and $K \in L^2(X\times X,\nu\times\nu)$, the integral operator $\mathcal{K}: L^2(X,\nu)\to L^2(X,\nu)$ is compact ([5, p.86]). Being K hermitian, $\mathcal{K}$ is selfadjoint. Applying the Spectral Theorem for compact selfadjoint operators ([5, p.93]), we can deduce that $\mathcal{K}$ is an $L^2(X,\nu)$-convergent series of the form

$$\mathcal{K}(f)=\sum_{n=1}^{\infty}\lambda_n(\mathcal{K})\langle f,\phi_n\rangle_2\phi_n,\quad f\in L^2(X,\nu),\tag{2.3}$$

where $\{\lambda_n(\mathcal{K})\}$ decreases to 0 and $\{\phi_n\}$ is $L^2(X,\nu)$-orthonormal. Next, we consider auxiliary kernels K_p, $p \ge 1$, given by the formula

$$K_p(x,y)=K(x,y)-\sum_{n=1}^{p}\lambda_n(\mathcal{K})\phi_n(x)\overline{\phi_n(y)},\quad x,y\in X.$$

Obviously, $K_p \in L^2(X\times X,\nu\times\nu)\cap C(X\times X)$ while standard computations show that $K_p \in \mathcal{A}(X,\nu)$. Lemma 2.2 in [6] reveals that $K_p(x,x)\ge 0$, $x\in X$, that is,

$$\sum_{n=1}^{p}\lambda_n(\mathcal{K})|\phi_n(x)|^2\le K(x,x),\quad x\in X.$$

The inequality

$$\left|\sum_{n=p}^{p+q}\lambda_n(\mathcal{K})\langle f,\phi_n\rangle_2\phi_n(x)\right|^2\le \lambda_1(\mathcal{K})\sup_{y\in Y}K(y,y)\sum_{n=p}^{p+q}|\langle f,\phi_n\rangle_2|^2,\quad x\in Y,$$

holds whenever Y is a compact subset of X and $q,\, p \geq 1$. As so, the convergence of the series in (2.3) is uniform on compact subsets of X. This takes care of (ii). From the Cauchy-Schwarz inequality we now obtain

$$\left|\sum_{n=p}^{p+q} \lambda_n(\mathcal{K})\phi_n(x)\overline{\phi_n(y)}\right|^2 \leq (K(x,x)+1)\sum_{n=p}^{p+q} \lambda_n(\mathcal{K})|\phi_n(y)|^2, \quad x, y \in X, \quad p,\, q \geq 1.$$

The Cauchy Criterion for convergence and the continuity of K imply that the series $\sum_{n=1}^{\infty} \lambda_n(\mathcal{K})\phi_n(x)\overline{\phi_n(y)}$ is convergent to a function in $C(X)$, when one of the variables is held fixed. However, due to (i) and (ii),

$$\int_X \left(\sum_{n=1}^{\infty} \lambda_n(\mathcal{K})\phi_n(x)\overline{\phi_n(y)}\right) f(y)\, d\nu(y) = \sum_{n=1}^{\infty} \lambda_n(\mathcal{K})\langle f, \phi_n\rangle_2 \phi_n(x) = \mathcal{K}(f)(x),$$

whenever $f \in L^2(X,\nu)$ and $x \in X$. Using this information with a convenient choice for f and recalling our assumption on X, we deduce that

$$\sum_{n=1}^{\infty} \lambda_n(\mathcal{K})\phi_n(x)\overline{\phi_n(y)} = K(x,y), \quad x, y \in X.$$

Dini's Theorem leads to

$$\sum_{n=1}^{\infty} \lambda_n(\mathcal{K})|\phi_n(x)|^2 = K(x,x), \quad x \in X, \tag{2.4}$$

with uniform and absolute convergence on compact subsets of X. Finally, the Cauchy Criterion for uniform convergence and the Cauchy-Schwarz inequality imply uniform and absolute convergence of the series on compact subsets of $X \times X$. The Monotone Convergence Theorem along with (2.4) resolves (iv). □

The assumptions listed in the previous theorem are to be assumed from now on. Theorem 2.4 provides basic information on decay rates for the eigenvalues of the integral operator $\mathcal{K}$, at least when K fits the description considered there. That we quote in a separated result.

Corollary 2.5. *Under the conditions stated in Theorem* 2.4*, it holds* $\lambda_n(\mathcal{K}) = o(n^{-1})$*, as* $n \to \infty$*.*

3. The square root of $\mathcal{K}$

In this section, we will list some of the properties the square root $\mathcal{K}^{1/2}$ of the integral operator $\mathcal{K}$ has, when K fits the assumptions in Theorem 2.4. Such properties will be used ahead in some key arguments. The existence of $\mathcal{K}^{1/2}$ of $\mathcal{K}$ is guaranteed by a well-known result from Hilbert space theory ([19, p.142]).

Lemma 3.1. *Let X and ν be as in Theorem 2.4 and $K \in \mathcal{A}(X, \nu)$. Consider the representation for K provided by Theorem 2.4-(iii). Then $\mathcal{K}^{1/2}$ coincides with the integral operator $\mathcal{S} : L^2(X, \nu) \to L^2(X, \nu)$, in which $S \in L^2PD(X, \nu)$ is the kernel*

$$S(x, y) := \sum_{n=1}^{\infty} \lambda_n(\mathcal{K})^{1/2} \phi_n(x) \overline{\phi_n(y)}, \quad x, y \in X. \tag{3.1}$$

Proof. Due to Theorem 2.4-(*iv*), it is easily seen that the series

$$\sum_{n=1}^{\infty} \lambda_n(\mathcal{K})^{1/2} \phi_n \otimes \overline{\phi_n},$$

in which $\phi_n \otimes \overline{\phi_n}(x, y) := \phi_n(x)\overline{\phi_n(y)}$, $x, y \in X$, converges in $L^2(X \times X, \nu \times \nu)$ (see Theorem 4.11 in [19]. Hence, Formula (3.1) defines an element S in $L^2(X \times X, \nu \times \nu)$. On the other hand, due to Theorem 2.4-(*ii*),

$$\begin{aligned} \int_X S(x, y) f(y)\, d\nu(y) &= \sum_{n=1}^{\infty} \lambda_n(\mathcal{K})^{1/2} \langle f, \phi_n \rangle_2 \phi_n(x) \\ &= \mathcal{K}^{1/2}(f)(x), \quad x \in X, \quad f \in L^2(X, \nu). \end{aligned}$$

Thus, $\mathcal{S} = \mathcal{K}^{1/2}$. The L^2-positive definiteness of S is clear. □

Lemma 3.2 below describes a crucial information regarding the range of $\mathcal{K}^{1/2}$.

Lemma 3.2. *Under the conditions stated in Lemma 3.1, the range of $\mathcal{K}^{1/2}$ is a subset of $C(X) \cap L^2(X, \nu)$.*

Proof. The proof uses the formula

$$\mathcal{K}^{1/2}(f)(x) = \sum_{n=1}^{\infty} \lambda_n(\mathcal{K})^{1/2} \langle f, \phi_n \rangle_2 \phi_n(x), \quad x \in X, \quad f \in L^2(X, \nu). \tag{3.2}$$

If $\lambda_n(\mathcal{K}) > 0$, Theorem 2.4-(*iii*) asserts that ϕ_n is an eigenfunction of $\mathcal{K}$ associated with the eigenvalue $\lambda_n(\mathcal{K})$. Hence, Theorem 2.4-(*i*) implies that ϕ_n is continuous. Thus, to reach the continuity of $\mathcal{K}^{1/2}$, it suffices to show that the series in (3.2) converges uniformly on compact subsets of X. But, that follows from the inequalities

$$\begin{aligned} \left| \sum_{n=p}^{p+q} \lambda_n(\mathcal{K})^{1/2} \langle f, \phi_n \rangle_2 \phi_n(x) \right|^2 &\leq \sum_{n=p}^{p+q} |\lambda_n(\mathcal{K})^{1/2} \phi_n(x)|^2 \sum_{n=p}^{p+q} |\langle f, \phi_n \rangle_2|^2 \\ &\leq \langle f, f \rangle_2 \sum_{n=p}^{p+q} \lambda_n(\mathcal{K}) |\phi_n(x)|^2, \quad x \in X, \quad p, q \geq 1, \end{aligned}$$

consequences of the Cauchy-Schwarz and Bessel inequalities. □

Lemma 3.3 establishes an integral connection between the kernels K and S associated with $\mathcal{K}$ and $\mathcal{S} = \mathcal{K}^{1/2}$ respectively.

Lemma 3.3. *Under the conditions stated in the previous lemmas, K can be recovered from S through the formula*

$$\int_X S(x,u)\overline{S(x,v)}\, d\nu(x) = K(v,u), \quad u,v \in X.$$

Proof. Due to Lemma 3.1, $S \in L^2(X \times X, \nu \times \nu)$. For $y \in X$, define

$$S_y^j := \sum_{n=1}^{j} \lambda_n(\mathcal{K})^{1/2}\, \overline{\phi_n(y)}\, \phi_n, \quad j = 1, 2, \ldots.$$

Since $S_y^j \in L^2(X,\nu)$, $j = 1, 2, \ldots$ and $S(\cdot, y) \in L^2(X,\nu)$, it is easily seen that

$$\langle S(\cdot,y) - S_y^j, S(\cdot,y) - S_y^j\rangle_2 = \sum_{n=j+1}^{\infty} \lambda_n(\mathcal{K})|\phi_n(y)|^2, \quad y \in X, \quad j = 1, 2, \ldots.$$

Due to the continuity of the inner product, it now follows that

$$\lim_{j\to\infty} \langle S_u^j, S_v^j\rangle_2 = \langle S(\cdot,u), S(\cdot,v)\rangle_2, \quad u,v \in X.$$

Meanwhile, the orthonormality of $\{\phi_n\}$ implies that

$$\lim_{j\to\infty} \langle S_u^j, S_v^j\rangle_2 = \lim_{j\to\infty} \sum_{n=1}^{j} \lambda_n(\mathcal{K})\phi_n(v)\overline{\phi_n(u)} = K(v,u), \quad u,v \in X.$$

By uniqueness, $\langle S(\cdot,u), S(\cdot,v)\rangle_2 = K(v,u)$, $u,v \in X$, and the result follows. □

4. Finite rank kernels

Let (X,d) be as in the statement of Theorem 2.4. In this section we will deal with the integral operator $\mathcal{F}$ generated by the kernel F given by the formula

$$F(x,y) := \sum_{n=1}^{\Gamma} \frac{1}{\nu(C_n)} \chi_{C_n}(x)\chi_{C_n}(y), \quad x,y \in X, \tag{4.1}$$

in which $\{C_n : n = 1, 2, \ldots, \Gamma\}$ is a family of subsets of X satisfying the following two requirements: $0 < \nu(\cup_{n=1}^{\Gamma} C_n) < \infty$ and $\nu(C_n \cap C_l) = 0$, $n \neq l$. The inequality is needed in order to guarantee that F is an element of $L^2(X \times X, \nu \times \nu)$ while the other condition enters in some orthonormality arguments (see the beginning of the proof of Lemma 4.1 below for example). The symbol χ_{C_n} will stand for the usual characteristic function of C_n.

Depending on the family $\{C_n : n = 1, 2, \ldots, \Gamma\}$, $\mathcal{F}$ can be used to construct a convenient finite rank approximation to $\mathcal{K}$, with respect to the trace norm, at least when $K \in \mathcal{A}(X,\nu)$. That will become clear at the end of the section when we estimate the sum of all eigenvalues of $\mathcal{K}$. The symbol I stands for the identity operator.

Lemma 4.1. *The following assertions hold:*

(i) *The integral operator $\mathcal{F} : L^2(X,\nu) \to L^2(X,\nu)$ is positive of rank at most Γ;*

(ii) *The operator $I - \mathcal{F}$ is positive.*

Proof. The set $\{\nu(C_n)^{-1/2}\chi_{C_n} : n = 1, 2, \dots, \Gamma\}$ being $L^2(X \times X, \nu \times \nu)$-orthonormal, we can write $\mathcal{F}$ in the form

$$\mathcal{F}(f) = \sum_{n=1}^{\Gamma} \left\langle f, \nu(C_n)^{-1/2}\chi_{C_n} \right\rangle_2 \nu(C_n)^{-1/2}\chi_{C_n}, \quad f \in L^2(X,\nu), \tag{4.2}$$

and assertion (i) follows. The same representation shows that $\mathcal{F}$ is positive, having 0 and 1 as the only possible eigenvalues. As so, it is an operator of norm 1. It is now clear that, for all $f \in L^2(X,\nu)$,

$$\langle (I - \mathcal{F})(f), f\rangle_2 = \langle f, f\rangle_2 - \langle \mathcal{F}(f), f\rangle_2 \geq \langle f, f\rangle_2 - \|\mathcal{F}\|\langle f, f\rangle_2 = 0, \tag{4.3}$$

and the proof is complete. □

In Lemma 4.2 below we will deal with the operator $\mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}$.

Lemma 4.2. *Let K be an element of $\mathcal{A}(X,\nu)$. The following assertions hold:*

(i) *$\mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}$ is an integral operator whose kernel is an element of $\mathcal{A}(X,\nu)$;*

(ii) *The number Γ is an upper bound for the rank of $\mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}$;*

(iii) *The operator $\mathcal{K} - \mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}$ is positive.*

Proof. Let us write $\mathcal{S} := \mathcal{K}^{1/2}$. Recalling the proof of Lemma 3.1, we deduce that

$$\mathcal{S}\mathcal{F}\mathcal{S}(f)(x) = \int_X S(x,u) \left(\int_X F(u,v) \left(\int_X S(v,y) f(y)\, d\nu(y) \right) d\nu(v) \right) d\nu(u),$$

whenever $f \in L^2(X,\nu)$ and $x \in X$. Due to Fubini's Theorem, we conclude that $\mathcal{S}\mathcal{F}\mathcal{S}$ is an integral operator on $L^2(X,\nu)$, with kernel $G \in L^2(X \times X, \nu \times \nu)$ given by the formula

$$G(x,y) = \int_X \int_X S(x,u)F(u,v)S(v,y)\, d\nu(u)\, d\nu(v), \quad x, y \in X,$$

or, alternatively,

$$G(x,y) = \int_{\cup_{n=1}^{\Gamma} C_n} \int_{\cup_{n=1}^{\Gamma} C_n} S(x,u)F(u,v)S(v,y)\, d\nu(u)\, d\nu(v), \quad x, y \in X. \tag{4.4}$$

Returning to the definition of F,

$$\begin{aligned} G(x,y) &= \sum_{n=1}^{\Gamma} \int_X S(x,u) \frac{\chi_{C_n}(u)}{\nu(C_n)^{1/2}}\, d\nu(u) \int_X S(v,y) \overline{\frac{\chi_{C_n}(v)}{\nu(C_n)^{1/2}}}\, d\nu(v) \\ &= \sum_{n=1}^{\Gamma} \int_X S(x,u) \frac{\chi_{C_n}(u)}{\nu(C_n)^{1/2}}\, d\nu(u) \overline{\int_X S(y,v) \frac{\chi_{C_n}(v)}{\nu(C_n)^{1/2}}\, d\nu(v)}, \quad x, y \in X. \end{aligned}$$

It follows that $\mathcal{SFS}$ has rank at most Γ. Lema 3.2 reveals that G is continuous while Lemma 4.1-(i) justifies

$$\langle \mathcal{SFS}(f), f\rangle_2 = \langle \mathcal{FS}(f), \mathcal{S}(f)\rangle_2 \geq 0, \quad f \in L^2(X,\nu).$$

In other words, $G \in L^2PD(X) \cap C(X \times X)$. To finish the proof, first we use the Cauchy-Schwarz inequality to obtain

$$0 \leq G(x,x) = \sum_{n=1}^{\Gamma} \left| \int_X S(x,u) \frac{\chi_{C_n}(u)}{\nu(C_n)^{1/2}}\, d\nu(u) \right|^2 \leq \sum_{n=1}^{\Gamma} \int_X |S(x,u)|^2 d\nu(u), \quad x \in X.$$

Due to Lemma 3.3, it follows that

$$0 \leq G(x,x) \leq K(x,x)\Gamma, \quad x \in X,$$

and, therefore, the function $x \in X \to G(x,x)$ belongs to $L^1(X,\nu)$. This takes care of (i) and (ii). From Lemma 4.1-(ii) and (4.3), we can write

$$\langle \mathcal{K}(f), f\rangle_2 = \langle \mathcal{S}(f), \mathcal{S}(f)\rangle_2 \geq \langle \mathcal{FS}(f), \mathcal{S}(f)\rangle_2 = \langle \mathcal{SFS}(f), f\rangle_2, \quad f \in L^2(X,\nu).$$

Assertion (iii) follows. □

In Lemma 4.3 below, we will deduce a formula that allows one to compare the traces of $\mathcal{K}$ and $\mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}$.

Lemma 4.3. *If K is an element of $\mathcal{A}(X,\nu)$ then*

$$tr(\mathcal{K}) - tr(\mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}) = \sum_{n=1}^{\Gamma} \frac{1}{\nu(C_n)} \int_{C_n}\int_{C_n} [K(u,u) - K(v,u)]\, d\nu(u)\, d\nu(v) + \int_{X\setminus(\cup_{n=1}^{\Gamma} C_n)} K(u,u)\, d\nu(u).$$

Proof. If $K \in \mathcal{A}(X,\nu)$ then, due to Lemma 4.2-(i), Theorem 2.4-(iv) can be applied to both $\mathcal{K}$ and $\mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}$. Hence,

$$tr(\mathcal{K}) - tr(\mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}) = \int_X K(x,x)\, d\nu(x) - \int_X G(x,x) d\nu(x).$$

We compute these two integrals separately. Employing (4.4), Fubini's Theorem and then Lemma 3.3, it is not hard to see that

$$\begin{aligned} \int_X G(x,x)\, d\nu(x) &= \int_X \int_X F(u,v)K(v,u)\, d\nu(u)\, d\nu(v) \\ &= \int_{\cup_{n=1}^{\Gamma} C_n} \int_{\cup_{n=1}^{\Gamma} C_n} F(u,v)K(v,u)\, d\nu(u)\, d\nu(v). \end{aligned}$$

The definition of F reveals that

$$\int_{\cup_{n=1}^{\Gamma} C_n} F(u,v)\, d\nu(u) = \sum_{n=1}^{\Gamma} \chi_{C_n}(v) = 1, \quad v \in \cup_{n=1}^{\Gamma} C_n, \text{ a.e.,}$$

so that

$$\int_{\cup_{n=1}^{\Gamma} C_n} K(u,u)\,d\nu(u) = \int_{\cup_{n=1}^{\Gamma} C_n} \left[\int_{\cup_{n=1}^{\Gamma} C_n} F(v,u)\,d\nu(v)\right] K(u,u)\,d\nu(u)$$
$$= \int_{\cup_{n=1}^{\Gamma} C_n} \left[\int_{\cup_{n=1}^{\Gamma} C_n} F(u,v)\,d\nu(u)\right] K(v,v)\,d\nu(v).$$

Hence,

$$\int_X K(u,u)\,d\nu(u) = \int_{X\setminus(\cup_{n=1}^{\Gamma} C_n)} K(u,u)\,d\nu(u) + \int_{\cup_{n=1}^{\Gamma} C_n} K(u,u)\,d\nu(u)$$
$$= \int_{X\setminus(\cup_{n=1}^{\Gamma} C_n)} K(u,u)\,d\nu(u)$$
$$+ \int_{\cup_{n=1}^{\Gamma} C_n} \int_{\cup_{n=1}^{\Gamma} C_n} F(u,v)K(v,v)\,d\nu(u)\,d\nu(v).$$

Thus, denoting $\mathcal{S} = \mathcal{K}^{1/2}$,

$$tr(\mathcal{K} - \mathcal{S}\mathcal{F}\mathcal{S}) = \int_{X\setminus(\cup_{n=1}^{\Gamma} C_n)} K(u,u)\,d\nu(u)$$
$$+ \int_{\cup_{n=1}^{\Gamma} C_n} \int_{\cup_{n=1}^{\Gamma} C_n} F(u,v)\,[K(v,v) - K(v,u)]\,d\nu(u)\,d\nu(v).$$

The formula in the statement of the lemma follows from (4.2). □

Proposition 4.4 below is an extension of a result on best approximation by finite rank operators, originally found in [15].

Proposition 4.4. *Let T be a compact self-adjoint operator on a Hilbert space $\mathcal{H}$ and consider its series representation*

$$T(f) = \sum_{n=1}^{\infty} \lambda_n(T)\langle f, \phi_n\rangle_{\mathcal{H}}\, \phi_n, \quad f \in \mathcal{H},$$

as given by the spectral theorem for such operators. If $R \in \mathcal{L}(\mathcal{H})$ has rank at most k then

$$\|T - R\|_{tr} \geq \|T - T_k\|_{tr}\,,$$

where $T_k \in \mathcal{L}(\mathcal{H})$ is the truncated sum

$$T_k(f) = \sum_{n=1}^{k} \lambda_n(T)\langle f, \phi_n\rangle_{\mathcal{H}}\phi_n, \quad f \in \mathcal{H}.$$

Proof. Since $T - R$ is compact and self-adjoint, we may consider its spectral representation

$$(T-R)(f) = \sum_{n=1}^{\infty} \lambda_n(T-R)\langle f, \psi_n\rangle_{\mathcal{H}}\psi_n, \quad f \in \mathcal{H}.$$

Defining $A_0 = R$ and

$$A_p(f) = R(f) + \sum_{n=1}^{p} \lambda_n(T-R)\langle f, \psi_n\rangle_{\mathcal{H}}\psi_n, \quad f \in \mathcal{H}, \quad p = 1, 2, \ldots,$$

it is easily seen that A_p has rank at most $j+p$ and

$$(T - A_p)(f) = \sum_{n=p+1}^{\infty} \lambda_n(T-R)\langle f, \psi_n\rangle_{\mathcal{H}}\psi_n, \quad f \in \mathcal{H}.$$

Using Theorem 2.5 in [9] it is now seen that

$$|\lambda_{p+1}(T-R)| = \|T - A_p\| \geq \|T - T_{j+p}\| \geq |\lambda_{j+p+1}(T)|, \quad p = 0, 1, \ldots,$$

and the proof follows. □

Next, using Proposition 4.4, we describe a method to estimate the eigenvalues of $\mathcal{K}$ using the family $\{C_n : n = 1, 2, \ldots, \Gamma\}$ behind the definition of F.

Theorem 4.5. *Let K be an element of $\mathcal{A}(X, \nu)$. If $\{C_n : n = 1, 2, \ldots, \Gamma\}$ is a family of subsets of X such that $0 < \nu(\cup_{n=1}^{\Gamma} C_n) < \infty$ and $\nu(C_n \cap C_l) = 0$, $n \neq l$, then*

$$\sum_{n=\Gamma+1}^{\infty} \lambda_n(\mathcal{K}) \leq \sum_{n=1}^{\Gamma} \frac{1}{\nu(C_n)} \int_{C_n}\int_{C_n} [K(u,u) - K(v,u)]\, d\nu(u)\, d\nu(v) + \int_{X\setminus(\cup_{n=1}^{\Gamma} C_n)} K(u,u)\, d\nu(u).$$

Proof. Consider the series representation for $\mathcal{K}$ as described in Theorem 2.4-(ii) and write T to denote the operator obtained from the series by truncating it at Γ:

$$T(f)(x) = \sum_{n=1}^{\Gamma} \lambda_n(\mathcal{K})\langle f, \phi_n\rangle_2 \phi_n, \quad f \in L^2(X, \nu).$$

Proposition 4.4 implies that

$$\sum_{n=\Gamma+1}^{\infty} \lambda_n(\mathcal{K}) = \|\mathcal{K} - T\|_{tr} \leq \|\mathcal{K} - \mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}\|_{tr},$$

in which F is the kernel described in (4.1). Since

$$\|\mathcal{K} - \mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}\|_{tr} = \mathrm{tr}\,(\mathcal{K}) - \mathrm{tr}\,(\mathcal{K}^{1/2}\mathcal{F}\mathcal{K}^{1/2}),$$

the inequality in the statement of the theorem follows from Lemma 4.3. □

An alternative for the inequality in Theorem 4.5 is provided below.

Theorem 4.6. *Let K be an element of $\mathcal{A}(X,\nu)$. If $\{C_n : n = 1,2,\ldots,\Gamma\}$ is a family of subsets of X such that $0 < \nu(\cup_{n=1}^{\Gamma} C_n) < \infty$ and $\nu(C_n \cap C_l) = 0$, $n \neq l$, then*

$$\sum_{n=\Gamma+1}^{\infty} \lambda_n(\mathcal{K}) \leq \sum_{n=1}^{\Gamma} \frac{1}{\nu(C_n)} \int_{C_n} \int_{C_n} \left[\frac{K(u,u)+K(v,v)}{2} - K(v,u) \right] d\nu(u)\, d\nu(v) + \int_{X\setminus(\cup_{n=1}^{\Gamma} C_n)} K(u,u)\, d\nu(u).$$

Proof. It is analogous to the proof of Theorem 4.5, but using a version of Lemma 4.3 leading to an inequality involving the kernel $2^{-1}(K(u,u)+K(v,v))-K(v,u)$. The details are left to the readers. □

5. Decay rates for the eigenvalues under Lipschitz conditions

Keeping the context described in Theorem 2.4, this section describes decay rates for the eigenvalues of the integral operator $\mathcal{K}$, at least when the kernel K comes from $\mathcal{A}(X,\nu)$ and satisfies a convenient Lipschitz condition. The rates hold when the metric space (X,d) fits in the description below.

Let q be a positive integer and t a positive real. The space (X,d) is said to be (q,t)-*compact* when there exist $x_0 \in X$ and positive real numbers a,b,c and r_0 for which the following condition holds: if $N \in \mathbb{Z}_+$ and $r \geq r_0$ there exist a family $\{C_n^r : n = 1,2,\ldots,k(N)\}$ of subsets of X, all having finite measure, such that

(i) $\nu(C_n^r \cap C_l^r) = \emptyset$, $n \neq l$;
(ii) $d(x,y) \leq ar^t N^{-t}$, $x,y \in C_n^r$, $n = 1,2,\ldots,k(N)$;
(iii) $k(N) \leq bN^q$;
(iv) $B[x_0, r\,c] := \{x \in X : d(x,x_0) \leq r\,c\} = \cup_{n=1}^{k(N)} C_n^r$.

Example. Let X be a measurable subset of $\mathbb{R}^m$ having positive Lebesgue measure. Set $a = \sqrt{m}$, $b = 1$, $c = 1/2$ and choose $x_0 \in X$. Clearly, $B[x_0, r/2]$ is a subset of the m-dimensional (closed) cube of edge r. Subdividing the cube in N^m (not necessarily closed) m-dimensional (disjoint) cubes Q_n^r, $n = 1,2,\ldots,N^m := k(N)$, then $C_n^r := Q_n^r \cap B[x_0, r/2]$ satisfy the conditions in the definition above with $t = 1$ and $q = m$. The number r_0 can be any positive real.

Example. Results in [7] and [17, p.219] reveal that a subset of the unit sphere S^{m-1} in $\mathbb{R}^m$, endowed with its usual Lebesgue measure, is $(m-1,1)$-compact. The numbers in the definition are now $a = \pi/2$, $c = 1$, and $r_0 = 2$. The constant b can be 10 while the point x_0 can be any point in S^{m-1}.

Example. A similar process can be applied to a subset X of a p-dimensional surface in $\mathbb{R}^m$, endowed with its surface measure. It can be shown that (X,d), in which d is the metric induced by the usual norm of $\mathbb{R}^m$, is $(m_1,1)$-compact for some $m_1 \leq m$. If X is a subset of a p-dimensional C^k-manifold M, endowed with some measure which is finite on balls, then using a Whitney-type Theorem ([10, p.54]) it can be shown that (X,d) is $(m,1)$-compact whenever m is large enough and

$d(x,y) := d_1(f(x), f(y))$, where $f : M \to \mathbb{R}^{m+p}$ is an embedding and d_1 is a metric in $f(M)$, induced by the usual norm of $\mathbb{R}^{m+p}$.

Next, we introduce the Lipschitz condition we will adopt. Let $\alpha > 0$ and $s \geq 0$ be constants. A kernel $K : X \times X \to \mathbb{C}$ belongs to the Lipschitz class $Lip^{\alpha,s}(X,\nu)$ when the following two conditions hold:

(i) There exist $\delta > 0$ and a locally integrable function $A : X \to [0, +\infty]$ so that

$$|K(x,x) - K(x,y)| \leq A(x)d(x,y)^{\alpha}, \quad x, y \in X, \quad d(x,y) \leq \delta; \tag{5.1}$$

(ii) There exists $B \geq 0$ such that

$$\limsup_{r\to\infty} r^{-s} \int_{B[y,r]} A(x)\, d\nu(x) \leq B, \quad y \in X. \tag{5.2}$$

The definition above is a weaker version of others in the literature (see [7, 12, 13]). For instance, the first inequality above is easily found in a nonlocal form such as

$$|K(x,y) - K(x,y')| \leq A(x)d(y,y')^{\alpha}, \quad x, y, y' \in X.$$

Theorem 5.1. *Let X be (q,t)-compact and $K \in \mathcal{A}(X,\nu) \cap Lip^{\alpha,s}(X,\nu)$. Assume there exist $\beta > 0$ and $C > 0$ such that*

$$\limsup_{r\to\infty} r^{\beta} \int_{X\setminus B[y,r]} K(x,x)\, d\nu(x) \leq C, \quad y \in X. \tag{5.3}$$

Define $\gamma := t\alpha\beta(\beta + s + t\alpha)^{-1}$. If N is large enough then there exists a constant $C_1 > 0$ such that

$$\sum_{n=k(N)+1}^{\infty} \lambda_n(\mathcal{K}) \leq \frac{C_1}{N^{\gamma}},$$

for some $k(N) \in \{0, 1, \ldots, bN^q\}$.

Proof. Let x_0, a, b, c, and r_0 be as in the definition of (q,t)-compactness and let δ, A and B as in the definition of the class $Lip^{\alpha,s}(X,\nu)$. Write $\mathcal{S} := \mathcal{K}^{1/2}$. Due to (5.2), there exists a $r_{x_0} > 0$ such that

$$\int_{B[x_0,r\,c]} A(x)d\nu(x) \leq Br^s, \quad r \geq r_{x_0}. \tag{5.4}$$

Without loss of generality we can assume that $r_0 > r_{x_0}$ and $r_0\, c > r_{x_0}$. For each $N \in \mathbb{Z}_+$ and $r \geq r_0$ consider families $\{C_n^r : n = 1, 2, \ldots, k(N)\}$ as described in the definition of (q,t)-compactness. Theorem 4.5 implies that

$$\sum_{n=k(N)+1}^{\infty} \lambda_n(\mathcal{K}) \leq \sum_{n=1}^{k(N)} \frac{1}{\nu(C_n^r)} \int_{C_n^r} \int_{C_n^r} [K(u,u) - K(v,u)]\, d\nu(u)\, d\nu(v) + \int_{X\setminus B[x_0,r\,c]} K(u,u)\, d\nu(u).$$

By increasing r_0 if necessary, we can use (5.3) to conclude that

$$\int_{X\setminus B[x_0,r\,c]} K(u,u)\,d\nu(u) \le \frac{Cc^{-\beta}}{r^\beta}.$$

If $ar^tN^{-t} < \delta$, which is always guaranteed when N is large enough, we can use (5.1) to write

$$\sum_{n=k(N)+1}^{\infty} \lambda_n(\mathcal{K}) \le \sum_{n=1}^{k(N)} \frac{1}{\nu(C_n^r)} \int_{C_n^r}\int_{C_n^r} A(u)d(u,v)^\alpha\,d\nu(u)\,d\nu(v) + \frac{Cc^{-\beta}}{r^\beta}.$$

It is now clear that

$$\begin{aligned}\sum_{n=k(N)+1}^{\infty} \lambda_n(\mathcal{K}) &\le \sum_{n=1}^{k(N)} \frac{1}{\nu(C_n^r)} \int_{C_n^r}\int_{C_n^r} A(u)a^\alpha\left(\frac{r}{N}\right)^{t\alpha} d\nu(u)\,d\nu(v) + \frac{Cc^{-\beta}}{r^\beta}\\ &\le a^\alpha\left(\frac{r}{N}\right)^{t\alpha}\sum_{n=1}^{k(N)}\int_{C_n^r} A(u)\,d\nu(u) + \frac{Cc^{-\beta}}{r^\beta}\\ &\le a^\alpha\left(\frac{r}{N}\right)^{t\alpha}\int_{B[x_0,r\,c]} A(u)\,d\nu(u) + \frac{Cc^{-\beta}}{r^\beta}.\end{aligned}$$

Recalling (5.4), we finally deduce that

$$\sum_{n=k(N)+1}^{\infty} \lambda_n(\mathcal{K}) \le a^\alpha\left(\frac{r}{N}\right)^{t\alpha} Br^s + \frac{Cc^{-\beta}}{r^\beta} = Ba^\alpha\frac{r^{\alpha t+s}}{N^{\alpha t}} + \frac{Cc^{-\beta}}{r^\beta}, \tag{5.5}$$

as long as $r \ge r_0$ and N is large enough. To conclude the proof, we will apply the above estimate using a special choice of r. Precisely, we will put $r = r(N) := N^{\alpha t/(\beta+\alpha t+s)}$. Since $\lim_{N\to\infty} r(N) = \infty$, $r(N) \ge r_0$ when N is large enough. Since

$$\lim_{N\to\infty}\frac{r(N)}{N} = 0,$$

the inequality $ar^tN^{-t} < \delta$ can be equally captured. Since $\sigma := \alpha t/(\beta+\alpha t+s)$ satisfies $\alpha t - \sigma(\alpha t+s) = \sigma\beta$, inequality (5.5) takes the form

$$\sum_{n=k(N)+1}^{\infty} \lambda_n(\mathcal{K}) \le \frac{Ba^\alpha + Cc^{-\beta}}{N^{\alpha t\beta/(\alpha t+\beta+s)}}. \qquad \square$$

A special case is as follows.

Theorem 5.2. *Let X be (q,t)-compact and $K \in \mathcal{A}(X,\nu)\cap Lip^{\alpha,s}(X,\nu)$. If either X is bounded or K vanishes outside of a bounded set and N is large enough then there exists a constant $C_1 > 0$ such that*

$$\sum_{n=k(N)+1}^{\infty} \lambda_n(\mathcal{K}) \le \frac{C_1}{N^{t\alpha}},$$

for some $k(N) \in \{0,1,\ldots,bN^q\}$.

Proof. Under either condition mentioned in the statement of the theorem, there exists $r_0 > 0$ such that

$$\int_{X\setminus B[y,r]} K(u,u)\, d\nu(u) = 0, \quad r > r_0, \quad y \in X.$$

Repeating the arguments used in the proof of Lemma 5.1 and adjusting r_0, if necessary, inequality (5.5) reduces itself to

$$\sum_{n=k(N)+1}^{\infty} \lambda_n(\mathcal{K}) \leq Ba^{\alpha} \frac{r^{\alpha t+s}}{N^{\alpha t}}$$

as long as $r \geq r_0$ and N is large enough. In particular,

$$\sum_{n=k(N)+1}^{\infty} \lambda_n(\mathcal{K}) \leq Ba^{\alpha} \frac{(r_0+1)^{\alpha t+s}}{N^{\alpha t}}$$

for N arbitrarily large. □

In order to re-phrase the previous results in a language a little bit more familiar, we will need a lemma ([7]).

Lemma 5.3. *Let $\{a_n\}$ be a non-increasing sequence of nonnegative real numbers. Let l, q and N_0 be nonnegative integers, p a positive integer at least 1 and $\gamma \in \mathbb{R}$. Suppose there exists a constant $C > 0$ satisfying the following property: if $N \geq N_0$, there exists $k(N) \leq pN^q$ such that*

$$\sum_{n=k(N)+l+1}^{\infty} a_n \leq \frac{C}{N^{\gamma}}.$$

Then, the set $\{n^{1+\gamma/q} a_n : n = 1, 2, \ldots\}$ is bounded. In particular,

$$a_n = O(n^{-1-\gamma/q}), \quad \textit{as } n \to \infty.$$

The main results of the paper are as follows.

Theorem 5.4. *Let X be (q,t)-compact and $K \in \mathcal{A}(X,\nu) \cap Lip^{\alpha,s}(X,\nu)$. If there exist $\beta > 0$ and $C \geq 0$ such that*

$$\limsup_{r\to\infty} r^{\beta} \int_{X\setminus B[y,r]} K(x,x)\, d\nu(x) \leq C, \quad y \in X, \tag{5.6}$$

then

$$\lambda_n(\mathcal{K}) = O(n^{-1-\gamma/q}), \quad \textit{as } n \to \infty, \tag{5.7}$$

where $\gamma := t\alpha\beta(\beta + s + t\alpha)^{-1}$.

Proof. This follows from Theorem 5.1 and Lemma 5.3. □

Theorem 5.5. *Let X be (q,t)-compact and $K \in \mathcal{A}(X,\nu) \cap Lip^{\alpha,s}(X,\nu)$. If for every $\beta > 0$ there exists $C = C(\beta) \geq 0$ such that*

$$\limsup_{r\to\infty} r^\beta \int_{X\setminus B[y,r]} K(x,x)\, d\nu(x) \leq C, \quad y \in X, \tag{5.8}$$

then

$$\lambda_n(\mathcal{K}) = o(n^{-1-\theta/q}), \quad \text{as } n \to \infty,$$

whenever $\theta \in [0, t\alpha)$.

Proof. The function

$$\gamma(\beta) := t\alpha \frac{\beta}{\beta + s + t\alpha}, \quad \beta \in [0,\infty),$$

is continuous with range $[0, t\alpha)$. As so, the previous theorem implies that

$$\lambda_n(\mathcal{K}) = O(n^{-1-\theta/q}), \quad \text{as } n \to \infty,$$

whenever $\theta \in [0, t\alpha)$. If

$$\lambda_n(\mathcal{K}) \neq o(n^{-1-\gamma_0/q}), \quad \text{as } n \to \infty,$$

for some $\gamma_0 \in [0, t\alpha)$, then there would exist $C > 0$ such that

$$\limsup_{n\to\infty} \{n^{-1-\gamma_0/q} \lambda_n(\mathcal{K})\} \geq C.$$

But this would imply in unbounded sequences $\{n^{1+\theta/q}\lambda_n(\mathcal{K})\}$ when $\theta \in (\gamma_0, t\alpha)$, a clear contradiction. □

Theorem 5.6. *Let X be (q,t)-compact and $K \in \mathcal{A}(X,\nu) \cap Lip^{\alpha,s}(X,\nu)$. If either X is bounded or K vanishes outside a bounded set then*

$$\lambda_n(\mathcal{K}) = O(n^{-1-t\alpha/q}), \quad \text{as } n \to \infty.$$

Proof. This follows from Theorem 5.2 and Lemma 5.3. □

The reader is advised that the results above generalize some of the results proved in [2] to the multi-dimensional case. There, the authors use a particular case of Theorem 4.6 to obtain decay rates for the eigenvalues of $\mathcal{K}$ under differentiability hypotheses. We intend to use Theorem 4.6 to investigate multi-dimensional versions of such context in a future work.

Lets return to the context of $\mathbb{R}^m$. If X is a finite union of convex subsets of $\mathbb{R}^m$ and the assumption $K \in Lip^{\alpha,s}(X,\mu)$ is changed to the existence and boundedness of $\partial K/\partial x$ then the mean value inequality and the continuity of K show that $K \in Lip^{1,m}(X_l,\mu)$, in which X_l is a convex component of X. In view of this, some of the proofs can be adapted to show that the estimate (5.7) holds with $\gamma := \beta(\beta+m+1)^{-1}$. Finally, we would like to observe that the existence of a $\beta > 0$ so that

$$\limsup_{|x|\to\infty} |x|^{\beta+m} K(x,x) < \infty$$

implies condition (5.6) in Theorem 5.4. Here, $|\cdot|$ stands for the usual norm in $\mathbb{R}^m$. A similar remark applies to condition (5.8) in Theorem 5.5.

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