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Integral Equations and Operator Theory

The Spectrum of the Wavelet Galerkin Operator

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Abstract. We give a complete description of spectrum of the wavelet Galerkin operator

$$R_{m_0,m_0}f(z) = \frac{1}{N}\sum_{w^N=z} |m_0|^2(w)f(w), \quad (z \in \mathbb{T})$$

associated to a a low-pass filter m_0 and a scale N, in the Banach spaces $C(\mathbb{T})$ and $L^p(\mathbb{T})$, $1 \le p \le \infty$.

1. Introduction

We begin with a short motivation of our study. For more background on wavelets and their connection to the wavelet Galerkin operator we refer the reader to [Dau92], [BraJo] or [HeWe]. The wavelet analysis studies functions $\psi \in L^2(\mathbb{R})$ with the property that

$$\left\{2^{\frac{j}{2}}\psi\left(2^j x - k\right) \mid j,k \in \mathbb{Z}\right\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. Such functions are called wavelets. The scale (2 here) can be also any integer $N \ge 2$. One way to construct wavelets is by multiresolutions. A multiresolution is a nest of subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ with the following properties:

(i) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$;
(ii) $f \in V_j$ if and only if $f(Nx) \in V_{j+1}, (j \in \mathbb{Z})$;
(iii)

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\};$$

(iv)

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R});$$

(v) There exists a function $\varphi \in V_0$ such that $\{\varphi(x-k) \mid k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

To build such a multiresolution one needs the function φ called scaling function (or father function or refinable function). The scaling function satisfies a scaling equation:

$$\frac{1}{\sqrt{N}}\varphi\left(\frac{x}{N}\right) = \sum_{k\in\mathbb{Z}} a_k\varphi(x-k), \quad (x \in \mathbb{R}),$$

a_k being some complex coefficients. The Fourier transform of the scaling equation is:

$$\sqrt{N}\widehat{\varphi}(N\xi) = m_0(\xi)\widehat{\varphi}(\xi), \quad (\xi \in \mathbb{R}),$$

where $m_0(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}$ is a 2π-periodic function called low-pass filter.

Thus, the scaling functions φ are determined by the low-pass filters m_0 and the construction of scaling functions has the low-pass filters as the starting point.

The multiresolution theory has shown that many of the properties of the scaling function φ can be expressed in terms of the wavelet Galerkin operator associated to the filter m_0:

$$R_{m_0,m_0}f(z) = \frac{1}{N}\sum_{w^N=z} |m_0|^2(w)f(w), \quad (z \in \mathbb{T}).$$

$\mathbb{T}$ is the unit circle, f is some measurable function on $\mathbb{T}$, and we will identify functions on $\mathbb{T}$ with 2π-periodic functions on $\mathbb{R}$.

For example, one needs the integer translates of the scaling function $\varphi(x-k)$, $k \in \mathbb{Z}$, to be orthonormal. To obtain this, a neccesary condition is the quadrature mirror filter condition:

$$\frac{1}{N}\sum_{w^N=z} |m_0|^2(w) = 1, \quad (z \in \mathbb{T}),$$

which can be rewritten as $R_{m_0,m_0}1 = 1$. In [Law91a] it is proved that the integer translates of the scaling function form an orthonormal set if and only if the constants are the only continuous functions that satisfy $R_{m_0,m_0}h = h$. So 1 has to be a simple eigenvalue for the operator $R_{m_0,m_0} : C(\mathbb{T}) \to C(\mathbb{T})$. Also, the regularity of the scaling function can be determined by the spectrum of R_{m_0,m_0} (see [Str96],[RoSh]).

We will impose some restrictions on m_0, restrictions that are custom in the setting of compactly supported wavelets:

$$m_0 \text{ is a Lipschitz function;} \tag{1.1}$$

$$m_0 \text{ has only a finite number of zeroes;} \tag{1.2}$$

$$m_0(0) = \sqrt{N}; \tag{1.3}$$

$$R_{m_0,m_0}1 = 1. \tag{1.4}$$

In fact, for compactly supported wavelets, m_0 is a trigonometric polynomial, but for our purpose we can assume more generally that m_0 is Lipschitz.

The wavelet Galerkin operator R_{m_0,m_0} bears several other names in the literature. It is also called the Ruelle operator because there are connections with the Ruelle-Perron-Frobenius theory for positive operators (see[Bal00]), or transfer operator. We will use these names in the sequel. The existence of fixed points and periodic points for Perron-Frobenius operators is also treated in [Nus01].

An extensive study of the spectral properties of the Ruelle operator can be found in [BraJo]. We will gather some results from [BraJo],[Dutb] and add some new ones to give a complete picture of the spectrum of this Ruelle operator in the Banach spaces $C(\mathbb{T})$ and $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, answering in this way some questions posed in [BraJo].

2. The Spectrum of R_{m_0,m_0}

In this section we present the results. We consider an integer $N \geq 2$ and a function m_0 on $\mathbb{T}$ that satisfies (1.1)-(1.4). To m_0 we associate the Ruelle operator R_{m_0,m_0} defined by

$$R_{m_0,m_0} f(z) = \frac{1}{N} \sum_{w^N = z} |m_0|^2 (w) f(w), \quad (z \in \mathbb{T}),$$

where f is a measurable function on $\mathbb{T}$. We will see that R_{m_0,m_0} is an operator on the spaces $C(\mathbb{T})$, $L^p(\mathbb{T})$ where $1 \leq p \leq \infty$, and we will describe the spectrum and the eigenvalue spectrum of this operator on these spaces.

Before we give the results, some definitions and notations are needed. We denote by $R = R_{m_0,m_0}$. For a function f on $\mathbb{T}$ and $\rho \in \mathbb{T}$

$$\alpha_\rho(f)(z) = f(\rho z), \quad (z \in \mathbb{T}).$$

For $\varphi \in L^1(\mathbb{R})$,

$$\operatorname{Per}(\varphi)(x) = \sum_{k \in \mathbb{Z}} \varphi(x + 2k\pi), \quad (x \in \mathbb{R}).$$

We call a set $\{z_1, \dots, z_p\}$ a cycle of length p, and denote this by $z_1 \to \cdots \to z_p \to z_1$, if $z_1^N = z_2, z_2^N = z_3, \dots, z_{p-1}^N = z_p, z_p^N = z_1$ and the points $z_1, \dots, z_p$ are distinct. We call $z_1 \to \cdots \to z_p \to z_1$ an m_0-cycle if $|m_0|(z_i) = \sqrt{N}$ for $i \in \{1, \dots, p\}$.

For a complex function f on $\mathbb{T}$ and a positive integer n,

$$f^{(n)}(z) = f(z) f\left(z^N\right) \dots f\left(z^{N^{n-1}}\right), \quad (z \in \mathbb{T}).$$

Theorem 2.1 (The spectrum of R on $C(T)$). *Let m_0 be a function satisfying* (1.1)–(1.4).

(i) *The spectral radius of the operator $R : C(\mathbb{T}) \to C(\mathbb{T})$ is equal to* 1.

(ii) *Each point $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ is an eigenvalue for R, having infinite multiplicity and the spectrum of R on $C(\mathbb{T})$ is the unit disk $\{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$.*

(iii) *(The peripheral spectrum) Let $C_1, \dots, C_n$ be the m_0-cycles,*

$$C_i = z_{1i} \to \cdots \to z_{p_i i} \to z_{1i}, \quad (i \in \{1, \dots, n\}).$$

Let $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then λ is an eigenvalue for R if and only if $\lambda^{p_i} = 1$ for some $i \in \{1, \dots, n\}$. The multiplicity of such a λ equals the cardinality of the set

$$\{i \in \{1, \dots, n\} \,|\, \lambda^{p_i} = 1\}.$$

A basis for the eigenspace corresponding to λ is obtained as follows:

For $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, p_i\}$ define

$$\varphi_{ki}(x) = \prod_{l=1}^{\infty} \frac{e^{-i\theta_i} \alpha_{z_{ki}} \left(m_0^{(p_i)}\right) \left(\frac{x}{N^{l p_i}}\right)}{\sqrt{N^{p_i}}}, \quad (x \in \mathbb{R}),$$

where

$$e^{i\theta_i} = \frac{m_0(z_{1i})}{|m_0|(z_{1i})} \cdots \frac{m_0(z_{p_i i})}{|m_0|(z_{p_i i})}.$$

Then

$$g_{ki} = \alpha_{z_{ki}^{-1}} \left(\mathrm{Per}\, |\varphi_{ik}|^2\right).$$

The basis for the eigenspace corresponding to the eigenvalue λ is

$$\{\sum_{k=1}^{p_i} \lambda^{-k+1} g_{ki} \,|\, i \in \{1, \dots, n\} \text{ with } \lambda^{p_i} = 1\}.$$

Moreover, the functions in this basis are Lipschitz (or trigonometric polynomials when m_0 is one).

Proof. (i) Take $f \in C(\mathbb{T})$ and $z \in \mathbb{T}$.

$$\begin{aligned} |Rf(z)| &\le \frac{1}{N} \sum_{w^N = z} |m_0(w)|^2 |f(w)| \\ &\le \|f\|_\infty \frac{1}{N} \sum_{w^N = z} |m_0(w)|^2 = \|f\|_\infty \end{aligned}$$

Therefore $\|Rf\|_\infty \le \|f\|_\infty$ so the spectral radius is less then 1. But condition (1.4) implies that 1 is an eigenvalue for R so the spectral radius is 1.

(ii) We begin with a lemma

Lemma 2.2. *If $z_0 \to \cdots \to z_{p-1} \to z_0$ is a cycle with p large enough, then there exists a continuous function $f \neq 0$ with $Rf = 0$, such that $f(z_0) = 1$, $f(z_i) = 0$ for $i \in \{1, \dots, p-1\}$.*

Proof. To be able to produce such a function, we will need some conditions on the cycle. We will need z_0 and $e^{\frac{2\pi i}{N}} z_0$ to be outside the set of zeroes of m_0. Because m_0 has only finitely many zeros, this can be achieved as long as p is big enough. We will also need $e^{\frac{2\pi i}{N}} z_0 \neq z_l$ for $l \in \{1, \dots, p-1\}$, but this is true because, otherwise, $z_1 = \left(e^{\frac{2\pi i}{N}} z_0\right)^N = z_l^N = z_{l+1}$ for some $l \in \{1, \dots, p-1\}$.

So, when the cycle is long enough we have that $z_0, e^{\frac{2\pi i}{N}} z_0$ are outside the set of zeroes of m_0 and also $e^{\frac{2\pi i}{N}} z_0 \neq z_l$ for all $l \in \{1, \ldots, p-1\}$. Then we can choose a small interval $[a, b]$ (on $\mathbb{T}$) centered at z_0, such that

$$[a, b] \cup [e^{\frac{2\pi i}{N}} a, e^{\frac{2\pi i}{N}} b] \text{ contains no zeroes of } m_0; \tag{2.1}$$

$$[a, b] \cup [e^{\frac{2\pi i}{N}} a, e^{\frac{2\pi i}{N}} b] \text{ contains no } z_l, l \in \{1, \ldots, p-1\}; \tag{2.2}$$

$$\text{The intervals } [e^{\frac{2k\pi}{N} i} a, e^{\frac{2k\pi}{N} i} b], k \in \{0, \ldots, N-1\} \text{ are disjoint.} \tag{2.3}$$

Define f on $[a, b]$ continuously, to be 1 at z_0 and 0 at a and b. Define f on $[e^{\frac{2\pi i}{N}} a, e^{\frac{2\pi i}{N}} b]$ by

$$f(z) = -\frac{1}{|m_0(z)|^2} |m_0|^2 \left(e^{\frac{-2\pi i}{N}} z\right) f\left(e^{\frac{-2\pi i}{N}} z\right), \quad (z \in [e^{\frac{2\pi i}{N}} a, e^{\frac{2\pi i}{N}} b])$$

and define f to be 0 everywhere else. f is well defined because of (2.1) and (2.3). f is continuous because it is 0 at $a, b, e^{\frac{2\pi i}{N}} a$ and $e^{\frac{2\pi i}{N}} b$. It is also clear that $f(z_0) = 1$ and $f(z_i) = 0$ for $i \in \{1, \ldots, p-1\}$ due to (2.2).

Now we check that $Rf = 0$ which amounts to verifying that

$$\sum_{k=0}^{N-1} |m_0|^2 \left(e^{\frac{2k\pi i}{N}} z\right) f\left(e^{\frac{2k\pi i}{N}} z\right) = 0, \quad (z \in \mathbb{T}) \tag{2.4}$$

The only interesting case is when for some k,

$$e^{\frac{2k\pi i}{N}} z \in [a, b] \cup [e^{\frac{2\pi i}{N}} a, e^{\frac{2\pi i}{N}} b].$$

So assume $e^{\frac{2k\pi i}{N}} z \in [a, b]$ for some $k \in \{0, \ldots, N-1\}$. Then

$$e^{\frac{2(k+1)\pi i}{N}} z \in [e^{\frac{2\pi i}{N}} a, e^{\frac{2\pi i}{N}} b]$$

and, using (2.3), $f\left(e^{\frac{2l\pi i}{N}} z\right) = 0$ for $l \in \{0, \ldots, N-1\} \setminus \{k, k+1\}$. (We use here notation modulo N that is $N = 0, N+1 = 1$ etc.) Having theese, (2.4) follows from the definition of f.

If

$$e^{\frac{2k\pi i}{N}} z \in [e^{\frac{2\pi i}{N}} a, e^{\frac{2\pi i}{N}} b]$$

then

$$e^{\frac{2(k-1)\pi i}{N}} \in [a, b]$$

and we can use the same argument as before to obtain (2.4). This concludes the proof of the lemma. $\square$

We return to the prof of our theorem. Take $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Choose a long enough cycle $z_0 \to \cdots \to z_{p-1} \to z_0$. Lemma (2.2) produces a function $f_{z_0} \in C(\mathbb{T})$ with $Rf_{z_0} = 0$, $f_{z_0}(z_i) = \delta_{0i}$ for $i \in \{0, \ldots, p-1\}$.

Define

$$h_{z_0}(z) = \sum_{n=0}^{\infty} \lambda^n f\left(z^{N^n}\right), \quad (z \in \mathbb{T}).$$

(For $\lambda = 0$ we make the convention $\lambda^0 = 1$.)

The series is uniformly convergent because $\left\|f_{z_0}\left(z^{N^n}\right)\right\|_\infty = \|f\|_\infty$ for all $n \geq 0$ and $|\lambda| < 1$, so h_{z_0} is continuous.

Also, if we use the fact that $R\left(f\left(z^{N^n}\right)\right) = f\left(z^{N^{n-1}}\right)$ for $n \geq 1$, which is a consequence of the definition of R and (1.4), we have:

$$\begin{aligned} Rh_{z_0}(z) &= Rf_{z_0}(z) + \sum_{n=1}^{\infty} \lambda^n R\left(f\left(z^{N^n}\right)\right) \\ &= \lambda \sum_{n=1}^{\infty} \lambda^{n-1} f\left(z^{N^{n-1}}\right) = \lambda h_{z_0} \end{aligned}$$

We evaluate h_{z_0} at the points of the cycle $z_0, z_1, \dots, z_{p-1}$. Note that

$$f_{z_0}\left(z_i^{N^n}\right) = f_{z_0}(z_{n+i}) = \begin{cases} 1 & \text{for} \quad n+i = 0 \mod p \\ 0 & \text{otherwise} \end{cases}$$

(again, we use notation mod p, $z_p = z_0, z_{p+1} = z_1$, etc.)

Hence,

$$h_{z_0}(z_0) = \sum_{m=0}^{\infty} \lambda^{mp} = \frac{1}{1-\lambda^p},$$

$$h_{z_0}(z_i) = \sum_{m=0}^{\infty} \lambda^{p-i+mp} = \frac{\lambda^{p-i}}{1-\lambda^p}, \quad (i \in \{1, \dots, p-1\}),$$

so

$$(h_{z_0}(z_0), \dots, h_{z_0}(z_{p-1})) = \frac{1}{1-\lambda^p}(1, \lambda^{p-1}, \lambda^{p-2}, \dots, \lambda^2, \lambda).$$

Now we make the same construction but considering the cycle starting from z_k. We obtain a function $f_{z_k} \in C(\mathbb{T})$ satisfying $Rf_{z_k} = 0$, $f_{z_k}(z_i) = \delta_{ki}$ and

$$h_{z_k}(z) = \sum_{n=0}^{\infty} \lambda^n f_{z_k}\left(z^{N^n}\right)$$

has the properties $h_{z_k} \in C(\mathbb{T})$, $Rh_{z_k} = \lambda h_{z_k}$ and, for example, for $k = 1$ we have the vector

$$(h_{z_1}(z_0), \dots, h_{z_1}(z_{p-1})) = \frac{1}{1-\lambda^p}(\lambda, 1, \lambda^{p-1}, \lambda^{p-2}, \dots, \lambda^2).$$

Note that this vector is obtained from the previous one (the one corresponding to z_0), after a cyclic permutation. In fact the matrix

$$(1-\lambda^p)\begin{pmatrix} h_{z_0}(z_0) & h_{z_0}(z_1) & \dots & h_{z_0}(z_{p-1}) \\ h_{z_1}(z_0) & h_{z_1}(z_1) & \dots & h_{z_1}(z_{p-1}) \\ \vdots & \vdots & & \vdots \\ h_{z_{p-1}}(z_0) & h_{z_{p-1}}(z_1) & \dots & h_{z_{p-1}}(z_{p-1}) \end{pmatrix}$$

is equal to

$$\begin{pmatrix} 1 & \lambda^{p-1} & \lambda^{p-2} & \dots & \lambda^3 & \lambda^2 & \lambda \\ \lambda & 1 & \lambda^{p-1} & \dots & \lambda^4 & \lambda^3 & \lambda^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \lambda^{p-2} & \lambda^{p-3} & \lambda^{p-4} & \dots & \lambda & 1 & \lambda^{p-1} \\ \lambda^{p-1} & \lambda^{p-2} & \lambda^{p-3} & \dots & \lambda^2 & \lambda & 1 \end{pmatrix}$$

Our goal is to prove that h_{z_k} are linearly independent. We can achieve this if we show that the matrix is nonsingular. For this, look at the entries below the diagonal. We note that, on each row, the part below the diagonal can be obtained from the previous row times λ. Therefore, if we subtract from the $p-1$-st row λ times the $p-2$-nd row, substract from the $p-2$-nd row λ times the $p-3$-rd row,..., substract from the 1-st row λ times the 0-th row, we obtain an upper triangular matrix having $1-\lambda^p$ on each diagonal entry and which has the same determinant as the initial one. Since $|\lambda| < 1$, this matrix will be nonsingular so $h_{z_0}, h_{z_1}, \dots, h_{z_{p-1}}$ are linearly independent eigenvectors that correspond to the eigenvalue λ. As p can be chosen as big as we want, the multiplicity of λ is infinite.

(iii) See [Dutb]. □

Theorem 2.3 (The spectrum of R on $L^\infty(\mathbb{R})$). *Let m_0 be a function on $\mathbb{T}$ satisfying (1.1)–(1.4).*

(i) *The spectral radius of the operator $R : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is equal to 1 and the spectrum of R is the unit disk $\{\lambda \in \mathbb{C} \mid |\lambda| \le 1\}$.*

(ii) *Each point $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$ is an eigenvalue for R of infinite multiplicity.*

Proof. (i) The argument used in the proof of theorem 2.1 applies here to obtain the spectral radius equal to 1 and the fact that the spectrum is the unit disk will follow from (ii).

(ii) If $|\lambda| < 1$ then the assertion follows trivialy from theorem 2.1 (ii). It remains to consider the case $|\lambda| = 1$. Define

$$\varphi(x) = \prod_{n=1}^{\infty} \frac{m_0\left(\frac{x}{N^n}\right)}{\sqrt{N}}, \quad (x \in \mathbb{R}).$$

φ is a well defined, continuous function in $L^2(\mathbb{R})$ and $\operatorname{Per}|\varphi|^2$ is a Lipschitz function on $\mathbb{T}$ (see [BraJo]). Also, $\operatorname{Per}|\varphi|^2(0) = 1$, $\varphi(0) = 1$ and

$$\varphi(x) = \frac{m_0\left(\frac{x}{N}\right)}{\sqrt{N}} \varphi\left(\frac{x}{N}\right), \quad (x \in \mathbb{R}),$$

(φ is the Fourier transform of a scaling function).

Now, consider a function $f \in L^\infty(\mathbb{R})$ with the property that $f(x) = \frac{1}{\lambda} f\left(\frac{x}{N}\right)$ a.e. on $\mathbb{R}$, and take $h_f = \operatorname{Per}\left(f|\varphi|^2\right)$. Clearly, $|h_f(z)| \le \|f\|_\infty \operatorname{Per}|\varphi|^2(z)$ for $z \in \mathbb{T}$ so h_f is an $L^\infty(\mathbb{T})$ function. We want to prove that

$$Rh_f = \lambda h_f.$$

We have

$$\begin{aligned}
h_f(x) &= \sum_{k\in\mathbb{Z}} f(x+2k\pi)|\varphi|^2(x+2k\pi)\\
&= \sum_{k\in\mathbb{Z}} \frac{1}{\lambda} f\left(\frac{x+2k\pi}{N}\right) \frac{1}{N}|m_0|^2\left(\frac{x+2k\pi}{N}\right)|\varphi|^2\left(\frac{x+2k\pi}{N}\right)\\
&= \frac{1}{N}\frac{1}{\lambda}\sum_{l=0}^{N-1}\sum_{m\in\mathbb{Z}} f\left(\frac{x+2(Nm+l)\pi}{N}\right)|m_0|^2\left(\frac{x+2(Nm+l)\pi}{N}\right)\cdot\\
&\cdot|\varphi|^2\left(\frac{x+2(Nm+l)\pi}{N}\right)\\
&= \frac{1}{N}\frac{1}{\lambda}\sum_{l=0}^{N-1}|m_0|^2\left(\frac{x+2l\pi}{N}\right)\sum_{m\in\mathbb{Z}} f|\varphi|^2\left(\frac{x+2k\pi}{N}+2m\pi\right)\\
&= \frac{1}{\lambda}Rh_f.
\end{aligned}$$

so, we have indeed $Rh_f = \lambda h_f$.

Next, we argue why the vector space

$$\left\{h_f \mid f\in L^\infty(\mathbb{R}), f(x)=\frac{1}{\lambda}f\left(\frac{x}{N}\right)\text{ a.e. on }\mathbb{R}\right\}$$

is infinite dimensional.

For this, we prove first that if $h_f = 0$ then $f = 0$. Indeed, if $h_f = 0$ then

$$f(x)|\varphi|^2(x) = -\sum_{k\in\mathbb{Z}\setminus\{0\}} f(x+2k\pi)|\varphi|^2(x+2k\pi). \tag{2.5}$$

We claim that the term on the right converges to 0 as $x \to 0$. We have

$$\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} f(x+2k\pi)|\varphi|^2(x+2k\pi)\right| \le$$

$$\le \|f\|_\infty \sum_{k\in\mathbb{Z}\setminus\{0\}}|\varphi|^2(x+2k\pi) = \|f\|_\infty\left(\mathrm{Per}\,|\varphi|^2(x)-|\varphi|^2(x)\right)\to 0,$$

because both $\mathrm{Per}\,|\varphi|^2$ and $|\varphi|^2$ are continuous and their value at 0 is 1. Then, using (2.5), we obtain $f(x)\to 0$ as $x\to 0$. But we know that $f(x)=\frac{1}{\lambda}f\left(\frac{x}{N}\right)$ a.e. on $\mathbb{R}$. So for a.e. x we have

$$f\left(\frac{x}{N^n}\right) = \lambda^n f(x),\quad \text{for all } n.$$

This implies that

$$\left|f\left(\frac{x}{N^n}\right)\right| = |f(x)|,\quad (n\in\mathbb{N})$$

and, coupled with the limit of f at 0, it entails that f is constant 0 almost everywhere.

Having these, we try to construct a set of p linearly independent functions h_f with $p \in \mathbb{N}$ arbitrary. Take p linearly independent functions $g_1, \dots, g_p$ in $L^\infty([-N,-1] \cup [1,N])$. Define f_i, $i \in \{1,\dots,p\}$ on $\mathbb{R}$ as follows: let $f_i(x) = g_i(x)$ on $[-N,-1] \cup [1,N]$ and extend it on $\mathbb{R}$ requiring that

$$\frac{1}{\lambda} f_i\left(\frac{x}{N}\right) = f_i(x), \quad (x \in \mathbb{R}).$$

That is, for $x \in [-\frac{1}{N^l}, -\frac{1}{N^{l+1}}] \cup [\frac{1}{N^{l+1}} \cup \frac{1}{N^l}]$

$$f_i(x) = \lambda^{l+1} g_i(N^{l+1}x),$$

for all $l \in \mathbb{Z}$. Since $|\lambda| = 1$, $f_1, \dots, f_p$ are in $L^\infty(\mathbb{R})$ and they are linearly independent because $g_1, \dots, g_p$ are. $h_{f_1}, \dots, h_{f_p}$ are linearly independent by the following argument: if for some complex constants $a_1, \dots, a_p$ we have $a_1 h_{f_1} + \cdots + a_p h_{f_p} = 0$ then $h_{a_1 f_1 + \cdots + a_p f_p} = 0$ so $a_1 f_1 + \cdots + a_p f_p = 0$ and $a_1 = \cdots = a_p = 0$ by linear independence. Since we proved that $Rh_{f_i} = \lambda_{f_i}$, $i \in \{1,\dots,p\}$, and since p is arbitrary, it follows that the multiplicity of the eigenvalue λ is infinite. □

Theorem 2.4 (The spectrum of R on $L^p(\mathbb{T})$). *Let m_0 be a function on $\mathbb{T}$ satisfying (1.1)–(1.4) and $1 \le p < \infty$.*

(i) *The spectral radius of the operator $R : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ is equal to $N^{\frac{1}{p}}$ and the spectrum of R is the disk $\{\lambda \in \mathbb{C} \mid |\lambda| \le N^{\frac{1}{p}}\}$.*

(ii) *Each point $\lambda \in \mathbb{C}$ with $|\lambda| < N^{\frac{1}{p}}$ is an eigenvalue for R of infinite multiplicity.*

(iii) *There are no eigenvalues of R with $|\lambda| = N^{\frac{1}{p}}$.*

Proof. (i) is proved in [BraJo] but we present here a different argument that we will need for (iii) also. Take $f \in L^p(\mathbb{T})$.

$$\begin{aligned}\|Rf\|_p &= \left(\int_0^{2\pi} \left|\frac{1}{N}\sum_{k=0}^{N-1} |m_0|^2 f\left(\frac{\theta+2k\pi}{N}\right)\right|^p d\theta\right)^{\frac{1}{p}} \\ &\le \left(\int_0^{2\pi} \left(\frac{1}{N}\sum_{k=0}^{N-1} |m_0|^2 |f|\left(\frac{\theta+2k\pi}{N}\right)\right)^p d\theta\right)^{\frac{1}{p}}\end{aligned}$$

Since

$$\frac{1}{N}\sum_{k=0}^{N-1} |m_0|^2 \left(\frac{\theta+2k\pi}{N}\right) = 1, \quad (\theta \in [0,2\pi])$$

and $x \mapsto x^p$ is convex, we can use Jensen's inequality:

$$\begin{aligned}\left(\frac{1}{N}\sum_{k=0}^{N-1} |m_0|^2 |f|\left(\frac{\theta+2k\pi}{N}\right)\right)^p &\le \frac{1}{N}\sum_{k=0}^{N-1} |m_0|^2 |f|^p \left(\frac{\theta+2k\pi}{N}\right) \\ &\le \sum_{k=0}^{N-1} |f|^p \left(\frac{\theta+2k\pi}{N}\right)\end{aligned}$$

For the last inequality we used the fact that $|m_0|^2 \le N$ which follows from (1.4).

Also, by a change of variable,

$$\left(\int_0^{2\pi}\sum_{k=0}^{N-1}|f|^p\left(\frac{\theta+2k\pi}{N}\right)d\theta\right)^{\frac{1}{p}} = \left(\sum_{k=0}^{N-1}N\int_{\frac{2k\pi}{N}}^{\frac{2(k+1)\pi}{N}}|f(\theta)|^p\,d\theta\right)^{\frac{1}{p}}$$
$$= N^{\frac{1}{p}}\int_0^{2\pi}|f(\theta)|^p\,d\theta.$$

Putting together the previous equalities and inequalities we obtain that $\|Rf\|_p \leq N^{\frac{1}{p}}\|f\|_p$. This implies that the norm and the spectral radius of the operator $R : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ are less than $N^{\frac{1}{p}}$. A result of R. Nussbaum (see [BraJo]) shows that every $\lambda \in \mathbb{C}$ with $1 < |\lambda| < N^{\frac{1}{p}}$ is an eigenvalue of R of infinite multiplicity. Also theorem 2.3 shows that all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ is an eigenvalue of R of infinite multiplicity. This establishes (i) and (ii).

It remains to prove that (iii) is valid. Suppose there is a function $f \in L^p(\mathbb{T})$ and $\lambda \in \mathbb{C}$ such that $|\lambda| = N^{\frac{1}{p}}$ and $Rf = \lambda f$. Then $\|Rf\|_p = N^{\frac{1}{p}}\|f\|_p$ so we have equalities in all inequalities that we used for proving (i). In particular, we have

$$\int_0^{2\pi}\frac{1}{N}\sum_{k=0}^{N-1}|m_0|^2|f|^p\left(\frac{\theta+2k\pi}{N}\right)d\theta = \int_0^{2\pi}\sum_{k=0}^{N-1}|f|^p\left(\frac{\theta+2k\pi}{N}\right)d\theta$$

and, since $\frac{|m_0|^2}{N} \leq 1$ the corresponding terms of the sums must be equal: for $k \in \{0,\dots,N-1\}$,

$$\int_0^{2\pi}\frac{|m_0|^2\left(\frac{\theta+2k\pi}{N}\right)}{N}|f|^p\left(\frac{\theta+2k\pi}{N}\right)d\theta = \int_0^{2\pi}|f|^p\left(\frac{\theta+2k\pi}{N}\right)d\theta$$

Therefore, utilizing again $|m_0|^2 \leq N$,

$$\frac{|m_0|^2\left(\frac{\theta+2k\pi}{N}\right)}{N}|f|^p\left(\frac{\theta+2k\pi}{N}\right) = |f|^p\left(\frac{\theta+2k\pi}{N}\right)$$

for almost every $\theta \in [0,2\pi]$ and for all $k \in \{0,\dots,N-1\}$. But this implies that $\frac{1}{N}|m_0|^2|f|^p = |f|^p$ almost everywhere on $\mathbb{T}$. However, m_0 is continuous and has finitely many zeroes and, because $\sum_{w^N=z}|m_0|^2(w) = N$ for all $z \in \mathbb{T}$, this implies that $|m_0|^2(z) = N$ for at most finitely many points so f must be 0 almost everywhere. In conclusion, there are no eigenvalues λ of modulus $N^{\frac{1}{p}}$ and the proof of the theorem is complete. □

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