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**Integral Equations and Operator Theory**

# Remarks on the Structure of Complex Symmetric Operators

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**Abstract.** A conjugation $C$ is antilinear isometric involution on a complex Hilbert space $\mathcal{H}$, and $T \in \mathcal{B}(\mathcal{H})$ is called complex symmetric if $T^* = CTC$ for some conjugation $C$. We use multiplicity theory to describe all conjugations commuting with a fixed positive operator. We expand upon a result of Garcia and Putinar to provide a factorization of complex symmetric operators which is based on the polar decomposition.

**Keywords.** Complex symmetric operator, factorization, conjugation.

## 1. Introduction

An operator $T$ on a complex Hilbert space $\mathcal{H}$ is called *complex symmetric* if $T$ has a symmetric matrix relative to some orthonormal basis for $\mathcal{H}$. Complex symmetric operators have been studied for many years in the finite dimensional setting. Recently, S.R. Garcia and M. Putinar have proven interesting results for this class of operators, primarily in the infinite dimensional case. They show that the class is surprisingly large and includes the normal operators, the Hankel operators, the compressed Toeplitz operators, and many integral operators. We refer the reader to [2, 3, 4] for details, including historical comments and references.

The polar decomposition of a complex symmetric operator is described in Theorem 2 of [4]. This result gives an outline of a description, up to unitary equivalence, of all complex symmetric operators. Our main goal in this note is to use spectral multiplicity theory to complete this description. Our discussion will emphasize an operator algebraic point of view.

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## 2. Terminology and preliminary results

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on the separable complex Hilbert space $\mathcal{H}$. Recall that a map $A : \mathcal{H} \to \mathcal{H}$ is *antilinear* if

$$A(\alpha x + \beta y) = \overline{\alpha} Ax + \overline{\beta} Ay$$

for $x, y \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$. Let $\mathcal{B}_{\mathrm{a}}(\mathcal{H})$ denote the collection of all bounded antilinear operators on $\mathcal{H}$. We will see in Proposition 2.2 a natural relation between $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_{\mathrm{a}}(\mathcal{H})$.

$C$ is a *conjugation* if $C \in \mathcal{B}_{\mathrm{a}}(\mathcal{H})$, $C^2 = I$, and $\langle Cx, Cy\rangle = \langle y, x\rangle\, \forall x, y \in \mathcal{H}$. That is, $C$ is an antilinear isometric involution. We say that $T \in \mathcal{B}(\mathcal{H})$ is *$C$-symmetric* if $T^* = CTC$. (Equivalently, $T^*C = CT$, or $TC = CT^*$.) Let $\mathcal{S}_{\mathrm{C}}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : T^* = CTC\}$ denote the collection of $C$-symmetric operators. T is *complex symmetric* provided $T \in \mathcal{S}_{\mathrm{C}}(\mathcal{H})$ for some conjugation $C$. We note that this definition agrees with the definition given in the introduction. In fact, (see [3]), $C \in \mathcal{B}_{\mathrm{a}}(\mathcal{H})$ is a conjugation iff there is a basis $\{e_n\}$ that is fixed by $C$; so $C(\sum \alpha_n e_n) = \sum \overline{\alpha_n} e_n$ for $(\alpha_n) \in \ell^2$. Thus, if $T \in \mathcal{B}(\mathcal{H})$ and $(T_{ij})$ is the matrix of T relative to this basis, then $T \in \mathcal{S}_{\mathrm{C}}(\mathcal{H})$ iff $T_{ij} = T_{ji}\ \forall i, j$.

**Remark 2.1.** If $C_1$ and $C_2$ are conjugations on $\mathcal{H}$, then there is a unitary operator $U \in \mathcal{B}(\mathcal{H})$ so that the map $T \to UTU^{-1}$ carries $\mathcal{S}_{C_1}(\mathcal{H})$ onto $\mathcal{S}_{C_2}(\mathcal{H})$. (Thus $\mathcal{S}_{C_1}(\mathcal{H})$ and $\mathcal{S}_{C_2}(\mathcal{H})$ are spatially linearly isomorphic.) In fact, if $\{e_n\}$ and $\{f_n\}$ are bases fixed by $C_1$ and $C_2$, respectively, one can take $U$ to be defined by $Ue_n = f_n$ $\forall n$.

We now list some elementary structural results for $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$ and $\mathcal{B}_{\mathrm{a}}(\mathcal{H})$. Some of these results appear at least implicitly in the work of Garcia and Putinar.

**Proposition 2.2.** *If $C$ is a conjugation on $\mathcal{H}$, then $\mathcal{B}_{\mathrm{a}}(\mathcal{H}) = \mathcal{B}(\mathcal{H})\, C$. Further, the map $\phi : T \to TC$ is a linear isometry of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}_{\mathrm{a}}(\mathcal{H})$ with inverse given by $\phi^{-1} : A \to AC$ for $A \in \mathcal{B}_{\mathrm{a}}(\mathcal{H})$.*

*Proof.* Since $C$ is an isometric bijection, if follows that $\phi$ is isometric. Clearly $T \in \mathcal{B}(\mathcal{H})$ implies $TC \in \mathcal{B}_{\mathrm{a}}(\mathcal{H})$. Also, if $A \in \mathcal{B}_{\mathrm{a}}(\mathcal{H})$, then $AC \in \mathcal{B}(\mathcal{H})$ and $(AC)C = A$, so $\phi$ is onto. The linearity of $\phi$ is trivial. □

**Remark 2.3.** Similar arguments show that $\psi : T \to CT$ is an antilinear isometry of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}_{\mathrm{a}}(\mathcal{H})$. Also, note that while $\mathcal{B}(\mathcal{H})$ is an algebra, $\mathcal{B}_{\mathrm{a}}(\mathcal{H})$ is only a Banach space. If $A, B \in \mathcal{B}_{\mathrm{a}}(\mathcal{H})$ then $AB \in \mathcal{B}(\mathcal{H})$, not $\mathcal{B}_{\mathrm{a}}(\mathcal{H})$.

**Proposition 2.4.** *If $C$ is a conjugation on $\mathcal{H}$, then $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$ is a weakly closed, *-closed subspace of $\mathcal{B}(\mathcal{H})$.*

*Proof.* If $T^* = CTC$, then $CT^*C = C(CTC)C = T$, so $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$ is *-closed. Next, if $S, T \in \mathcal{S}_{\mathrm{C}}(\mathcal{H})$ and $\beta \in \mathbb{C}$, then $C(S + \beta T)C = CSC + \overline{\beta} CTC = S^* + \overline{\beta} T^*$ and $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$ is a subspace. Now let $\{T_\alpha\}$ be a net in $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$ so that $T_\alpha \to T$ weakly. For $x, y \in \mathcal{H}$, $\langle T_\alpha^* x, y\rangle = \langle CT_\alpha Cx, y\rangle = \langle Cy, T_\alpha Cx\rangle$. Taking limits, we get that $\langle T^* x, y\rangle = \langle CTCx, y\rangle = \langle Cy, TCx\rangle$. Thus, $T \in \mathcal{S}_{\mathrm{C}}(\mathcal{H})$ and $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$ is weakly closed. □

Note that if $dim\,\mathcal{H} \geq 2$, then $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$ is not an algebra. To see this fact, it suffices to choose two symmetric matrices whose product is not symmetric. However, the next proposition shows that $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$ does contain many algebras. Given a family $\mathcal{F}$ of operators on $\mathcal{B}(\mathcal{H})$, let $\mathcal{W}(\mathcal{F})$ denote the weakly closed unital algebra generated by $\mathcal{F}$.

**Proposition 2.5.** *If $\mathcal{F}$ is a commuting family of operators in $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$, then $\mathcal{W}(\mathcal{F}) \subset \mathcal{S}_{\mathrm{C}}(\mathcal{H})$.*

*Proof.* We may as well assume that $I \in \mathcal{F}$. Then $\mathcal{W}(\mathcal{F})$ is the weakly closed span of all finite products of elements in $\mathcal{F}$, so it will suffice to show that if $A$ and $B$ are commuting operators in $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$, then $AB \in \mathcal{S}_{\mathrm{C}}(\mathcal{H})$. This is easy. Suppose that $AC = CA^*$, $BC = CB^*$ and $AB = BA$. Then

$$(BA)C = ABC = ACB^* = CA^*B^* = C(BA)^*,$$

and $AB \in \mathcal{S}_{\mathrm{C}}(\mathcal{H})$. □

We note that the conclusion of this proposition can be stengthened slightly. Suppose $\mathcal{F}$ is as in Proposition 2.5 and let $\mathcal{F}^* = \{A^* : A \in \mathcal{F}\}$. Then the weak closure of the subspace $\mathcal{W}(\mathcal{F}) + \mathcal{W}(\mathcal{F}^*)$ lies in $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$. That is, the weakly closed, *-closed subspace generated by $\mathcal{W}(\mathcal{F})$ is in $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$. For the proof, apply Propositions 2.4 and 2.5.

## 3. Structure of complex symmetric operators

In [4, Theorem 2], the polar decompostion of a complex symmetric operator is described. This theorem and the discussion following its proof yields the following structure theorem.

**Theorem 3.1.** *If $C$ is a conjugation on $\mathcal{H}$, then*

$$\mathcal{S}_{\mathrm{C}}(\mathcal{H}) = \{CJP : \ P \text{ is a positive operator and } J \text{ is a conjugation commuting with } P\}.$$

The factorization in Theorem 3.1 is related to polar decomposition as follows. Garcia and Putinar show that if $T \in \mathcal{S}_{\mathrm{C}}(\mathcal{H})$, then $T = CJP$, where $P$ is the positive factor in the polar decomposition of $T$. If the kernel of $T$ is trivial, then the partially isometric factor in the polar decomposition of $T$ is the unitary operator $CJ$, while in general $CJ$ is a unitary extension of the partially isometric factor.

Our goal now is to give a complete description of the conjugations that commute with a fixed positive operator. We will use spectral multiplicity theory, so we begin with some brief remarks on normal operators. If $N \in \mathcal{B}(\mathcal{H})$ is normal, then there is a compactly supported measure $\nu$ and a function $\phi \in L^\infty(\nu)$ so that $N \cong M_\phi$, where $M_\phi$ denotes multiplication of $\phi$ on $L^2(\nu)$ and $\cong$ denotes unitary equivalence. As noted earlier, $N$ is complex symmetric ([2]). In fact, if $K$ is the conjugation on $L^2(\nu)$ given by $Kf = \overline{f}$, then $M_\phi \in \mathcal{S}_K(L^2(\nu))$. The following is now immediate from Proposition 2.5.

**Corollary 3.2.** *If $N$ is a normal operator in $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$, then $\mathcal{W}(\{N, N^*, I\})$, the von Neumann algebra generated by $N$, lies in $\mathcal{S}_{\mathrm{C}}(\mathcal{H})$.*

*Proof.* Take $\mathcal{F} = \{N, N^*, I\}$ in Proposition 2.5. □

Next we describe the antilinear operators that commute with a normal operator.

**Lemma 3.3.** *Suppose that $K$ is a conjugation and that the normal operator $N$ lies in $\mathcal{S}_K(\mathcal{H})$. Then $\{A \in \mathcal{B}_{\mathrm{a}}(\mathcal{H}) : AN = NA\} = \{TK : T \in \mathcal{B}(\mathcal{H})$ and $TN = NT^*\}$.*

*Proof.* We apply Proposition 2.2. If $A \in \mathcal{B}_{\mathrm{a}}(\mathcal{H})$, then $A = TK$ for some $T \in \mathcal{B}(\mathcal{H})$. If also $AN = NA$, then $T(N^*K) = T(KN) = N(TK)$, so $TN^* = NT$. Now, reverse that argument. If $T \in \mathcal{B}(\mathcal{H})$ and $TN^* = NT$, and if $A = TK$, then $A \in \mathcal{B}_{\mathrm{a}}(\mathcal{H})$ and $AN = TKN = TN^*K = NTK = NA$. □

**Proposition 3.4.** *Suppose that $P$ is a positive operator, that $K$ is a conjugation, and that $P \in \mathcal{S}_K(\mathcal{H})$. Then $J$ is a conjugation commuting with $P$ iff $J \in \{UK : U \in \mathcal{S}_K(\mathcal{H}), U$ is unitary, and $UP = PU\}$.*

*Proof.* Suppose that $J$ is a conjugation and that $JP = PJ$. By Lemma 3.3, $J = UK$ for some $U \in \mathcal{B}(\mathcal{H})$ with $UP = PU$. But then $U = JK$, so $U$ is unitary. Also, $I = J^2 = (UK)^2 = U(KUK)$, so $U^* = KUK$ and $U \in \mathcal{S}_K(\mathcal{H})$. It is easy to check that the argument can be reversed to finish the proof. □

The last proposition says that to find the conjugations commuting with a positive operator $P$, fix a conjugation $K$ so that $P \in \mathcal{S}_K(\mathcal{H})$ and then find the unitary operators in $\mathcal{S}_K(\mathcal{H})$ that commute with $P$. For this we apply spectral multiplicity theory to $P$ (see Chapter IX, Section 10 of [1]).

Suppose first that $P$ has multiplicity one. Then there is a Borel measure $\mu$ whose support is the spectrum of $P$ so that $P \cong P_\mu$ on $L^2(\mu)$, where $P_\mu : f(t) \to tf(t)$ is multiplication by the independent variable. Then $\{P_\mu\}'$, the commutant of $P_\mu$, is $\{M_\phi : \phi \in L^\infty(\mu)\}$ [1, Corollary 6.9]. Define the conjugation $K_\mu$ on $L^2(\mu)$ by $K_\mu f = \overline{f}$, $f \in L^2(\mu)$. Then $P_\mu \in \mathcal{S}_{K_\mu}(L^2(\mu))$. In fact, Corollary 3.2 gives that $\{P_\mu\}' = \mathcal{W}(\{P_\mu\}) \subset \mathcal{S}_{K_\mu}(L^2(\mu))$. Note that $U \in \{P_\mu\}'$ is unitary iff $U = M_\phi$ for some $\phi \in L^\infty(\mu)$ with $|\phi| = 1$ $\mu$ a.e. Using Proposition 3.4 we see that

$$\{J : JP_\mu = P_\mu J\} = \{M_\phi K_\mu : \phi \in L^\infty(\mu) \text{ and } |\phi| = 1 \ \mu \ a.e.\}. \tag{1}$$

Next we suppose that $P$ has uniform multiplicity $n$, $1 < n \leq \infty$. Thus there is a Borel measure $\mu$ so that $P \cong P_\mu^{(n)}$, where $P_\mu^{(n)}$ is the direct sum of $n$ copies of $P_\mu$, acting on $L^2(\mu)^{(n)}$, the direct sum of $n$ copies of $L^2(\mu)$. $K_\mu^{(n)}$ is a conjugation on $L^2(\mu)^{(n)}$, and $P_\mu^{(n)} \in \mathcal{S}_{K_\mu^{(n)}}(L^2\mu^{(n)})$. It is well-known (see [1, Proposition 6.1]) and easy to check that $\{P_\mu^{(n)}\}' = \mathcal{M}_n(\{P_\mu\}')$, the $n \times n$ matrices with entries in $\{P_\mu\}'$.

Suppose that $U \in \{P_\mu^{(n)}\}'$ is unitary. Then $U = (M_{\phi_{ij}})$, an $n \times n$ operator matrix with $\phi_{ij} \in L^\infty(\mu)\ \forall i, j$. Also $U^* = (M_{\overline{\phi_{ji}}})$. An elementary matrix computation shows that $K_\mu^{(n)} U K_\mu^{(n)} = (K_\mu M_{\phi_{ij}} K_\mu) = (M_{\overline{\phi_{ij}}})$. Thus $U \in \mathcal{S}_{K_\mu^{(n)}}(L^2(\mu)^{(n)})$ iff $U^* = (M_{\overline{\phi_{ij}}})$ iff $\overline{\phi_{ji}} = \overline{\phi_{ij}}\ \forall i, j$. That is, $U$ has a symmetric operator matrix.

If we now apply Proposition 3.4, we get the following result.

**Proposition 3.5.** *$J$ is a conjugation commuting with $P_\mu^{(n)}$ if and only if $J$ has the form $J = (M_{\phi_{ij}})K_\mu^{(n)}$, where $(M_{\phi_{ij}})$ is a unitary operator matrix such that $\phi_{ij} = \phi_{ji} \in L^\infty(\mu)$, $\forall i, j$.*

Now consider any positive operator $P$. Following [1, Theorem 10.20], there are mutually singular measures $\mu_\infty, \mu_1, \mu_2, \cdots$ so that, up to unitary equivalence,

$$P = \left(\bigoplus_{n=1}^{\infty} P_{\mu_n}^{(n)}\right) \oplus P_{\mu_\infty}^{(\infty)} \text{ on } \mathcal{H} = \left(\bigoplus_{n=1}^{\infty} L^2(\mu_n)^{(n)}\right) \oplus L^2{\mu_\infty}^{(\infty)} \tag{2}$$

and

$$\{P\}' = \left(\bigoplus_{n=1}^{\infty} \{P_{\mu_n}^{(n)}\}'\right) \oplus \{P_{\mu_\infty}^{(\infty)}\}'. \tag{3}$$

Then $P \in \mathcal{S}_K(\mathcal{H})$, where $K = \left(\bigoplus_{n=1}^\infty K_{\mu_n}^{(n)}\right) \oplus K_{\mu_\infty}^{(\infty)}$. Also, $J$ is a conjugation commuting with $P$ iff $J = (\bigoplus_{n=1}^\infty J_n) \oplus J_\infty$ is a direct sum of conjugations. For each n, $J_n = U_n K_{\mu_n}^{(n)}$ where $U_n$ is a symmetric unitary operator matrix described in Proposition 3.5..

This completes the description, up to unitary equivalence, of the complex symmetric operators.

We close with an observation of how the complex symmetric operators sit between $\mathcal{B}(\mathcal{H})$ and the set of normal operators. First note that each unitary $U$ is the product of conjugations. This follows immediately from Theorem 3.1. So suppose $T \in \mathcal{B}(\mathcal{H})$ has polar decomposition $T = UP$ and that $U$ is unitary. (For example, suppose $T$ is invertible.) Then $T = CJP$ for some conjugations $C$ and $J$.

But if $T$ is complex symmetric, Theorem 3.1 shows that the above factorization can be achieved with the additional condition that $J$ commutes with $P$.

If $T$ is normal, we show that we can choose both $C$ and $J$ to commute with $P$.

**Proposition 3.6.** $\{T \in \mathcal{B}(\mathcal{H}) : T$ *is normal* $\}$
$= \{CJP : P$ *is positive and $C$ and $J$ are conjugations commuting with $P$* $\}$.

*Proof.* Supose $T$ is normal. If $\mathcal{H}_0 = \ker T$, then $\mathcal{H}_0$ reduces $T$ so if $\mathcal{H}_1 = \mathcal{H}_0^\perp$ and if $T_1 = T_{|\mathcal{H}_1}$, then $T_1$ is normal, so we have $T_1 = C_1 J_1 P$ for conjugations $C_1, J_1$ on $\mathcal{H}_1$ with $J_1$ commuting with $P$. Clearly $C_1 J_1$ also commutes with $P$, so $C_1 P J_1 = C_1 J_1 P = P C_1 J_1$, and $C_1$ commutes with $P$. Now extend $C_1$ and $J_1$ to be conjugations on all of $\mathcal{H}$.

The other inclusion is trivial to check. □

Thus there is a natural sense in which the complex symmetric operators sit halfway between the normal operators and $\mathcal{B}(\mathcal{H})$.

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