**Integral Equations
and Operator Theory**

Hankel and Toeplitz Transforms on H^1: Continuity, Compactness and Fredholm Properties

M. Papadimitrakis and J. A. Virtanen

Abstract. We revisit the boundedness of Hankel and Toeplitz operators acting on the Hardy space H^1 and give a new proof of the old result stating that the Hankel operator H_a is bounded if and only if a has bounded logarithmic mean oscillation. We also establish a sufficient and necessary condition for H_a to be compact on H^1. The Fredholm properties of Toeplitz operators on H^1 are studied for symbols in a Banach algebra similar to $C + H^\infty$ under mild additional conditions caused by the differences in the boundedness of Toeplitz operators acting on H^1 and H^2.

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1. Introduction and main results

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane $\mathbb{C}$ and $\mathbb{T} = \partial \mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ be the unit circle. The usual Lebesgue spaces for $\mathbb{T}$ are denoted by $L^p = L^p(\mathbb{T})$ and we write

$$f(\zeta) \sim \sum_{n=-\infty}^{+\infty} \widehat{f}(n)\zeta^n$$

for the Fourier series of a function f in L^1. The Hardy spaces for $\mathbb{T}$ are defined by

$$H^p = \left\{ f \in L^p \, : \, \widehat{f}(n) = 0 \text{ for } n < 0 \right\}$$

and their variants by $H_0^p = \left\{ f \in L^p \, : \, \widehat{f}(n) = 0 \text{ for } n \leq 0 \right\}$. We also define the spaces $\overline{H^p} = \left\{ f \in L^p \, : \, \widehat{f}(n) = 0 \text{ for } n > 0 \right\}$ and the corresponding variants $\overline{H_0^p} = \left\{ f \in L^p \, : \, \widehat{f}(n) = 0 \text{ for } n \geq 0 \right\}$.

The M. Riesz Theorem says that the Riesz projection P, defined by

$$Pf(\zeta) \sim \sum_{n=0}^{+\infty} \widehat{f}(n)\zeta^n$$

for every $f(\zeta) \sim \sum_{n=-\infty}^{+\infty} \widehat{f}(n)\zeta^n$, is a bounded operator $L^p \to H^p$ when $1 < p < \infty$; note, however, that the operator P is not bounded either on L^1 or L^∞. We also define a related operator $P_1 : L^p \to H_0^p$ by $P_1 f(\zeta) \sim \sum_{n=1}^{+\infty} \widehat{f}(n)\zeta^n$ and denote the complementary projection of P by $Q : L^p \to \overline{H_0^p}$, $Qf(\zeta) \sim \sum_{n=-\infty}^{-1} \widehat{f}(n)\zeta^n$. We say that Pf is the analytic part and Qf is the antianalytic part of f.

The Toeplitz operator T_a with symbol $a \in L^2$ is defined by

$$T_a f = P(af)$$

and the Hankel operator H_a by

$$H_a f = P(aJf),$$

where J is the "flip operator" defined by

$$Jf(\zeta) = \overline{\zeta}f(\overline{\zeta}) \sim \sum_{n=-\infty}^{+\infty} \widehat{f}(-n-1)\zeta^n.$$

Both operators T_a and H_a are obviously well defined for analytic polynomials, i.e. for finite sums $f(\zeta) = \sum_{n=0}^{N} \widehat{f}(n)\zeta^n$. The set of analytic polynomials is dense in each H^p $(1 \leq p < +\infty)$ and there are classical results which specify, for every particular value of p, the necessary and sufficient conditions on the symbol a so that these operators are extended as bounded or even compact operators on H^p. It is easy to see that T_a is not compact whenever a is not the zero function. The situation is described by the following Theorems 1.1–1.5.

Theorem 1.1. *Let $1 < p < +\infty$. Then T_a is bounded on H^p if and only if $a \in L^\infty$.*

Theorem 1.2. *(Nehari, for $p = 2$) Let $1 < p < +\infty$. Then H_a is bounded on H^p if and only if $P_1 a \in$ BMO.*

Theorem 1.3. *(Hartman, for $p = 2$) Let $1 < p < +\infty$. Then H_a is compact on H^p if and only if $P_1 a \in$ VMO.*

In the case of the space H^1 the results are slightly more complicated.

Theorem 1.4. *(Stegenga, 1976, for real or antianalytic a; Janson-Peetre-Semmes, 1984; Tolokonnikov, 1987; Cima-Stegenga, 1987) The Toeplitz operator T_a with symbol a is bounded on H^1 if and only if $a \in L^\infty$ and $Qa \in \mathrm{BMO}_{\log}$.*

Theorem 1.5. *(Janson-Peetre-Semmes, 1984; Tolokonnikov, 1987; Cima-Stegenga, 1987) The Hankel operator H_a is bounded on H^1 if and only if $P_1 a \in \mathrm{BMO}_{\log}$.*

The purpose of the first part of the article is to give a new proof of Theorem 1.5 together with a precise estimate of the operator norm of H_a and to prove the analogous result about the compactness of H_a, that is, we prove the following two theorems.

Theorem 1.6. *The Hankel operator H_a is bounded on H^1 if and only if $P_1 a \in \mathrm{BMO}_{\log}$, in which case*

$$\|H_a\|_{H^1 \to H^1} \quad \asymp \quad \|P_1 a\|_{\mathrm{BMO}_{\log}},$$

where $A \asymp B$ means that there are two positive numerical constants c_1 and c_2 so that $c_1 \leq \frac{A}{B} \leq c_2$ for all values of the independent variables in A and B.

Theorem 1.7. *The Hankel operator H_a is compact on H^1 if and only if $P_1 a \in \mathrm{VMO}_{\log}$.*

The second part of this article deals with spectral properties of Toeplitz operators. The case of continuous symbols was recently studied in [14]. Here we consider symbols that are not necessarily continuous. The motivation comes from the well-known result on the Fredholm properties of Toeplitz operators on H^p $(1 < p < \infty)$ with $a \in C + \overline{H^\infty}$, due to Douglas [6] when $p = 2$. This suggests the following theorem, which is indeed the best we can hope for because of the differences in boundedness and compactness of the operators determined by the underlying spaces H^1 and H^p.

Theorem 1.8. *Let $a \in \mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log} := \mathcal{V} + (\overline{H^\infty} \cap \mathrm{BMO}_{\log})$, where $\mathcal{V} = C \cap \mathrm{VMO}_{\log}$. Then the following conditions are equivalent:*

(1) T_a *is Fredholm on H^1, that is, $\ker T_a$ and $\operatorname{coker} T_a$ are both of finite dimension;*

(2) *a is invertible in the algebra $\mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$;*

(3) *a is bounded away from zero, that is, there are $\epsilon > 0$ and $\delta > 0$ such that*

$$|a(z)| \geq \epsilon \quad \text{for } 1 - \delta < |z| < 1,$$

where $a(z)$ for $z \in \mathbb{D}$ is defined via the harmonic extension—see (1) below; in this case for any $1 - \delta < r < 1$

$$\operatorname{Ind} T_a := \dim \ker T_a - \dim \operatorname{coker} T_a = -\operatorname{ind} a_r$$

where $a_r(\zeta) = a(r\zeta)$ for all $\zeta \in \mathbb{T}$ and $\operatorname{ind} a_r$ is the winding number of the function a_r.

2. Preliminaries

In this section we consider some (known) results from harmonic analysis. The Poisson extension of $f \in L^1$ at $z \in \mathbb{D}$ is given by

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta| \tag{1}$$

and the Szegö projection of f at z by

$$Pf(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

For $1 \leq p < +\infty$ and every $f \in L^p$ the limit $\lim_{r \to 1-} f(r\zeta) = f(\zeta)$ holds for almost every $\zeta \in \mathbb{T}$ and also in the L^p sense. Since $f(z)$ is a harmonic function of $z \in \mathbb{D}$, it is also called the harmonic extension of f in $\mathbb{D}$.

On the other hand, for $1 \leq p < +\infty$ and every $f \in L^p$, the limit

$$\lim_{r \to 1-} Pf(r\zeta)$$

exists for almost every $\zeta \in \mathbb{T}$ and, when $1 < p < +\infty$, this limit is equal to $Pf(\zeta)$ (where P is the Riesz projection) in both the almost everywhere sense and in the L^p sense. In the case $p = 1$, the limit $Pf(\zeta) = \lim_{r \to 1-} Pf(r\zeta)$ serves as the definition of the function Pf which, as is well known, belongs to the space $L^{1,w}$ of weak-L^1 functions. In all cases $Pf(z)$ is an analytic function of $z \in \mathbb{D}$.

If $1 < p < +\infty$ and $f \in L^p$, the Poisson extension of $Pf \in L^p$ at every $z \in \mathbb{D}$ is equal to $Pf(z)$:

$$Pf(z) = \frac{1}{2\pi} \int_{\mathbb{T}} Pf(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta|,$$

while if $1 \leq p < +\infty$ and $f \in H^p$, then (obviously) $Pf(\zeta) = f(\zeta)$ and

$$Pf(z) = f(z) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta| = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for every $z \in \mathbb{D}$.

We next consider the space of functions of bounded mean oscillation and its important (logarithmic) subspaces. A function f is in BMO if $f \in L^1$ and

$$\|f\|_* = \sup_I \frac{1}{|I|} \int_I |f(\zeta) - f_I| |d\zeta| < +\infty,$$

where the supremum is taken over all arcs I of $\mathbb{T}$, $f_I = \frac{1}{|I|} \int_I f(\zeta) |d\zeta|$ and $|I|$ is the length of I. The space BMO is a Banach space under the norm $\|f\|_{\text{BMO}} = |\widehat{f}(0)| + \|f\|_*$. We also have the space BMOA of analytic functions in BMO, defined as $\text{BMOA} = \text{BMO} \cap H^1 = \{f \in \text{BMO} : \widehat{f}(n) = 0 \text{ for } n < 0\}$.

It is well known that $L^\infty \subseteq \text{BMO} \subseteq L^p$ for every $p < +\infty$ and that for every $f \in \text{BMO}$

$$\|f\|_* \; \asymp \; \left(\sup_{z \in \mathbb{D}} \int_{\mathbb{T}} |f(\zeta) - f(z)|^2 \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta| \right)^{\frac{1}{2}}. \tag{2}$$

The subspace VMO of BMO contains by definition all functions $f \in L^1$ for which

$$\lim_{\delta \to 0+} \sup_{I, \; |I| < \delta} \frac{1}{|I|} \int_I |f(\zeta) - f_I||d\zeta| = 0.$$

We also define $\text{VMOA} = \text{VMO} \cap H^1$. The space VMO is the closure in the space BMO of the set of all polynomials (or, equivalently, of all continuous functions). Also, $f \in \text{BMO}$ belongs to VMO if and only if $\lim_{r \to 1-} \|f_r - f\|_{\text{BMO}} = 0$, where the function f_r is defined by $f_r(\zeta) = f(r\zeta)$.

Somewhat less known are the spaces $\text{BMO}_{\log}$ and $\text{VMO}_{\log}$ and their variants $\text{BMOA}_{\log}$ and $\text{VMOA}_{\log}$. These are defined as follows. A function f is in $\text{BMO}_{\log}$ if $f \in L^1$ and

$$\|f\|_{**} = \sup_I \frac{\log \frac{4\pi}{|I|}}{|I|} \int_I |f(\zeta) - f_I||d\zeta| < +\infty,$$

where, again, the supremum is taken over all arcs I of $\mathbb{T}$. The space $\text{BMO}_{\log}$ is a Banach space under the norm $\|f\|_{\text{BMO}_{\log}} = |\widehat{f}(0)| + \|f\|_{**}$. We define $\text{BMOA}_{\log} = \text{BMO}_{\log} \cap H^1$.

It is obvious that $\text{BMO}_{\log} \subseteq \text{BMO}$. The following estimate

$$\|f\|_{**} \; \asymp \; \left(\sup_{z \in \mathbb{D}} \log^2 \frac{2}{1 - |z|^2} \int_{\mathbb{T}} |f(\zeta) - f(z)|^2 \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta| \right)^{\frac{1}{2}} \tag{3}$$

where $f \in \text{BMO}_{\log}$, requires a similar consideration as in the case of the space BMO, starting with the analogue of the John-Nirenberg theorem. The proofs do not seem to have been recorded anywhere but they are almost straightforward and, in any case, these facts have been used many times in the literature.

The logarithmic Lipschitz space $\text{Lip}_{\log}$ is defined by

$$\text{Lip}_{\log} = \left\{ f : \mathbb{T} \to \mathbb{C} \; : \; \sup_{\zeta, \eta \in \mathbb{T}} \log \frac{4}{|\zeta - \eta|} |f(\zeta) - f(\eta)| < \infty \right\}.$$

This is a space of continuous functions under the norm

$$\|f\|_{\text{Lip}_{\log}} = |\widehat{f}(0)| + \sup_{\zeta, \eta \in \mathbb{T}} \log \frac{4}{|\zeta - \eta|} |f(\zeta) - f(\eta)|.$$

The space $\text{Lip}_{\log}$ is continuously imbedded in $\text{BMO}_{\log}$ and the main result of [9] is:

$$\text{BMO}_{\log} = \{f + Pg \; : \; f, g \in \text{Lip}_{\log}\}. \tag{4}$$

In particular, if $h \in \text{BMO}_{\log}$, there are $f, g \in \text{Lip}_{\log}$ such that $h = f + Pg$ and

$$\|f\|_{\text{Lip}_{\log}} + \|g\|_{\text{Lip}_{\log}} \leq c \|h\|_{\text{BMO}_{\log}}.$$

where c is a positive numerical constant.

The subspace $\text{VMO}_{\log}$ of $\text{BMO}_{\log}$ contains by definition all functions $f \in L^1$ for which

$$\lim_{\delta \to 0+} \sup_{I, \; |I| < \delta} \frac{\log \frac{4\pi}{|I|}}{|I|} \int_I |f(\zeta) - f_I||d\zeta| = 0.$$

We also define $\mathrm{VMOA}_{\log} = \mathrm{VMO}_{\log} \cap H^1$. The following two results will be needed several times.

Theorem 2.1. *For the logarithmic* VMO *space, we have the following characterization:*

$$\mathrm{VMO}_{\log} = \{f + Pg \,:\, f, g \in \mathrm{lip}_{\log}\}, \tag{5}$$

where $\mathrm{lip}_{\log}$ *stands for the so-called vanishing logarithmic Lipschitz space defined by*

$$\mathrm{lip}_{\log} = \left\{f \in \mathrm{Lip}_{\log} \,:\, \lim_{\delta \to 0+} \sup_{|\zeta - \eta| < \delta} \log \frac{4}{|\zeta - \eta|} \, |f(\zeta) - f(\eta)| = 0\right\}.$$

Theorem 2.2. *For* $f \in \mathrm{BMO}_{\log}$, *the following conditions are equivalent:*

(1) $f \in \mathrm{VMO}_{\log}$;
(2) $\lim_{\eta \to 1} \|\tau_\eta f - f\|_{\mathrm{BMO}_{\log}} = 0$, *where* $\tau_\eta f(\zeta) = f(\zeta\overline{\eta})$ *for* $\eta, \zeta \in \mathbb{T}$;
(3) $\lim_{r \to 1-} \|f_r - f\|_{\mathrm{BMO}_{\log}} = 0$, *where* $f_r(\zeta) = f(r\zeta)$ *for* $\zeta \in \mathbb{T}$.

The following descriptions are also useful:

$$\{a \in L^\infty \,:\, Qa \in \mathrm{BMO}_{\log}\} = \mathrm{Lip}_{\log} + H^\infty \tag{6}$$

and

$$\{a \in L^\infty \,:\, Qa \in \mathrm{VMO}_{\log}\} = \mathrm{lip}_{\log} + H^\infty. \tag{7}$$

These can be verified by means of the characterizations in (4) and (5); for example, if $a = l + h \in \mathrm{lip}_{\log} + H^\infty$, then $a \in L^\infty$ and $Qa = Ql \in \mathrm{VMO}_{\log}$, and conversely if $a \in L^\infty$ and $Qa \in \mathrm{VMO}_{\log}$, then $Qa = f + Pg$ for some $f, g \in \mathrm{lip}_{\log}$, so $Qa = Qf$, which implies that $a - f \in H^\infty$ and we can write $a = f + (a - f) \in \mathrm{lip}_{\log} + H^\infty$.

For each arc I we define $S(I) = \{z \in \mathbb{D} : 0 < 1 - |z| < \frac{|I|}{2\pi}, \frac{z}{|z|} \in I\}$, called the Carleson "square" with base I. A positive Borel measure μ in $\mathbb{D}$ is called a Carleson measure if

$$\sup_I \frac{\mu(S(I))}{|I|} < +\infty,$$

where the supremum is taken over all arcs I of $\mathbb{T}$. It is known that μ is a Carleson measure if and only if there is a constant c so that

$$\int\!\!\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \;\leq\; c \int_{\mathbb{T}} |f(\zeta)|^2 |d\zeta|, \qquad f \in L^2, \tag{8}$$

and that, if c is the smallest constant for which this inequality holds,

$$c \;\asymp\; \sup_I \frac{\mu(S(I))}{|I|}. \tag{9}$$

In this connection, we have a function $f \in L^1$ in BMO if and only if the Borel measure $|\nabla f(z)|^2 (1 - |z|^2) dm(z)$, where dm is the area measure, is a Carleson measure and

$$\|f\|_* \;\asymp\; \left(\sup_I \frac{1}{|I|} \int\!\!\int_{S(I)} |\nabla f(z)|^2 (1 - |z|^2) dm(z)\right)^{\frac{1}{2}}. \tag{10}$$

Similarly, $f \in L^1$ is in VMO if and only if

$$\lim_{\delta \to 0+} \sup_{I,\, |I| < \delta} \frac{1}{|I|} \int \int_{S(I)} |\nabla f(z)|^2 (1 - |z|^2) dm(z) \;\; = \;\; 0. \tag{11}$$

Of course, in the case of $f \in H^1$ we may replace $\nabla f(z)$ by $f'(z)$ in the above characterizations of BMO and VMO.

Analogously, for functions f in $\text{BMO}_{\log}$, we have

$$\|f\|_{**} \;\; \asymp \;\; \Big(\sup_I \frac{\log^2 \frac{4\pi}{|I|}}{|I|} \iint_{S(I)} |\nabla f(z)|^2 (1 - |z|^2) dm(z) \Big)^{\frac{1}{2}}. \tag{12}$$

Note also that there exists a positive numerical constant c so that for every $f \in$ BMOA and every $z \in \mathbb{D}$:

$$|f(z)| \;\; \leq \;\; c\|f\|_{\text{BMO}} \log \frac{2}{1 - |z|^2}. \tag{13}$$

Conversely, there exists a positive numerical constant c so that for every $z \in \mathbb{D}$ there exists an $f \in$ BMOA with

$$\|f\|_{\text{BMO}} \;\; = \;\; 1, \qquad |f(z)| \;\; \geq \;\; c \log \frac{2}{1 - |z|^2}. \tag{14}$$

Also, if $f \in$ VMOA then

$$\lim_{|z| \to 1-} \frac{|f(z)|}{\log \frac{2}{1-|z|^2}} \;\; = \;\; 0. \tag{15}$$

Finally, we shall use the inequality

$$|\langle f, g \rangle| \;\; \leq \;\; c\|f\|_{H^1} \|g\|_{\text{BMOA}},$$

where the binary form $\langle \cdot, \cdot \rangle$ is defined by

$$\langle f, g \rangle \;\; = \;\; \lim_{r \to 1-} \frac{1}{2\pi} \int_{\mathbb{T}} f_r(\zeta) g(\overline{\zeta}) |d\zeta| \;\; = \;\; \lim_{r \to 1-} \frac{1}{2\pi i} \int_{\mathbb{T}} f(r\zeta) g(\overline{\zeta}) \overline{\zeta} d\zeta.$$

The Fefferman-Stein duality which is induced by this binary form says that BMOA is isomorphic to $(H^1)^*$. It is also true that, under the same binary form, H^1 is isomorphic to $(\text{VMOA})^*$.

3. Proof of Theorem 1.6

Proof. Before proceeding to the proof, note that the part $a - P_1 a$ of a plays no role in the Hankel operator H_a. Indeed, for all analytic polynomials f the function $(a - P_1 a)Jf$ is antianalytic and, hence, $H_a f = H_{P_1 a} f$. We may thus suppose in all that follows that $a = P_1 a$ or in other words that

$$\widehat{a}(n) = 0, \qquad n \leq 0. \tag{16}$$

We recall that BMOA is isomorphic to the dual space of H^1 and it is easy to see that, formally at least, the dual operator to H_a on H^1 is H_a on BMOA. This means that

$$\langle H_a f, g \rangle \quad = \quad \langle f, H_a g \rangle$$

for all analytic polynomials f and all $g \in$ BMOA. Hence, we need to prove that H_a is bounded on BMOA if and only if $a \in \mathrm{BMO_{log}}$ and that

$$\|H_a\|_{\mathrm{BMOA}\to\mathrm{BMOA}} \quad \asymp \quad \|a\|_{\mathrm{BMO_{log}}}$$

under the assumption (16).

Sufficiency. Let $a \in \mathrm{BMO_{log}}$ satisfy (16) and take an arbitrary $f \in$ BMOA. Then

$$H_a f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{a(\zeta)\overline{\zeta} f(\overline{\zeta})}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{b(\zeta)\overline{g(\zeta)}}{\zeta - z} d\zeta,$$

where we set $b(\zeta) = \overline{\zeta} a(\zeta)$ and $g(\zeta) = \overline{f(\overline{\zeta})}$. It is obvious that $b \in \mathrm{BMOA_{log}}$ with $\|b\|_{\mathrm{BMO_{log}}} \asymp \|a\|_{\mathrm{BMO_{log}}}$ and that $g \in$ BMOA with $\|g\|_{\mathrm{BMO}} = \|f\|_{\mathrm{BMO}}$.

Throughout, the symbol c denotes a numerical constant, not necessarily the same at each occurrence. We have

$$
\begin{aligned}
(H_a f)'(z) &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{b(\zeta)\overline{g(\zeta)}}{(\zeta - z)^2} d\zeta \\
&= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{b(\zeta)\overline{(g(\zeta) - g(z))}}{(\zeta - z)^2} d\zeta + \overline{g(z)} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{b(\zeta)}{(\zeta - z)^2} d\zeta \\
&= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(b(\zeta) - b(z))\overline{(g(\zeta) - g(z))}}{(\zeta - z)^2} d\zeta + \overline{g(z)} b'(z).
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality together with (2) and (3), we get

$$
\begin{aligned}
|(H_a f)'(z)|^2 &\leq c \int_{\mathbb{T}} \frac{|b(\zeta) - b(z)|^2}{|\zeta - z|^2} |d\zeta| \int_{\mathbb{T}} \frac{|g(\zeta) - g(z)|^2}{|\zeta - z|^2} |d\zeta| + c|g(z)|^2 |b'(z)|^2 \\
&\leq c\|b\|_{**}^2 \|g\|_*^2 \frac{1}{(1 - |z|^2)^2 \log^2 \frac{2}{1-|z|^2}} + c|g(z)|^2 |b'(z)|^2.
\end{aligned}
$$

This, for every arc I of $\mathbb{T}$, implies

$$
\begin{aligned}
&\frac{1}{|I|} \int\int_{S(I)} |(H_a f)'(z)|^2 (1 - |z|^2) dm(z) \\
&\leq c\|b\|_{**}^2 \|g\|_*^2 \frac{1}{|I|} \int\int_{S(I)} \frac{1}{(1 - |z|^2)\log^2 \frac{2}{1-|z|^2}} dm(z) \\
&\quad + c\frac{1}{|I|} \int\int_{S(I)} |g(z) - g(z_I)|^2 |b'(z)|^2 (1 - |z|^2) dm(z) \\
&\quad + c|g(z_I)|^2 \frac{1}{|I|} \int\int_{S(I)} |b'(z)|^2 (1 - |z|^2) dm(z) \\
&= A + B + C, \tag{17}
\end{aligned}
$$

where z_I is the point in the middle of the internal side of $S(I)$, defined by $1 - |z_I| = \frac{|I|}{2\pi}$ and $\frac{z_I}{|z_I|}$ =midpoint of I.

Let us first estimate the term A. A direct calculation of the integral, using polar coordinates, gives

$$
\begin{aligned}
A &\leq c \frac{1}{\log \frac{4\pi}{|I|}} \|b\|_{**}^2 \|g\|_*^2 \\
&\leq c \frac{1}{\log \frac{4\pi}{|I|}} \|a\|_{\mathrm{BMO_{log}}}^2 \|f\|_{\mathrm{BMO}}^2.
\end{aligned}
\tag{18}
$$

Observing that $|1 - \overline{z_I} z| \asymp |I|$ for all $z \in S(I)$ and considering the Borel measure $d\mu(z)$ which is equal to $|b'(z)|^2 (1 - |z|^2) dm(z)$ on $S(I)$ and equal to zero on $\mathbb{D} \setminus S(I)$, we find using (8) and (9) that

$$
\begin{aligned}
B &\leq c|I| \int\!\!\int_{S(I)} \frac{|g(z) - g(z_I)|^2}{|1 - \overline{z_I} z|^2} |b'(z)|^2 (1 - |z|^2) dm(z) \\
&\leq c|I| \sup_J \frac{\mu(S(J))}{|J|} \int_{\mathbb{T}} \frac{|g(\zeta) - g(z_I)|^2}{|1 - \overline{z_I} \zeta|^2} |d\zeta| \\
&\leq c \sup_J \frac{\mu(S(J))}{|J|} \int_{\mathbb{T}} |g(\zeta) - g(z_I)|^2 \frac{1 - |z_I|^2}{|\zeta - z_I|^2} |d\zeta| \\
&\leq c\|g\|_*^2 \sup_J \frac{\mu(S(J))}{|J|}.
\end{aligned}
\tag{19}
$$

Estimating $\frac{\mu(S(J))}{|J|} = \frac{\mu(S(J) \cap S(I))}{|J|}$, we observe that we need only consider arcs J having nonempty intersection with I. In the case $|J| > |I|$, $\frac{\mu(S(J))}{|J|} \leq \frac{\mu(S(I))}{|I|}$. If $|J| \leq |I|$, then $J \subseteq 3I$, where $3I$ is the arc with the same midpoint as I and with length three times the length of I. Hence, in both cases we get using (12)

$$
\begin{aligned}
\sup_J \frac{\mu(S(J))}{|J|} &\leq \sup_{J \subseteq 3I} \frac{1}{|J|} \int\!\!\int_{S(J)} |b'(z)|^2 (1 - |z|^2) dm(z) \\
&\leq c \sup_{J \subseteq 3I} \frac{1}{\log^2 \frac{4\pi}{|J|}} \|b\|_{**}^2 \\
&\leq c \frac{1}{\log^2 \frac{4\pi}{|I|}} \|b\|_{**}^2.
\end{aligned}
$$

Therefore (19) implies

$$
\begin{aligned}
B &\leq c \frac{1}{\log^2 \frac{4\pi}{|I|}} \|b\|_{**}^2 \|g\|_*^2 \\
&\leq c \frac{1}{\log^2 \frac{4\pi}{|I|}} \|a\|_{\mathrm{BMO_{log}}}^2 \|f\|_{\mathrm{BMO}}^2.
\end{aligned}
\tag{20}
$$

Finally, (12) and (13) imply

$$
\begin{aligned}
C &\leq c\|b\|_{**}^2\|g\|_{\mathrm{BMO}}^2 \\
&\leq c\|a\|_{\mathrm{BMO}_{\log}}^2\|f\|_{\mathrm{BMO}}^2.
\end{aligned}
\tag{21}
$$

Now, estimates (18), (20) and (21) together with (17) imply

$$
\frac{1}{|I|}\int\int_{S(I)}|(H_af)'(z)|^2(1-|z|^2)dm(z) \;\leq\; c\|a\|_{\mathrm{BMO}_{\log}}^2\|f\|_{\mathrm{BMO}}^2
$$

and, taking the supremum over all arcs I and using (10),

$$
\|H_af\|_* \;\leq\; c\|a\|_{\mathrm{BMO}_{\log}}\|f\|_{\mathrm{BMO}}.
$$

On the other hand,

$$
\begin{aligned}
|\widehat{H_af}(0)|^2 &\leq c\int_{\mathbb{T}}|H_af(\zeta)|^2|d\zeta| \;=\; c\int_{\mathbb{T}}|a(\zeta)|^2|f(\zeta)|^2|d\zeta| \\
&\leq c\Big(\int_{\mathbb{T}}|a(\zeta)|^4|d\zeta|\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{T}}|f(\zeta)|^4|d\zeta|\Big)^{\frac{1}{2}} \\
&\leq c\|a\|_{\mathrm{BMO}}^2\|f\|_{\mathrm{BMO}}^2 \;\leq\; c\|a\|_{\mathrm{BMO}_{\log}}^2\|f\|_{\mathrm{BMO}}^2.
\end{aligned}
$$

The last two estimates show that

$$
\|H_af\|_{\mathrm{BMO}} \;\leq\; c\|a\|_{\mathrm{BMO}_{\log}}\|f\|_{\mathrm{BMO}}
$$

and hence

$$
\|H_a\|_{\mathrm{BMOA}\to\mathrm{BMOA}} \;\leq\; c\|a\|_{\mathrm{BMO}_{\log}}.
$$

Necessity, step 1. Here we make the *a priori* assumption that $a\in BMO_{\log}$ (and, that a satisfies (16)) and we set $b(\zeta)=\overline{\zeta}a(\zeta)$ as before.

If $|\widehat{b}(0)|\geq\frac{1}{2}\|b\|_{BMO_{\log}}$, then

$$
\begin{aligned}
\|H_a\|_{BMOA\to BMOA} &\geq \|H_a1\|_{BMO} \;=\; \|b\|_{BMO} \\
&\geq |\widehat{b}(0)|\geq\frac{1}{2}\|b\|_{BMO_{\log}} \\
&\geq c\|a\|_{BMO_{\log}}.
\end{aligned}
$$

If $|\widehat{b}(0)|<\frac{1}{2}\|b\|_{BMO_{\log}}$, then $\|b\|_{**}\geq\frac{1}{2}\|b\|_{BMO_{\log}}$ and based on (12) we find an arc I such that

$$
c\|a\|_{BMO_{\log}}^2\leq c\|b\|_{BMO_{\log}}^2 \;\leq\; \frac{\log^2\frac{4\pi}{|I|}}{|I|}\int\int_{S(I)}|b'(z)|^2(1-|z|^2)dm(z).
\tag{22}
$$

Through (14) we find an $f\in\mathrm{BMOA}$ and the corresponding $g(\zeta)=\overline{f(\overline{\zeta})}$ so that

$$
\|f\|_{\mathrm{BMO}} \;=\; 1\,, \qquad c\log\frac{4\pi}{|I|} \;\leq\; |f(\overline{z_I})| \;=\; |g(z_I)|.
\tag{23}
$$

The trivial variant of inequality (17) together with the estimates (18), (20), (22) and (23) imply

$$
\begin{aligned}
\|H_a\|^2_{\mathrm{BMOA}\to\mathrm{BMOA}} \;&\geq\; \|H_a f\|^2_{\mathrm{BMO}} \\
&\geq\; \frac{1}{|I|}\int\int_{S(I)} |(H_a f)'(z)|^2 (1-|z|^2)\,dm(z) \\
&\geq\; C - A - B \;\geq\; c\Big(1 - c\,\frac{1}{\log\frac{4\pi}{|I|}}\Big)\|a\|^2_{\mathrm{BMO_{log}}}.
\end{aligned}
$$

Hence, if $|I|$ is smaller than a certain positive numerical constant we find that

$$
\|H_a\|_{\mathrm{BMOA}\to\mathrm{BMOA}} \;\geq\; c\|a\|_{\mathrm{BMO_{log}}}.
$$

On the other hand, if $|I|$ is larger than the same positive numerical constant, then

$$
\begin{aligned}
\|H_a\|^2_{\mathrm{BMOA}\to\mathrm{BMOA}} \;&\geq\; \|H_a 1\|^2_{\mathrm{BMO}} \;=\; \|b\|^2_{\mathrm{BMO}} \\
&\geq\; c\frac{1}{|I|}\int\int_{S(I)} |b'(z)|^2 (1-|z|^2)\,dm(z) \\
&\geq\; c\frac{\log^2\frac{4\pi}{|I|}}{|I|}\int\int_{S(I)} |b'(z)|^2 (1-|z|^2)\,dm(z) \\
&\geq\; c\|a\|^2_{\mathrm{BMO_{log}}}.
\end{aligned}
$$

We conclude that if a is assumed to be in $\mathrm{BMO_{log}}$ and satisfy (16) then

$$
\|H_a\|_{\mathrm{BMOA}\to\mathrm{BMOA}} \;\geq\; c\|a\|_{\mathrm{BMO_{log}}}
$$

and, by the usual duality,

$$
\|H_a\|_{H^1\to H^1} \;\geq\; c\|a\|_{\mathrm{BMO_{log}}}.
$$

Lemma 3.1. *If H_a is bounded on H^1, then for every $f \in H^1$ and all $r < 1$ we have $H_{a_r} f = r(H_a f_r)_r$.*

Proof. The operator H_{a_r} is bounded on H^1 since a_r is smooth. Verifying the equality involves a straightforward calculation using Fourier series. □

Necessity, step 2. Applying the *a priori* estimate of step 1 to the functions a_r we have

$$
\|H_{a_r}\|_{H^1\to H^1} \;\geq\; c\|a_r\|_{\mathrm{BMO_{log}}}. \tag{24}
$$

The lemma of Fatou with (12) implies that

$$
\|a\|_{\mathrm{BMO_{log}}} \;\leq\; \liminf_{r\to 1-} \|a_r\|_{\mathrm{BMO_{log}}}. \tag{25}
$$

Now, Lemma 3.1 implies that for all $f \in H^1$

$$
\begin{aligned}
\|H_{a_r} f\|_{H^1} \;&\leq\; r\|H_a f_r\|_{H^1} \\
&\leq\; \|H_a\|_{H^1\to H^1}\|f_r\|_{H^1} \\
&\leq\; \|H_a\|_{H^1\to H^1}\|f\|_{H^1}
\end{aligned}
$$

and hence

$$\|H_{a_r}\|_{H^1 \to H^1} \quad \leq \quad \|H_a\|_{H^1 \to H^1}. \tag{26}$$

Relations (24), (25) and (26) complete the necessity part of Theorem 1.6. □

4. Proof of Theorem 1.7

Proof. Let $a \in \mathrm{VMO}_{\log}$ satisfy (16). Let $r < 1$ and take $f_n \in H^1$ with $\|f_n\|_{H^1} \leq 1$. Choosing a subsequence, we may assume that there is a function $f \in H^1$ so that $(f_n)_r \to f$ in H^1. Since H_a is bounded, we get $H_a(f_n)_r \to H_a f$ in H^1 and, hence, $H_{a_r} f_n = r \big(H_a(f_n)_r \big)_r \to r(H_a f)_r$ in H^1. Therefore, H_{a_r} is compact on H^1. Finally, $\|H_{a_r} - H_a\|_{H^1 \to H^1} \leq c\|a_r - a\|_{\mathrm{BMO}_{\log}} \to 0$ as $r \to 1-$ and, hence, H_a is compact on H^1.

Let $a \in \mathrm{BMO}_{\log}$ satisfy (16). It is a consequence of the proof of Theorem 1.6 that H_a is bounded on VMOA. Indeed, taking any $f \in \mathrm{VMOA}$, (17) together with (12), (15), (18) and (20) imply that

$$\sup_{I, |I| < \delta} \frac{1}{|I|} \int \int_{S(I)} |(H_a f)'(z)|^2 (1 - |z|^2) dm(z)$$

$$\leq c \frac{1}{\log \frac{4\pi}{\delta}} \|a\|^2_{\mathrm{BMO}_{\log}} \|f\|^2_{\mathrm{BMO}} + c\|a\|^2_{\mathrm{BMO}_{\log}} \sup_{I, |I| < \delta} \frac{|f(z_I)|^2}{\log^2 \frac{4\pi}{|I|}}$$

$$\to 0$$

as $\delta \to 0+$. Therefore, (11) implies that $H_a f \in \mathrm{VMOA}$.

Now, if we assume that H_a is compact on H^1 then it is also compact on VMOA, since H^1 is isomorphic to $(\mathrm{VMOA})^*$. To get a contradiction we suppose that a does not belong to $\mathrm{VMO}_{\log}$. Then there exist some $\delta > 0$ and $r_n \to 1-$ such that

$$\|a_{r_n} - a\|_{\mathrm{BMO}_{\log}} \quad \geq \quad \delta.$$

This implies

$$\|H_{a_{r_n}} - H_a\|_{\mathrm{BMO}_{\log}} \quad \geq \quad c\delta$$

and we can choose $f_n \in H^1$ with $\|f_n\|_{H^1} \leq 1$, so that

$$\begin{aligned}
c\delta \quad &\leq \quad \|H_{a_{r_n}} f_n - H_a f_n\|_{H^1} \\
&= \quad \|r_n \big(H_a(f_n)_{r_n} \big)_{r_n} - H_a f_n\|_{H^1} \\
&= \quad \|\big(H_a(f_n)_{r_n} \big)_{r_n} - H_a f_n\|_{H^1} + o(1) \\
&\leq \quad \|\big(H_a(f_n)_{r_n} \big)_{r_n} - (H_a f_n)_{r_n}\|_{H^1} + \|(H_a f_n)_{r_n} - H_a f_n\|_{H^1} + o(1) \\
&\leq \quad \|H_a(f_n)_{r_n} - H_a f_n\|_{H^1} + \|(H_a f_n)_{r_n} - H_a f_n\|_{H^1} + o(1).
\end{aligned}$$

Taking a subsequence, if necessary, we may assume that there is a $v \in H^1$ so that $H_a f_n \to v$ in H^1. Therefore

$$
\begin{aligned}
c\delta &\leq \|H_a((f_n)_{r_n} - f_n)\|_{H^1} + \|(H_a f_n)_{r_n} - v_{r_n}\|_{H^1} + \|v_{r_n} - v\|_{H^1} \\
&\quad + \|v - H_a f_n\|_{H^1} + o(1) \\
&= \|H_a((f_n)_{r_n} - f_n)\|_{H^1} + o(1).
\end{aligned}
$$

If we choose $h_n \in \mathrm{VMOA}$ with $\|h_n\|_{\mathrm{BMOA}} = 1$ and

$$
c\|H_a((f_n)_{r_n} - f_n)\|_{H^1} \leq \big\langle H_a((f_n)_{r_n} - f_n), h_n \big\rangle,
$$

we have

$$
\begin{aligned}
c\delta &\leq \big\langle H_a((f_n)_{r_n} - f_n), h_n \big\rangle + o(1) \\
&= \langle (f_n)_{r_n} - f_n, H_a h_n \rangle + o(1) \\
&= \langle f_n, (H_a h_n)_{r_n} - H_a h_n \rangle + o(1).
\end{aligned}
$$

Since H_a is compact on VMOA, taking a subsequence once more we see that there is a $w \in \mathrm{VMOA}$ so that $H_a h_n \to w$ in VMOA. Hence

$$
\begin{aligned}
c\delta &\leq \langle f_n, (H_a h_n)_{r_n} - w_{r_n} \rangle + \langle f_n, w_{r_n} - w \rangle + \langle f_n, w - H_a h_n \rangle + o(1) \\
&\leq c\|(H_a h_n)_{r_n} - w_{r_n}\|_{\mathrm{BMO}} + c\|w_{r_n} - w\|_{\mathrm{BMO}} + c\|w - H_a h_n\|_{\mathrm{BMO}} + o(1) \\
&\leq c\|w_{r_n} - w\|_{\mathrm{BMO}} + o(1) \\
&= o(1).
\end{aligned}
$$

This is false and hence $a \in \mathrm{VMO}_{\log}$. $\qquad\square$

5. Fredholmness of Toeplitz operators

We start by proving the equivalence of the criteria (2) and (3) for Fredholmness in Theorem 1.8. We use the symbol χ_n for the functions

$$
\chi_n(\zeta) = \zeta^n, \qquad \zeta \in \mathbb{T}.
$$

Lemma 5.1. *The functions in $\mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$ can be approximated in the space $L^\infty \cap \mathrm{BMO}_{\log}$ by functions of the form $\chi_n h$ with $n \geq 0$ and $h \in \overline{H^\infty} \cap \mathrm{BMO}_{\log}$.*

Proof. Let $v + b \in \mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$. According to (3) of Theorem 2.2, there are trigonometric polynomials p_k such that $\|v - p_k\|_{L^\infty \cap \mathrm{BMO}_{\log}} \to 0$. Since

$$
p_k + b \in \big\{ \chi_n h \ : \ n \geq 0, \ h \in \overline{H^\infty} \cap \mathrm{BMO}_{\log} \big\},
$$

the proof is complete. $\qquad\square$

Proposition 5.2. *Let $a \in \mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$. Then a is invertible in $\mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$ if and only if a is bounded away from zero, that is, there are $\epsilon > 0$ and $\delta > 0$ such that*

$$
|a(z)| \geq \epsilon \quad \text{for } 1 - \delta < |z| < 1.
$$

Proof. If a is invertible in $\mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$, then it is obviously invertible in $C + \overline{H^\infty}$ and thus bounded away from zero according to [7, Theorem 6.45].

By the preceding lemma, there are $N \to +\infty$ and corresponding $h_N \in \overline{H^\infty} \cap \mathrm{BMO}_{\log}$ such that $\|a - \chi_N h_N\|_{L^\infty \cap \mathrm{BMO}_{\log}} \to 0$. By [7, Theorem 6.45], h_N is invertible in $\overline{H^\infty}$ with N sufficiently large. As $h_N \in \mathrm{BMO}_{\log}$, so is its inverse. Thus, $\chi_N h_N$ is invertible in $\mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$. Now $\chi_{-N} h_N^{-1} \to a^{-1}$ in $L^\infty \cap \mathrm{BMO}_{\log}$ and so a^{-1} is in the closed space $\mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$. $\qquad\square$

It remains to show that the two conditions above are indeed sufficient and necessary for Fredholmness. This follows from Theorem 5.6 and Proposition 5.9 below.

Let us first consider two basic results for quite general symbols that are needed in what follows.

Proposition 5.3. *Let* $a, b \in L^\infty \cap \mathrm{BMO}_{\log}$. *Then*

$$T_{ab} = T_a T_b + H_a H_{\tilde{b}}, \tag{27}$$

$$H_{ab} = T_a H_b + H_a T_{\tilde{b}}, \tag{28}$$

where $\tilde{b}(\zeta) = b(1/\zeta)$, $\zeta \in \mathbb{T}$.

Proof. See, e.g., [3, Proposition 2.14]. $\qquad\square$

The next theorem gives a necessary condition for Fredholmness—cf. the well-known theorem of Simonenko in the case of $1 < p < \infty$.

Theorem 5.4. *Let* $a \in L^\infty \cap \mathrm{BMO}_{\log}$. *If* T_a *is Fredholm, then*

$$\operatorname{ess\,inf}_{\zeta \in \mathbb{T}} |a(\zeta)| > 0;$$

in particular, the symbol is invertible in $L^\infty \cap \mathrm{BMO}_{\log}$.

Proof. Suppose that T_a is Fredholm but $\operatorname{ess\,inf}_{\zeta \in \mathbb{T}} |a(\zeta)| = 0$.

We consider a small $\epsilon > 0$ and decompose $a = u + iv$ into real and imaginary parts. Define $u_\epsilon = \max(u, \epsilon) + \min(u, -\epsilon)$ and v_ϵ by the analogous formula. Now the function $a_\epsilon = u_\epsilon + iv_\epsilon$ is equal to 0 on a set of positive measure and $\|a - a_\epsilon\|_{L^\infty \cap BMO_{\log}} \to 0$ as $\epsilon \to 0+$. This implies that $\|T_a - T_{a_\epsilon}\| \to 0$ as $\epsilon \to 0+$ and, hence, that T_{a_ϵ} is Fredholm if ϵ is small enough.

If $T_{a_\epsilon} f = P(a_\epsilon f) = 0$, then $Q(a_\epsilon f) = a_\epsilon f - P(a_\epsilon f) = 0$ on a set of positive measure and, hence, $Q(a_\epsilon f) = 0$. Therefore, $a_\epsilon f = 0$ and, if ϵ is small enough (so that $a_\epsilon \neq 0$) we find that $f = 0$ on a set of positive measure. This implies that $f = 0$ and we conclude that T_{a_ϵ} is one-to-one. The same is true for the dual operator $(T_{a_\epsilon})^* = T_{\tilde{a}_\epsilon}$. Therefore, T_{a_ϵ} is invertible.

Since T_{a_ϵ} is invertible, there is some f so that $T_{a_\epsilon} f = P(a_\epsilon f) = 1$. Then $Q(a_\epsilon f) = a_\epsilon f - P(a_\epsilon f) = -1$ on a set of positive measure and, hence, $Q(a_\epsilon f) = -1$ which is clearly impossible. $\qquad\square$

Remark 5.5. *We do not know whether Fredholmness of T_a, when $a \in L^\infty$ and $Qa \in \mathrm{BMO}_{\log}$, implies invertibility of the symbol in this symbol class, which is optimal in the sense of boundedness.*

We next turn our attention to the relation between the symbol class $\mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$ and the space

$$\mathcal{A}^1 = \left\{ a \in L^\infty \cap \mathrm{BMO}_{\log} : H_a \in \mathcal{K}(H^1) \right\}$$

according to the following result.

Theorem 5.6. $\mathcal{A}^1 = \mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}.$

Proof. If $a = v + h$ for some $v \in \mathcal{V}$ and $h \in \overline{H^\infty} \cap \mathrm{BMO}_{\log}$, then

$$H_a = H_v + H_h = H_v,$$

which is compact according to Theorem 1.7. On the other hand, if H_a is compact, then $P_1 a \in \mathrm{VMO}_{\log}$ according to Theorem 1.7. Therefore, (7) implies that

$$a \in \mathrm{lip}_{\log} + \overline{H^\infty} \cap \mathrm{BMO}_{\log} \subseteq \mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}. \qquad \square$$

Proposition 5.7. *The space $\mathcal{A}^1$ is a closed subalgebra of $L^\infty \cap \mathrm{BMO}_{\log}$.*

Proof. The fact that the space is an algebra follows from Proposition 5.3. Suppose that $a_n \to a$ in $L^\infty \cap \mathrm{BMO}_{\log}$ with $a_n \in \mathcal{A}^1$. Then

$$\|H_a - H_{a_n}\| = \|H_{a-a_n}\| \leq c \, \|P_1(a - a_n)\|_{\mathrm{BMO}_{\log}} \to 0$$

(see Theorem 1.6). Thus, H_a is compact. $\qquad \square$

Corollary 5.8. *The space $\mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$ is a Banach algebra.*

Proof. This is immediate from the preceding two results. It can also be proved directly. $\qquad \square$

The proof of the following theorem is based on an argument of Böttcher and Silberman [2, ch. IV] when $1 < p < \infty$.

Proposition 5.9. *Let $a \in \mathcal{A}^1$. Then T_a is Fredholm on H^1 if and only if a is invertible in $\mathcal{A}^1$.*

Proof. If a is invertible, then formula (27) shows that $T_{a^{-1}}$ is a regularizer of T_a, and so T_a is Fredholm.

If T_a is Fredholm, then Theorem 5.4 implies that a is invertible in $L^\infty \cap \mathrm{BMO}_{\log}$. Since T_a has a regularizer, say R, we can write

$$RT_a = I + K,$$

where K is compact. Therefore, by (28),

$$0 = H_{aa^{-1}} = T_a H_{a^{-1}} + H_a T_{\tilde{a}^{-1}}.$$

This implies

$$H_{a^{-1}} = -K H_{a^{-1}} - R H_a T_{\tilde{a}^{-1}}$$

and, hence, $a^{-1} \in \mathcal{A}^1$. $\qquad \square$

6. Index formula

For analytic symbols, the Fredholm properties of Toeplitz operators on H^1 are well understood:

Theorem 6.1. *For $a \in \overline{H^\infty}$, the Toeplitz operator T_a on H^1 is Fredholm if and only if a is bounded away from zero, in which case*

$$\operatorname{Ind} T_a = -\operatorname{ind} a_r.$$

Proof. See [15, Theorem 10]. □

Our aim in this section is to show that the preceding formula also holds for invertible symbols in the algebra $\mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$. We start with a preliminary lemma.

Lemma 6.2. *If $v \in \mathcal{V}$ and $f \in L^\infty \cap \mathrm{BMO}_{\log}$, then*

$$\|(vf)_r - v_r f_r\|_{L^\infty \cap \mathrm{BMO}_{\log}} \longrightarrow 0 \qquad (29)$$

as $r \to 1$.

Proof. From Lemma 2.61 of [3] it follows that $\|(vf)_r - v_r f_r\|_\infty \to 0$. Therefore, it is enough to show that $\|(vf)_r - v_r f_r\|_{\mathrm{BMO}_{\log}} \to 0$. Also, since

$$\|(vf)_r - v_r f_r\|_{**} \leq \|(vf)_r - v f_r\|_{**} + \|v f_r - v_r f_r\|_{**},$$

and $\|v - v_r\|_{\mathrm{BMO}_{\log}} \to 0$ according to Theorem 2.2, it is sufficient to prove that

$$\|(vf)_r - v f_r\|_{**} \to 0.$$

For a function $g : \mathbb{T} \to \mathbb{C}$, we write

$$g_\eta(\zeta) = g(\zeta \overline{\eta})$$

when $\zeta, \eta \in \mathbb{T}$. Then

$$(vf)_r(\zeta) - (v f_r)(\zeta) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - r^2}{|1 - r\overline{\eta}|^2} (v_\eta(\zeta) - v(\zeta)) f_\eta(\zeta) |d\eta|$$

and we need to estimate the expression

$$\begin{aligned}
J := {} & \frac{\log \frac{4\pi}{|I|}}{|I|} \int_I |(vf)_r(\zeta) - (v f_r)(\zeta) - ((vf)_r - v f_r)_I| \, |d\zeta| \\
= {} & \frac{\log \frac{4\pi}{|I|}}{|I|} \int_I \left| \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - r^2}{|1 - r\overline{\eta}|^2} \Big((v_\eta(\zeta) - v(\zeta)) f_\eta(\zeta) \right. \\
& \left. \qquad\qquad - \frac{1}{|I|} \int_I (v_\eta(\theta) - v(\theta)) f_\eta(\theta) |d\theta| \Big) d\eta \right| |d\zeta| \\
\leq {} & \frac{\log \frac{4\pi}{|I|}}{|I|} \int_I \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|I|} \int_I \frac{1 - r^2}{|1 - r\overline{\eta}|^2} \Big| (v_\eta(\zeta) - v(\zeta)) f_\eta(\zeta) \\
& \qquad\qquad - (v_\eta(\theta) - v(\theta)) f_\eta(\theta) \Big| |d\theta| |d\eta| |d\zeta|.
\end{aligned}$$

Write

$$(v_\eta(\zeta) - v(\zeta))f_\eta(\zeta) - (v_\eta(\theta) - v(\theta))f_\eta(\theta)$$
$$= \left[(v_\eta(\zeta) - v(\zeta))f_\eta(\zeta) - (v_\eta - v)_I f_\eta(\zeta)\right] + \left[(v_\eta - v)_I f_\eta(\zeta) - (v_\eta - v)_I f_\eta(\theta)\right]$$
$$+ \left[(v_\eta - v)_I f_\eta(\theta) - (v_\eta(\theta) - v(\theta))f_\eta(\theta)\right] =: J_1 + J_2 + J_3. \quad (30)$$

Then

$$J \le \frac{\log\frac{4\pi}{|I|}}{|I|} \int_I \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|I|} \int_I \frac{1 - r^2}{|1 - r\overline{\eta}|^2} (|J_1| + |J_2| + |J_3|)|d\theta||d\eta||d\zeta|.$$

Let us first consider J_1. We have

$$\frac{\log\frac{4\pi}{|I|}}{|I|} \int_I \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|I|} \int_I \frac{1 - r^2}{|1 - r\overline{\eta}|^2} |J_1| |d\theta||d\eta||d\zeta|$$

$$\le \frac{\|f\|_\infty}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{|1 - r\overline{\eta}|^2} \frac{\log\frac{4\pi}{|I|}}{|I|} \int_I |v_\eta(\zeta) - v(\zeta) - (v_\eta - v)_I| |d\zeta||d\eta|$$

$$\le \frac{\|f\|_\infty}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{|1 - r\overline{\eta}|^2} \|v_\eta - v\|_{**} |d\eta|$$

$$= \frac{\|f\|_\infty}{2\pi} \int_{0 < |1-\eta| < \delta} \frac{1 - r^2}{|1 - r\overline{\eta}|^2} \|v_\eta - v\|_{**} |d\eta|$$

$$+ \frac{\|f\|_\infty}{2\pi} \int_{\delta < |1-\eta| < 2} \frac{1 - r^2}{|1 - r\overline{\eta}|^2} \|v_\eta - v\|_{**} |d\eta|.$$

Now given $\epsilon > 0$, there is $\delta > 0$ (according to Theorem 2.2) and $r < 1$ such that the above sum of two integrals can be estimated above by

$$\frac{\|f\|_\infty}{2\pi} 2\pi\epsilon + \frac{\|f\|_\infty}{2\pi} \epsilon 2 \|v\|_{**}.$$

Similarly the part made of J_3 can be shown to be as small as we wish provided that r is sufficiently close to 1.

It remains to consider J_2. Note that

$$(v_\eta - v)_I f_\eta(\zeta) - (v_\eta - v)_I f_\eta(\theta)$$
$$= (v_\eta - v)_I (f_\eta(\zeta) - (f_\eta)_I) + (v_\eta - v)_I ((f_\eta)_I - f_\eta(\theta))$$

and that it is sufficient to consider only one of the terms in the equality above. Since, by the choice of $\delta > 0$ and $r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{|1 - r\overline{\eta}|^2} |v_\eta - v|_I \frac{\log\frac{4\pi}{|I|}}{|I|} \int_I |f_\eta(\zeta) - (f_\eta)_I| |d\zeta||d\eta|$$

$$\le \frac{\|f\|_{**}}{2\pi} \Big(\int_{0 < |1-\eta| < \delta} \frac{1 - r^2}{|1 - r\overline{\eta}|^2} |v_\eta - v|_I |d\eta|$$

$$+ \int_{\delta < |1-\eta| < 2} \frac{1 - r^2}{|1 - r\overline{\eta}|^2} |v_\eta - v|_I |d\eta|$$

can be made arbitrarily small, the proof is complete. $\square$

Theorem 6.3. *If* $a, b \in \mathcal{V} + \overline{H^\infty} \cap \mathrm{BMO}_{\log}$, *then*

$$\|(ab)_r - a_r b_r\|_{L^\infty \cap \mathrm{BMO}_{\log}} \longrightarrow 0 \ \ as \ r \to 1$$

Proof. The statement follows easily from the preceding lemma and the fact $(hg)_r = h_r g_r$ for $h, g \in \overline{H^\infty}$. $\square$

Proof of the index formula. According to Lemma 5.1 and the general theory of Fredholm operators, there is a function $\chi_n h$ ($n \geq 0$ and $h \in \overline{H^\infty}$) that has the same index as a and generates a Toeplitz operator that is Fredholm of the same index as T_a. Using Theorems 6.1 and 6.3, and well-known properties of the index (of Fredholm operators and of continuous functions), and [14, Lemma 5] saying that $\mathrm{Ind}\, T_{\chi_n} = -n$, we can conclude that for r sufficiently close to 1, we have

$$\begin{aligned}
\mathrm{Ind}\, T_{\chi_n h} &= \mathrm{Ind}\, T_{\chi_n} + \mathrm{Ind}\, T_h \quad \text{(Atkinson)} \\
&= -n - \mathrm{ind}\, h_r \\
&= -\mathrm{ind}((\chi_n)_r h_r) \\
&= -\mathrm{ind}(\chi_n h)_r \\
&= -\mathrm{ind}\, a_r.
\end{aligned}$$
      $\square$

Proof of Theorem 1.8. Indeed this is an immediate consequence of Propositions 5.2 and 5.9, Theorem 5.6, and the preceding proof of the index formula. $\square$

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M. Papadimitrakis
Department of Mathematics
University of Crete
Knossos Avenue
71409 Iraklion
Greece
e-mail: `papadim@math.uoc.gr`

J. A. Virtanen
Department of Mathematics
University of Crete
Knossos Avenue
71409 Iraklion
Greece
Current address:
Department of Mathematics
University of Helsinki
Gustaf Hällströmin katu 2 b
00014 Helsinki
Finland
e-mail: `jani.virtanen@helsinki.fi`