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Integral Equations and Operator Theory

On Two Variable Jordan Block (II)

Rongwei Yang

Abstract. On the Hardy space over the bidisk $H^2(D^2)$, the Toeplitz operators T_{z_1} and T_{z_2} are unilateral shifts of infinite multiplicity. A closed subspace M is called a submodule if it is invariant for both T_{z_1} and T_{z_2}. The two variable Jordan block $(S_1,\ S_2)$ is the compression of the pair $(T_{z_1},\ T_{z_2})$ to the quotient $H^2(D^2)\ominus M$. This paper defines and studies its defect operators. A number of examples are given, and the Hilbert-Schmidtness is proved with good generality. Applications include an extension of a Douglas-Foias uniqueness theorem to general domains, and a study of the essential Taylor spectrum of the pair $(S_1,\ S_2)$. The paper also estabishes a clean numerical estimate for the commutator $[S_1^*,\ S_2]$ by some spectral data of S_1 or S_2. The newly-discovered core operator plays a key role in this study.

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1. Introduction

Let $H^2(D)$ be the Hardy space over the unit disk. Multiplication by coordinate function w on $H^2(D)$ is the unilateral shift, and its invariant subspace is of the form $\theta H^2(D)$, where θ is an inner function. The compression $S(\theta)$ of the unilateral shift to the quotient space $H^2(D)\ominus\theta H^2(D)$ is called a Jordan block. To be precise,

$$S(\theta)f = P_\theta wf, \quad f \in H^2(D) \ominus \theta H^2(D),$$

where P_θ is the projection from $H^2(D)$ onto $H^2(D) \ominus \theta H^2(D)$. Study of the unilateral shift and the Jordan block is a solid foundation for the development of non-selfadjoint operator theory (cf. [3], [20]). A similar foundation for multivariable operator theory, on the other hand, has yet to be discovered. One natural approach in this quest is to study multivariable analogues of the unilateral shift and the Jordan block, and this usually demands different ideas and techniques for different settings (cf. [4]). This paper studies a two variable analogue of the Jordan block in the setting of the Hardy space over the bidisk $H^2(D^2)$.

On the Hardy space $H^2(D^2)$ with coordinate function $z = (z_1, z_2)$, the Toeplitz operators T_{z_1} and T_{z_2} (or simply denoted by T_1 and T_2, respectively,) are unilateral shifts of infinite multiplicity. A closed subspace $M \subset H^2(D^2)$ is called a *submodule* over the bidisk algebra $A(D^2)$ if it is invariant under multiplication by functions in $A(D^2)$, i.e., $pf \in M$ for every $f \in M$ and every $p \in A(D^2)$. Clearly, M is a submodule if and only if it is invariant for both T_1 and T_2.

Given a submodule M, its orthogonal complement $N = H^2(D^2) \ominus M$ also has a module structure over $A(D^2)$, with module action defined by $p \longrightarrow S_p$, where $p \in A(D^2)$ and

$$S_p f = P_N p f, \quad f \in N.$$

Here P_E stands for the orthogonal projection from $H^2(D^2)$ onto a subspace E. The pair $(S_{z_1},\ S_{z_2})$ (denoted by $(S_1,\ S_2)$ for simplicity) is a natural two variable analogue of the classical Jordan block, and for this reason we shall call it a two variable Jordan block in $H^2(D^2)$. The two variable Jordan block has very rich structure and, like the one variable case, reveals much information about M. Some related early work can be found in [8], [9], [15], [27] and the references therein. This paper is organized as follows.

Section 2 is preparation.

Section 3 defines and studies the *defect operators* for $(S_1,\ S_2)$. The defect operators are very useful associates of $(S_1,\ S_2)$. For example, they can be used to extend Douglas and Foias's uniqueness theorem in [8]. The main goal of this section is to reveal connections between the defect operators and the *core operator*.

Properties of the defect operators are used in Section 4 to study the essential Taylor spectrum of $(S_1,\ S_2)$. We will show that if the core operator is compact then $\sigma_e(S_1,\ S_2)$ is a subset of ∂D^2.

In Section 5, based on some spectral data of S_1 or S_2, a clean estimate of the Hilbert-Schmidt norm of $[S_1^*,\ S_2]$ will be established. This work in some way is reminiscent of the spirit of the Berger-Shaw theorem.

2. Preliminaries

We first fix a few notations and introduce some key elements in the study. Throughout the paper we let $R(A)$ denote the range of a bounded linear operator A and let $|A|_2$ denote its Hilbert-Schmidt norm, i.e., $|A|_2^2 = trA^*A$. It is well-known (cf. [11]) that for bounded linear operators X and Y,

$$|XAY|_2 \leq \|X\|\|Y\||A|_2. \tag{2.1}$$

In this paper, we let $\sigma_c(A)$ denote the collection of complex numbers w such that $A - wI$ either does not have closed range or has infinite dimensional kernel. Clearly, $\sigma_c(A)$ is a subset of the essential spectrum $\sigma_e(A)$.

We let $K(\lambda, z)$ denote the reproducing kernel $\frac{1}{(1-\overline{\lambda_1}z_1)(1-\overline{\lambda_2}z_2)}$ for $H^2(D^2)$, and we let $k_\lambda(z) = \frac{\sqrt{1-|\lambda_1|^2}\sqrt{1-|\lambda_2|^2}}{(1-\overline{\lambda_1}z_1)(1-\overline{\lambda_2}z_2)}$ be the normalized reproducing kernel for $H^2(D^2)$. The reproducing kernel for a closed subspace M is denoted by $K^M(\lambda, z)$. For a submodule M, the *core function* $G^M(\lambda, z)$ is

$$G^M(\lambda, z) := \frac{K^M(\lambda, z)}{K(\lambda, z)} = (1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)K^M(\lambda, z),$$

and the *core operator* on $H^2(D^2)$ is defined by

$$C^M(f)(z) := \int_{T^2} G^M(\lambda, z)f(\lambda)dm(\lambda), \quad z \in D^2,$$

where $dm(\lambda)$ is the normalized Lebesgue measure on the torus T^2. For simplicity, we neglect the "M" in our writing of G^M and C^M when no confusion shall result. It is known that $C = 0$ on N, and on M, C is a bounded self-adjoint operator with $\|C\| = 1$ (cf. [14]).

For a submodule M, we let $(R_1,\ R_2)$ denote the restriction of $(T_{z_1},\ T_{z_2})$ to M. So it is clear that $(R_1,\ R_2)$ is a pair of commuting isometries. One relation between the core operator and the pair $(R_1,\ R_2)$ is displayed in the formula (cf. [14])

$$C = 1 - R_1R_1^* - R_2R_2^* + R_1R_2R_1^*R_2^*. \tag{2.2}$$

And it follows from (2.2) (cf. [28]) that C^2 is unitarily equivalent to

$$\begin{pmatrix} [R_1^*,\ R_1][R_2^*,\ R_2][R_1^*,\ R_1] & 0 \\ 0 & [R_2^*,\ R_1]^*[R_2^*,\ R_1] \end{pmatrix}. \tag{2.3}$$

Moreover, if C is Hilbert-Schmidt, then (cf. [24])

$$|[R_1^*,\ R_1][R_2^*,\ R_2]|_2^2 = |[R_2^*,\ R_1]|_2^2 + 1.$$

For simplicity, we let Σ_0 stand for $|[R_1^*,\ R_1][R_2^*,\ R_2]|_2^2$. By the preceding facts,

$$\Sigma_0 \geq 1, \quad \text{and} \quad |C|_2^2 = 2\Sigma_0 - 1 \geq 1.$$

It is worth pointing out that Σ_0 is an invariant for M under unitary equivalence of submodules.

A submodule M is said to be *Hilbert-Schmidt* if C^M is Hilbert-Schmidt. It follows from (2.3) and Theorem 2.3 of [24] that if either $\sigma_c(S_1)$ or $\sigma_c(S_2)$ is not the full closed unit disk then M is Hilbert-Schmidt. We will revisit this fact in Section 5. Almost all known submodules are Hilbert-Schmidt, for example, if $I = (p_1, p_2, \ldots, p_n)$ is an ideal of the polynomial ring $\mathbf{C}[z]$, then its closure in $H^2(D^2)$ (for which we denote by $[p_1, p_2, \ldots, p_n]$) is a Hibert-Schmidt submodule. Of course, there are plenty of Hilbert-Schmidt submodules that are not generated by polynomials.

Let $Aut(D^2)$ be the group of biholomorphic self maps of D^2. It is well-known (cf. [17], [18]) that every $x \in Aut(D^2)$ is of the form

$$x(z) = \left(b_1 \frac{a_1 - z_{\sigma(1)}}{1 - \overline{a_1} z_{\sigma(1)}},\ b_2 \frac{a_2 - z_{\sigma(2)}}{1 - \overline{a_2} z_{\sigma(2)}} \right), \tag{2.4}$$

for some unique $(a_1,\ a_2) \in D^2$, $(b_1,\ b_2) \in T^2$ and a permutation σ on $(1,\ 2)$. For the studies in this paper, it is sufficient to look at the group elements with $b_1 = b_2 = 1$ and $\sigma(i) = i,\ i = 1,\ 2$. For convenience, we let $x_i(z_i) = \frac{a_i - z_i}{1 - \overline{a_i} z_i},\ i = 1,\ 2$, i.e., $x(z) = (x_1(z_1),\ x_2(z_2))$. One easily checks that $x(x(z)) = z$.

Now consider the action L of $Aut(D^2)$ on $H^2(D^2)$ defined by

$$(L_x f)(z) = f(x(z)), \quad x \in Aut(D^2).$$

It is well-known that L_x is bounded and invertible. The following property is proved in [28].

Proposition 2.1. *For every submodule M and every $x \in Aut(D^2)$, $L_x(M)$ is a submodule and*

$$C^{L_x(M)} = L_x C^M L_x^*.$$

Since k_a does not vanish on $\overline{D^2}$, multiplication by k_a is an invertible module action on $H^2(D^2)$. If we let operator U_x be defined by

$$U_x(f)(z) := k_a(z) f(x(z)), \quad f \in H^2(D^2),$$

where x and a are as in (2.4), then U_x is a unitary. It is also easy to see that $U_x(M) = L_x(M)$. Furthermore, since

$$U_x(H^2(D^2) \ominus M) = U_x(H^2(D^2)) \ominus U_x(M) = H^2(D^2) \ominus U_x(M),$$

$U_x(H^2(D^2) \ominus M)$ is a quotient module. Setting $M' = U_x(M)$ and $N' = U_x(N)$, one checks easily that

$$U_x^* P_{M'} U_x = P_M, \quad U_x^* P_{N'} U_x = P_N.$$

We let the pairs $(R_1,\ R_2)$ on M' and $(S_1,\ S_2)$ on N' be denoted by $(R_1',\ R_2')$ and $(S_1',\ S_2')$, respectively, and verify that for every $f \in H^2(D^2) \ominus M$,

$$\begin{aligned} S_i' U_x f &= P_{N'} \left(z_i k_a f(x(\cdot)) \right) \\ &= P_{N'} \left(k_a x_i(x_i(z_i)) f(x(\cdot)) \right) \\ &= P_{N'} \left(U_x(x_i(z_i) f) \right) \\ &= U_x U_x^* P_{N'} U_x (x_i(z_i) f) \\ &= U_x P_N (x_i(z_i) f) \\ &= U_x x_i(S_i) f, \quad i = 1,\ 2, \end{aligned} \tag{2.5}$$

which means $(S_1',\ S_2')$ is unitarily equivalent to $(x_1(S_1),\ x_2(S_2))$.

Likewise, for every $g \in M$,

$$\begin{aligned} R_i' U_x g &= z_i k_a(z) g(x(z)) \\ &= k_a(z) x_i(x_i(z_i)) g(x(z)) \\ &= U_x\left(x_i(z_i) g\right) \\ &= U_x\left(x_i(R_i) g\right), \quad i = 1,\ 2, \end{aligned} \tag{2.6}$$

which shows that $(R_1',\ R_2')$ is unitarily equivalent to $(x_1(R_1),\ x_2(R_2))$.

We summarize preceding observations in the following lemma. Notations are as above.

Lemma 2.2. *For every $x \in Aut(D^2)$,*

(a) *$(S_1',\ S_2')$ is unitarily equivalent to $(x_1(S_1),\ x_2(S_2))$.*
(b) *$(R_1',\ R_2')$ is unitarily equivalent to $(x_1(R_1),\ x_2(R_2))$.*

Proposition 2.1 and Lemma 2.2 are useful for the rest of the paper.

3. The defect operators for $(S_1,\ S_2)$

For a contraction F acting on a Hilbert space, its defect operators are $D_F = (1 - F^*F)^{1/2}$ and $D_{F^*} = (1 - FF^*)^{1/2}$, and the associated *characteristic operator function* is

$$\Theta_F(\lambda) = [-F + \lambda D_{F^*}(1 - \lambda F^*)^{-1} D_F]|_{\mathcal{D}_F}, \quad \lambda \in D,$$

where $\mathcal{D}_F = \overline{R(F)}$. The defect operators and the characteristic operator function are key elements in functional model theory, in which the characteristic operator function gives rise to a representation of F by a model that looks like a Jordan block in a vector-valued Hardy space (cf. [7], [20]). F and Θ_F are connected in many ways. A spectral connection will be used Section 5.

In multivariable settings, however, there is not a universal definition of defect operator for tuples of operators, and different tuples may demand different definitions for a meaningful study. For tuples acting on a reproducing kernel Hilber space, a good definition is through the so-called hereditary functional calculus (cf. [1]). In $H^2(D^2)$, since

$$\frac{1}{K(\lambda, z)} = 1 - \overline{\lambda_1} z_1 - \overline{\lambda_2} z_2 + \overline{\lambda_1 \lambda_2} z_1 z_2,$$

the hereditary functional calculus $\frac{1}{K}(A)$ of a pair of commuting operators $A = (A_1,\ A_2)$ is given by

$$\Delta_A := \frac{1}{K}(A) = I - A_1^* A_1 - A_2^* A_2 + A_1^* A_2^* A_1 A_2.$$

So for $A^* = (A_1^*,\ A_2^*)$,

$$\Delta_{A^*} = \frac{1}{K}(A^*) = I - A_1 A_1^* - A_2 A_2^* + A_1 A_2 A_1^* A_2^*.$$

In this paper, the operators Δ_A and Δ_A^* are called the defect operators for $A = (A_1,\ A_2)$.

Lemma 3.1. *If $A = (A_1,\ A_2)$ is a pair of commuting contractions, then Δ_A is a contraction.*

Proof. Since Δ_A is selfadjoint, it is sufficient to show that $-I \leq \Delta_A \leq I$. In fact, one checks easily that

$$I + \Delta_A = (I - A_1^* A_1) + (I - A_2^* A_2) + A_1^* A_2^* A_1 A_2 \geq 0,$$

and

$$I - \Delta_A = A_1^* A_1 + A_2^*(I - A_1^* A_1) A_2 \geq 0. \qquad \square$$

One observes that when M is a submodule in $H^2(D^2)$ and $R = (R_1,\ R_2)$, $\Delta_R = 0$ and Δ_{R^*} is indeed the core operator of M (cf. (2.2)). For the rest of the paper, given a nontrivial quotient module N we let ϕ denote the function $P_N 1$. Clearly, $0 < \|\phi\| \leq 1$. As a special case of the results in [28], Δ_{S^*} is always a rank one operator, and in fact

$$\Delta_{S^*} = \phi \otimes \phi. \tag{3.1}$$

(3.1) is a very useful fact. For one example, it gives rise to a very clean proof of the irreducibility of $(S_1,\ S_2)$ (cf. [28]). Here we give another example. In the study of $(S_1,\ S_2)$, a natural question is whether the pair completely determines the quotient space N on which it is defined. In other words, if $(S_1,\ S_2)$ is unitarily equivalent to $(S_1',\ S_2')$ on N', then how close are N and N'? In [8] Douglas and Foias showed (in the polydisk setting) that it is the case only if $N = N'$. (3.1) enables one to give a more concise proof of this fact.

First, one checks that

$$\|\Delta_{S^*}\| = \|\phi\|^2.$$

Moreover,

$$\begin{aligned} K(\lambda, S)\Delta_{S^*}\frac{\phi}{\|\phi\|} &= \|\phi\| K(\lambda, S) P_N 1 \\ &= \|\phi\| P_N K(\lambda, \cdot) \\ &= \|\phi\| K^N(\lambda, \cdot), \end{aligned}$$

so it follows that

$$|K(\lambda, S)\Delta_{S^*}|_2^2 = \|\phi\|^2 K^N(\lambda, \lambda).$$

If $(S_1,\ S_2)$ is unitarily equivalent to $(S_1',\ S_2')$ on N' and $\phi' = P_{N'} 1$, then above calculations imply $\|\phi\| = \|\phi'\|$ and also $K^N(\lambda, \lambda) = K^{N'}(\lambda, \lambda),\ \lambda \in D^2$, which concludes that $N = N'$ (cf. [10]).

Remark. Clearly, the validity of this argument is not limited to the bidisk. To put this observation in a general setting, we let Ω be a complex domain in $\mathbf{C^n}$ and let $\mathcal{H}$ be a Hilbert space of holomorphic functions over Ω with a reproducing kernel $K^{\mathcal{H}}(\lambda, z)$. Two additional conditions are needed.

1. The reciprocal $1/K^{\mathcal{H}}$ is a polynomial.

2. $1 \in \mathcal{H}$, and with action defined by the multiplication of functions, $\mathcal{H}$ is a module over the polynomial ring $C[z_1, z_2, \ldots, z_n]$.

Now we let N be a quotient module in $\mathcal{H}$ and define

$$S_i f = P_N z_i f,\ i = 1,\ 2,\ \ldots,\ n,\ f \in N,$$

where P_N is the orthogonal projection from $\mathcal{H}$ onto N. In this setting, one also has

$$\Delta_{S^*} := \frac{1}{K^{\mathcal{H}}}(S^*) = P_N 1 \otimes P_N 1$$

(cf. Corollary 4.3 of [28]), and it is easy to check that

$$K^{\mathcal{H}}(\lambda, S) P_N 1 = K^N(\lambda, \cdot).$$

So the above argument applies. We summarize this observation as

Corollary 3.2. *With the notations above, $(S_1, S_2, \ldots, S_n)$ on N is unitarily equivalent to $(S_1, S_2, \ldots, S_n)'$ on N' only if $N = N'$.*

Corollary 3.2 extends Douglas and Foias's result to a more general domain. We point out that when Ω is the unit ball B_n and $K^{\mathcal{H}}(\lambda, z) = (1 - \bar{\lambda}_1 z_1 - \bar{\lambda}_2 z_2 - \cdots - \bar{\lambda}_n z_n)^{-1}$, this fact is implied by Arveson's dilation theorem (cf. [2]).

Now we continue our study in $H^2(D^2)$. The operator Δ_S, as oppose to Δ_{S^*}, can be of arbitrary rank, and hence contains more information about N. We shall focus our study on Δ_S in the rest of this section. We calculate two examples first.

Example 1. Let p_1 and p_2 be two one variable inner functions and consider $M = p_1(z_1)H^2(D^2) + p_2(z_2)H^2(D^2)$. Then M is a submodule, and on $H^2(D^2) \ominus M$, $S_1^* S_2 = S_2 S_1^*$ (cf. [15]). Moreover, N can be decomposed as

$$N = (H^2(D) \ominus p_1 H^2(D)) \otimes (H^2(D) \ominus p_2 H^2(D)),$$

and with respect to this decomposition,

$$S_1 = S(p_1) \otimes I, \quad S_2 = I \otimes S(p_2),$$

where I stands for the identity operator on the respective spaces. So in this case the two variable Jordan block is essentially a pair of two classical Jordan blocks. It is then not hard to check that

$$\begin{aligned} \Delta_S &= (I - S_1^* S_1)(I - S_2^* S_2) \\ &= [(T_1^* p_1)(T_2^* p_2)] \otimes [(T_1^* p_1)(T_2^* p_2)]. \end{aligned}$$

So Δ_S in this case is also of rank 1.

Δ_S can be surjective in other cases.

Example 2. Consider $M = [z_1^2, z_1z_2, z_2^2]$. Then $H^2(D^2) \ominus M = span\{1,\ z_1,\ z_2\}$, and with respect to this orthonormal basis,

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

One then easily verifies that

$$\Delta_S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For an eigenvalue ζ of an operator F, we let $E_\zeta(F)$ denote the corresponding eigenspace. It turns out that $E_1(\Delta_S)$ and $E_{-1}(\Delta_S)$ can be readily calculated. If $f \in H^2(D^2) \ominus M$ such that $\Delta_S f = f$, then

$$S_1^*S_1 f + S_2^*(I - S_1^*S_1)S_2 f = 0,$$

which implies

$$S_1 f = 0, \quad (I - S_1^*S_1)^{1/2} S_2 f = 0.$$

Multiplying the second equation by $(I - S_1^*S_1)^{1/2}$, we have $(I - S_1^*S_1)S_2 f = 0$. Since

$$ker(I - S_1^*S_1) = \{g \in N |\ z_1 g \in N\},$$

we get

$$z_1 S_2 f = S_1 S_2 f = S_2 S_1 f = 0,$$

from which it follows that $S_2 f = 0$. These observations show the inclusion $E_1(\Delta_S) \subset kerS_1 \cap kerS_2$. Since the inclusion $kerS_1 \cap kerS_2 \subset E_1(\Delta_S)$ is obvious, we prove the equality

$$E_1(\Delta_S) = kerS_1 \cap kerS_2. \tag{3.2}$$

If $f \in H^2(D^2) \ominus M$ such that $\Delta_S f = -f$, then

$$(I - S_1^*S_1)f + (I - S_2^*S_2)f + S_1^*S_2^*S_1S_2 f = 0,$$

and this implies that

$$(I - S_1^*S_1)f = 0, \quad (I - S_2^*S_2)f = 0, \quad S_1S_2 f = 0,$$

which means that

$$z_1 f \in N, \quad z_2 f \in N, \quad z_1z_2 f \in M.$$

It is not hard to see that this is true if and only if $z_1z_2 f \in M \ominus M_0$. In conclusion we have

$$E_{-1}(\Delta_S) = \{f \in N |\ z_1z_2 f \in M \ominus M_0\}. \tag{3.3}$$

Of course, there are many submodules M for which $kerS_1 \cap kerS_2 = \{0\}$ and $M \ominus M_0$ contains no nontrivial function with factor z_1z_2. Other eigenvalues of Δ_S, unfortunately, are usually difficult to compute. However, the trace $tr\Delta_S$ is easy to determine when N is finite dimensional. From the definition of the defect operators,

$$\Delta_S - \Delta_{S^*} = -[S_1^*,\ S_1] - [S_2^*,\ S_2] + [(S_1S_2)^*,\ (S_1S_2)],$$

and hence

$$tr\Delta_S = tr\Delta_{S^*} = \|\phi\|^2.$$

Clearly, Δ_S has at least one positive eigenvalue when N is finite dimensional. Situations are different if N is infinite dimensional, for instance we will see a submodule in Example 3 for which $\Delta_S \leq 0$.

In general, it is difficult to determine whether Δ_S is trace class, let alone computing its trace. The Hilbert-Schmidtness of Δ_S, on the other hand, can be established for most submodules. The proof makes a good use of the core operator.

For a submodule M, we let $D_i = P_N T_{z_i}^*|_M,\ i = 1,\ 2$. Clearly, D_i is nontrivial only on $M \ominus z_iM$. It is indicated in [23, 24] that the following identities hold:

$$S_i^*S_i + D_iD_i^* = I, \quad i = 1,\ 2, \tag{3.4}$$

$$[R_2^*,\ R_1] = D_1^*D_2, \quad \text{and} \quad [S_2^*,\ S_1] = -D_2D_1^*. \tag{3.5}$$

We then calculate that

$$\begin{aligned}\Delta_S &= I - S_1^*S_1 - S_2^*S_2 + S_1^*S_2^*S_1S_2 \\ &= (I - S_1^*S_1)(I - S_2^*S_2) + S_1^*[S_2^*,\ S_1]S_2 \\ &= D_1D_1^*D_2D_2^* + S_1^*[S_2^*,\ S_1]S_2.\end{aligned}$$

By (3.5),

$$\Delta_S = D_1[R_2^*,\ R_1]D_2^* - S_1^*D_2D_1^*S_2.$$

Furthermore, D_1^* is a map from $H^2(D^2) \ominus M$ into $M \ominus z_1M$, so $D_1^* = [R_1^*,\ R_1]D_1^*$ on N; and D_2 is equal to 0 on z_2M, so $D_2 = D_2[R_2^*,\ R_2]$. Hence we can write

$$D_2D_1^* = D_2[R_2^*,\ R_2][R_1^*,\ R_1]D_1^*.$$

Therefore,

$$\Delta_S = D_1[R_2^*,\ R_1]D_2^* - S_1^*D_2[R_2^*,\ R_2][R_1^*,\ R_1]D_1^*S_2. \tag{3.6}$$

The following corollary follows immediately from (2.3) and (3.6)

Corollary 3.3. *For a submodule M,*

(1) *if C is finite rank, then so is Δ_S, and $rank\Delta_S \leq rankC$;*
(2) *if C is compact, then Δ_S is compact;*
(3) *if C is Hilbert-Schmidt, then Δ_S is Hilbert-Schmidt.*

Moreover, when C is Hilbert-Schmidt, (3.6) implies an estimate of $|\Delta_S|_2$ by $|C|_2$. In fact, by (2.1)

$$\begin{aligned}|\Delta_S|_2 &\leq |[R_2^*,\ R_1]|_2 + |[R_2^*,\ R_2][R_1^*,\ R_1]|_2 \\ &\leq \sqrt{2}(|[R_2^*,\ R_1]|_2^2 + |[R_2^*,\ R_2][R_1^*,\ R_1]|_2^2)^{1/2} \\ &= \sqrt{2}|C|_2.\end{aligned} \tag{3.7}$$

The next theorem improves the estimate.

Theorem 3.4. *If M is a Hilbert-Schmidt submodule, then Δ_S is Hilbert-Schmidt with*

$$tr\Delta_S^2 \le 2\|\phi\|^2 + trC^2 - 1.$$

Proof. First of all, for every $p \in A(D^2)$ and $f \in N$, we have $S_p^* f = P_N \bar{p} f$ and

$$S_p f = \int_{T^2} K^N(\lambda,\ z) p(\lambda) f(\lambda) dm(\lambda).$$

Then

$$\begin{aligned}\Delta_S f &= f - P_N(\bar{z_1}\int_{T^2} K^N(\lambda,\ z)\lambda_1 f(\lambda)dm(\lambda)) - P_N(\bar{z_2}\int_{T^2} K^N(\lambda,\ z)\lambda_2 f(\lambda)dm(\lambda)) \\ &\quad + P_N(\overline{z_1 z_2}\int_{T^2} K^N(\lambda,\ z)\lambda_1\lambda_2 f(\lambda)dm(\lambda)) \\ &= P_N\left(\int_{T^2} K^N(\lambda,\ z)(1 - \lambda_1\bar{z_1} - \lambda_2\bar{z_2} + \lambda_1\lambda_2\overline{z_1 z_2})f(\lambda)dm(\lambda)\right).\end{aligned}$$

If we let $\hat{G^N} = \frac{K^N}{K}$ and for $g \in L^2(T^2)$ define

$$\hat{C^N} g(z) = \int_{T^2} \hat{G^N}(\lambda, z) g(\lambda) dm(\lambda),$$

then clearly

$$\Delta_S = P_N \hat{C^N}|_N. \tag{3.8}$$

We continue to check that on $D^2 \times D^2$

$$\begin{aligned}|\hat{G^N}(\lambda, z)| &= \left| \frac{K^N}{K}(\lambda, z) \right| \\ &= \left| \frac{K - K^M}{K}(\lambda, z) \right| \\ &= |1 - G^M(\lambda, z)|.\end{aligned}$$

When C^M is Hilbert-Schmidt, G^M has nontangential boundary value to almost every point in T^2 and its boundary value function is in $L^2(T^2 \times T^2)$ (cf. [14]). So $\hat{G^N}$ also has the these properties. Furthermore

$$\begin{aligned}\int_{T^2} |\hat{G^N}(\lambda, z)|^2 dm(\lambda)dm(z) &= \int_{T^2} |1 - G^M(\lambda, z)|^2 dm(\lambda)dm(z) \\ &= \int_{T^2} 1 - G^M(\lambda, z) - G^M(z, \lambda) + |G^M(\lambda, z)|^2 dm(\lambda)dm(z) \\ &= 1 - 2G^M(0,0) + \|G^M\|_2^2 \\ &= 1 - 2K^M(0,0) + \|G^M\|_2^2 \\ &= 2(1 - \|P_M 1\|^2) - 1 + \|G^M\|_2^2 \\ &= 2\|\phi\|^2 + \|G^M\|_2^2 - 1.\end{aligned}$$

Since $\hat{G^N}$ is the integral kernel of $\hat{C^N}$, $\hat{C^N}$ is Hilbert-Schmidt with

$$|\hat{C^N}|_2^2 = \|\hat{G^N}\|_2^2 = 2\|\phi\|^2 + \|G^M\|_2^2 - 1.$$

So by (3.8) and the fact $tr(C)^2 = \|G\|_2^2$, we have

$$tr\Delta_S^2 \leq 2\|\phi\|^2 + trC^2 - 1. \qquad \Box$$

Example 3. If $M = \psi H^2(D^2)$, where ψ is an inner function in $H^2(D^2)$, then $[R_2^*,\ R_1] = 0$ by [12]. So by (3.6) we have

$$\Delta_S = -S_1^* D_2 [R_2^*,\ R_2][R_1^*,\ R_1] D_1^* S_2.$$

Since in this case

$$[R_2^*,\ R_2][R_1^*,\ R_1]f =< f,\ \psi > \psi, \quad f \in M,$$

we have for every $g \in H^2(D^2) \ominus M$

$$\begin{aligned}\Delta_S g &= -S_1^* D_2 (< D_1^* S_2 g,\ \psi > \psi)\\ &= - < g,\ S_2^* D_1 \psi > S_1^* D_2 \psi.\end{aligned}$$

One verifies that

$$S_2^* D_1 \psi = S_1^* D_2 \psi = T_1^* T_2^* \psi,$$

and concludes that

$$\Delta_S = -(T_1^* T_2^* \psi) \otimes (T_1^* T_2^* \psi).$$

So

$$tr\Delta_S^2 = \|T_1^* T_2^* \psi\|^4 \leq (1 - |\psi(0)|^2)^2.$$

In this case, $trC^2 = 1$ (cf. [14]) and $\phi = 1 - \overline{\psi(0)}\psi$, so

$$2\|\phi\|^2 + trC^2 - 1 = 2(1 - |\psi(0)|^2).$$

4. The essential Taylor spectrum of $(S_1,\ S_2)$

For a classical Jordan block $S(\theta)$, the essential spectrum $\sigma_e(S(\theta))$ is a subset of T and the Fredholm index $ind(S(\theta))$ is always equal to 0. For a general two variable Jordan block $(S_1,\ S_2)$, its essential Taylor spectrum $\sigma_e(S_1,\ S_2)$ may not be a subset of ∂D^2. For instance, there are quotient modules on which $(S_1,\ S_2)$ is not a Fredholm pair. In this section, however, we will show that when the core operator is compact the essential Taylor spectrum $\sigma_e(S_1,\ S_2)$ is a subset of ∂D^2. This generalized the work in [21].

Let $A = (A_1,\ A_2)$ be a pair of commuting operators acting on a Hilbert space H. One good way to study the Fredholness of $A = (A_1,\ A_2)$ is through the matrix

$$\hat{A} = \begin{pmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{pmatrix} \tag{4.1}$$

on $H \oplus H$. It is well known (cf. [6]) that the pair A is Fredholm if and only if $\hat{A}$ is Fredholm on $H \oplus H$, and in this case

$$ind(A_1,\ A_2) = ind\hat{A}.$$

We now use this technique to study the Fredholmness of $(S_1,\ S_2)$. First, we check that

$$\hat{S}\hat{S}^* = \begin{pmatrix} S_1S_1^* + S_2S_2^* & 0 \\ 0 & S_1^*S_1 + S_2^*S_2 \end{pmatrix}. \qquad (4.2)$$

Since $I - \Delta_{S^*}$ is a positive Fredholm operator by (3.1) and

$$S_1S_1^* + S_2S_2^* = I - \Delta_{S^*} + S_1S_2S_1^*S_2^*,$$

$S_1S_1^* + S_2S_2^*$ is Fredholm. If C^M is compact, then the Fredholmness of $S_1^*S_1 + S_2^*S_2$ follows similarly from Corollary 3.3. Therefore, $\hat{S}\hat{S}^*$ is Fredholm. It is then clear that $\hat{S}$ has a closed range with a finite codimension. So to prove the Fredholmness of $\hat{S}$, we only need to check if $ker\hat{S}$ is finite dimensional. It is easy to see from (4.1) that

$$ker\hat{S} = \{(f_1, f_2) \in N \oplus N|\ z_1f_1 + z_2f_2 \in M,\ -\bar{z}_2f_1 + \bar{z}_1f_2 \perp H^2(D^2).\} \qquad (4.3)$$

For later use, we let B_1 be the operator from $ker\hat{S}$ to M which maps (f_1, f_2) to $z_1f_1 + z_2f_2$. We now check that B_1 is injective. In fact, if $B_1(f_1, f_2) = 0$, then $f_1 = z_2g$ and $f_2 = -z_1g$ for some $g \in H^2(D^2) \ominus M$. The fact that $-\bar{z}_2f_1 + \bar{z}_1f_2 \perp H^2(D^2)$ then implies $g = 0$, and hence $f_1 = f_2 = 0$.

Multiplying $-\bar{z}_2f_1 + \bar{z}_1f_2$ by z_1z_2, we have

$$-z_1f_1 + z_2f_2 \in H^2(D^2) \ominus z_1z_2H^2(D^2). \qquad (4.4)$$

(4.4) is equivalent to the existence of g_1 and g_2 in $H^2(D)$ with $g_1(0) = g_2(0) = 0$ such that

$$-z_1f_1 + z_2f_2 = g_1(z_1) + g_2(z_2). \qquad (4.5)$$

So in fact,

$$g_1(z_1) = -z_1f_1(z_1, 0),\quad g_2(z_2) = z_2f_2(0, z_2). \qquad (4.6)$$

If we set $h = z_1f_1 + z_2f_2$, then $g_1 = -R(0)h$ and $g_2 = L(0)h$, and by (4.5) and (4.6),

$$2z_1f_1 = h - L(0)h + R(0)h,\quad 2z_2f_2 = h + L(0)h - R(0)h. \qquad (4.7)$$

Since $z_if_i \perp z_iM,\ i = 1,\ 2$, we have $h - g_1(z_1) \perp z_1M,\ h + g_2 \perp z_2M$, and hence $(I + L(0) + R(0))h$ is orthogonal to both z_1M and z_2M. For simplicity, we denote $z_1M + z_2M$ by M_0. It can be seen from the work in [24] (cf. [24] Section 4) that when C^M is compact M_0 is closed, and hence it is a submodule. It follows from our previous arguments that $P_M(I + L(0) + R(0))h \in M \ominus M_0$.

To continue with the discussion, we consider the operator $B_0 = P_M(I + L(0) + R(0))P_M$ defined on M. Clearly B_0 is selfadjoint. Since for every $h \in M$,

$$\begin{aligned}\langle B_0h,\ h\rangle &= \langle (I + L(0) + R(0))h,\ h\rangle \\ &= \|h\|^2 + \|L(0)h\|^2 + \|R(0)h\|^2 \\ &\geq \|h\|^2,\end{aligned}$$

B is invertible. We summarize these observations in the following lemma.

Lemma 4.1. *The operator B_0B_1 is an injective map from $ker\hat{S}$ to $M \ominus M_0$.*

Therefore, $dim(ker\hat{S}) \leq dim(M \ominus M_0)$. When C^M is compact, $M \ominus M_0$ is finite dimensional by [14] Corollary 3.4, and hence so is $ker\hat{S}$. We thus obtain the following fact.

Corollary 4.2. *If C^M is compact, then $(S_1,\ S_2)$ is Fredholm.*

To study $\sigma_e(S_1,\ S_2)$, we need to look further at the Fredholmness of $(S_1 - a_1I,\ S_2 - a_2I)$ for $(a_1, a_2) \in D^2$. This, in fact, can be easily achieved through the action of $Aut(D^2)$ mentioned in Section 2.

Theorem 4.3. *If C^M is compact, then $\sigma_e(S_1,\ S_2) \subset \partial D^2$.*

Proof. The proof is a routine argument based on Corollary 4.2. For $(a_1, a_2) \in D^2$, we let $x_i(z_i) = \frac{a_i - z_i}{1 - \bar{a}_i z_i},\ i = 1,\ 2$, and thus $x = (x_1, x_2) \in Aut(D^2)$. Let $M' = U_x(M)$. Since C^M is compact, so is $C^{M'}$ by Proposition 2.1, and hence by Corollary 4.2 $(S_1',\ S_2')$ is Fredholm. It then follows from Lemma 2.2(a) that $(x_1(S_1),\ x_2(S_2))$ is Fredholm. Since $x_i(S_i) = (1 - \bar{a}_i S_i)^{-1}(a_i - S_i),\ i = 1,\ 2$, $(S_1 - a_1,\ S_2 - a_2)$ is also Fredholm (cf. [6]). This shows that $\sigma_e(S_1,\ S_2) \subset \mathbf{C}^2 \setminus D^2$. The theorem then follows from the fact that $(S_1,\ S_2)$ is a pair of contractions. □

We point out that in many cases $\sigma_e(S_1,\ S_2)$ is not a subset of T^2. For example it is easy to find quotient modules on which S_1 is a strict contraction. The index of $(S_1,\ S_2)$ can be determined with a mild condition. For example, it follows from [13] that if M contains a bounded function that does not vanish at $(0, 0)$ then $ind(S_1,\ S_2) = 0$. If we assume C^M is Hilbert-Schmidt, then we can also show that $ind(S_1,\ S_2) = 0$. These facts prompt the following conjecture.

Conjecture. If C^M is compact then $\sigma_e(S_1,\ S_2)$ is a proper subset of ∂D^2.

In fact, we suspect $\sigma_e(S_1,\ S_2)$ has measure 0 in ∂D^2. Since for a classical Jordan block $S(\theta)$, the essential spectrum $\sigma_e(S(\theta))$ is determined by the inner function θ, we also suspect that $\sigma_e(S_1,\ S_2)$ may be an important invariant of the functions in M.

5. An estimate of trC^2 and $||[S_1^*,\ S_2]||_2$

As indicated by our previous studies, the core operator is connected with many other key elements, and it is compact in most cases. So an esitmate of its "size" will certainly be useful. For the two variable Jordan block, the cross commutator $[S_1^*,\ S_2]$ is mostly compact, though S_1 and S_2 themselves are in general not essentially normal. In fact, the compactness of the core operator implies the compactness of $[S_1^*,\ S_2]$. As we have remarked early that if either $\sigma_c(S_1)$ or $\sigma_c(S_2)$ is not the full closed disk, then the core operator is Hilbert-Schmidt. So a challenging question is whether one can use some spectral data of S_1 or S_2 to estimate trC^2 and $|[S_1^*,\ S_2]|_2$.

We first do some preparations. For any bounded linear operator F on a Hilbert space H, the so-called minimum modulus

$$\gamma(F) := \inf\{\|Fx\| :\ x \in (kerF)^{\perp}, \|x\| = 1\}$$

measures the norm of F's "partial inverse". Clearly, F has closed range if and only if $\gamma(F) > 0$. When F is invertible,

$$\begin{aligned}\gamma^{-1}(F) &= \frac{1}{\inf_{\|x\|=1} \|Fx\|} \\ &= \sup_{\|x\|=1} \frac{\|x\|}{\|Fx\|} \\ &= \sup_{\|x\|=1} \frac{\|F^{-1}Fx\|}{\|Fx\|} \\ &= \|F^{-1}\|. \end{aligned} \tag{5.1}$$

When F has a closed range, the restriction $F :\ kerF^{\perp} \longrightarrow R(F)$ is invertible, and hence there is an inverse $F' :\ R(F) \longrightarrow kerF^{\perp}$ such that $F'F = I - P_{kerF}$. Moreover, it follows from (5.1) that $\|F'\| = \gamma^{-1}(F)$.

Lemma 5.1. *If A and F are bounded linear operators on a Hilbert space H such that $0 \notin \sigma_c(F)$ and FA is Hilber-Schmidt, then A is Hilbert-Schmidt with*

$$|A|_2^2 \leq \gamma^{-2}(F)|FA|_2^2 + \|A\|^2 dimkerF.$$

Proof. Using the notations above, we write

$$A = (F'F + P_{kerF})A = F'FA + P_{kerF}A.$$

Since $R(F')$ is orthogonal to $kerF$, for every $x \in H$ we have

$$\|Ax\|^2 = \|F'FAx\|^2 + \|P_{kerF}Ax\|^2,$$

and it follows that

$$\begin{aligned}|A|_2^2 &= |F'FA|_2^2 + |P_{kerF}A|_2^2 \\ &\leq \|F'\|^2|FA|_2^2 + \|A\|^2|P_{kerF}|_2^2 \\ &= \gamma^{-2}(F)|FA|_2^2 + \|A\|^2 dim(kerF).\end{aligned}$$

□

The following lemma is crucial. Its proof is based on an improvement of some ideas contained in [24], in particular two key facts will be used. The first is the equivalence bewteen $L(\lambda)|_{M\ominus z_1 M}$ and the characteristic function $\Theta_{S_1}(\lambda)$, and the second is the Hilbert-Schmidtness of $L(\lambda)$ when restricted to $M \ominus z_2 M$. Also, we ecall that Σ_0 stands for $|[R_1^*,\ R_1][R_2^*,\ R_2]|_2^2$.

Lemma 5.2. *For every submodule,*

$$\Sigma_0 \leq \gamma^{-2}(S_1) + dim(kerS_1).$$

Proof. The inequality holds trivially if $\gamma(S_1) = 0$ or S_1 has infinite dimensional kernel. So we assume S_1 has a closed range and a finite dimensional kernel. First of all, it is indicated in [23] that $L(\lambda)|_{M\ominus z_1 M}$ is equivalent to the characteristic operator function $\Theta_{S_1}(\lambda)$ for S_1 in the sense that there are constant unitaries $U,\ V, W$ such that

$$L(\lambda)|_{M\ominus z_1 M} = (U\Theta_{S_1}(\lambda)V) \oplus W, \tag{5.2}$$

where W is nonzero only if M contains nontrivial functions independent of z_1. If S_1 has a closed range and a finite dimensional kernel then so is $\Theta_{S_1}(0)$ with $dim(kerS_1) = dim(\Theta_{S_1}(0))$ (cf. [20]), and hence by (5.2) $L(0)|_{M\ominus z_1 M}$ has a closed range with

$$dim(L(0)|_{M\ominus z_1 M}) = dim(\Theta_{S_1}(0)) = dim(kerS_1). \tag{5.3}$$

Since for every $f \in M \ominus z_2 M$,

$$L(0)[R_1^*,\ R_1][R_2^*,\ R_2]f = L(0)(f - z_1 R_1^* f) = L(0)f,$$

and $L(0)$ is Hilbert-Schmidt on $M \ominus z_2 M$ with $|L(0)|_{M\ominus z_2 M}|_2 \leq 1$ (cf. [22]), $L(0)[R_1^*,\ R_1][R_2^*,\ R_2]$ is Hilbert-Schmidt, and

$$|L(0)[R_1^*,\ R_1][R_2^*,\ R_2]|_2 = |L(0)|_{M\ominus z_2 M}|_2 \leq 1. \tag{5.4}$$

So it follows from Lemma 5.1

$$\Sigma_0 \leq \gamma^{-2}(L(0)|_{M\ominus z_1 M}) + dim(ker(L(0)|_{M\ominus z_1 M})). \tag{5.5}$$

Since

$$L(0)|_{M\ominus z_1 M} = U\Theta_{S_1}(0)V \oplus W,$$

$\gamma(L(0)|_{M\ominus z_1 M}) = \gamma(\Theta_{S_1}(0))$. Using the expressions in the beginning of Section 3, we have

$$\Theta_{S_1}(0) = -S_1|_{D_{S_1}} \quad \text{and} \quad kerS_1 \subset D_{S_1},$$

and it follows that

$$\gamma(\Theta_{S_1}(0)) = inf\{\|S_1 f\| :\ \|f\| = 1,\ f \in D_{S_1} \ominus kerS_1\} \geq \gamma(S_1).$$

Combining the observations above with (5.3), we see that the theorem follows directly from (5.5). □

A parallel argument based on S_2 will prove the same inequality in Lemma 5.2 with S_1 replaced by S_2.

Theorem 5.3. *For every submodule,*

(a) $trC^2 \leq 2\gamma^{-2}(S_1) + 2dim(kerS_1) - 1$;
(b) $|[S_2^*,\ S_1]|_2^2 \leq \gamma^{-2}(S_1) + dim(kerS_1)$.

Proof. (a) follows directly from Lemma 5.2 and the remarks following (2.3).

For (b), using (3.5) and the remarks leading to (3.6), we have

$$[S_2^*,\ S_1] = -D_2[R_2^*,\ R_2][R_1^*,\ R_1]D_1^*.$$

So

$$|[S_2^*,\ S_1]|_2^2 \leq |[R_2^*,\ R_2][R_1^*,\ R_1]|_2^2 = \Sigma_0,$$

and the inequality follows from Lemma 5.2. □

Example 4. Let us consider $M = [z_1 - \theta(z_2)]$, where θ is an inner function. It is shown in [16] that there is a unitary

$$U:\ H^2(D^2) \ominus M \longrightarrow \big(H^2(D) \ominus \theta H^2(D)\big) \otimes L_a^2(D),$$

such that $US_1 = (I \otimes B)U$, where B is the Bergman shift on the classical Bergman space $L_a^2(D)$. So $\gamma(S_1) = \gamma(B) = \frac{1}{\sqrt{2}}$, and by Theorem 5.3

$$trC^2 \leq 3 \quad and \quad |[S_1^*,\ S_2]|_2 \leq \sqrt{2}.$$

In particular, when $\theta(z_2) = z_2$, S_1 is unitarily equivalent to B and it is known that in this case (cf. [14])

$$trC^2 = \frac{\pi^2}{3} - 1 \approx 2.29.$$

Moreover, $S_1 = S_2$ in this case and it is easy to see that

$$|[S_1^*,\ S_2]|_2 = |[B^*,\ B]|_2 \approx 0.538.$$

One interesting fact follows directly from Theorem 5.3. If S_1 (or S_2) is an isometry, then $\gamma(S_1) = 1$, and by Theorem 5.3(a), $trC^2 \leq 1$. So it follows from (2.3) and the remarks after it that $[R_2^*,\ R_1] = 0$. This happens only if $M = \psi H^2(D^2)$ for some inner function ψ (cf. [12]).

Clearly, if S_1 is invertible then Theorem 5.3 takes the cleaner form

$$trC^2 \leq 2\|S_1^{-1}\|^2 - 1, \quad \text{and} \quad |[S_1^*,\ S_2]|_2 \leq \|S_1^{-1}\|. \tag{5.6}$$

When working with examples of submodules, one sees that it is not rare that $0 \in \sigma_c(S_1)$ but instead $\zeta \notin \sigma_c(S_1)$ for some nonzero $\zeta \in D$. In this case, the following corollary generalizes Theorem 5.3.

Corollary 5.4. *Let M be a submodule and $\zeta \in D$. Then*

$$\frac{(1-|\zeta|)^2}{(1+|\zeta|)^2}|C|_2^2 \leq 2(1+|\zeta|)^2\gamma^{-2}(S_1 - \zeta I) + 2dim(S_1 - \zeta I) - 1.$$

Proof. For $\zeta \in D$, we let $x_1(z_1) = \frac{\zeta - z_1}{1 - \bar{\zeta} z_1}$ and $x_2(z_2) = z_2$, then

$$U_x(f)(z_1, z_2) := \frac{\sqrt{1-|\zeta|^2}}{1 - \bar{\zeta} z_1} f(x_\zeta(z_1), z_2), \quad f \in H^2(D^2),$$

We let $M' = U_x(M)$ $(= L_x(M))$ and $N' = H^2(D^2) \ominus M'$, and denote the core operator on M' by C'. By Theorem 5.3(a),

$$trC'^2 \leq 2\gamma^{-2}(S_1') + 2dim(kerS_1') - 1,$$

and it follows from Proposition 2.1 and (2.5) that

$$|L_x C L_x^*|_2^2 \leq 2\gamma^{-2}(x_1(S_1)) + 2dim(kerx_1(S_1)) - 1. \tag{5.7}$$

We now let $\psi_\zeta = \frac{\sqrt{1-|\zeta|^2}}{1-\bar{\zeta}z_1}$ and let T_{ψ_ζ} denotes the multiplication by ψ_ζ on $H^2(D^2)$. Then $T_{\psi_\zeta} L_x$ is the unitary U_x. So for the left-hand side of (5.7) we have that

$$\begin{aligned} |L_x C L_x^*|_2 &= \frac{\|T_{\psi_\zeta}\| |L_x C L_x^*|_2 \|T_{\psi_\zeta}^*\|}{\|T_{\psi_\zeta}\|^2} \\ &\geq \frac{|U_x C U_x^*|_2}{\|T_{\psi_\zeta}\|^2} \\ &= \frac{1-|\zeta|}{1+|\zeta|}|C|_2. \end{aligned}$$

For the righ-hand side of (5.7), we first note that $x_1(S_1) = (1-\bar{\zeta}S_1)^{-1}(S_1 - \zeta I)$, so clearly $dim(kerx_1(S_1)) = dim(S_1 - \zeta I)$, and moreover

$$\gamma(x_1(S_1)) \geq \frac{\gamma(S_1 - \zeta I)}{\|I - \bar{\zeta}S_1\|} \geq \frac{\gamma(S_1 - \zeta I)}{1+|\zeta|}.$$

Combining these estimates, we have

$$\frac{(1-|\zeta|)^2}{(1+|\zeta|)^2}|C|_2^2 \leq 2(1+|\zeta|)^2\gamma^{-2}(S_1 - \zeta I) + 2dim(S_1 - \zeta I) - 1.$$

□

Clearly, when $\zeta = 0$ we return to Theorem 5.3.

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Rongwei Yang
Department of Mathematics and Statistics
SUNY at Albany
Albany, NY 12222
U.S.A.
e-mail: `ryang@@math.albany.edu`

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