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Integral Equations and Operator Theory

Eigenvalue Distribution of Positive Definite Kernels on Unbounded Domains

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Abstract. We study eigenvalues of positive definite kernels of L^2 integral operators on unbounded real intervals. Under the assumptions of integrability and uniform continuity of the kernel on the diagonal the operator is compact and trace class. We establish sharp results which determine the eigenvalue distribution as a function of the smoothness of the kernel and its decay rate at infinity along the diagonal. The main result deals at once with all possible orders of differentiability and all possible rates of decay of the kernel. The known optimal results for eigenvalue distribution of positive definite kernels in compact intervals are particular cases. These results depend critically on a 2-parameter differential family of inequalities for the kernel which is a consequence of positivity and is a differential generalization of diagonal dominance.

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1. Introduction and definitions

Given an interval $I \subset \mathbb{R}$, a linear operator $K : L^2(I) \to L^2(I)$ is integral if there exists a measurable function $k(x, y)$ on $I \times I$ such that for all $\phi \in L^2(I)$

$$\phi \longmapsto K(\phi) = \int_I k(x, y)\, \phi(y)\, dy \tag{1.1}$$

almost everywhere. The function $k(x, y)$ is called the kernel of K. If $k(x, y) = \overline{k(y, x)}$ a.e. in I^2 then K is self-adjoint. If in addition K satisfies the condition

$$\int_I \int_I k(x, y)\, \phi(y)\overline{\phi(x)}\ dx\, dy \geq 0 \tag{1.2}$$

for all $\phi \in L^2(I)$, then it is a positive operator and the corresponding kernel $k(x, y)$ is called a *positive definite kernel*.

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This paper shall deal exclusively with positive integral operators and the corresponding positive definite kernels. Its purpose is the study of the asymptotic behavior of eigenvalues of K in the case where I is unbounded.

The case where I is a compact interval has been thoroughly studied. We next describe what is known in this case.

Integral operators with $L^2(I^2)$ kernels are compact. For self-adjoint operators standard spectral methods yield the bilinear expansion for the kernel

$$k(x,y) = \sum_{n\geq 1} \lambda_n \, \phi_n(x) \, \overline{\phi_n(y)}, \tag{1.3}$$

where $\lambda_n \in \mathbb{R}$ are the eigenvalues of K repeated according to multiplicity, ordered non-increasingly by absolute value and accumulating only at 0. The $\{\phi_n\}_{n\geq 1}$ are the corresponding $L^2(I)$-orthonormal eigenfunctions spanning the range of K and equality is in the $L^2(I)$ sense. If the operator is positive, $k(x,y)$ is a positive definite kernel and $\lambda_n \geq 0$, so the eigenvalue sequence $\{\lambda_n\}_{n\geq 1}$ is non-increasing.

The asymptotic behavior of the eigenvalue sequence $\{\lambda_n\}_{n\geq 1}$ is closely related to smoothness properties of the kernel $k(x,y)$. If k is continuous, the classical theorem of Mercer (see e.g. [22]) asserts that eigenfunctions are continuous, convergence of the series (1.3) is absolute and uniform and the operator K is trace class with

$$\text{tr }(K) = \int_I k(x,x)\, dx = \sum_{n\geq 1} \lambda_n \tag{1.4}$$

from which the basic eigenvalue estimate $\lambda_n = o(1/n)$ is derived.

For general (not necessarily positive definite) kernels it was shown by Weyl [23] that if $k(x,y)$ is C^1 then $\lambda_n = o(1/n^{3/2})$. This estimate may be improved when k is a positive definite kernel, as shown by Reade [16], to $\lambda_n = o(1/n^2)$. It may also be shown that if a positive definite kernel k, in addition to continuity, satisfies a Lipschitz condition of order α, $0 < \alpha \leq 1$, then $\lambda_n = O(1/n^{1+\alpha})$, and that this estimate is best possible as a power of n. More generally, positive definite C^p kernels satisfy $\lambda_n = o(1/n^{p+1})$ [18]. In fact the optimal estimates are slightly sharper: $\lambda_n = o(1/n^{p+1})$ for odd p and $\sum_1^\infty n^p \lambda_n < +\infty$ for even p; see Ha [10] and Reade [19]. Cochran and Lukas [8] and Chang and Ha [7] derive the corresponding results for the decay rate of eigenvalues when a suitable higher-order derivative is Lip^α.

Comparatively with the case where I is compact, little is known about eigenvalues of positive definite kernels in unbounded domains. There are fundamental reasons for this: integral operators in unbounded domains are in general non-compact, so there will be no pure eigenvalue spectrum but in general also a continuous part.

The abstract theory of eigenvalue distribution for integral operators uses operator ideals and interpolation between Besov spaces; see e.g. Birman and Solomyak [1], Gohberg-Krein [9], König [12], Pietsch [15] and references therein. While this approach allows for more precise estimation of eigenvalue asymptotics (determining Lorentz space summability of $\{\lambda_n\}$), its results are not directly applicable to our context. In fact, most of the results are valid only in bounded domains; for unbounded domains compactness of the operators must be externally forced. Thus Pietsch [15] and Birman and Solomyak [1] achieve this with parametrically weighted kernels. If the weights are sufficiently strong to ensure that the resulting kernel and its derivatives decay sufficiently fast at ∞, eigenvalue estimates may be derived.

It should however be noted that there are no results in this theory specifically for positive definite kernels in unbounded domains. From what has been described for the compact case, it is to be expected that restriction to this class of kernels yields results which improve on the general estimates. Indeed, although a straightforward comparison is much more delicate than in the compact case (see, e.g., Pietsch's 10-parameter eigenvalue theorem in [15]), this paper shows that one can say much more in this case than follows from the general theory. For instance, we show below that under very mild assumptions (integrability on the diagonal), positivity is sufficient to ensure compactness of the operator, thus totally dispensing weight factors.

2. Preliminaries: classes $\mathcal{S}_n$ and $\mathcal{A}_n$

The purpose of this paper is the study of the eigenvalue distribution of positive integral operators in the case where I is an unbounded interval in $\mathbb{R}$. For this purpose, it will be essential to restrict to the following classes of kernels. Note first of all that if $k(x, y)$ is a continuous positive definite kernel then $\forall x \in I\ k(x, x) \geq 0$ and $\forall x,\ y \in I\ |k(x, y)|^2 \leq k(x, x)k(y, y)$; we refer to this property as the diagonal dominance inequality for positive definite kernels.

In all this section $I \subset \mathbb{R}$ is only assumed to be a topologically closed interval; the definitions and results below apply whether I is bounded or not.

Definition 2.1. A function $k(x, y) : I^2 \to \mathbb{C}$ is said to belong to class $\mathcal{A}_0(I)$ if:

1. $k(x, y)$ is continuous in I^2;
2. $k(x, x) \in L^1(I)$;
3. $k(x, x)$ is uniformly continuous in I.

Remark 2.2. If I is compact, a kernel is in $\mathcal{A}_0(I)$ if and only if it is continuous in I^2. Less obviously, if I is unbounded then a positive definite kernel $k \in \mathcal{A}_0(I)$ if and only if $k(x, y)$ is continuous in I^2, $k(x, x) \in L^1(I)$ and $k(x, x) \to 0$ as $|x| \to +\infty$; see [2].

The following summarizes the essential properties of $\mathcal{A}_0(I)$ positive definite kernels. For compact I these follow from the classical Mercer theorem (see e.g. [22]); for unbounded I they are proved in [2].

If $k(x, y)$ is a positive definite kernel in class $\mathcal{A}_0(I)$, then the associated integral operator K defined by (1.1) is Hilbert-Schmidt, therefore compact, so it has a pure eigenvalue spectrum $\{\lambda_n\}_{n\geq 1}$ with $\lambda_n \geq 0$ forming a non-increasing sequence converging to 0. Eigenfunctions ϕ_n associated with nonzero eigenvalues are uniformly continuous and so vanish at infinity in I is unbounded. Moreover, Mercer's theorem holds in this wider context: the bilinear eigenseries (1.3) for the kernel is absolutely and uniformly convergent and the operator $K : L^2(I) \to L^2(I)$ is trace class with trace given by (1.4), whence the eigenvalue sequence satisfies $\lambda_n = o(1/n)$.

Class $\mathcal{A}_0(I)$ seems to be the most general class of positive definite kernels (in bounded or unbounded domains) for which Mercer's theorem holds; see counterexamples in [2] as well as more general results in Novitskii [13]. It is therefore natural to adopt it as the starting point for the study of eigenvalue distribution of positive definite kernels in unbounded domains.

Remark 2.3. Throughout this paper the diagonal $\{(x, y) \in I^2 : y = x\}$ will play a prominent role in determining the behavior of $k(x, y)$. We abbreviate reference to this set simply as "the diagonal".

The following definitions are useful in the study of properties arising from differentiability of the kernel k. If x is a boundary point of I, a limit at x will mean the one-sided limit as $y \to x$ with $x \in I$.

Definition 2.4. Let $I \subset \mathbb{R}$ be an interval. A function $k : I^2 \to \mathbb{C}$ is said to *be of class* $\mathcal{S}_n(I)$ if, for every $m_1 = 0, 1, \ldots n$ and $m_2 = 0, 1, \ldots n$, the partial derivatives $\dfrac{\partial^{m_1+m_2}}{\partial y^{m_2} \partial x^{m_1}} k(x, y)$ exist and are continuous in I^2.

Definition 2.5. Let $n \geq 1$ be an integer. A function $k : \mathbb{R}^2 \to \mathbb{C}$ is said to belong to class $\mathcal{A}_n(I)$ if $k \in \mathcal{S}_n(I)$ and

$$k(x, y), \; \frac{\partial^2}{\partial y \partial x} k(x, y), \; \ldots \; \frac{\partial^{2n}}{\partial y^n \partial x^n} k(x, y)$$

are in class $\mathcal{A}_0(I)$.

Remark 2.6. Observe, in analogy with Remark 2.2, that if I is compact $\mathcal{S}_n(I) \subset \mathcal{A}_n(I)$.

If I is compact the contents of Theorem 2.7 below are essentially proved by Kadota [11]; the extension to unbounded I is proved in [5]. Here $H^n(I)$ denotes the Sobolev Hilbert space $W^{n,2}(I)$.

Theorem 2.7. *Let $k(x, y)$ be a positive definite kernel in $\mathcal{A}_n(I)$ with eigenseries expansion* (1.3)*. Then the following statements hold.*

1. *If $\lambda_i \neq 0$, ϕ_i is in $C^n(I) \cap H^n(I)$;*
2. *each k_m is a positive definite kernel in class $\mathcal{A}_{n-m}(I)$ and*

$$k_m(x, y) = \frac{\partial^{2m} k}{\partial y^m \partial x^m}(x, y) = \sum_{i \geq 1} \lambda_i \phi_i^{(m)}(x) \overline{\phi_i^{(m)}(y)} \tag{2.1}$$

uniformly and absolutely in I^2 for each $m = 0, 1, \ldots, n$;

3. *the $L^2(I)$ integral operator K_m with kernel k_m is trace class with*

$$\operatorname{tr}(K_m) = \mathcal{K}_m = \int_I k_m(x,x)\,dx. \tag{2.2}$$

Particular attention will be devoted, in § 4, to positive definite kernels in class $\mathcal{A}_0(I) \cap \mathcal{S}_n(I)$. Observe that, trivially from the definitions, $\mathcal{A}_n(I) \subset \mathcal{A}_0(I) \cap \mathcal{S}_n(I)$.

3. Preparatory results

We next present methods introduced by Ha [10] and Reade [16]. Although adaptations of these methods have been developed in [6] to the context of integral operators defined in unbounded domains, in this paper we will only need results relative to operators defined in the compact interval $[0, L]$. Proofs may be found in these papers and will be omitted.

3.1. Best approximations

Let k be a continuous positive definite kernel in $[0, L]^2$ and K be the associated positive integral operator. It follows from the general theory of compact operators in Hilbert space that, if R is the operator with kernel $\sum_{n=1}^{N} \lambda_n \, \phi_n(x) \, \overline{\phi_n(y)}$, then R is the best approximation to K in the operator norm by symmetric operators of rank $\leq N$, the minimum distance being $\|K - R\|_{\text{op}} = \lambda_{n+1}$ (see e.g. Gohberg and Krein [9], Theorem III.6.1). Also $\sum_{n=1}^{N} \lambda_n \, \phi_n(x) \, \overline{\phi_n(y)}$ is the best approximation to $k(x, t)$ by $L^2([0, L])$ symmetric kernels of rank $\leq N$ which generate compact integral operators, the minimum distance being $(\sum_{n=N+1}^{\infty} \lambda_n^2)^{1/2}$ (see [22] for a version for integral operators or [9] for linear operators in Hilbert space).

Lemma 3.1. *If $k(x, y)$ is continuous in $[0, L]^2$, then $\sum_{n=1}^{N} \lambda_n \, \phi_n(x) \, \overline{\phi_n(y)}$ is the best approximation in the trace norm by $L^2([0, L])$ symmetric kernels of rank $\leq N$.*

3.2. Square roots

Any positive operator K in Hilbert space has a unique positive square root S [21]. This fact implies that if K is a positive operator with continuous kernel k satisfying the bilinear eigenfunction expansion (1.3), the corresponding square root operator S is an $L^2([0, L])$ positive integral operator. Since K is trace class, standard arguments imply that the positive definite kernel $s(x, y)$ of S satisfies the bilinear expansion

$$s(x, y) = \sum_{n \geq 1} \lambda_n^{1/2} \, \phi_n(x) \, \overline{\phi_n(y)}, \tag{3.1}$$

where the last equality is in the sense of L^2 convergence. In general, of course, $s(x, y)$ will not be continuous, so the corresponding operator S will not be trace class. However, the following holds.

Lemma 3.2. *If $k(x,y)$ is continuous in $[0,L]$ and $s(x,y)$ is the kernel of the corresponding positive square root operator, then for any $f \in L^2([0,L])$*

$$Sf(x) = \int_0^L s(x,y) f(y)\, dy$$

is a continuous function of x.

3.3. A class of finite rank operators

We now define a class of finite rank operators to be used in the approximation of a positive operator K with continuous kernel k defined in the interval $[0,L]$, $L>0$.

Let $N>0$ be an integer and $L>0$ be a positive real number. We define $R^{N,L}$ to be the $L^2([0,L])$ operator with kernel

$$r^{N,L}(u,v) = \frac{N}{L} \sum_{n=1}^{N} \psi_n^{N,L}(u)\, \psi_n^{N,L}(v),$$

where

$$\psi_n^{N,L}(x) = \begin{cases} 1 & \text{if } (n-1)\frac{L}{N} < x \leq n\frac{L}{N} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $R^{N,L}$ is an orthogonal projection in $L^2([0,L])$. It is thus a positive operator of rank N with $0 \leq R^{N,L} \leq I$. Its spectrum is $\{0, 1\}$, the eigenvalue 1 having multiplicity N and the corresponding orthogonal (unnormalized) eigenfunctions being the $\psi_n^{N,L}$.

Given an operator $K \in L^2([0,L])$ with continuous kernel k and square root S, it follows that $0 \leq SR^{N,L}S \leq K$; since by Lemma 3.2 ([16], Lemma 3) $SR^{N,L}S$ has a continuous kernel, it is trace class. It then follows (see [16], [6] for details) that

$$\|K - SR^{N,L}S\|_{\mathrm{tr}} = \sum_{n=1}^{N} \frac{N}{L} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} (k(u,u) - k(u,v))\, du\, dv \tag{3.2}$$

$$= \sum_{n=1}^{N} \frac{N}{L} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \left(\frac{k(u,u) - k(u,v) - k(v,u) + k(v,v)}{2} \right) du\, dv \tag{3.3}$$

Equations (3.2) and (3.3) will be used in § 4 in the proof of our main results.

Remark 3.3. Since self-adjointness of the operator K implies conjugate symmetry of the kernel, it is immediate to conclude that the contribution of the imaginary part of k to the integral $\int_0^L \int_0^L r^{N,L}(u,v) k(v,u)\, du\, dv$ is zero. The same observation obviously applies to integration in any square symmetric with respect to the diagonal. Consequently, we may regard $k(u,v)$ in (3.2) and (3.3) as being real-valued without any loss of generality.

3.4. Eigenvalues of symmetric derivatives

The evaluation of the rate of decay of eigenvalues of the operators K_m, m a positive integer, whose kernels are the symmetric derivatives $k_m(x,y) = \dfrac{\partial^{2m}}{\partial y^m \partial x^m} k(x,y)$ of the class $\mathcal{S}_m([0,L])$ positive definite kernel k plays a key role in the study of the eigenvalues of the operator K with kernel k (see e.g. [8], [10]). Recall that, according to Theorem 2.7, k_m is a continuous positive definite kernel defined on the compact set $[0,L]^2$. Conventions and properties, in particular Mercer's theorem described in § 1, are thus applicable to k_m. A straightforward adaptation of a result of Ha [10] for positive definite kernels defined in $[0,1]^2$ yields the following upper bound for the eigenvalues of an operator K with positive definite kernel in $\mathcal{S}_m([0,L])$:

Lemma 3.4. *Let $k : [0,L]^2 \to \mathbb{C}$ be a positive definite kernel in class $\mathcal{S}_m([0,L])$, $m \geq 1$. Let $\{\lambda_n(k)\}_{n\in\mathbb{N}}$ (resp. $\{\lambda_n(k_m)\}_{n\in\mathbb{N}}$) be the sequence of eigenvalues of the integral operator with kernel k (resp. k_m). Then*

$$\lambda_{2n}(k) \leq L^{2m}\left(\frac{4}{\pi^2}\right)\frac{\lambda_n(k_m)}{(2n-4m-1)^{2m}}$$

for every $n \geq 2m+1$.

Given a trace class kernel k, we define its N-tail $T_k(N) = \sum_{n=N+1}^{\infty} \lambda_n(k)$. We now state and prove a useful consequence of Lemma 3.4.

Corollary 3.5. *Let $k : [0,L]^2 \to \mathbb{C}$ be a positive definite kernel in class $\mathcal{S}_m([0,L])$, $m \geq 1$. Then there exists $N_0 \in \mathbb{N}$ and a real positive constant C such that*

$$T_k(2N+1) \leq C\left(\frac{L}{N}\right)^{2m} T_{k_m}(N)$$

for $N > N_0$.

Proof. According to Lemma 3.4 we may write, for every $n > N_0 = 2m+1$,

$$\begin{aligned}\lambda_{2n}(k) &\leq \frac{n^{2m}}{(2n-4m-1)^{2m}} L^{2m}\frac{4}{\pi^2}\frac{\lambda_n(k_m)}{n^{2m}} \\ &\leq \frac{1}{4^m}\left(\frac{L}{n}\right)^{2m}\frac{4}{\pi^2}\lambda_n(k_m) \\ &= \frac{C}{2}\left(\frac{L}{n}\right)^{2m}\lambda_n(k_m),\end{aligned}$$

where $C = \dfrac{8}{4^m \pi^2}$ depends only on m. Since $\lambda_n(k)$ and $\lambda_n(k_m)$ are positive non-increasing sequences, using the above inequalities we obtain

$$\begin{aligned}
T_k(2N+1) &= \sum_{n=2N+2}^{\infty} \lambda_n(k) \\
&= \sum_{n=N+1}^{\infty} \lambda_{2n}(k) + \lambda_{2n+1}(k) \leq 2 \sum_{n=N+1}^{\infty} \lambda_{2n}(k) \\
&\leq 2 \sum_{n=N+1}^{\infty} \frac{C}{2} \left(\frac{L}{n}\right)^{2m} \lambda_n(k_m) \\
&= C\, L^{2m} \sum_{n=N+1}^{\infty} \frac{\lambda_n(k_m)}{n^{2m}} \\
&\leq C \frac{L^{2m}}{N^{2m}} \sum_{n=N+1}^{\infty} \lambda_n(k_m) \\
&= C \left(\frac{L}{N}\right)^{2m} T_{k_m}(N)
\end{aligned}$$

for $N > N_0$. This finishes the proof. □

4. Asymptotic distribution of eigenvalues

This section is devoted to the proof of our main results. We take without loss of generality $I = [0, +\infty[$ as our model unbounded interval; the adaptations to other types of unbounded intervals are trivial. We mention however that the case $I = \mathbb{R}$ is particularly significant in view of Fourier transforms, see Remark 4.19 and Corollary 4.20.

We begin by establishing some basic lemmas and definitions. Suppose k is a positive definite kernel in class $\mathcal{A}_0([0, +\infty[)$. Let $L > 0$ and consider the restriction k^L of k to the compact square $[0, L]^2$. In view of the definition of class $\mathcal{A}_0(I)$, both k and k^L are positive definite kernels associated with trace class operators on the corresponding intervals. Using the notation introduced in 3.4, we have:

Lemma 4.1. *Let $k \in \mathcal{A}_0([0, +\infty[)$ and k^L be the restriction of k to $[0, L]^2$. Then there is $N_0 \in \mathbb{N}$ such that, for $N > N_0$, we have*

$$T_k(N) \leq T_{k^L}(N) + \int_L^{\infty} k(x, x)\, dx.$$

Proof. Since both K and K^L are trace class operators we have by (1.4)

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n(k) &= \int_0^{\infty} k(x,x)\,dx = \int_0^{L} k(x,x)\,dx + \int_L^{\infty} k(x,x)\,dx \\ &= \sum_{n=1}^{\infty}\lambda_n(k^L) + \int_L^{\infty} k(x,x)\,dx.\end{aligned} \tag{4.1}$$

Suppose $\int_L^{\infty} k(x,x)\,dx = 0$. Since k is continuous and non-negative along the diagonal it follows that $k(x,x) \equiv 0$ for $x \geq L$. Since k is positive definite, we have by diagonal dominance $|k(x,y)|^2 \leq k(x,x)k(y,y)$, and therefore the support of k is contained in $[0,L]^2$. Direct calculation then shows that the restriction to $[0,L]$ determines a mapping $\varphi_n \to \varphi_n^L$ from the set of eigenfunctions of k to the set of eigenfunctions of k^L which is one-to-one and preserves the associated eigenvalues $\lambda_n(k)$. Hence k and k^L have the same spectrum (including multiplicities) and $T_k(N) = T_{k^L}(N)$ for all N. The same conclusion may be derived using the principle of related operators ([12], [14]), since inclusion and truncation in this case act as a relation between the operators $K : L^2([0,\infty[) \to L^2([0,\infty[)$ and $K^L : L^2([0,L]) \to L^2([0,L])$, both of which are compact.

Suppose now $\int_L^{\infty} k(x,x)\,dx > 0$. Then (4.1) implies that $\sum_{n=1}^{\infty}\lambda_n(k) > \sum_{n=1}^{\infty}\lambda_n(k^L)$. Thus there exists $N_0 \in \mathbb{N}$ such that, for $N > N_0$,

$$\sum_{n=1}^{N}\lambda_n(k) \geq \sum_{n=1}^{N}\lambda_n(k^L). \tag{4.2}$$

From (4.1) and (4.2) we conclude that for $N > N_0$

$$T_k(N) \leq T_{k^L}(N) + \int_L^{\infty} k(x,x)\,dx,$$

finishing the proof. □

The following result is a central tool in the study of the decay rate of eigenvalues of a positive definite kernel k in class $\mathcal{A}_0([0,+\infty[) \cap \mathcal{S}_n([0,+\infty[)$.

Lemma 4.2. *Let $\beta > 1$, $q > 0$, $A > 0$, $B > 0$, $x_0 > 0$, $\delta > 0$ be real numbers. Suppose $f : [0,+\infty[\to [0,+\infty[$ is a continuous function satisfying the condition $\int_L^{\infty} f(x)\,dx \leq \frac{B}{L^{\beta-1}}$ for $L \geq x_0$. Let $R(L,N)$ be defined by*

$$R(L,N) = A\left(\frac{L}{N}\right)^{q} L + \int_L^{\infty} f(x)\,dx \tag{4.3}$$

whenever $\frac{L}{N} < \delta$ and write $\gamma = \frac{(q+1)\beta}{q+\beta}$. Then there exist $N_0 \in \mathbb{N}$, $D > 0$ and an increasing sequence $L(N) \to +\infty$ such that $L(N)/N$ is decreasing and convergent to zero and such that the inequality

$$R(L(N),N) \leq \frac{D}{N^{\gamma-1}} \tag{4.4}$$

holds for $N \geq N_0$.

Proof. Define $L(N) = N^\theta$ with $\theta = \dfrac{q}{q+\beta}$. Observe that, since $0 < \theta < 1$, $L(N)$ in an increasing sequence converging to $+\infty$ and $\dfrac{L(N)}{N}$ is a decreasing sequence converging to 0. Choose N_0 such that $N_0^\theta = L(N_0) > x_0$ and $N_0^{\theta-1} = \dfrac{L(N_0)}{N_0} < \delta$. Then $L(N) > x_0$, $\dfrac{L(N)}{N} < \delta$ and $\int_{L(N)}^\infty f(x)\,dx \leq \dfrac{B}{L(N)^{\beta-1}}$ whenever $N \geq N_0$. Thus, according to (4.3) we may write

$$R(L(N),N) = A\left(\frac{L(N)}{N}\right)^q L(N) + \int_{L(N)}^\infty f(x)\,dx \tag{4.5}$$

$$\leq A N^{(\theta-1)q+\theta} + B\,N^{\theta(1-\beta)}. \tag{4.6}$$

Performing the corresponding calculations we derive from (4.6) that, for $D = A + B$,

$$R(L(N),N) \leq \frac{D}{N^{\gamma-1}}$$

for $N \geq N_0$, proving the statement. □

The optimality of the estimate provided by Lemma 4.2 is the issue of the next result.

Proposition 4.3. *Suppose that $\int_L^\infty f(x)\,dx \sim 1/L^{\beta-1}$ as $L \to +\infty$ while keeping the remaining hypotheses of Lemma 4.2. Then, for any positive sequence $L(N)$ such that $L(N)/N < \delta$, there exists a subsequence $L(N')$ and constants N_0, $C > 0$ such that $R(L(N'),N') \geq \dfrac{C}{(N')^{\gamma-1}}$ for $N' \geq N_0$. In particular, the exponent γ in (4.4) cannot be improved for any such sequence $L(N)$.*

Proof. The assertion is trivially verified if $L(N)$ does not converge to ∞. In this case, $L(N)$ admits a bounded subsequence $L(N') < L_0$ for some $L_0 > 0$ and we have

$$R(L(N'),N') = A\left(\frac{L(N')}{N'}\right)^q L(N') + \int_{L(N')}^\infty f(x)\,dx$$

$$\geq \int_{L_0}^\infty f(x)\,dx \equiv C > 0 \tag{4.7}$$

since f is, by hypothesis, a non-negative continuous function and $C = 0$ would imply $f(x) = 0$ for all $x > L_0$, contradicting the hypothesis $\int_L^\infty f(x)\,dx \sim 1/L^{\beta-1}$. Since $\gamma > 1$, from (4.7) we immediately derive

$$R(L(N'),N') \geq \frac{C}{(N')^{\gamma-1}} \quad \text{for } N' \geq 1.$$

Suppose now that $L(N) \to +\infty$. From the hypothesis we derive, in particular, that there exist constants $b > 0$, $x_0 > 0$ such that $\int_L^\infty f(x)\,dx \geq \dfrac{b}{L^{\beta-1}}$ whenever

$L \geq x_0$. Since $L(N) \to +\infty$, we have $L(N) > x_0$ for sufficiently large N and hence, from (4.3),

$$R(L(N), N) = A \left(\frac{L(N)}{N} \right)^q L(N) + \int_{L(N)}^{\infty} f(x)\, dx$$
$$\geq A \left(\frac{L(N)}{N} \right)^q L(N) + \frac{b}{L(N)^{\beta-1}}. \tag{4.8}$$

Recalling the definition of θ and γ, we rewrite (4.8) in the form

$$N^{\gamma-1} R(L(N), N) \geq A \left(\frac{L(N)}{N^\theta} \right)^{q+1} + b \left(\frac{N^\theta}{L(N)} \right)^{\beta-1}. \tag{4.9}$$

Suppose $L(N')$ is a subsequence of $L(N)$ such that $\frac{L(N')}{(N')^\theta}$ converges either to 0 or to $+\infty$. Then from (4.9) we conclude that $(N')^{\gamma-1} R(L(N'), N') \to +\infty$, which implies the assertion of the proposition. On the other hand, if $L(N)$ does not admit such a subsequence it follows that $C_1 < \frac{L(N)}{N^\theta} < C_2$ for some C_1, $C_2 > 0$ and all sufficiently large N. From (4.9) we then derive that for sufficiently large N, say $N \geq N_0$, $R(L(N), N) \geq \frac{C}{N^{\gamma-1}}$ for $C = A\, C_1^{q+1} + b \left(\frac{1}{C_2} \right)^{\beta-1}$, which completes the proof. □

Remark 4.4. Notice that the condition $\int_L^\infty f(x)\, dx \leq \frac{B}{L^{\beta-1}}$ for some $B > 0$ and every $L \geq x_0 > 0$ is implied by the somewhat less general requirement $f(x) \leq \frac{B'}{x^\beta}$ for all $x \geq x_0$ with $B' = B\,(\beta - 1)$. The hypothesis of Lemma 4.2 may thus be seen as a generalized condition on the rate of decay of $f(x)$ as $x \to +\infty$. A similar observation applies to the comparison of the hypothesis of Proposition 4.3 with the condition $f(x) \sim \frac{1}{x^\beta}$.

The following uniform continuity and Lipschitz conditions will be useful in the sequence.

Definition 4.5. Let $k : [0, +\infty[\to \mathbb{C}$. We say that $k(x, y)$ *is uniformly continuous with respect to* y *on the diagonal* if for every $\epsilon > 0$ there is $\delta > 0$ such that $|k(x, x) - k(x, y)| < \epsilon$ whenever $|x - y| < \delta$.

Definition 4.6. Let $k : [0, +\infty[\to \mathbb{C}$ and $\alpha \in]0, 1]$. We say that $k(x, y)$ *is* α*-Lipschitz (written Lip*$^\alpha$*) with respect to* y *on the diagonal* if there is a positive constant A such that $|k(x, x) - k(x, y)| \leq A|x - y|^\alpha$ for every $(x, y) \in [0, +\infty[^2$.

Remark 4.7. It is clear that if k is Lip$^\alpha$ with respect to y on the diagonal then it is uniformly continuous with respect to y on the diagonal. It is also clear that each of these conditions is implied by their respective counterpart on the plane. More specifically, uniform continuity on $[0, +\infty[^2$ implies the condition in definition 4.5 and Lip$^\alpha$ on $[0, +\infty[^2$ implies the condition in definition 4.6.

Remark 4.8. If k is a positive definite kernel in class $\mathcal{A}_0([0,+\infty[)$, standard arguments together with diagonal dominance and the fact that $k(x,x)\to 0$ as $x\to+\infty$ easily show that k is uniformly continuous on the diagonal [2] and, in particular, satisfies the condition in definition 4.5.

Remark 4.9. It is easily seen that functions satisfying conjugate symmetry $k(x,y)=\overline{k(y,x)}$ on $[0,+\infty[^2$ and the conditions of definitions 4.5 or 4.6 will automatically satisfy the analogues of these with respect to the variable x. Properties of this kind for the partial derivatives $\dfrac{\partial k}{\partial x}, \dfrac{\partial k}{\partial y}$ can also be seen to arise from conjugate symmetry of k. Incidentally, none of these will play any relevant part in the sequence.

We are now ready to prove our main results. They describe how, for a kernel in class $\mathcal{A}_0([0,+\infty[)\cap\mathcal{S}_n([0,+\infty[)$ — and, in particular, in class $\mathcal{A}_n([0,+\infty[)$ — uniform continuity or Lip^α continuity on the diagonal together with the rate of decay of $k(x,x)$ at infinity allow us to control the rate of decay of the eigenvalues.

Theorem 4.10. *Let $m\geq 0$ and suppose $k(x,y)$ is a positive definite kernel in class $\mathcal{A}_0([0,+\infty[)\cap\mathcal{S}_m([0,+\infty[)$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the sequence of eigenvalues of the integral operator with kernel k. Then the following statements hold.*

1.1 *Suppose $k_m(x,y)$ is uniformly continuous with respect to y on the diagonal. Then:*
   - i) *If $\beta>1$, $\int_L^\infty k(x,x)\,dx = O\big(1/L^{\beta-1}\big)$ (resp. $\int_L^\infty k(x,x)dx = o\left(1/L^{\beta-1}\right)$) as $L\to+\infty$ and $\gamma=\dfrac{(2m+1)\beta}{2m+\beta}$, then $\lambda_n = O\left(1/n^\gamma\right)$ (resp. $\lambda_n = o(1/n^\gamma)$).*
   - ii) *If $\int_L^\infty k(x,x)\,dx = O\left(1/L^{\beta-1}\right)$ as $L\to+\infty$ for all $\beta>1$, then $\lambda_n = o\left(1/n^\gamma\right)$ for all $\gamma<2m+1$.*
   - iii) *If $k(x,x)$ has compact support, then $\lambda_n = o\left(1/n^{2m+1}\right)$.*

1.2 *Suppose $k_m(x,y)$ is Lip^α with respect to y on the diagonal. Then:*
   - i) *If $\beta>1$, $\int_L^\infty k(x,x)dx = O\big(1/L^{\beta-1}\big)$ as $L\to\infty$ and $\gamma=\dfrac{(2m+\alpha+1)\beta}{2m+\alpha+\beta}$, then $\lambda_n = O\left(1/n^\gamma\right)$.*
   - ii) *If $\int_L^\infty k(x,x)\,dx = O\left(1/L^{\beta-1}\right)$ as $L\to+\infty$ for all $\beta>1$, then $\lambda_n = o\left(1/n^\gamma\right)$ for all $\gamma<2m+\alpha+1$.*
   - iii) *If $k(x,x)$ has compact support, then $\lambda_n = O\left(1/n^{2m+\alpha+1}\right)$.*

2.1 *Suppose $k_m(x,y)$ is continuously differentiable with respect to x and that $\dfrac{\partial k_m}{\partial x}$ is uniformly continuous with respect to y on the diagonal. Then:*
   - i) *If $\beta>1$, $\int_L^\infty k(x,x)dx = o\left(1/L^{\beta-1}\right)$ (resp. $\int_L^\infty k(x,x)\,dx = o\left(1/L^{\beta-1}\right)$) as $L\to+\infty$ and $\gamma=\dfrac{(2m+2)\beta}{2m+1+\beta}$, then $\lambda_n = O\left(1/n^\gamma\right)$ (resp. $\lambda_n = o(1/n^\gamma)$).*
   - ii) *If $\int_L^\infty k(x,x)\,dx = O\left(1/L^{\beta-1}\right)$ as $L\to+\infty$ for all $\beta>1$, then $\lambda_n = o\left(1/n^\gamma\right)$ for all $\gamma<2m+2$.*
   - iii) *If $k(x,x)$ has compact support, then $\lambda_n = o\left(1/n^{2m+2}\right)$.*

2.2 *Suppose $k_m(x,y)$ is continuously differentiable with respect to x and that $\dfrac{\partial k_m}{\partial x}$ is Lip$^\alpha$ with respect to y on the diagonal. Then:*

i) *If $\beta > 1$, $\int_L^\infty k(x,x)dx = O\big(1/L^{\beta-1}\big)$ as $L \to \infty$ and $\gamma = \dfrac{(2m+\alpha+2)\beta}{2m+\alpha+1+\beta}$, then $\lambda_n = O\left(1/n^\gamma\right)$.*

ii) *If $\int_L^\infty k(x,x)\,dx = O\left(1/L^{\beta-1}\right)$ as $L \to +\infty$ for all $\beta > 1$, then $\lambda_n = o\left(1/n^\gamma\right)$ for all $\gamma < 2m+\alpha+2$.*

iii) *If $k(x,x)$ has compact support, then $\lambda_n = O\left(1/n^{2m+\alpha+2}\right)$.*

Proof. Let $m \geq 0$ and suppose k is a positive definite kernel in class $\mathcal{A}_0([0,+\infty[) \cap \mathcal{S}_m([0,+\infty[)$. Let k^L be the restriction of k to the interval $[0,L]^2$. By Lemma 4.1 there exists $N_0 \in \mathbb{N}$ such that, for $N > N_0$, we have

$$T_k(N) \leq T_{k^L}(N) + \int_L^\infty k(x,x)\,dx. \tag{4.10}$$

In particular, we may write

$$T_k(2N+1) \leq T_{k^L}(2N+1) + \int_L^\infty k(x,x)\,dx. \tag{4.11}$$

According to Corollary 3.5, inequality (4.11) implies, for $m \geq 1$,

$$T_k(2N+1) \leq C\left(\frac{L}{N}\right)^{2m} T_{k_m^L}(N) + \int_L^\infty k(x,x)\,dx \tag{4.12}$$

for some $C > 0$ and sufficiently large N.

For $m \geq 0$ we now use the results of § 3 in the approximation of the operator K_m^L with kernel k_m^L by finite rank operators. Let S be the square root of K_m^L. Defining $R^{N,L}$ as in § 3.3 and recalling (3.2) and (3.3), we have according to Lemma 3.1

$$\begin{aligned} T_{k_m^L}(N) &\leq \|K_m^L - SR^{N,L}S\|_{\mathrm{tr}} \\ &= \sum_{n=1}^N \frac{N}{L} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \left(k_m^L(u,u) - k_m^L(u,v)\right)\,du\,dv \qquad (4.13) \\ &= \frac{1}{2}\sum_{n=1}^N \frac{N}{L} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \left(k_m^L(u,u) - k_m^L(u,v) - k_m^L(v,u) + k_m^L(v,v)\right)\,du\,dv \qquad (4.14) \end{aligned}$$

where, according to Remark 3.3, we may without loss of generality regard k_m^L as being real valued.

We are now ready to prove statement 1.1 i). If $m = 0$ we have by hypothesis $\gamma = 1$ and the assertion reduces to the already known fact that, for a kernel k in class $\mathcal{A}_0$, $\lambda_n(k) = o(1/n)$.

Suppose that $m \geq 1$. Since by hypothesis k_m is in class $\mathcal{A}_0([0,+\infty[) \cap \mathcal{S}_m([0,+\infty[)$ and is uniformly continuous with respect to y on the diagonal, for every $\epsilon > 0$ there exists $\delta > 0$, independent of L, such that $|k_m^L(u,u) - k_m^L(u,v)| < \epsilon$

for all $u, v \in [0, L]$ satisfying $|v - u| < \delta$. For every positive L and every positive integer N such that $L/N < \delta$ we then have, after performing the relevant calculations from (4.13),

$$T_{k_m^L}(N) \leq \|K_m^L - SR^{N,L}S\|_{\mathrm{tr}} \leq N \cdot \frac{N}{L}\left(\frac{L}{N}\right)^2 \epsilon = L\,\epsilon. \tag{4.15}$$

Therefore, according to (4.12), we derive for N sufficiently large that

$$T_k(2N+1) \leq \epsilon\, C \left(\frac{L}{N}\right)^{2m} L + \int_L^\infty k(x,x)\,dx. \tag{4.16}$$

We now use the hypothesis on the decay rate of $k(x,x)$ at infinity. Suppose there exist $\beta > 1$, $B > 0$, $x_0 > 0$ such that $\int_L^\infty k(x,x)\,dx \leq B/L^{\beta-1}$ whenever $L \geq x_0$. We recall Lemma 4.2 setting $q = 2m$ and $A = \epsilon\, C$ and define a sequence of finite rank operators $SR^{N,L}S$ by setting $L = L(N) = N^\theta$ with $\theta = \frac{2m}{2m+\beta}$. Recall, from Lemma 4.2, that $L(N)$ is an increasing sequence with $L(N) \to +\infty$ and such that $L(N)/N$ is decreasing and convergent to 0. Therefore, there exists N_0 such that $L(N) > x_0$ and $L(N)/N < \delta$ for every $N > N_0$. Therefore we derive from (4.16) and Lemma 4.2 that, for sufficiently large N,

$$T_k(2N+1) \leq \frac{\epsilon\, C + B}{N^{\gamma-1}} = \frac{D}{N^{\gamma-1}}$$

where $\gamma = \dfrac{(2m+1)\beta}{2m+\beta}$.

Now, if $\int_L^\infty k(x,x)\,dx \leq B/L^{\beta-1}$ whenever $L \geq x_0$, the above condition is verified for some fixed $D > 0$, which implies that $\lambda_n = O(1/n^\gamma)$. If $\int_L^\infty k(x,x)\,dx = o(1/L^{\beta-1})$ as $L \to \infty$, the same condition holds for arbitrary $D > 0$ and we derive in this case the stronger conclusion that $\lambda_n = o(1/n^\gamma)$. This finishes the proof of statement 1.1 i).

Consider now the hypothesis of statement 1.2 i). Suppose k_m is Lip^α with respect to y on the diagonal. Choose $L > 0$ and $N \in \mathbb{N}$. Then $|u - v| < L/N$ for all u, $v \in [(n-1)\frac{L}{N}\,, n\frac{L}{N}]$. Since k_m is Lip^α, (4.13) implies

$$T_{k_m^L}(N) \leq \|K_m^L - SR^{N,L}S\|_{\mathrm{tr}} \leq A\left(\frac{L}{N}\right)^\alpha L. \tag{4.17}$$

In the case $m = 0$, we derive from (4.10) and (4.17) that, for sufficiently large N,

$$\begin{aligned} T_k(N) &\leq T_{k^L}(N) + \int_L^\infty k(x,x)\,dx \\ &\leq A\left(\frac{L}{N}\right)^\alpha L + \int_L^\infty k(x,x)\,dx. \end{aligned} \tag{4.18}$$

Similarly, if $m \geq 1$, we derive from (4.11), (4.12) and (4.17), for sufficiently large N,

$$T_k(2N+1) \leq A\,C\left(\frac{L}{N}\right)^{2m+\alpha} L + \int_L^\infty k(x,x)\,dx. \tag{4.19}$$

To use the hypothesis on the decay rate of $k(x,x)$ at infinity we suppose, as in the previous case, that there exist $\beta > 1$, $B > 0$ and $x_0 > 0$ such that $\int_L^\infty k(x,x)\,dx \le B/L^{\beta-1}$ whenever $L \ge x_0$. We recall Lemma 4.2 taking $q = 2m+\alpha$, $\delta = \infty$, and A replaced with AC in the case $m > 1$. As in the proof of 1.1 i), define a sequence of finite rank operators $SR^{N,L}S$ taking $L \equiv L(N) = N^\theta$, where in this case $\theta = \frac{2m+\alpha}{2m+\alpha+\beta}$. According to Lemma 4.2 we then derive from (4.18) and (4.19) that, for sufficiently large N,

$$T_k(N) \le \frac{A+B}{N^{\gamma-1}} \quad \text{if } m = 0$$

and

$$T_k(2N+1) \le \frac{AC+B}{N^{\gamma-1}} \quad \text{if } m \ge 1,$$

where $\gamma = \dfrac{(2m+\alpha+1)\beta}{2m+\alpha+\beta}$ for $m \ge 0$. These conditions together imply $\lambda_n = O(1/n^\gamma)$, completing the proof of statement 1.2 i).

To prove statements 2.1. i) and 2.2 i) we start by rewriting the integrand in (4.14) in a more convenient way. Suppose k_m is continuously differentiable with respect to x in $[0,+\infty[^2$. We set

$$g(u,v) = k_m^L(u,u) - k_m^L(u,v) - k_m^L(v,u) + k_m^L(v,v). \tag{4.20}$$

For u, v, $t \in [0,L]$ define $\varphi(t) = k_m^L(t,v) - k_m^L(t,u)$. Notice that φ is in $C^1([0,L])$ and that $g(u,v) = \varphi(v) - \varphi(u)$. Hence there exists t_0 between u and v such that

$$\begin{aligned} g(u,v) &= \varphi'(t_0)(v-u) \\ &= \left(\frac{\partial k_m^L}{\partial x}(t_0,v) - \frac{\partial k_m^L}{\partial x}(t_0,u)\right)(v-u) \\ &= \left(\frac{\partial k_m^L}{\partial x}(t_0,v) - \frac{\partial k_m^L}{\partial x}(t_0,t_0)\right)(v-u) + \left(\frac{\partial k_m^L}{\partial x}(t_0,t_0) - \frac{\partial k_m^L}{\partial x}(t_0,u)\right)(v-u) \end{aligned} \tag{4.21}$$

We now prove statement 2.1 i). Suppose $\dfrac{\partial k_m}{\partial x}$ is uniformly continuous with respect to y on the diagonal. Then, for every $\epsilon > 0$ there exists $\delta > 0$, independent of L, such that

$$\left|\frac{\partial k_m^L}{\partial x}(t_0,v) - \frac{\partial k_m^L}{\partial x}(t_0,t_0)\right| < \epsilon$$

for all t_0, $v \in [0,L]$ satisfying $|v - t_0| < \delta$. Then, for every positive L and every $N \in \mathbb{N}$ such that $L/N < \delta$, we derive from (4.21) that if $|v-u| < L/N$ we have $g(u,v) \le 2\epsilon L/N$, whence from (4.14)

$$T_{k_m^L}(N) \le \epsilon \frac{L^2}{N}. \tag{4.22}$$

Therefore, it follows from (4.10), (4.11) and (4.22) that

$$\begin{aligned} T_k(N) &\le T_{k^L}(N) + \int_L^\infty k(x,x)\,dx \\ &\le \epsilon \frac{L^2}{N} + \int_L^\infty k(x,x)\,dx, \end{aligned} \tag{4.23}$$

$$\begin{aligned} T_k(2N+1) &\le C\left(\frac{L}{N}\right)^{2m} T_{k_m^L}(N) + \int_L^\infty k(x,x)\,dx \\ &\le \epsilon\, C\left(\frac{L}{N}\right)^{2m+1} L + \int_L^\infty k(x,x)\,dx \end{aligned} \tag{4.24}$$

for some $C > 0$, N sufficiently large and $m \ge 1$.

Using the hypothesis on the decay rate of $k(x,x)$ we now proceed as in the proof of statement 1.1 i). Suppose there exist $\beta > 1$, $B > 0$, $x_0 > 0$ such that $\int_L^\infty k(x,x)\,dx \le B/L^{\beta-1}$ whenever $L \ge x_0$. Recall Lemma 4.2 with $q = 2m+1$, $A = \epsilon$ if $m = 0$, $A = \epsilon\, C$ if $m \ge 1$. Define a sequence of finite rank operators $SR^{N,L}S$ by setting $L = L(N) = N^\theta$, where $\theta = \frac{2m+1}{2m+1+\beta}$. According to Lemma 4.2 we then derive from (4.23) and (4.24) that, for sufficiently large N,

$$T_k(N) \le \frac{\epsilon + B}{N^{\gamma-1}} = \frac{D_1}{N^{\gamma-1}} \ \text{ if } m = 0$$

and

$$T_k(2N+1) \le \frac{\epsilon\, C + B}{N^{\gamma-1}} = \frac{D_2}{N^{\gamma-1}} \ \text{ if } m \ge 1$$

where $\gamma = \dfrac{(2m+2)\beta}{2m+1+\beta}$.

Finally observe that if $\int_L^\infty k(x,x)\,dx = O(1/L^{\beta-1})$ as $L \to +\infty$, the above conditions are satisfied for fixed D_1 and D_2, which implies $\lambda_n = O(1/n^\gamma)$. If $\int_L^\infty k(x,x)\,dx = o(1/L^{\beta-1})$ as $L \to +\infty$, the same conditions hold for arbitrary D_1 and D_2, implying $\lambda_n = o(1/n^\gamma)$. This finishes the proof of statement 2.1 i).

We now focus on the proof of statement 2.2 i). Suppose $\dfrac{\partial k_m}{\partial x}$ is continuous on $[0,+\infty[^2$ and Lip^α with respect to y on the diagonal, $\alpha \in]0,1]$. Choose $L \in \mathbb{R}^+$ and $N \in \mathbb{N}$. Then $|u-v| < L/N$ for all u, v in $[(n-1)\frac{L}{N}, n\frac{L}{N}]$, $n = 1,\dots,N$. Since $\dfrac{\partial k_m}{\partial x}$ is Lip^α with respect to y on the diagonal we derive from (4.20) and (4.21) that

$$g(u,v) \le A\left(\frac{L}{N}\right)^{1+\alpha}$$

and, from (4.14),

$$T_{k_m^L}(N) \le A\left(\frac{L}{N}\right)^{1+\alpha} L. \tag{4.25}$$

Therefore, according to (4.10), (4.12) and (4.25) we have

$$\begin{aligned} T_k(N) &\leq T_{k^L}(N) + \int_L^\infty k(x,x)\,dx \\ &\leq A\left(\frac{L}{N}\right)^{1+\alpha} L + \int_L^\infty k(x,x)\,dx, \end{aligned} \tag{4.26}$$

$$\begin{aligned} T_k(2N+1) &\leq C\left(\frac{L}{N}\right)^{2m} T_{k_m^L}(N) + \int_L^\infty k(x,x)\,dx \\ &\leq A\left(\frac{L}{N}\right)^{2m+\alpha+1} L + \int_L^\infty k(x,x)\,dx. \end{aligned} \tag{4.27}$$

We proceed as in the proof of the previous statements. Using the hipothesis on the decay rate of $k(x,x)$ at infinity and taking $q = 2m + \alpha + 1$ and replacing A by AC if $m \geq 1$, $\delta = \infty$ in Lemma 4.2, we define the sequence of finite rank operators $SR^{N,L}S$ by setting $L = L(N) = N^\theta$, where $\theta = \frac{2m+1+\alpha}{2m+1+\alpha+\beta}$. Then, from Lemma 4.2, (4.26) and (4.27), it follows that for sufficiently large N

$$T_k(N) \leq \frac{A+B}{N^{\gamma-1}} \text{ if } m = 0$$

and

$$T_k(2N+1) \leq \frac{AC+B}{N^{\gamma-1}} = \text{ if } m \geq 1$$

where $\gamma = \dfrac{(2m+1+\alpha)\beta}{2m+1+\alpha+\beta}$. Both conditions imply $\lambda_n = O(1/n^\gamma)$, completing the proof of statement 2.2 i).

We now prove statements 1.1 ii), 1.2 ii), 2.1 ii) and 2.2 ii). Notice that the hypothesis on the decay rate of $k(x,x)$, namely $\int_L^\infty k(x,x)\,dx = O(1/L^{\beta-1})$ as $L \to +\infty$ for all $\beta > 1$, is common to these four statements. We concentrate on the proof of 1.1 ii), which is based on the contents of statement 1.1 i). Suppose k_m is uniformly continuous with respect to y on the diagonal and that $\int_L^\infty k(x,x)\,dx = O(1/L^{\beta-1})$ as $L \to +\infty$ for all $\beta > 1$. Then, according to 1.1 i), $\lambda_n = O(1/n^\gamma)$ for all $\gamma < 2m+1$. This fact actually implies the stronger statement that $\lambda_n = o(1/n^\gamma)$ for every $\gamma < 2m+1$. In fact, if there were $\gamma_0 < 2m+1$ such that $n^{\gamma_0}\lambda_n \to C$ for some $C > 0$, then λ_n would not be $O(1/n^\gamma)$ for $\gamma_0 < \gamma < 2m+1$, contradicting the previous result. Thus under this hypothesis $\lambda_n = o(1/n^\gamma)$ for every $\gamma < 2m+1$, proving statement 1.1. ii).

The proofs of statements 1.2 ii), 2.1 ii) and 2.2 ii) are derived in the exact same way respectively from 1.2 i), 2.1 i) and 2.2 i), so details are omitted.

Finally, we prove statements 1.1 iii), 1.2 iii), 2.1 iii) and 2.2 iii). Notice that the hypothesis that $k(x,x)$ has compact support is common to these four statements. We concentrate on the proof of 1.1 iii). For $m = 0$ we have $\gamma = 1$ and the common assertion reduces to the already known fact that for a positive definite kernel k in $\mathcal{A}_0([0,+\infty[)$, $\lambda_n(k) = o(1/n)$. Let then $m \geq 1$. Choose $L > 0$ such that supp $k(x,x) \subset [0,L]$; with this choice we obviously have $\int_L^{+\infty} k(x,x)\,dx = 0$. Fix L and proceed as in the proof of 1.1 i), considering (4.11), (4.12), (4.13), (4.15), (4.16). From this last equation we derive in this case that

$$T_k(2N+1) \le \epsilon\, C \left(\frac{L}{N}\right)^{2m} L$$

for sufficiently large N. Since L and C are fixed and $\epsilon > 0$ is arbitrary, this implies $T_k(2N+1) = o(1/N^{2m})$ as $N \to +\infty$. For $m \ge 0$ it then follows that $\lambda_n = o(1/n^{2m+1})$, as asserted.

To prove 1.2 iii) we fix L as above and proceed as in the proof of 1.2 i), writing (4.10), (4.11), (4.12), (4.13), (4.17), (4.18) and (4.19). From the last two equations we derive in this case that

$$\begin{aligned} T_k(N) &\le A \left(\frac{L}{N}\right)^{\alpha} L, \\ T_k(2N+1) &\le A\,C \left(\frac{L}{N}\right)^{2m+\alpha} L \end{aligned}$$

for N sufficiently large and $m \ge 1$. Since A, C and L are fixed, it follows that $\lambda_n = O(1/n^{2m+\alpha+1})$ for $m \ge 0$, as asserted.

The proof of 2.1. iii) follows along the same lines. We fix L as above and proceed as in the proof of 2.1 i), considering (4.10), (4.11), (4.12),(4.14), (4.20), (4.21), (4.22), (4.23) and (4.24). The last two equations yield in this case

$$\begin{aligned} T_k(N) &\le \epsilon \frac{L^2}{N}, \\ T_k(2N+1) &\le \epsilon\, C \left(\frac{L}{N}\right)^{2m+1} L \end{aligned}$$

for N sufficiently large and $m \ge 1$. Since C and L are fixed and ϵ is arbitrary this implies that $\lambda_n = o(1/n^{2m+\alpha+1})$, for $m \ge 0$, as asserted.

Finally, to prove 2.2.iii) we fix L as above and proceed as in the proof of 2.2.i), writing (4.10), (4.11), (4.12),(4.14), (4.20), (4.21), (4.25), (4.26) and (4.27). The last two equations yield

$$\begin{aligned} T_k(N) &\le A \left(\frac{L}{N}\right)^{1+\alpha} L, \\ T_k(2N+1) &\le A\,C \left(\frac{L}{N}\right)^{2m+\alpha+1} L \end{aligned}$$

for N sufficiently large and $m \ge 1$. Since A, C and L are fixed, it follows that $\lambda_n = O(1/n^{2m+\alpha+2})$ for $m \ge 0$, as asserted. This finishes the proof. □

Corollary 4.11. *Suppose $k(x,y)$ is a positive definite kernel in $\mathcal{A}_0([0,+\infty[)$. Suppose furthermore that k is of class $\mathbb{C}^\infty([0,\infty[^2)$ and that k_m is uniformly continuous with respect to y on the diagonal for every $m \in \mathbb{N}$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be* *the sequence of eigenvalues of the integral operator with kernel k. If* $\beta > 1$ *and* $\int_L^\infty k(x,x)\,dx = O(1/L^{\beta-1})$ *as* $L \to +\infty$*, then* $\lambda_n = o\,(1/n^\gamma)$ *for all* $\gamma < \beta$*.*

Proof. According to statement 1.1 i) of Theorem 4.10, $\lambda_n = o\,(1/n^\gamma)$ for every γ of the form $\gamma = \dfrac{(2m+1)\beta}{2m+\beta}$ with $m \geq 0$. Since, for fixed β, $\lim\limits_{m\to\infty} \dfrac{(2m+1)\beta}{2m+\beta} = \beta$, an argument similar to the one used in the proof of statement 1.1 ii) implies that $\lambda_n = o\,(1/n^\gamma)$ for all $\gamma < \beta$. □

Remark 4.12. Observe that Corollary 4.11 cannot be improved, since its assertion is not true for $\gamma = \beta$ even in the weaker form $\lambda_n = O\left(1/n^\beta\right)$, as the counterexample $\lambda_n = \log n/n^\gamma$ shows. A similar observation can be applied to statements 1.1 ii), 1.2 ii), 2.1 ii) and 2.2 ii) of Theorem 4.10.

Remark 4.13. As a consequence of Remark 4.8 it is immediate to recognize that the assertion of Corollary 4.11 is valid, in particular, for positive definite kernels lying in class $\mathcal{A}_m$ for all $m \geq 0$.

Corollary 4.14. *Let* $k(x,y)$ *be a positive definite kernel in class* $\mathcal{A}_0([0,+\infty[)$*. Suppose k is of class* C^p *in* $[0,+\infty[^2$ *and that the partial derivatives up to order p are uniformly continuous with respect to y on the diagonal, and let* $\{\lambda_n\}_{n\in\mathbb{N}}$ *be the sequence of eigenvalues of the integral operator with kernel k. If* $\beta > 1$ *and* $\int_L^\infty k(x,x)\,dx = O(1/L^{\beta-1})$ *(resp.* $\int_L^\infty k(x,x)\,dx = o(1/L^{\beta-1})$*) as* $L \to +\infty$*, then* $\lambda_n = O\,(1/n^\gamma)$ *(resp.* $\lambda_n = o\,(1/n^\gamma)$ *) for* $\gamma = \dfrac{(p+1)\beta}{p+\beta}$*.*

Proof. For p even (resp. p odd) the hypotheses are easily seen to imply those of statement 1.1 i) (resp. 1.2.i)) of Theorem 4.10. Setting $p = 2m$ (resp. $p = 2m+1$) yields the result. □

Corollary 4.15. *Let* $k(x,y)$ *be a positive definite kernel in class* $\mathcal{A}_0([0,+\infty[)$*. Suppose k is of class* C^p *in* $[0,+\infty[^2$ *and that the partial derivatives up to order p are satisfy an* α*-Lipschitz condition with respect to y on the diagonal and let* $\{\lambda_n\}_{n\in\mathbb{N}}$ *be the sequence of eigenvalues of the integral operator with kernel k. If* $\beta > 1$ *and* $\int_L^\infty k(x,x)\,dx = O(1/L^{\beta-1})$ *as* $L \to +\infty$*, then* $\lambda_n = O\,(1/n^\gamma)$ *for* $\gamma = \dfrac{(p+1+\alpha)\beta}{p+\alpha+\beta}$.

Proof. For p even (resp. p odd) the hypotheses are easily seen to imply those of statements 2.1 i) and 2.2 i) in Theorem 4.10. Setting $p = 2m$ (resp. $p = 2m+1$) yields the result. □

Remark 4.16. A few interesting observations can be made from the study of limiting cases in the formulas for the exponent γ given by Theorem 4.10.

Suppose m and α are fixed and consider the parameter β which controls the rate of decay of the kernel k along the diagonal for $1 < \beta < +\infty$.

If $\beta \to +\infty$, the limiting values obtained for γ from formulas in items i) of statements 1.1, 1.2, 2.1 and 2.2 of Theorem 4.10 coincide with the exponent determining the bound for the decay rate of eigenvalues given in items ii) and iii) of the corresponding statements. These values ultimately reflect the fact that, in the case of kernels decaying rapidly on the diagonal and, in particular, in the case of kernels with compact support, the regularity assumed for k determines the upper bound for the decay rate exponent of eigenvalues.

The limiting case $m \to +\infty$ is the subject of corollary 4.11 and is, in a way, symmetric to the case above. It shows that operators with indefinitely differentiable kernels have eigenvalue distributions whose decay rate exponent bound is determined by the decay rate exponent of k along the diagonal.

For fixed m and β we finally observe that, in consonance with the interpretation of α as an index of intermediate differentiability, the limiting cases $\alpha = 0$ and $\alpha = 1$ in statements 1.2 and 2.2 produce the corresponding expected upper bounds for the decay rate exponent γ given by statements 1.1 and 2.1.

Remark 4.17. Some observations are relevant to the discussion of the hypothesis of Theorem 4.10.

We first note that, as indeed in the definition of class $\mathcal{A}_0$, the essential requirements on the behavior of k in the hypothesis of Theorem 4.10 may be restricted to the diagonal with no consequence on the proofs, a fact which is not apparent in the previous literature.

Secondly, observe that, in view of Remark 4.9 and conjugate symmetry of kernel $k(x, y)$, the hypothesis on uniform and Lipschitz continuity with respect to y on the diagonal assumed for partial derivatives of k could have been replaced with similar hypothesis with respect to x on the diagonal for conveniently chosen partial derivatives of k. Stronger conditions not specifying the variable x or y can naturally be imposed but with no advantage to the proofs.

Finally we note that uniform continuity requirements, which trivially derive from the $\mathcal{S}_n$ condition in the compact domain case, must be explicitly imposed in the case of unbounded domains.

Remark 4.18. Clearly, the last assertions (items iii)) of statements 1.1, 1.2, 2.1 and 2.2 in Theorem 4.10 bear a direct connection to the formally identical results known for positive trace class operators defined on a compact interval, namely those which apply to C^p or $C^{p+\alpha}$ kernels described in § 1. In fact, if transcribed to the case of kernels defined on compact domains, these assertions require somewhat weaker (yet sufficient) versions of the above referred conditions (see Remark 4.17).

The first part of the proof of Lemma 4.1 clarifies this connection by establishing the equivalence of the study of eigenvalues of the operator with compactly supported kernel k defined on $[0, \infty[^2$ and the operator whose kernel is the restriction of k to a square $[0, L]^2$ containing the support of k. The fact that the steps taken in our proofs of the referred assertions collapse into (a combination of) the proofs in [16], [8] and [7] can be seen as a consequence of this equivalence.

On the other hand, the same authors show that the results obtained in this case are optimal. This is done by explicitly constructing kernels verifying the required assumptions of differentiability and Lipschitz continuity whose eigenvalues attain the bounds for decay rate established by the theorems. The results in Theorem 4.10 are thus known to be optimal in the limit cases corresponding to compactly supported kernels. These facts and the contents of Proposition 4.3 strongly suggest that the remaining statements in Theorem 4.10 are also optimal.

Remark 4.19. Our results are stated and proved for the unbounded interval $I = [0, +\infty[$ for convenience only. They are valid, with the obvious rephrasing, for $L^2(I)$ integral operators K and respective positive definite kernels $k(x, y)$ for the other types of unbounded intervals in $\mathbb{R}$.

Of particular significance (see below) is the case where $I = \mathbb{R}$. In this case we are dealing with $L^2(\mathbb{R})$ positive definite kernels; class $\mathcal{A}_0(\mathbb{R})$ kernels are continuous in $\mathbb{R}^2$, with $k(x, x) \in L^1(\mathbb{R})$ and $k(x, x) \to 0$ as $|x| \to +\infty$. All the results and proofs carry through simply by replacing the required asymptotic behavior of $k(x, x)$ as $x \to +\infty$ by the corresponding requirement as $|x| \to +\infty$.

There is a close connection between positive definiteness of a continuous $L^2(\mathbb{R})$ kernel $k(x, y)$ and that of its Fourier transform $\hat{k}(\nu_1, \nu_2)$. More specifically, it is possible to show that if k is in class $\mathcal{A}_0(\mathbb{R})$ as defined in Remark 4.19, then its "rotated" Fourier transform $\tilde{k}(\nu_1, \nu_2) = \hat{k}(\nu_1, -\nu_2)$ is a positive definite kernel with the same eigenvalues λ_n as k and whose associated eigenfunctions are the Fourier transforms of the corresponding eigenfunctions of k. Moreover, if $k^{1/2}(x, x) \in L^1(\mathbb{R})$ then the $L^2(\mathbb{R})$ integral operator $\tilde{K}$ with kernel $\tilde{k}$ is trace class with the same trace as K; see [3] for details.

If k is in class $\mathcal{A}_0(\mathbb{R})$, a sufficient condition for $k^{1/2}(x, x) \in L^1(\mathbb{R})$ may be formulated in terms of the asymptotic behavior of $k(x, x)$ as $k(x, x) = O(1/x^\beta)$ for some $\beta > 2$. The following result is an immediate consequence of Corollaries 4.14 and 4.15 and Remark 4.4.

Corollary 4.20. *Suppose $k(x, y)$ is a positive definite kernel in class $\mathcal{A}_0(\mathbb{R})$ with $k(x, x) = O(1/x^\beta)$ as $|x| \to +\infty$ for some $\beta > 1$ and let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the sequence of eigenvalues of the integral operator with kernel k with associated eigenfunctions ϕ_n. Let $\hat{k}(\nu_1, \nu_2)$ be the double Fourier transform of $k(x, y)$, $\tilde{k}(\nu_1, \nu_2) = \hat{k}(\nu_1, -\nu_2)$ and $\hat{\phi}_n$ be the Fourier transform of ϕ_n. Then the following statements hold.*

(i) *$\tilde{k}$ is a positive definite kernel with $L^2(\mathbb{R})$ eigenfunction expansion*

$$\tilde{k}(\nu_1, \nu_2) = \sum_{n\geq 1} \lambda_n \, \hat{\phi}_n(\nu_1) \, \overline{\hat{\phi}_n(\nu_2)}. \tag{4.28}$$

(ii) *If k is of class $C^p(\mathbb{R})$ and the partial derivatives up to order p are uniformly continuous on the diagonal, then $\lambda_n = O(1/n^\gamma)$, where $\gamma = \dfrac{(p+1)\beta}{p+\beta}$.*

(iii) *If k is of class $C^p(\mathbb{R})$ and the partial derivatives up to order p satisfy an α-Lipschitz condition on the diagonal, then $\lambda_n = O(1/n^\gamma)$ with $\gamma = \dfrac{(p+1+\alpha)\beta}{p+\alpha+\beta}$.*

If $\beta > 2$ we have, in addition, that $\tilde{k}$ is in class $\mathcal{A}_0(\mathbb{R})$, the series (4.28) is absolutely and uniformly convergent and the operator $\tilde{K} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with kernel

$\tilde{k}$ is trace class with

$$tr\ \tilde{K} = tr\ K = \int_{-\infty}^{+\infty} k(x,x)\,dx = \int_{-\infty}^{+\infty} \tilde{k}(\nu,\nu)\,d\nu = \sum_{n\geq 1} \lambda_n.$$

Proof. The hypothesis imply that k is a Mercer-like kernel. Then statements i), ii) and iii) follow, in view of remark 4.4, from corollaries 4.14 and 4.15 and propositions 4.1 and 4.2 in [3]. The hypothesis that $k(x,x) = O(1/|x|^\beta)$ for some $\beta > 2$ implies, in addition, that $k^{1/2}(x,x) \in L^1(\mathbb{R})$, from which the last statement derives by direct application of theorem 4.4 in [3].

Remark 4.21. It is clearly seen that all conclusions in corollary 4.20 still hold if the hypothesis that $k : \mathbb{R}^2 \to \mathbb{C}$ be a continuous positive definite kernel satisfying $k(x,x) = O(1/|x|^\beta)$ as $|x| \to +\infty$ for some $\beta > 1$ is replaced with the assumption that k is a Mercer-like kernel defined on $\mathbb{R}^2$ satisfying $\int_L^\infty k(x,x)\,dx = O\left(1/L^{\beta-1}\right)$ for some $\beta > 1$ and if condition $\beta > 2$ is replaced with the hypothesis that $k^{1/2}(x,x) \in L^1(\mathbb{R})$ or, more generally, that $k(x,y) \in L^1(\mathbb{R}^2)$. Once again the version presented, although somewhat weaker, underlines how the behavior of the kernel on the diagonal controls events.

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