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Integral Equations and Operator Theory

The Essential Norm of Hankel Operators on the Weighted Bergman Spaces with Exponential Type Weights

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To my sons, Samy and Nassim.

Abstract. Let $AL^2_\phi(\mathbb{D})$ denote the closed subspace of $L^2(\mathbb{D}, e^{-2\phi}d\lambda)$ consisting of analytic functions in the unit disc $\mathbb{D}$. For certain class of subharmonic functions $\phi : \mathbb{D} \to \mathbb{R}$ and $f \in L^2(\mathbb{D})$, it is shown that the essential norm of Hankel operator $H_f : AL^2_\phi(\mathbb{D}) \to L^2_\phi(\mathbb{D})$ is comparable to the distance norm from H_f to compact Hankel operators.

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1. Introduction and statement of main result

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and $d\lambda$ be its Lebesgue measure. For a subharmonic function $\phi : \mathbb{D} \to \mathbb{R}$, let $L^2_\phi(\mathbb{D})$ be the Hilbert space of measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{L^2_\phi} := \Big(\int_{\mathbb{D}} |f|^2 e^{-2\phi} d\lambda\Big)^{\frac{1}{2}} < +\infty$$

Let $AL^2_\phi(\mathbb{D})$ be the closed subspace of $L^2_\phi(\mathbb{D})$ consisting of analytic functions. Let P_ϕ be the orthogonal projection of $L^2_\phi(\mathbb{D})$ onto $AL^2_\phi(\mathbb{D})$:

$$P_\phi g(z) := \int_{\mathbb{D}} K_\phi(z, w) g(w) e^{-2\phi(w)} d\lambda$$

where K_ϕ is the reproduced kernel of P_ϕ. Let $L^\infty_\phi(\mathbb{D})$ be the space of measurable functions f on $\mathbb{D}$ such that $e^{-\phi} f \in L^\infty(\mathbb{D})$ and $H^\infty_\phi(\mathbb{D})$ be the subspace of $L^\infty(\mathbb{D})$

consisting of analytic functions. Given $f \in L^2(\mathbb{D})$, it is possible to define, for some weights ϕ, the Hankel operator H_f on $H_\phi^\infty(\mathbb{D})$ by

$$H_f g = fg - P_\phi(fg)$$

For certain subharmonic functions ϕ on $\mathbb{D}$, already defined on $\mathbb{C}$ by Oleinik[14] and Oleinik-Perel'man [15], Lin and Rochberg [7] find necessary and sufficient conditions involving f such that the Hankel operator H_f is bounded or compact on $AL_\phi^2(\mathbb{D})$. Our aim is to estimate the essential norm of H_f :

$$\|H_f\|_e := \inf\{\|H_f - K\| : K \text{ compact operator}\}$$

The first estimate was established by Hartman-Adamyan-Arov-Krein for the Hardy space (see [2]).

Theorem 1.1. *Let $f \in L^\infty(\partial\mathbb{D})$ and H_f be the Hankel operator defined on the Hardy space $H^2(\mathbb{D})$ by $H_f g = fg - S(fg)$ where S is the Szegö projection on $L^2(\partial\mathbb{D})$ onto $H^2(\mathbb{D})$. Then*

$$\begin{aligned}\|H_f\|_e &= \inf\{\|H_f - K\| : K \textit{ compact Hankel operator}\}\\ &= dist_{L^\infty(\partial\mathbb{D})}(f, C(\partial\mathbb{D}) + H^\infty(\mathbb{D}))\end{aligned}$$

Later Lin and Rochberg [6] proved similar results for the Hankel operator on the weighted Bergman space $AL^2(\mathbb{D}, (1-|z|^2)^s d\lambda), s > -1$.

Theorem 1.2. *Let $f \in L^2(\mathbb{D})$. Then*
(1) $\|H_f\|_e \sim \inf\{\|H_f - K\| : K$ *is compact Hankel operator*$\}$
(2) $\|H_f\|_e \sim dist_{BDA}(f, VDA)$, *where* $dist_{BDA}(f, VDA) = \inf_{h\in VDA} \|f-h\|_{BDA}$.

Similar results for the Hankel operator on the Bergman space of strongly pseudoconvex domains in $\mathbb{C}^n$ were proved in [1].

The subject of this paper is to prove the corresponding version for Hankel operator on the Bergman space $AL_\phi^2(\mathbb{D})$ for some class of subharmonic functions ϕ on $\mathbb{D}$ introduced by Oleinik [14] and Oleinik-Perel'man [15].

Definition 1.3. For $\phi \in C^2(\mathbb{D})$ and $\Delta\phi > 0$ put $\tau(z) := (\Delta\phi(z))^{-1/2}$ where Δ is the Laplace operator. We call $\phi \in \mathcal{D}$ if the following conditions hold.
(1) $\exists\, C_1 > 0$ such that $|\tau(z) - \tau(w)| \le C_1|z-w| \quad \forall z, w \in \mathbb{D}$
(2) $\exists\, C_2 > 0$ such that $\tau(z) \le C_2(1-|z|) \quad \forall z \in \mathbb{D}$
(3) $\exists\, 0 < C_3 < 1$ and $a > 0$ such that $\tau(z) \le \tau(w) + C_3|z-w|$ for $|z-w| > a\tau(w)$.

Some examples of functions in class $\mathcal{D}$ are as follows :
(i) $\phi_1(z) = -\frac{A}{2}\log(1-|z|^2), A > 2$. The corresponding weight $e^{-2\phi_1}$ is the standard weight $(1-|z|^2)^A$ for $A > 2$.
(ii) $\phi_2(z) = \frac{1}{2}\big(-A\log(1-|z|^2) + \frac{B}{(1-|z|^2)}\big), A \ge 0, B > 0$. The corresponding weight $e^{-2\phi_2}$ is the exponential weight $(1-|z|^2)^A e^{-B/(1-|z|^2)}, A \ge 0, B > 0$.
(iii) $\phi_1 + h$ and $\phi_2 + h$ where ϕ_1 and ϕ_2 are as in (i) and (ii) respectively and $h \in C^2(\mathbb{D})$ can be any harmonic function on $\mathbb{D}$. Let $\alpha \in]0, \frac{1}{16}\min(C_1^{-1}, C_2^{-1})[$

fixed. For $z \in \mathbb{D}$ and f measurable on $\mathbb{D}$, let

$$F_\alpha(z) := \inf\Big\{\Big(\frac{1}{|D(\alpha\tau(z))|)}\int_{D(\alpha\tau(z))}|f-k|^2 d\lambda\Big)^{1/2} : k \text{ analytic on } D(\alpha\tau(z))\Big\}$$

where $D(\alpha\tau(z)) := \{w \in \mathbb{D}, |w-z| \leq \alpha\tau(z)\}$ and $|D(\alpha\tau(z))| = \lambda(D(\alpha\tau(z)))$. The function space $\mathrm{BDA}_\alpha(\mathbb{D})$, bounded distance to analytic, is defined by

$$\mathrm{BDA}_\alpha(\mathbb{D}) = \{f : \sup_{z\in\mathbb{D}} F_\alpha(z) < +\infty\}$$

The function space $\mathrm{VDA}_\alpha(\mathbb{D})$, vanishing distance to analytic, is defined by

$$\mathrm{VDA}_\alpha(\mathbb{D}) = \{f : \limsup_{|z|\to 1} F_\alpha(z) = 0\}$$

In theorem 1.2 the function spaces BDA and VDA are defined with respect to hyperbolic discs $D(z)$ with fixed raduis. For $f \in \mathrm{BDA}_\alpha(\mathbb{D})$ let $\|f\|_{BDA_\alpha} := \sup_{z\in\mathbb{D}} F_\alpha(z)$. The main result is the following theorem.

Theorem 1.4 (Main Theorem). *Let $f \in L^2(\mathbb{D})$ and $\phi \in \mathcal{D}$. Suppose that $H^\infty_\phi(\mathbb{D})$ is dense on $AL^2_\phi(\mathbb{D})$. Let H_f defined on $H^\infty_\phi(\mathbb{D})$ by $H_f g = fg - P_\phi(fg)$. Then*

(1) $\|H_f\|_e \sim \inf\{\|H_f - K\| : K$ *compact Hankel operator*$\}$,

(2) $\|H_f\|_e \sim \inf_{h\in VDA_\alpha} \|f-h\|_{BDA_\alpha}$ *for some* $\alpha \in]0, \frac{1}{16}\min(C_1^{-1}, C_2^{-1})[$.

On Bergman space with weight $\phi_s(z) = \frac{s}{2}\log(1-|z|^2)$ $(s > 2)$, the explicit formula of the reproduced kernel or its local behaviour play a crucial role in the estimates . Generally in $AL^2_\phi(\mathbb{D})$ the reproduced kernel $K_\phi(z,w)$ is not explicit. Using Hörmander's estimates for $\bar\partial$ operator on $L^2_\phi(\mathbb{D})$ [4], Lin and Rochberg [7] constructed an extremal function $k_w(z) \in AL^2_\phi(\mathbb{D})$ which play role of $K_\phi(z,w)$ in local estimates and have the same behaviour as $K_\phi(z,w)$ at the boundary. In our case, we will modify this construction to obtain a family $(k_w)_{w\in\partial\mathbb{D}}$ for which $k_w(z)$ converge to zero at each point $z \in \mathbb{D}$ as w goes to $\partial\mathbb{D}$. Instead of the usual Hörmander's estimates for $\bar\partial$ operator we use the L^2 estimates for $\bar\partial \circ \mu$ for some function μ, introduced by Ohsawa-Takegoshi [9] and generalized by Ohsawa [10,11,12,13]. In the sequel the letter C design a constant which may change values in estimates but independently of main variables.

2. Preliminary results

Let μ be a locally finite nonnegative Borel measure on the unit disk $\mathbb{D}$, $d\lambda$ be the area measure on $\mathbb{D}$ and $\phi\mathbb{D} \to \mathbb{R}$ be subharmonic function. Let $L^2_{\phi,\mu}(\mathbb{D})$ be the space of all measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{\phi,\mu} = \Big(\int_{\mathbb{D}} |f|^2 e^{-2\phi} d\mu\Big)^{1/2} < \infty$$

Let $L^2_\phi(\mathbb{D})$ denote $L^2_{\phi,d\lambda}(\mathbb{D})$ and $AL^2_\phi(\mathbb{D})$ be the closed subspace of $L^2_\phi(\mathbb{D})$ consisting of analytic functions.

Definition 2.1. μ is called a Carleson measure on $AL^2_\phi(\mathbb{D})$ if the inclusion map from $AL^2_\phi(\mathbb{D})$ to $L^2_{\phi,\mu}(\mathbb{D})$ is a bounded linear map.

Definition 2.2. μ is called a vanishing Carleson measure on $AL^2_\phi(\mathbb{D})$ if the inclusion map from $AL^2_\phi(\mathbb{D})$ to $L^2_{\phi,\mu}(\mathbb{D})$ is a compact linear map.

Necessary and sufficient conditions for which μ is a Carleson measure or a vanishing Carleson measure are given by the following theorems.

Theorem 2.3. *Let $\phi \in \mathcal{D}$. Then μ is a Carleson measure if and only if there are $C > 0$ and $\alpha \in]0, \frac{1}{16}\min(C_1^{-1}, C_2^{-1})[$ such that*

$$\sup_{w\in\mathbb{D}} \frac{1}{\tau(w)^2}\mu\{z \in \mathbb{D} : |z-w| \leq \alpha\tau(w)\} \leq C$$

Proof. See Theorem 2.4 of [7]. □

Theorem 2.4. *Let $\phi \in \mathcal{D}$. Then μ is a vanishing Carleson measure if and only if there exists a constant $\alpha \in]0, \frac{1}{16}\min(C_1^{-1}, C_2^{-1})[$ such that*

$$\limsup_{|w|\to 1} \frac{1}{\tau(w)^2}\mu\{z \in \mathbb{D} : |z-w| \leq \alpha\tau(w)\} = 0$$

Proof. See Theorem 2.9 in [7]. □

Lemma 2.5. *Let $\phi \in \mathcal{D}$. There exists a sequence $(z_j) \subset \mathbb{D}$ such that*

(1) $z_j \notin D(\alpha\tau(z_k)), j \neq k,$
(2) $\cup_j D(\alpha\tau(z_j)) = \mathbb{D},$
(3) $\tilde{D}(\alpha\tau(z_j) \subset D(3\alpha\tau(z_j)),$ *where*

$$\tilde{D}(\alpha\tau(z_j) = \cup_{z\in D(\alpha\tau(z_j))} D(\alpha\tau(z)), \quad j = 1, 2, \ldots$$

(4) $\{D(3\alpha\tau(z_j))\}$ *is a covering of $\mathbb{D}$ with multiplicity N.*

Proof. See Lemma of covering in [14]. □

Lemma 2.6. *Let Ω be a domain in complex plane. Let ϕ be a real valued function in $C^2(\Omega)$ such that $\Delta\phi > 0$. Then for every measurable function f on Ω satisfying the condition*

$$\int_\Omega \frac{|f|^2}{\Delta\phi} e^{-2\phi} d\lambda < \infty$$

there exists $u \in L^2_\phi(\Omega)$ such that $\bar{\partial}u = f$ and

$$\int_\Omega |u|^2 e^{-2\phi} d\lambda \leq \int_\Omega \frac{|f|^2}{\Delta\phi} e^{-2\phi} d\lambda.$$

Proof. See Theorem 2.2.1$'$ in [4]. □

The key Lemma for estimates of the essential norm of H_f is the following.

Lemma 2.7 (Key Lemma). *Let $\phi \in \mathcal{D}$ and suppose that $H^\infty_\phi(\mathbb{D})$ is dense in $AL^2_\phi(\mathbb{D})$. Then for each $w \in \mathbb{D}$, there exists an analytic function $k_w(z) \in H^\infty_\phi(\mathbb{D})$ satisfying the following conditions.*

(1) $\|k_w\|_{L^2_\phi} \leq C$,
(2) $k_w(z) \to 0$ *as* $|w| \to 1$ *for every* $z \in \mathbb{D}$,
(3) *there exists* $\gamma_0 \in]0, 1/8[$ *such that*

$$|k_w(z)|^2 e^{-2\phi(z)} \geq \frac{C}{\tau(w)^2}, \quad \text{for } |z-w| \leq \gamma_0 \beta \tau(w),$$

where $\beta = \min(C_1^{-1}, C_2^{-1})/2$ *and the constants C's in* (1) *and* (3) *are independent of* w.

The existence of analytic functions satisfying (1) and (3) of lemma 2.7 was proved in [14] for $\mathbb{C}$ and in [7] for $\mathbb{D}$ using L^2 estimates of $\bar\partial$ operator (lemma 2.4). The key point for the proof of lemma 2.5 is the replacement, in the $\bar\partial$-equation, of $\bar\partial$ by $\bar\partial$ composed with a scalar function on the right. These are the famous L^2-estimates of Ohsawa-Takegoshi [9] for $\bar\partial \circ \mu$ operator. They introduced a way of producing the curvature term without the contribution of the metric. This is impossible by the usual L^2-estimates of Hörmander for $\bar\partial$ operator. This fact is explained by Siu in [16]. Here, we state the L^2 existence theorem for $\bar\partial \circ \mu$ operator on $\Omega \subset\subset \mathbb{C}$.

Proposition 2.8. *Let* Ψ *and* η *are* C^2 *functions on* Ω, *equipped with the usual metric, such that* $\eta > 0$ *and bounded on* Ω. *Suppose that*

$$\eta \Delta \Psi - \Delta \eta - \eta^{-1} |d\eta|^2 \geq c(z)$$

everywhere on Ω *for some positive measurable function* $c(z)$ *on* Ω. *Then for every function* $f \in L^2_\Psi(\Omega)$ *there exists* $g \in L^2_\Psi(\Omega)$ *such that* $\bar\partial(\sqrt{\eta} g) = f$ *and*

$$\int_\Omega |g|^2 e^{-2\Psi} d\lambda \leq \int_\Omega \frac{|f|^2}{c(z)} e^{-2\Psi} d\lambda$$

provided that the right integral is finite.

Proof. See Theorem 1.7 in [11] or Proposition 3.1 in [3]. □

For the proof of key lemma 2.7 we need the following two lemmas.

Lemma 2.9. *Let* $\phi \in \mathcal{D}$. *Let* $\beta = \min(C_1^{-1}, C_2^{-1})/2$ *where* C_1 *and* C_2 *are the constants of* ϕ *in Definition* 1.3. *For any fixed* $w \in \mathbb{D}$, *let* $\rho(w) = \beta\tau(w)$ *and* Φ *be a function analytic in* $|z-w| \leq \rho(w)$ *and continuous to the boundary such that* $u := Re\Phi = \phi$ *on the circle* $|z-w| = \rho(w)$. *Then* $0 \leq u(z) - \phi(z) \leq \beta^2$ *for* $|z-w| \leq \rho(w)$.

Proof. See Lemma 1 and lemma 2 of [15]. □

Lemma 2.10. *Let $\phi \in \mathcal{D}$. For z and w in $\mathbb{D}$ let $\eta_w(z) = \delta^2\rho(w)^2 + |z-w|^2$ where $\rho(w) = \beta\tau(w)$. There exist $\delta > 0$ and $C > 0$ such that if $\phi_w(z) := \phi(z) - \frac{5}{4}\log \eta_w(z)$ then for all z, w in $\mathbb{D}$ we have*

$$\eta_w(z)\Delta\phi_w(z) - \Delta\eta_w(z) - \frac{|d\eta_w(z)|^2}{\eta_w(z)} \geq C.$$

Proof. An easy computation shows that $\frac{5\eta_w}{4}\Delta_c \log \eta_w = \frac{5\delta^2\rho(w)^2}{4\eta_w}$, $\Delta_c \eta_w = 1$ and $\frac{|\partial\eta_w|^2}{\eta_w} = \frac{|z-w|^2}{\eta_w}$ where $\Delta_c = \frac{\partial^2}{\partial z \partial \bar{z}}$ is the complex Laplacian. Hence

$$\begin{aligned}\frac{5\eta_w}{4}\Delta_c \log \eta_w + \Delta_c \eta_w + \frac{|\partial\eta_w|^2}{\eta_w} &= \frac{5\delta^2\rho(w)^2}{4\eta_w} + 1 + \frac{|z-w|^2}{\eta_w} \\ &= \frac{\delta^2\rho(w)^2}{4\eta_w} + 2 \leq \frac{9}{4}\end{aligned}$$

Let $b > \max(a, 9\beta^{-1}(1-C_3^2)^{-1/2})$ and $\delta > 3(1+bC_1\beta)\beta^{-1}$. If $|z-w| > b\rho(w)$,

$$\begin{aligned}|z-w|^2\Delta_c\phi(z) &= \frac{|z-w|^2}{4}\Delta\phi(z) \\ &\geq \frac{|z-w|^2}{4(\tau(w) + C_3|z-w|)^2} \\ &\geq \frac{1}{4(b^{-1}\beta^{-1} + C_3)^2} \\ &\geq \frac{1}{4(9b^{-2}\beta^{-2} + \frac{C_3^2}{9})} > \frac{9}{4}\end{aligned}$$

hence

$$\begin{aligned}\eta_w\Delta_c\phi_w - \Delta_c\eta_w - \frac{|\partial\eta_w|^2}{\eta_w} &\geq \eta_w\Delta_c\phi - \frac{9}{4} \\ &\geq |z-w|^2\Delta_c\phi - \frac{9}{4} \\ &\geq \frac{1}{4(9b^{-2}\beta^{-2} + \frac{C_3^2}{9})} - \frac{9}{4} > 0\end{aligned}$$

Now if $|z-w| \leq b\rho(w)$,

$$\begin{aligned}\delta^2\rho(w)^2\Delta_c\phi(z) &= \frac{\delta^2\rho(w)^2}{4}\Delta\phi(z) \\ &\geq \frac{\delta^2\rho(w)^2}{4(\tau(w) + C_1|z-w|)^2} \\ &\geq \frac{\delta^2\beta^2}{4(1+bC_1\beta)^2} > \frac{9}{4}\end{aligned}$$

hence

$$\begin{aligned}\eta_w \Delta_c \phi_w - \Delta_c \eta_w - \frac{|\partial \eta_w|^2}{\eta_w} &\geq \eta_w \Delta_c \phi - \frac{9}{4} \\ &\geq \delta^2 \rho(w)^2 \Delta_c \phi - \frac{9}{4} \\ &\geq \frac{\delta^2 \beta^2}{4(1 + bC_1\beta)^2} - \frac{9}{4} > 0\end{aligned}$$

Since $\Delta = 4\Delta_c$ and $|d\eta| = 2|\partial \eta_w|$, the lemma follows. □

2.1. Proof of the Key lemma

Proof. For any fixed $w \in \mathbb{D}$, let the function $\gamma(z) = \theta(|z-w|/\rho(w))$, where $\theta \in C^\infty(\mathbb{R}), 0 \leq \theta(t) \leq 1, \theta(t) = 1$ for $0 \leq t \leq 1/2, \theta(t) = 0$ for $t \geq 1$ and $|\theta^{'}(t)| \leq 3$. Let us consider the function the function

$$\mathcal{F}_w(z) = \gamma(z) e^{\Phi(z)} - (z-w)\sqrt{\eta_w(z)} g_w(z)$$

where $\eta_w(z) = \delta^2 \rho(w)^2 + |z-w|^2$ and Φ as in Lemma 2.9. The function g_w is chosen in such manner that $\bar{\partial} \mathcal{F}_w = 0$ on $\mathbb{D}$. For g_w we obtain the $\bar{\partial} \circ \sqrt{\eta_w}$ equation

$$\frac{\partial(\sqrt{\eta_w} g_w)}{\partial \bar{z}}(z) = (z-w)^{-1} \frac{\partial \gamma(z)}{\partial \bar{z}} e^{\Phi(z)} \equiv h(z) \tag{2.1}$$

It's clear that h is a smooth function with support in $I_w := \{\rho(w)/2 \leq |z-w| \leq \rho(w)\}$. Let δ be as in Lemma as in Lemma 2.10 and for $z \in \mathbb{D}$ let

$$\phi_w(z) = \phi(z) - \frac{5}{4} \log \eta_w(z)$$

Then by lemma 2.10 there exists constant $C > 0$ such that

$$\eta_w(z) \Delta \phi_w(z) - \Delta \eta_w(z) - \frac{|d\eta_w(z)|^2}{\eta_w(z)} \geq C \tag{2.2}$$

Now applying Proposition 2.8 for ϕ_w, we obtain a solution g_w of (2.1) such that

$$\begin{aligned}\int_{\mathbb{D}} |g_w(z)|^2 e^{-2\phi_w(z)} d\lambda(z) &\leq \frac{1}{C} \int_{\mathbb{D}} |h(z)|^2 e^{-2\phi_w(z)} d\lambda(z) \\ &\leq C \int_{I_w} |(z-w)^{-1} \frac{\partial \gamma}{\partial \bar{z}}(z) e^{\Phi(z)}|^2 e^{-2\phi_w(z)} d\lambda(z) \\ &\leq C \int_{I_w} \frac{1}{\rho(w)^4} \rho(w)^5 e^{2u - 2\phi} d\lambda \\ &\leq C\rho(w)^3, \quad \text{since} \quad u - \phi \leq \beta^2.\end{aligned}$$

Now let us bound the L^2_ϕ norm of $\mathcal{F}_w$.

$$\begin{aligned}
\|\mathcal{F}_w\|^2_{L^2_\phi} &= \int_{\mathbb{D}} |\mathcal{F}_w(z)|^2 e^{-2\phi(z)} \\
&\le 2\Big(\int_{\mathbb{D}} \gamma^2(z) e^{2u(z)-2\phi(z)} d\lambda + \int_{\mathbb{D}} |(z-w)|^2 \eta_w(z) |g_w(z)|^2 e^{-2\phi(z)} d\lambda\Big) \\
&\le 2\Big(\int_{|z-w|\le\rho(w)} \gamma^2(z) e^{2u(z)-2\phi(z)} d\lambda + \int_{\mathbb{D}} \eta_w^2(z) |g_w(z)|^2 e^{-2\phi(z)} d\lambda\Big) \\
&\le C\rho(w)^2 + 2\int_{\mathbb{D}} \eta_w^{-1/2}(z) |g_w(z)|^2 e^{-2\phi_w(z)} d\lambda \\
&\le C\rho(w)^2 + \frac{2}{\delta\rho(w)} \int_{\mathbb{D}} |g_w(z)|^2 e^{-2\phi_w(z)} d\lambda \\
&\le C\rho(w)^2.
\end{aligned}$$

Now let us bound $|\mathcal{F}_w(z)|$ below on $|z-w| \le \gamma\rho(w)$ for some $\gamma > 0$ which will be chosen later. If $|z-w| \le \rho(w)/2$, then $\gamma(z) \equiv 1$, therefore

$$|\mathcal{F}_w(z)|^2 = e^{2u(z)} |1 - (z-w)\sqrt{\eta_w} g_w(z) e^{-\Phi(z)}|^2$$

Consequently, it is sufficient to bound $|(z-w)\sqrt{\eta_w} g_w(z) e^{-\Phi(z)}|$ above with an upper bound less than 1. Since $h \equiv 0$ for $|z-w| \le \rho(w)/2$, the function $\eta_w^{1/2} g_w e^{-\Phi}$ is analytic in $|z-w| \le \rho(w)/2$. Put $I_z := \{\xi \in \mathbb{D} : 2\rho(w)/8 \le |\xi - z| \le 3\rho(w)/8\}$. For $\gamma_0 \in]0, \frac{1}{8}[$ and $|z-w| \le \gamma_0\rho(w)$, apply the Cauchy formula to $\eta_w^{1/2} g_w e^{-\Phi}$ on the circle $|\xi - z| = t$ and then integrate both side with respect to t from $2\rho(w)/8$ to $3\rho(w)/8$, we obtain

$$\begin{aligned}
|\eta_w^{1/2} g_w e^{-\Phi}|(z) &\le \frac{8}{\rho(w)} . \frac{1}{2\pi} \int_{I_z} \Big| \frac{\eta_w^{1/2}(\xi) g_w(\xi) e^{-\Phi(\xi)}}{\xi - z} \Big| d\lambda(\xi) \\
&\le \frac{8}{\rho(w)} . 1 \frac{2}{\pi} . \frac{8}{2\rho(w)} \int_{I_z} |\eta_w^{1/2}(\xi) g_w(\xi) e^{-\Phi(z)}| d\lambda(\xi) \\
&= \frac{16}{\pi\rho(w)^2} \int_{I_z} \frac{1}{|\xi - w|} |(\xi - w)\eta_w^{1/2}(\xi) g_w(\xi)| e^{-u(\xi)} d\lambda(\xi) \\
&\le \frac{16}{\pi\rho(w)^2} . \frac{8}{\rho(w)} \int_{I_z} |(\xi - w)\eta_w^{1/2}(\xi) g_w(\xi)| e^{-u(\xi)} d\lambda(\xi) \\
&\le \frac{48}{\sqrt{\pi}\rho(w)^2} \Big(\int_{I_z} |(\xi - w)\eta_w^{1/2}(\xi) g_w(\xi)|^2 e^{-2u(\xi)} d\lambda(\xi)\Big)^{1/2} \\
&\le \frac{48}{\sqrt{\pi}\rho(w)^2} \Big(\int_{I_z} \eta_w^2(\xi) |g_w(\xi)|^2 e^{-2u(\xi)} d\lambda(\xi)\Big)^{1/2}.
\end{aligned}$$

Thus

$$\begin{aligned}|\eta_w^{1/2} g_w e^{-\Phi}|(z) &\le \frac{48}{\sqrt{\pi}\rho(w)^2}\Big(\int_{I_z} \eta_w^2(\xi)|g_w(\xi)|^2 e^{-2\phi(\xi)} d\lambda(\xi)\Big)^{1/2} \text{ (since } \phi - u \le 0)\\ &\le \frac{48}{\sqrt{\pi}\rho(w)^2}.\sqrt{C}\rho(w)\\ &= \frac{48\sqrt{C}}{\sqrt{\pi}}.\frac{1}{\rho(w)}.\end{aligned}$$

Now choose $\gamma_0 \in]0, 1/8[$ such that $\frac{48\sqrt{C}\gamma_0}{\sqrt{\pi}} < 1$. Then for $|z - w| \le \gamma_0\rho(w)$ we have

$$|(z-w)\sqrt{\eta_w(z)}g_w(z)e^{-\Phi(z)}| \le \gamma_0\rho(w)\frac{48\sqrt{C}}{\sqrt{\pi}}.\frac{1}{\rho(w)} < 1.$$

Therefore for $|z - w| \le \gamma_0\rho(w) = \gamma_0\beta\tau(w)$, we have

$$|\mathcal{F}_w(z)|^2 e^{-2\phi(z)} \ge Ce^{2u(z)-2\phi(z)} \ge C$$

since $u - \phi \ge 0$ by Lemma 2.9.

For $w \in \mathbb{D}$ fixed and $z \in \mathbb{D}$ let

$$f_w(z) = \frac{\mathcal{F}_w(z)}{\rho(w)}.$$

Then f_w satisfies the properties
(i) $\|f_w\|_{L^2_\phi} \le C$,
(ii) $\exists\ \gamma_0 \in]0, 1/8[$ such that $|f_w(z)|^2 e^{-2\phi(z)} \ge C/\rho(w)^2$ for $|z - w| \le \gamma_0\beta\tau(w)$.

Now we show that $f_w(z) \to 0$ as $|w| \to 1$ at each $z \in \mathbb{D}$. Since

$$f_w(z) = \frac{1}{\rho(w)}\theta(\frac{|z-w|}{\rho(w)})e^{\Phi(z)} - (z-w)\sqrt{\delta^2\rho(w)^2 + |z-w|^2}\frac{g_w(z)}{\rho(w)}$$

and $|z-w| > \rho(w)$ if $|w| \sim 1$, then $\theta(\frac{|z-w|}{\rho(w)}) = 0$. Hence we need only to show that

$$\lim_{|w|\to 1} \frac{g_w(z)}{\rho(w)} = 0$$

Since $\sqrt{\eta_w} g_w$ is analytic near z and η_w has an uniform upper bound in z and w, by mean value inequality

$$\begin{aligned}
|\eta_w^{1/2}(z) g_w(z)| &\le \frac{C}{\tau(z)^2} \int_{D(z,\tau(z)/4C_1)} |\eta_w^{1/2}(\xi) g(\xi)| d\lambda(\xi) \\
&\le \frac{C}{\tau(z)^2} \Big(\int_{D(z,\tau(z)/4C_1)} |g_w(\xi)|^2 e^{-2\phi_w(\xi)} d\lambda(\xi) \Big)^{1/2} \times \\
&\quad \Big(\int_{D(z,\tau(z)/4C_1)} \frac{e^{2\phi(\xi)}}{\eta_w(\xi)^{5/2}} d\lambda(\xi) \Big)^{1/2} \\
&\le \frac{C}{\tau(z)^2} \rho(w) \sqrt{\rho(w)} \Big(\int_{D(z,\tau(z)/4C_1)} \frac{e^{2\phi(\xi)}}{|\xi - w|^5} d\lambda(\xi) \Big)^{1/2} \\
&\le \frac{C}{\tau(z)^2} \rho(w) \sqrt{\rho(w)} \tau(z)^{-3/2} \sup_{\xi \in D(z,\tau(z)/4C_1)} e^{2\phi(\xi)}
\end{aligned}$$

since the two inequalities $|\xi - w| \ge ||\xi - z| - |z - w||$ and $C_1|z-w| \ge |\tau(z) - \tau(w)| \ge \tau(z)/2$ if $|w| \sim 1$ show $|\xi - w| \ge \tau(z)/4C_1$. Since $\eta_w^{1/2}(z) \ge |z - w|$ for all $w, z \in \mathbb{D}$ we conclude that $\tau(w)^{-1} g_w(z) \to 0$ as $|w| \to 1$. Thus $f_w(z) \to 0$ as $|w| \to 1$.

Finally we show that we can choose f_w in $H^\infty_\phi(\mathbb{D})$. Since $H^\infty_\phi(\mathbb{D})$ is dense in $AL^2_\phi(\mathbb{D})$, for any fixed w there exists a sequence $f_w^n \in H^\infty_\phi(\mathbb{D})$ such that $\|f_w^n - f_w\|_{L^2_\phi} \to 0$ as $n \to \infty$. Since ϕ is continuous in $\mathbb{D}$ by the mean value theorem and the Hölder inequality, $f_w^n(z) \to f_w(z)$ uniformly on compact subset of $\mathbb{D}$. Then there exists a constant $N_1(w) > 0$ such that when $n > N_1(w)$ we have

$$|e^{-\phi(z)} f_w^n(z) - e^{-\phi(z)} f_w(z)| < \frac{1}{2} \Big(\frac{C}{\tau(w)^2}\Big)^{1/2} \quad \text{for} |z - w| \le \gamma_0 \beta \tau(w),$$

where C and γ_0 are constants as in (ii). Hence

$$\begin{aligned}
|e^{-\phi(z)} f_w^n(z)| &\ge |e^{-\phi(z)} f_w(z)| - \frac{1}{2} \Big(\frac{C}{\tau(w)^2}\Big)^{1/2} \\
&= \frac{1}{2} \Big(\frac{C}{\tau(w)^2}\Big)^{1/2} \quad \text{for} n > N_1(w) \text{ and } |z - w| \le \gamma_0 \beta \tau(w).
\end{aligned}$$

Also, since $\|f_w^n - f_w\|_{L^2_\phi} \to 0$ as $n \to \infty$, there exists another constant $N_2(w) > 0$ such that when $n > N_2(w)$ we have

$$\|f_w^n - f_w\|_{L^2_\phi} \le \tau(w)$$

Let $N(w) = \max(N_1(w), N_2(w)) + 1$. We define $k_w(z)$ by

$$k_w(z) = f_w^{N(w)}(z)$$

Then $k_w \in H^\infty_\phi(\mathbb{D})$ and satisfies the condition (1) and (3) of Lemma 2.5. By the mean value theorem and Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |f_w^{N(w)}(z) - f_w(z)| &\leq \frac{C}{\tau(z)^2} \int_{D(z,\tau(z)/4)} |f_w^{N(w)}(\xi) - f_w(\xi)| d\lambda(\xi) \\ &\leq C(z)\tau(w) \sup_{\xi \in D(z,\tau(z)/4)} e^{\phi(\xi)} \end{aligned}$$

Hence $|k_w(z)| \leq B(z)\tau(w) + |f_w(z)|$ and it's follows that $\lim_{|w|\to 1} k_w(z) = 0$. Thus k_w satisfies the condition (2) of Lemma 2.7. This completes the proof. □

3. The essential norm of Hankel operator on the weighted Bergman space

The following theorem is our first result about essential norm of H_f.

Theorem 3.1. *Let $\phi \in \mathbb{D}$ and suppose that $H^\infty_\phi(\mathbb{D})$ is dense on $AL^2_\phi(\mathbb{D})$. Let $f \in L^2(\mathbb{D})$ and H_f defined on $H^\infty_\phi(\mathbb{D})$ by $H_f g = fg - P_\phi(fg)$. The following quantities are equivalent.*

(1) $\|H_f\|_e$,

(2) $\limsup_{|w|\to 1} \|H_f(k_w)\|_{L^2_\phi}$ *where $(k_w)_{w\in\mathbb{D}}$ as in Lemma 2.5,*

(3) $\limsup_{|w|\to 1} F_\alpha(w)$ *for some $\alpha \in]0, \frac{1}{16}\min(C_1^{-1}, C_2^{-1})[$ where*

$$F_\alpha(z) := \inf\Big\{\Big(\frac{1}{|D(\alpha\tau(z))|)} \int_{D(\alpha\tau(z))} |f-h|^2 d\lambda\Big)^{1/2} : h \text{ analytic on } D(\alpha\tau(z))\Big\},$$

(4) $\inf_{f=f_1+f_2, f_2\in C^1(\mathbb{D})} \big[\limsup_{|w|\to 1}(G_\alpha(w) + (\Delta\phi(w))^{-1/2}|\bar\partial f_2(w)|)\big]$ *for some* $\alpha \in]0, \frac{1}{16}\min(C_1^{-1}, C_2^{-1})[$ *where* $G_\alpha(w) = \Big(\frac{1}{|D(\alpha\tau(w))|}\int_{D(\alpha\tau(w))} |f_1|^2 d\lambda\Big)^{1/2}$.

Proof. The proof of theorem follows the cycle

$$\begin{aligned} &(a) : (1) \geq C(2), \quad (b) : (2) \geq C(3) \\ &(c) : (3) \geq C(4), \quad (d) : (4) \geq C(1). \end{aligned}$$

Proof of (a). Let $(k_w)_{w\in\mathbb{D}}$ be the sequence of functions of lemma 2.7 and $K : AL^2_\phi(\mathbb{D}) \to L^2_\phi(\mathbb{D})$ be a compact operator. Then

$$\begin{aligned} \|H_f(k_w)\|_{L^2_\phi} - \|K(k_w)\|_{L^2_\phi} &\leq \|(H_f - K)(k_w)\|_{L^2_\phi} \\ &\leq \|k_w\|_{L^2_\phi}\|H_f - K\| \\ &\leq C\|H_f - K\| \end{aligned}$$

Since $k_w \to 0$ weakly as $|w| \to 1$ and K is a compact operator, we have

$$\limsup_{|w|\to 1} \|H_f(k_w)\|_{L^2_\phi} \leq C\|H_f\|_e.$$

Proof of (b). By lemma 2.7, the functions k_w and $\frac{1}{k_w}$ are analytic on $D(\alpha\tau(w))$. This implies

$$\begin{aligned}
\|H_f(k_w)\|^2_{L^2_\phi} &= \int_{\mathbb{D}} |fk_w - P_\phi(fk_w)|^2 e^{-2\phi} d\lambda \\
&\geq \int_{D(\alpha\tau(w))} |f - \frac{P_\phi(fk_w)}{k_w}|^2 |k_w|^2 e^{-2\phi} d\lambda \\
&\geq \frac{C}{\tau(w)^2} \int_{D(\alpha\tau(w))} |f - \frac{P_\phi(fk_w)}{k_w}|^2 d\lambda \\
&\geq CF^2_\alpha(w)
\end{aligned}$$

Thus

$$\limsup_{|w|\to 1} \|H_f(k_w)\|_{L^2_\phi} \geq C \limsup_{|w|\to 1} F_\alpha(w)$$

Proof of (c). By the proof of (2) $\Rightarrow$ (3) in theorem 4.1 in [7](see also [8]), there is a decomposition $f = f_1 + f_2$ of f with $f_2 \in C^1(\mathbb{D})$ such that for $w \in \mathbb{D}$:

$$\begin{aligned}
G_\alpha(w) = \frac{1}{|D(\alpha\tau(w))|} \int_{D(\alpha\tau(w))} |f_1|^2 d\lambda &\leq C \sup\{F_\alpha(z)^2 : z \in D(3\alpha\tau(w))\} \\
\frac{|\bar\partial f_2(w)|^2}{\Delta\phi(w)} &\leq C \sup\{F_\alpha(z)^2 : z \in D(3\alpha\tau(w))\}
\end{aligned}$$

Hence

$$\limsup_{|w|\to 1} \Big[G_\alpha(w) + (\Delta\phi(w))^{-1/2}|\bar\partial f_2(w)|\Big] \leq C \limsup_{|w|\to 1} F_\alpha(w)$$

Proof of (d). Let $f = f_1 + f_2$ be a decomposition of f with $f_2 \in C^1(\mathbb{D})$. Then

$$\|H_f\|_e = \|H_{f_1+f_2}\|_e \leq \|H_{f_1}\|_e + \|H_{f_2}\|_e$$

So we need to prove :

$$\begin{aligned}
\|H_{f_1}\|_e &\leq C \limsup_{|w|\to 1} G_\alpha(w) \\
\|H_{f_2}\|_e &\leq C \limsup_{|w|\to 1} (\Delta\phi(w))^{-1/2}|\bar\partial f_2(w)|
\end{aligned}$$

We may suppose that $\limsup_{|w|\to 1} G_\alpha(w)$ and $\limsup_{|w|\to 1}(\Delta\phi(w))^{-1/2}|\bar\partial f_2(w)|$ are finite. Since $G_\alpha(w)$ and $(\Delta\phi(w))^{-1/2}|\bar\partial f_2(w)|$ are continuous on $\mathbb{D}$, then $G_\alpha(w)$ and $(\Delta\phi(w))^{-1/2}|\bar\partial f_2(w)|$ are bounded on $\mathbb{D}$. Let $r \in]0,1[$ and χ_r be the characteristic function of $D_r = \{z \in \mathbb{D} : |z| < r\}$.We consider the operator of multiplication $M_{\chi_r f_1}$ from $AL^2_\phi(\mathbb{D})$ to $L^2_\phi(\mathbb{D})$ defined by $M_{\chi_r f_1}(g) = \chi_r f_1 g$. Since $\chi_r f_1$ has compact support and G_α is bounded, $M_{\chi_r f_1}$ is compact : let $(g_n) \subset AL^2_\phi$ be a sequence tending weakly to zero. Then $\|g_n\|_{L^2_\phi}$ is bounded and g_n converge uniformly to zero on compact sets in $\mathbb{D}$. Then $\forall\epsilon > 0, \exists N > 0$ such that $e^{-\phi(z)}|g_n(z)| < \epsilon, \forall z \in D_r$

and $n > N$:

$$\begin{aligned}\|M_{\chi_r f_1} g_n\|^2_{L^2_\phi} &= \int_{\mathbb{D}} |\chi_r f_1|^2 |g_n^2 e^{-2\phi} d\lambda \\ &\le \epsilon^2 \int_{D_r} |f_1|^2 d\lambda \\ &\le \epsilon^2 \sum_{j=0}^{N_r} \int_{D_r \cap D(3\alpha\tau(z_j)} |f_1|^2 d\lambda \quad ((z_j)\text{as in lemma 2.5}) \\ &\le \epsilon^2 \sum_{j=0}^{N_r} |D(3\alpha\tau(z_j)| \Big(\frac{1}{|D(3\alpha\tau(z_j))|} \int_{D(3\alpha\tau(z_j))} |f_1|^2 d\lambda \Big) \\ &\le \epsilon^2 \alpha^2 C_2^2 N_r \sup_{w\in\mathbb{D}} G_\alpha^2(w)\end{aligned}$$

Hence $\lim_{n\to\infty} \|M_{\chi_r f_1} g_n\|^2_{L^2_\phi} = 0$. Since $H_{\chi_r f_1} = (I - P_\phi) M_{\chi_r f_1}$, the operator $H_{\chi_r f_1}$ is compact and

$$\|H_{f_1}\|_e \le \|H_{(1-\chi_r)f_1}\|$$

So we need only to give an upper bound of $\|H_{(1-\chi_r)f_1}\|^2$. Let $g \in AL^2_\phi$, then

$$\begin{aligned}\|H_{(1-\chi_r)f_1} g\|^2_{L^2_\phi} &= \|(I-P) M_{(1-\chi_r f_1} g\|^2_{L^2_\phi} \\ &\le \int_{\mathbb{D}} |g|^2 |(1-\chi_r) f_1|^2 e^{-2\phi} d\lambda \\ &\le C \sup_{w\in\mathbb{D}} \Big(\frac{1}{|D(\alpha\tau(w))|} \int_{D(\alpha\tau(w))} |(1-\chi_r) f_1|^2 d\lambda \Big) \int_{\mathbb{D}} |g|^2 e^{-2\phi} d\lambda\end{aligned}$$

since $|(1-\chi_r)f_1|^2 d\lambda$ is a Carleson measure on AL^2_ϕ ($|f_1|^2 d\lambda$ is a Carleson measure by Theorem 2.3). Since $\tau(w) \le C_2(1-|w|)$ and $\alpha C_2 < 1$, we have

$$\begin{aligned}\|H_{(1-\chi_r)f_1}\|^2 &\le C \sup_{\frac{r-\alpha C_2}{1-\alpha C_2} < |w| < 1} \Big(\frac{1}{|D(\alpha\tau(w))|} \int_{D(\alpha\tau(w))} |f_1|^2 d\lambda \Big) \\ &\le C \sup_{s(r) < |w| < 1} (G_\alpha(w))^2\end{aligned}$$

and then

$$\|H_{f_1}\|_e \le C \limsup_{|w|\to 1} G_\alpha(w)$$

For $f_2 \in C^1(\mathbb{D})$ and $g \in AL^2_\phi$: $H_{f_2} g = f_2 g - P_\phi(f_2 g)$ is the solution of $\bar\partial u = g \bar\partial f_2$ with minimal L^2_ϕ norm. We can write

$$\begin{aligned}H_{f_2} g &= \int_{\mathbb{D}} S_\phi(z,w) g(w) \bar\partial f_2(w) e^{-2\phi(w)} d\lambda \\ &= \int_{\mathbb{D}} S_\phi(z,w) \chi_r g \bar\partial f_2 e^{-2\phi} d\lambda + \int_{\mathbb{D}} S_\phi(z,w)(1-\chi_r) g \bar\partial f_2 e^{-2\phi} d\lambda \\ &= T_1 g + T_2 g\end{aligned}$$

where $S_\phi(z,w)$ is the reproduced kernel of L^2_ϕ minimal solution. The operator T_1 is compact : let $(g_n) \subset AL^2_\phi$ be a sequence tending weakly to zero. Then $\|g_n\|_{L^2_\phi}$ is bounded and g_n converge uniformly to zero on compact sets in $\mathbb{D}$. Then $\forall\epsilon > 0, \exists N > 0$ such that $e^{-\phi(z)}|g_n(z)| < \epsilon$ for $|z| \leq r$. Then

$$\|T_1 g_n\|^2_{L^2_\phi} = \Big\| \int_{\mathbb{D}} S_\phi(z,w)\chi_r g\bar{\partial} f_2 e^{-2\phi} d\lambda \Big\|^2_{L^2_\phi}$$

Since $\int_{\mathbb{D}} S_\phi(z,w)\chi_r g_n \bar{\partial} f_2 e^{-2\phi} d\lambda$ is the minimal solution of $\bar{\partial} u = \chi_r g_n \bar{\partial} f_2$, by lemma 2.6, there exists a solution U_n of $\bar{\partial} u = \chi_r g_n \bar{\partial} f_2$ such that

$$\int_{\mathbb{D}} |U_n|^2 e^{-2\phi} d\lambda \leq C \int_{D_r} |g_n|^2 |\bar{\partial} f_2|^2 / \Delta\phi e^{-2\phi} d\lambda$$

This implies

$$\|T_1 g_n\|_{L2_\phi} \leq \|U_n\|_{L^2_\phi} \leq C\epsilon \sup_{w\in\mathbb{D}} \frac{|\bar{\partial} f_2|}{\Delta\phi(w)^{1/2}}$$

Hence $\lim_{n\to\infty} \|T_1 g_n\|_{L^2_\phi} = 0$ i.e T_1 is compact. By definition of essential norm we have

$$\|H_{f_2}\|_e = \|T_2\|_e \leq \|T_2\|$$

Since

$$T_2 g = \int_{\mathbb{D}} S_\phi(z,w)(1-\chi_r) g \bar{\partial} f_2 e^{-2\phi} d\lambda$$

is the minimal solution of $\bar{\partial} u = (1-\chi_r)\bar{\partial} f_2$, by lemma 2.6 there exists a solution V of $\bar{\partial} u = (1-\chi_r)\bar{\partial} f_2$ such that

$$\int_{\mathbb{D}} |V|^2 e^{-2\phi} d\lambda \leq C \int_{\mathbb{D}} (1-\chi_r)^2 |g|^2 \frac{|\bar{\partial} f_2|^2}{\Delta\phi} e^{-2\phi} d\lambda$$

Hence

$$\begin{aligned} \|T_2 g\|^2_{L^2_\phi} &\leq C \int_{\mathbb{D}} (1-\chi_r)^2 |g|^2 \frac{|\bar{\partial} f_2|^2}{\Delta\phi} e^{-2\phi} d\lambda \\ &\leq C \sup_{r<|w|<1} \frac{|\bar{\partial} f_2|^2}{\Delta\phi(w)} \|g\|^2_{L^2_\phi} \end{aligned}$$

Thus

$$\|H_{f_2}\|_e \leq C \limsup_{|w|\to 1} \frac{|\bar{\partial} f_2|}{\Delta\phi(w)^{1/2}}$$

Combining above inequalities, we have

$$\|H_f\|_e \leq \|H_{f_1}\|_e + \|H_{f_2}\|_e \leq C \limsup_{|w|\to 1} \Big[G_\alpha(w) + (\Delta\phi(w))^{-1/2} |\bar{\partial} f_2(w)| \Big]. \qquad \square$$

As a consequence of Theorem 3.1 is the following compactness criteria of Hankel operator on $AL^2_\phi(\mathbb{D})$ [7].

Theorem 3.2. *Let $\phi \in \mathcal{D}$ and $f \in L^2(\mathbb{D})$. Suppose that $H_\phi^\infty(\mathbb{D})$ is dense in $AL_\phi^2(\mathbb{D})$. Then the following properties are equivalent.*

(a) *H_f is (extend to) a compact operator from AL_ϕ^2 to L_ϕ^2.*
(b) *$F_\alpha(w) \to 0$ as $|w| \to 1$ for some $\alpha \in]0, \frac{1}{16}\min(C_1^{-1}, C_2^{-1})[$.*
(c) *There is a decomposition $f = f_1 + f_2$ with $f_2 \in C^1(\mathbb{D})$ such that $G_\alpha(w) \to 0$ and $(\Delta\phi(w))^{-1/2}|\bar{\partial} f_2(w)| \to 0$ as $|w| \to 1$ for some $\alpha \in]0, \frac{1}{16}\min(C_1^{-1}, C_2^{-1})[$.*

When we have Theorem 3.1, we can go further to get the corresponding Lin-Rochberg theorem (stated in section 1) for the Hankel operator on $AL_\phi^2(\mathbb{D})$.

Theorem 3.3. *Let $f \in L^2(\mathbb{D})$ and $\phi \in \mathcal{D}$. Suppose that $H_\phi^\infty(\mathbb{D})$ is dense on $AL_\phi^2(\mathbb{D})$. Let H_f defined on $H_\phi^\infty(\mathbb{D})$ by $H_f g = fg - P_\phi(fg)$. Then*

(1) *$\|H_f\|_{ess} \sim \inf\{\|H_f - K\| : K$ compact Hankel operator$\}$,*
(2) *$\|H_f\|_{ess} \sim d_\alpha(f, VLDA_\alpha(\mathbb{D}))$ for some $\alpha \in]0, \frac{1}{16}\min(C_1^{-1}, C_2^{-1})[$.*

Proof. (1) By the definition of essential norm, it is obvious that

$$\|H_f\|_{ess} \leq \inf\{\|H_f - K\| : K \text{ compact Hankel operator}\}$$

So we need to prove

$$\inf\{\|H_f - K\| : K \text{ compact Hankel operator}\} \leq C\|H_f\|_{ess}$$

By theorem 3.1, we have

$$\|H_f\|_{ess} \sim \inf_{f=f_1+f_2, f_2 \in C(\mathbb{D})} (\limsup_{|w|\to 1} G_\alpha(w) + \limsup_{|w|\to 1} (\Delta\phi(w))^{-1/2}|\bar{\partial} f_2(w)|)$$

where $G_\alpha(w) = \left(\frac{1}{|D(\alpha\tau(w))|}\int_{D(\alpha\tau(w))}|f_1|^2 d\lambda(w)\right)^{1/2}$. So we only need to prove :

$$\begin{aligned} &\inf\{\|H_f - K\| : K \text{compact Hankel operator}\} \\ &\quad \leq C \inf_{f=f_1+f_2, f_2 \in C(\mathbb{D})} (\limsup_{|w|\to 1} G_\alpha(w) + \limsup_{|w|\to 1} (\Delta\phi(w))^{-1/2}|\bar{\partial} f_2(w)|) \end{aligned} \quad (3.1)$$

We will prove (3.1) by proving that there is a constant C such that for any decomposition $f = f_1 + f_2$ with $f_2 \in C^1(\mathbb{D})$ the following is true

$$\begin{aligned} &\inf\{\|H_f - K\| : K \text{ compact Hankel operator}\} \\ &\quad \leq C(\limsup_{|w|\to 1} G_\alpha(w) + \limsup_{|w|\to 1} (\Delta\phi(w))^{-1/2}|\bar{\partial} f_2(w)|) \end{aligned}$$

To prove (3.1), as before we may assume that $\limsup_{|w|\to 1} G_\alpha(w) < +\infty$ and $\limsup_{|w|\to 1}(\Delta\phi(w))^{-1/2}|\bar{\partial} f_2(w)| < +\infty$ and this implies that $\sup_{\mathbb{D}} G_\alpha < +\infty$ and $\sup_{\mathbb{D}}(\Delta\phi)^{-1/2}|\bar{\partial} f_2| < +\infty$ since both G_α and $(\Delta\phi)^{-1/2}|\bar{\partial} f_2|$ are continuous in $\mathbb{D}$.

For $f_1, \forall r \in]0, 1[$, let χ_r be the characteristic function of the set $\{z : |z| \leq r\}$. Since $\chi_r f_1$ has compact support and $\sup_{\mathbb{D}} G_\alpha$ is finite, the operator $H_{\chi_r f_1}$ is compact (see the proof of Theorem 3.1 (d)). Now for $f_2 \in C^1(\mathbb{D}), \forall r \in]0, 1[$ let $\sigma_r \in C_0^\infty(\mathbb{D})$

such that $\sigma_r = 1$ on $\{z : |z| \leq r\}$. Let $\psi_r \in C^1(\overline{\mathbb{D}})$ such that $\bar\partial\psi_r = \sigma_r\bar\partial f_2$. Since $\psi_r \in C^1(\overline{\mathbb{D}})$ the operator H_{ψ_r} is compact (see Zhu's book [17]). Hence

$$\begin{aligned}\inf\{\|H_f - K\| : K \text{ compact Hankel operator}\} &\leq \|H_{f_1+f_2} - H_{\chi_r f_1+\psi_r}\| \\ &\leq \|H_{f_1} - H_{\chi_r f_1}\| + \|H_{f_2} - H_{\Psi_r}\|\end{aligned}$$

By the proof of Theorem 3.1(d), we have

$$\|H_{f_1} - H_{\chi_r f_1}\| \leq \sup_{s(r)<|w|<1} G_\alpha(w),$$

where $s(r) \to 1$ as $r \to 1$. Also the operator $H_{f_2-\psi_r}$ is bounded and

$$\begin{aligned}\|H_{f_2} - H_{\psi_r}\| &\leq C \sup_{w\in\mathbb{D}}(\Delta\phi(w))^{-1/2}|\bar\partial(f_2 - \psi_r)(w)| \\ &\leq C \sup_{w\in\mathbb{D}}(\Delta\phi(w))^{-1/2}(1 - \sigma_r(w))|\bar\partial f_2| \\ &\leq C \sup_{r<|w|<1}(\Delta\phi(w))^{-1/2}|\bar\partial f_2|\end{aligned}$$

Thus for any decomposition $f = f_1 + f_2$ with $f_2 \in C^1(\mathbb{D})$ and $\sup_{\mathbb{D}} G_\alpha < \infty$ we have

$$\begin{aligned}&\inf\{\|H_f - K\| : K \text{ compact Hankel operator}\} \\ &\quad\leq C(\limsup_{|w|\to1} G_\alpha(w) + \limsup_{|w|\to1}(\Delta\phi(w))^{-1/2}|\bar\partial f_2(w)|)\end{aligned}$$

Hence

$$\inf\{\|H_f - K\| : K \text{ compact Hankel operator}\} \leq C\|H_f\|_{ess}$$

This completes the proof. □

4. Remarks

The Theorem 3.1 can be extended to any bounded domain Ω with C^1 boundary in the complex plane. In the definition 1.3 of $\mathcal{D}$ we replace the condition (2) $\tau(z) \leq C_2(1 - |z|)$ by (2) $\tau(z) \leq C_2 d(z, \mathbb{C} \setminus \Omega)$. For $w \in \Omega$, let $D(\alpha\tau(w)) = \{z \in \Omega : |z - w| \leq \alpha\tau(w)\}$ and BDA_α and VDA_α are the corresponding spaces. The method employed in the proof of Theorem 2.2 works without change to prove the corresponding theorem for $AL^2_\phi(\Omega)$: the covering Lemma 2.5 is valid in this case [14] and all Lemmas 2.9, 2.10 are true for Ω as stated for the unit disc. Hence the key Lemma 2.5 is true in this case. Following [8] (theorem 5) and [7] (theorem 3.1) we have

Theorem 4.1. *Let Ω be a bounded domain in the complex plane with C^1 boundary. Let $\phi \in \mathcal{D}$. Let P_ϕ denote the projection from $L^2_\phi(\Omega)$ to $AL^2_\phi(\Omega)$. Suppose that $H^\infty_\phi(\Omega)$ is dense in $AL^2_\phi(\Omega)$. Let $f \in L^2(\Omega)$ and let H_f be defined on $H^\infty_\phi(\Omega)$ by $H_f g = fg - P_\phi(fg)$. Then the following are equivalent:*

(1) *H_f is bounded in the L^2_ϕ norm.*

(2) *The function* $F_\alpha(w)$ *defined by*

$$F_\alpha(w)^2 = \inf\left\{\frac{1}{|D(\alpha\tau(w))|}\int_{D(\alpha\tau(w))}|f-h|^2 d\lambda : h \text{ analytic in } D(\alpha\tau(w))\right\}$$

is bounded for some $\alpha \in]0, \min(C_1^{-1}, C_2^{-1})/16[$.

(3) f *admits a decomposition* $f = f_1 + f_2$ *where* $f_2 \in C^1(\Omega)$ *and satisfies*

$$\frac{\bar\partial f_2}{(\Delta\phi)^{1/2}} \in L^\infty(\Omega),$$

while f_1 *satisfies the following condition : the function* $G_\alpha(w)$ *defined by*

$$G_\alpha(w)^2 = \frac{1}{|D(\alpha\tau(w))|}\int_{D(\alpha\tau(w))}|f_1|^2 d\lambda$$

is bounded for some $\alpha \in]0, \min(C_1^{-1}, C_2^{-1})/16[$.

For the essential norm of H_f we have

Theorem 4.2. *Let* Ω *be a bounded domain in the complex plane with* C^1 *boundary. Let* $\phi \in \mathcal{D}$. *Let* P_ϕ *denote the projection from* $L^2_\phi(\Omega)$ *to* $AL^2_\phi(\Omega)$. *Suppose that* $H^\infty_\phi(\Omega)$ *is dense in* $AL^2_\phi(\Omega)$. *Let* $f \in L^2(\Omega)$ *and let* H_f *be defined on* $H^\infty_\phi(\Omega)$ *by* $H_f g = fg - P_\phi(fg)$. *Then*

(1) $\|H_f\|_{ess} \sim \inf\{\|H_f - K\| : K \text{ compact Hankel operator}\}$,

(2) $\|H_f\|_e \sim \inf_{h\in VDA_\alpha}\|f-h\|_{BDA_\alpha}$ *for some* $\alpha \in]0, \frac{1}{16}\min(C_1^{-1}, C_2^{-1})[$.

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