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Integral Equations and Operator Theory

Finite Interval Convolution Operators with Transmission Property

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Abstract. We study convolution operators in Bessel potential spaces and (fractional) Sobolev spaces over a finite interval. The main purpose of the investigation is to find conditions on the convolution kernel or on a Fourier symbol of these operators under which the solutions inherit higher regularity from the data. We provide conditions which ensure the transmission property for the finite interval convolution operators between Bessel potential spaces and Sobolev spaces. These conditions lead to smoothness preserving properties of operators defined in the above-mentioned spaces where the kernel, cokernel and, therefore, indices do not depend on the order of differentiability. In the case of invertibility of the finite interval convolution operator, a representation of its inverse is presented in terms of the canonical factorization of a related Fourier symbol matrix function.

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1. Introduction

It is well known that the theory of convolution equations

$$\varphi(x) + \int_0^a k_a(x-y)\varphi(y)\,dy = f(x), \qquad x \in]0,a[, \tag{1.1}$$

where $k_a \in \mathbb{L}_1(]-a,a[)$, on semi-infinite intervals ($a=\infty$) is rather well developed. In particular, the solvability theory of (1.1) is well known (see, e.g., [11, 12, 15, 16, 21, 29, 30]) for various classes of kernel functions and different space settings of Besov-Triebel-Lizorkin type (also weighted spaces).

The situation is completely different for convolution equations on finite intervals (e.g., when $0 < a < +\infty$), which one encounters in several applications [28]. A part of the problems arises due to the difficulty of application of the Wiener-Hopf method [33] to such convolution equations (cf. [11, 14, 25, 26]). Another part of problems arises when we try to relate such equations to equations on the half-line: the appearance of semi-almost periodic terms leads to particular and cumbersome difficulties (see, e.g., [2, 3, 4, 7, 9, 13, 19, 28]).

Here, we will work in the setting of Bessel potential spaces $\mathbb{H}_p^s(\mathbb{R})$ and (fractional) Sobolev spaces $\mathbb{W}_p^s(\mathbb{R})$. Using the Fourier transformation $\mathcal{F}$, the space $\mathbb{H}_p^s(\mathbb{R})$, with $s \in \mathbb{R}$ and $p \in]1, +\infty[$, is defined as the space of tempered distributions φ such that

$$\left\| \varphi \,|\mathbb{H}_p^s(\mathbb{R}) \right\| = \left\| \mathcal{F}^{-1} \lambda^s \cdot \mathcal{F} \varphi \,|\mathbb{L}_p(\mathbb{R}) \right\| < +\infty, \tag{1.2}$$

for $\lambda(\xi) = (1 + \xi^2)^{1/2}$, $\xi \in \mathbb{R}$. As is well known, if $s \geq 0$, $\mathbb{W}_p^s(\mathbb{R})$ is the space of elements in $\mathbb{L}^p(\mathbb{R})$ such that

$$\left\| \varphi | \mathbb{W}_p^s(\mathbb{R}) \right\|^p = \sum_{h=0}^{[s]} \left\| D^h \varphi \right\|_{\mathbb{L}^p(\mathbb{R})}^p + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\left| D^{[s]} \varphi(x) - D^{[s]} \varphi(y) \right|^p}{|x - y|^{1 + p\{s\}}} dx dy < \infty$$

(where D denotes differentiation, $[s]$ is the largest integer less or equal to s and $s = [s] + \{s\}$). For $s < 0$, $\mathbb{W}_p^s(\mathbb{R}) = \left(\mathbb{W}_q^{-s}(\mathbb{R}) \right)'$, where $1/p + 1/q = 1$ assuming $p, q \in]1, +\infty[$ (throughout this paper).

Moreover, we denote by $\widetilde{\mathbb{H}}_p^s(]0, a[)$ the closed subspace of $\mathbb{H}_p^s(\mathbb{R})$ consisting of those distributions which are supported in $[0, a]$. $\mathbb{H}_p^s(]0, a[)$ denotes the space of generalized functions on $]0, a[$ which have extensions into $\mathbb{R}$ that belong to $\mathbb{H}_p^s(\mathbb{R})$. The space $\mathbb{H}_p^s(]0, a[)$ is endowed with the norm of the quotient space $\mathbb{H}_p^s(\mathbb{R}) / \widetilde{\mathbb{H}}_p^s(\mathbb{R} \backslash [0, a])$. Analogous spaces are considered if we start with $\mathbb{W}_p^s$. For $1/p - 1 < s < 1/p$ the spaces $\widetilde{\mathbb{H}}_p^s(]0, a[)$ and $\mathbb{H}_p^s(]0, a[)$ as well as the corresponding spaces for $\mathbb{W}_p^s$ can be identified. In particular, these definitions are valid for the Lebesgue spaces and we use the notation $\mathbb{L}_p(\mathbb{R}_+)$ for the space $\mathbb{H}_p^0(\mathbb{R}_+)$.

For a Banach space $\mathbb{Y}$, by $[\mathbb{Y}]^n$ we denote the Banach space of n-tuples $y = (y_1, \ldots, y_n)$, with $y_1, \ldots, y_n \in \mathbb{Y}$, endowed with the norm

$$\|y\| = \sum_{j=1}^{n} \|y_j | \mathbb{Y}\| . \tag{1.3}$$

From now on, throughout the paper, we take $0 < a < +\infty$ and use the abbreviation $\mathbb{X}_p^s$ to represent indistinctly $\mathbb{H}_p^s$ and $\mathbb{W}_p^s$.

First we recall a well-known boundedness property of the operator $\mathcal{K}_a$ in the left hand side of (1.1) for the case of the "tilde"-space domains. For this purpose, let $r_{\mathbb{R} \to]0,a[}$ denote the restriction operator from $\mathbb{R}$ to $]0, a[$ (acting between corresponding spaces), let $k \in \mathbb{L}_1(\mathbb{R})$ denote any extension of k_a from $]-a, a[$ to the full line and $\widehat{k} = \mathcal{F}k$. Now let, for appropriate function spaces,

$$W_\sigma^0 = \mathcal{F}^{-1} \sigma \cdot \mathcal{F} \tag{1.4}$$

denote the translation invariant operator on the real axis $\mathbb{R}$. The operator in the left hand side of (1.1) can be written in the form

$$\mathcal{K}_a = r_{\mathbb{R}\to]0,a[} W^0_{1+\widehat{k}} \tag{1.5}$$

and does not depend on the particular choice of the extension $k = \ell k_a \in \mathbb{L}_1(\mathbb{R})$ of k_a (one can take even the extension by zero $k = \ell_0 k_a$ where $k(x) = 0$ for $|x| > a$).

Proposition 1.1. *If there is a positive constant C such that*

$$\left\| W^0_{1+\widehat{k}}\psi | \mathbb{X}^s_p(\mathbb{R}) \right\| \leq C \left\| \psi | \mathbb{X}^s_p(\mathbb{R}) \right\| \quad \textit{for all} \;\; \psi \in \mathbb{X}^s_p(\mathbb{R})$$

(which reads: $1+\widehat{k}$ is a p-multiplier; see [11, 31]*) then*

$$\left\| \mathcal{K}_a\varphi | \mathbb{X}^s_p(]0,a[) \right\| \leq C \left\| \varphi | \widetilde{\mathbb{X}}^s_p(]0,a[) \right\| \quad \textit{for all} \;\; \varphi \in \widetilde{\mathbb{X}}^s_p(]0,a[)\,.$$

Proof. Let $\varphi \in \widetilde{\mathbb{X}}^s_p(]0,a[)$. The result follows directly from the definition of the norms in $\mathbb{X}^s_p(]0,a[)$ and $\widetilde{\mathbb{X}}^s_p(]0,a[)$ together with estimates for convolutions of $\mathbb{L}_1(\mathbb{R})$ and $\mathbb{L}_p(\mathbb{R})$ functions:

$$\begin{aligned} \left\| \mathcal{K}_a\varphi | \mathbb{X}^s_p(]0,a[) \right\| &= \left\| r_{\mathbb{R}\to]0,a[}\mathcal{F}^{-1}(1+\widehat{k})\cdot\mathcal{F}\varphi | \mathbb{X}^s_p(]0,a[) \right\| \\ &= \inf_{\ell} \left\| \ell\, r_{\mathbb{R}\to]0,a[}\mathcal{F}^{-1}(1+\widehat{k})\cdot\mathcal{F}\varphi | \mathbb{X}^s_p(\mathbb{R}) \right\| \\ &\leq C \left\| \varphi | \widetilde{\mathbb{X}}^s_p(]0,a[) \right\|, \end{aligned}$$

where $\ell\psi$ stands for any extension of ψ into $\mathbb{X}^s_p(\mathbb{R})$ and the infimum is taken with respect to all possible extensions. □

As a consequence the operator

$$\mathcal{K}_a : \widetilde{\mathbb{X}}^s_p(]0,a[) \to \mathbb{X}^s_p(]0,a[), \qquad s \in \mathbb{R}, \quad p \in]1,+\infty[, \tag{1.6}$$

is bounded in the present space setting. Further it is known (see [11, Theorem 8.8], [6, §2] and [24]) that the Fredholm property and its characteristics (defect numbers and index) depend on the smoothness parameter s and on p as well.

Let us assume that equation (1.1) has a solution $\varphi \in \widetilde{\mathbb{X}}^s_p(]0,a[)$ for a given $f \in \mathbb{X}^s_p(]0,a[)$. Now if the right hand side has an additional smoothness $f \in \mathbb{X}^{s+m}_p(]0,a[)$, $m = 1,2,\ldots$, for ensuring the same additional smoothness for the solution $\varphi \in \widetilde{\mathbb{X}}^{s+m}_p(]0,a[)$ we must impose m orthogonality conditions on f.

We can choose another option: consider a space setting different from (1.6) in order to obtain a result which is independent of the smoothness order. Such results are important for several reasons, including applications in numerical methods (see, e.g., [23]).

For this purpose we change the space setting (1.6) to the following one

$$\mathcal{K}_a : \mathbb{X}^s_p(]0,a[) \to \mathbb{X}^s_p(]0,a[), \tag{1.7}$$

which is common in the theory of pseudodifferential equations on manifolds with boundary (see [5, 12, 18, 27, 29]). Conditions on the symbols of the operators or on the kernel which ensure boundedness of pseudodifferential operators in "non-tilde" space settings are known in the literature as a transmission property.

Therefore we have first to find conditions for the kernel function k_a which ensures the boundedness of the operator in (1.7). This will lead to a priori smoothness $\varphi \in \mathbb{X}_p^{s+m}(]0,a[)$ of the solution, whenever the right-hand side is given in the same space, $f \in \mathbb{X}_p^{s+m}(]0,a[)$, and without imposing any orthogonality conditions on f.

Besides the boundedness and a priori smoothness, we will look for the Fredholm property and a representation of the inverse of $\mathcal{K}_a$ provided it exists.

2. Relations with Wiener-Hopf Operators

In this section we present some auxiliary results. In particular, we will present relations between the finite interval convolution operator $\mathcal{K}_a$ and corresponding Wiener-Hopf operators in the form of operator matrix identities. This will help us later to extract and transfer information from the Wiener-Hopf operators to our initial operator $\mathcal{K}_a$.

Theorem 2.1. *Assume that we have non-critical space orders:* $1 < p < \infty$, $s - 1/p \in \mathbb{R}\backslash\mathbb{Z}$ *and* $s = s' + s'' > -1 + 1/p$ *with* $s' \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ *and* $s'' \in]-1+1/p, 1/p[$.

Let $k \in \mathbb{L}_1(\mathbb{R})$ *and* $W = W_{1+\widehat{k},\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}(1+\widehat{k}) \cdot \mathcal{F}\ell_0 : \mathbb{X}_p^s(\mathbb{R}_+) \to \mathbb{X}_p^s(\mathbb{R}_+)$ *be bounded as a restriction* ($s > 0$) *or as a continuous extension* ($s < 0$), *respectively, from* $\mathbb{L}_p(\mathbb{R}_+)$ *where* ℓ_0 *denotes the extension by zero into the full line. Then finite interval convolution operator* $\mathcal{K}_a$ *(see* (1.7)*) is equivalent after extension to the Wiener-Hopf operator*

$$W_{\Psi,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Psi \cdot \mathcal{F} \begin{bmatrix} \ell & 0 \\ 0 & \ell_0 \end{bmatrix} : \mathbb{X}_p^s(\mathbb{R}_+) \times \left(r_{\mathbb{R}\to\mathbb{R}_+}\widetilde{\mathbb{X}}_p^s(\mathbb{R}_+)\right) \to [\mathbb{X}_p^s(\mathbb{R}_+)]^2, \tag{2.1}$$

where $\ell : \mathbb{X}_p^s(\mathbb{R}_+) \to \mathbb{X}_p^s(\mathbb{R})$ *denotes any extension (i.e. the operator is independent of that choice) and*

$$\Psi = \begin{bmatrix} \left(\frac{\lambda_-}{\lambda_+}\right)^{-s'} \tau_{-a} & 0 \\ 1+\widehat{k} & \tau_a \end{bmatrix}, \tag{2.2}$$

with $\lambda_\pm(\xi) = \xi \pm i$ *and* $\tau_{\pm a}(\xi) = e^{\pm ia\xi}$, *for* $\xi \in \mathbb{R}$. *This means, by definition* [1], *that there are additional Banach spaces* $\mathbb{Y}$ *and* $\mathbb{Z}$ *and invertible bounded operators* E *and* F *so that*

$$\begin{bmatrix} \mathcal{K}_a & 0 \\ 0 & I_{\mathbb{Y}} \end{bmatrix} = E \begin{bmatrix} W_{\Psi,\mathbb{R}_+} & 0 \\ 0 & I_{\mathbb{Z}} \end{bmatrix} F. \tag{2.3}$$

In the present case, the extension by $I_{\mathbb{Z}}$ *can be omitted.*

Proof. According to the symmetric space setting, $\mathcal{K}_a$ is equivalent to a general Wiener-Hopf operator [29]

$$\widetilde{\mathcal{K}_a} = PW_{|P\mathbb{X}^s_p(\mathbb{R}_+)}$$

where the projector P projects along $r_{\mathbb{R}\to\mathbb{R}_+}T_a\widetilde{\mathbb{X}}^s_p(\mathbb{R}_+)$ (with $T_a = \mathcal{F}^{-1}\tau_a\cdot\mathcal{F}$) and $W = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}(1+\widehat{k})\cdot\mathcal{F}\ell_0$.

We can take the complementary projector as

$$Q = I - P = r_{\mathbb{R}\to\mathbb{R}_+}T_a\Lambda_+^{-s'}\ell_0 r_{\mathbb{R}\to\mathbb{R}_+}\Lambda_+^{s'}T_{-a}\ell$$

because

$$\begin{aligned}\mathbb{X}^s_p(\mathbb{R}_+) = r_{\mathbb{R}\to\mathbb{R}_+}\mathbb{X}^s_p(\mathbb{R}) &= r_{\mathbb{R}\to\mathbb{R}_+}\left(T_a\Lambda_+^{-s'}\ell_0\mathbb{X}^{s''}_p(\mathbb{R}_+)\oplus T_a\Lambda_+^{-s'}\ell_0\mathbb{X}^{s''}_p(\mathbb{R}_-)\right)\\ &= \mathrm{Im\,Q}\oplus\,\mathrm{Ker\,Q}\end{aligned}$$

where $\Lambda^s_\pm = \mathcal{F}^{-1}\lambda^s_\pm\cdot\mathcal{F}$.

It is known (e.g. from [8, formula (4.6)]) that $\widetilde{\mathcal{K}_a}$ is equivalent after extension to

$$T = PW + Q : \mathbb{X}^s_p(\mathbb{R}_+)\to\mathbb{X}^s_p(\mathbb{R}_+) \tag{2.4}$$

which has the form of a *paired operator* on $\mathbb{X}^s_p(\mathbb{R}_+)$.

Now, rewriting Q in the form of the following factorization

$$Q = W_1\,W_2 = \left(r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\lambda_+^{-s'}\tau_a\cdot\mathcal{F}\ell_0\right)\left(r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\lambda_+^{s'}\tau_{-a}\cdot\mathcal{F}\ell\right)$$

we obtain, by the extension method of [8, (4.6)-(4.12)], that $T = PW + W_1\,W_2$ is equivalent after extension to

$$\begin{bmatrix} W_2 & 0\\ W & W_1\end{bmatrix} : \mathbb{X}^s_p(\mathbb{R}_+)\times\mathbb{X}^{s''}_p(\mathbb{R}_+)\to\mathbb{X}^{s''}_p(\mathbb{R}_+)\times\mathbb{X}^s_p(\mathbb{R}_+) \tag{2.5}$$

which obviously represents a bounded 2×2 matrix Wiener-Hopf operator with the symbol

$$\Psi_0 = \begin{bmatrix}\lambda_+^{s'}\tau_{-a} & 0\\ 1+\widehat{k} & \lambda_+^{-s'}\tau_a\end{bmatrix}.$$

At the end, lifting into the $\mathbb{X}^s_p(\mathbb{R}_+)$ setting yields equivalence with

$$\begin{aligned}W_{\Psi,\mathbb{R}_+} = \begin{bmatrix} r_{\mathbb{R}\to\mathbb{R}_+}\Lambda_-^{-s'}\ell_0 & 0\\ 0 & I_{\mathbb{X}^s_p(\mathbb{R}_+)}\end{bmatrix}\begin{bmatrix} W_2 & 0\\ W & W_1\end{bmatrix}\begin{bmatrix} I_{\mathbb{X}^s_p(\mathbb{R}_+)} & 0\\ 0 & r_{\mathbb{R}\to\mathbb{R}_+}\Lambda_+^{s'}\ell_0\end{bmatrix}\\ : \mathbb{X}^s_p(\mathbb{R}_+)\times\left(r_{\mathbb{R}\to\mathbb{R}_+}\widetilde{\mathbb{X}}^s_p(\mathbb{R}_+)\right)\to[\mathbb{X}^s_p(\mathbb{R}_+)]^2\end{aligned} \tag{2.6}$$

and the three factors amalgamate into the form of (2.1). The fact that E and F in (2.3) are bounded invertible operators results from the assumption that W is bounded. This is needed in (2.4) and (2.5) to guarantee that the relations (in brief)

$$PWP + Q = (I - PWQ)(PW + Q)$$

$$\begin{bmatrix} PW+Q & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -W_1 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & W_2 \end{bmatrix} \begin{bmatrix} W_2 & 0 \\ W & W_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ -W_2(W-I) & I \end{bmatrix} \tag{2.7}$$

have bounded entries $I - PWQ$ and $-W_2(W-I)$, respectively (all other terms contributing to E and F are bounded anyway). □

Remark 2.2. First let us note that the operator $W_{\Psi,\mathbb{R}_+}$ does not depend on the particular extension $\ell\psi_1 \in \mathbb{X}^s_p(\mathbb{R})$ that is taken for the first component $\psi_1 \in \mathbb{X}^s_p(\mathbb{R}_+)$.

Secondly, what happens, if we drop the boundedness of W from the assumptions in Theorem 2.1 ? Following the proof we find that (2.3) remains true in the sense of a so-called *algebraic equivalence after extension relation* [9, 22] where E and F are not necessarily bounded but densely defined and injective operators with dense images.

Proposition 2.3. *Under the same assumptions, the operator* $W_{\Psi,\mathbb{R}_+}$*, introduced in* (2.1)*, is equivalent to the Wiener-Hopf operator*

$$W_{\Phi,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Phi\cdot\mathcal{F}\begin{bmatrix} \ell & 0 \\ 0 & \ell_0 \end{bmatrix} : \mathbb{X}^s_p(\mathbb{R}_+)\times\mathbb{X}^{s''}_p(\mathbb{R}_+) \to \mathbb{X}^s_p(\mathbb{R}_+)\times\mathbb{X}^{s''}_p(\mathbb{R}_+),$$

where

$$\Phi(\xi) = \begin{bmatrix} 1+\widehat{k} & \tau_a\,\widehat{k}\,\lambda_+^{-s'} \\ \tau_{-a}\,\widehat{k}\,\lambda_+^{s'} & -1+\widehat{k} \end{bmatrix}, \tag{2.8}$$

is an invertible matrix of elements in the Wiener algebra provided $\widehat{k}\lambda_+^{s'} \in \mathcal{F}\mathbb{L}_1(\mathbb{R})$.

Proof. For $s = 0$, this result can be found e.g. in [13] (see also [25]).

Let us consider the following auxiliary bounded linear operators

$$W_{G_+,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}G_+\cdot\mathcal{F}\begin{bmatrix} \ell & 0 \\ 0 & \ell_0 \end{bmatrix} : \mathbb{X}^s_p(\mathbb{R}_+)\times\left(r_{\mathbb{R}\to\mathbb{R}_+}\widetilde{\mathbb{X}}^s_p(\mathbb{R}_+)\right) \to \mathbb{X}^s_p(\mathbb{R}_+)\times\mathbb{X}^{s''}_p(\mathbb{R}_+),$$

$$W_{G_-,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}G_-\cdot\mathcal{F}\begin{bmatrix} \ell & 0 \\ 0 & \ell_0 \end{bmatrix} : \mathbb{X}^s_p(\mathbb{R}_+)\times\mathbb{X}^{s''}_p(\mathbb{R}_+) \to \mathbb{X}^s_p(\mathbb{R}_+)\times\mathbb{X}^s_p(\mathbb{R}_+),$$

with

$$G_+ = \begin{bmatrix} 1 & \tau_a \\ 0 & -\lambda_+^{s'} \end{bmatrix},$$

$$G_- = \begin{bmatrix} \tau_{-a}\lambda_-^{-s'}\lambda_+^{s'} & -\lambda_-^{-s'} \\ 1 & 0 \end{bmatrix}.$$

Evidently the operators $W_{G_\pm,\mathbb{R}_+}$ are bounded invertible. Moreover, the structure of $W_{G_+,\mathbb{R}_+}$ and $W_{G_-,\mathbb{R}_+}$ allows to prove by a straightforward computation the identity

$$W_{G_-,\mathbb{R}_+} W_{\Phi,\mathbb{R}_+} W_{G_+,\mathbb{R}_+} = W_{\Psi,\mathbb{R}_+}, \tag{2.9}$$

which provides the equivalence between $W_{\Psi,\mathbb{R}_+}$ and $W_{\Phi,\mathbb{R}_+}$. □

3. The Boundedness of Convolution Operators with Transmission Property

Let us recall a result from [12] (we will apply these definitions later). The spaces defined below ensure a transmission property for the corresponding convolution operators on the half-line.

Let $1 < p < \infty$, $s > -1 + 1/p$ and define

$$\mathbb{TX}_p^s(\mathbb{R}) \;=\; \left\{\varphi \in \mathbb{L}_1(\mathbb{R}) : \left\|\varphi|\mathbb{TX}_p^s(\mathbb{R})\right\| = \|\varphi|\mathbb{L}_1(\mathbb{R})\| + \|\varphi\|_{\mathbb{X}}^{(s,p)} < +\infty\right\}, \tag{3.1}$$

where

$$\|\varphi\|_{\mathbb{X}}^{(s,p)} = \begin{cases} 0, & \text{if } \; -1+1/p < s < 1/p \\ \left\|r_{\mathbb{R}\to\mathbb{R}_+}\varphi|\mathbb{X}_p^{s-1}(\mathbb{R}_+)\right\|, & \text{if } \; \begin{array}{l} n-1+1/p < s < n+1/p, \\ n = 1,2,\dots \end{array} \\ \inf_{\nu>s}\left\|r_{\mathbb{R}\to\mathbb{R}_+}\varphi|\mathbb{X}_p^{\nu-1}(\mathbb{R}_+)\right\|, & \text{if } \; s = n+1/p,\; n = 0,1,\dots \end{cases}$$

Next we define similar spaces which ensure a transmission property for the kernels on the interval $]-a,a[$. Let again $1 < p < \infty$, $s > -1 + 1/p$ and define

$$\mathbb{TX}_p^s(]-a,a[) \;=\; \Big\{\varphi \in \mathbb{L}_1(]-a,a[) :$$
$$\left\|\varphi|\mathbb{TX}_p^s(]-a,a[)\right\| = \|\varphi|\mathbb{L}_1(]-a,a[)\| + \|\varphi\|_{\mathbb{X}_a}^{(s,p)} < +\infty\Big\}$$

where

$$\|\varphi\|_{\mathbb{X}_a}^{(s,p)} = \begin{cases} 0, & \text{if } \; -1+1/p < s < 1/p \\ \left\|\varphi|\mathbb{X}_p^{s-1}(]-a,a[)\right\|, & \text{if } \; \begin{array}{l} n-1+1/p < s < n+1/p, \\ n = 1,2,\dots \end{array} \\ \inf_{\nu>s}\left\|\varphi|\mathbb{X}_p^{\nu-1}(]-a,a[)\right\|, & \text{if } \; s = n+1/p,\; n = 0,1,\dots \end{cases}$$

Lemma 3.1. *Let $k_a \in \mathbb{TX}_p^s(]-a,a[)$.*

i. *There exists $k \in \mathbb{TX}_p^s(\mathbb{R})$ such that $k_a = r_{\mathbb{R}\to]-a,a[}k$.*

ii. *Further there exists a continuous linear extension operator*

$$E_{p,a}^s : \mathbb{TX}_p^s(]-a,a[) \to \mathbb{TX}_p^s(\mathbb{R})$$

with the property i. *where $k = E_{p,a}^s k_a$.*

iii. *Moreover all extensions can be chosen such that* $\operatorname{supp} k \subset [-a-\epsilon, a+\epsilon]$, *for any given* $\epsilon > 0$.

Proof. Proposition i. follows from the extensibility of $\mathbb{X}_p^s(]-a,a[)$ distributions into $\mathbb{X}_p^s(\mathbb{R})$ [31], e.g. with any compact support $\mathcal{K} \supset\supset]-a,a[$ such that $k \in \mathbb{L}_1(\mathbb{R})$, because $\mathbb{L}_p(\mathcal{K}) \subset \mathbb{L}_1(\mathcal{K})$ for bounded measurable $\mathcal{K}$. Continuous extension operators can be constructed by the same argument, since they exist for $\mathbb{X}_p^s(]-a,a[)$. □

Remark 3.2. Explicit formulas for possible extension operators can be found for instance in [31], namely extension operators of Fichtenholz-Hestenes or Triebel-Lizorkin type. On the other hand, note that $k \in \mathbb{TX}_p^s(\mathbb{R})$ is not sufficient to have $k_a = r_{\mathbb{R}\to]-a,a[}k \in \mathbb{TX}_p^s(]-a,a[)$.

Thus, let us agree that if $k_a \in \mathbb{TX}_p^s(]-a,a[)$ then its extension $k = \ell k_a$ to the real axis ($k_a = r_{\mathbb{R}\to]-a,a[}k$), belongs to the appropriate spaces:

i. to $\mathbb{L}_1(\mathbb{R})$ for $-1+1/p < s < 1/p$;
ii. to $\mathbb{X}_p^{s-1}(\mathbb{R})$ for $n-1+1/p < s < n+1/p$, $n = 1,2,\ldots$;
iii. to $\mathbb{X}_p^{\nu-1}(\mathbb{R})$ for $s = n+1/p,\ n = 0,1,\ldots$ and for some $\nu > s$

such that the corresponding norm in (3.1) is finite.

Theorem 3.3. *Let* $k_a \in \mathbb{TX}_p^s(]-a,a[)$ *and* $k \in \mathbb{L}_1(\mathbb{R})$ *such that* $k_{|]-a,a[} = k_a$, *with the above properties. Then* $\mathcal{K}_a \in \mathcal{L}\left(\mathbb{X}_p^s(]0,a[)\right)$ *and, moreover, the estimate*

$$\left\|\mathcal{K}_a\varphi|\mathbb{X}_p^s(]0,a[)\right\| \leq C\left(1+\left\|k_a|\mathbb{TX}_p^s(]-a,a[)\right\|\right)\left\|\varphi|\mathbb{X}_p^s(]0,a[)\right\|, \tag{3.2}$$

holds for some positive constant C *and all* $\varphi \in \mathbb{X}_p^s(]0,a[)$.

Proof. According to Lemma 3.1 we may put $k = E_{p,a}^s k_a$, since $\mathcal{K}_a$ does not depend on the choice of the extension of k_a. The operator $W_{1+\widehat{k},\mathbb{R}_+} : \mathbb{X}_p^s(\mathbb{R}_+) \to \mathbb{X}_p^s(\mathbb{R}_+)$ is bounded, see [12]. Further, for the non-critical space orders, the two operators are related by (2.1)-(2.3) (with $\mathbb{Z} = \{0\}$) which yields that

$$\begin{aligned}
\|\mathcal{K}_a\| &\leq \|E\|\,\|W_{\Psi,\mathbb{R}_+}\|\,\|F\| \\
&\leq C_1\left(1+\|W_{1+\widehat{k},\mathbb{R}_+}\|\right) \\
&\leq C_2\left(1+\|k|\mathbb{TX}_p^s(\mathbb{R})\|\right) \\
&\leq C\left(1+\|k_a|\mathbb{TX}_p^s(]-a,a[)\|\right).
\end{aligned}$$

Herein, C_1 contains (as factors) the norms of E, F and the norms of the operators due to the diagonal terms of (2.2) which are all bounded. The next estimate is taken from [12, theorems 14 and 15] and the final one uses the boundedness of $E_{p,a}^s$ (but taking into account the last part of Remark 3.2).

We will now be concerned with the critical space orders (s,p) so that $1 < p < \infty$ and $s - 1/p \in \mathbb{N}_0$. In this case, we can look for the corresponding spaces $\mathbb{X}_p^s$ (over any of the real sets considered above) as a complex interpolation space resulting from an interpolation couple of spaces of the same nature but with non-critical space orders (s_0, p) and (s_1, p). Namely, using the notation of Triebel [32, §§1.9, 2.4, 4.3] (for the present spaces), we have

$$\mathbb{X}_p^s = \left[\mathbb{X}_p^{s_0}, \mathbb{X}_p^{s_1}\right]_\theta$$

with $0 < \theta < 1$, $0 < s = (1-\theta)s_0 + \theta s_1 \notin \mathbb{N}$, $1 < p < \infty$. Thus, taking into account our initial convolution operator defined between spaces with critical orders

$$\mathcal{K}_a = r_{\mathbb{R}\to]0,a[}\mathcal{F}^{-1}(1+\widehat{k}) \cdot \mathcal{F}\ell : \mathbb{X}_p^s(]0,a[) \to \mathbb{X}_p^s(]0,a[)$$

and choosing non-critical orders (s_0, p) and (s_1, p) near of the critical one (s, p) such that, e.g., $s_0 < s < s_1$, it follows from the first part of the proof that if we take $k_a \in \mathbb{TX}_p^{s_1}(]-a,a[)$ and $k \in \mathbb{L}_1(\mathbb{R})$ (with $k_{|]-a,a[} = k_a$), then $\mathcal{K}_a \in \mathcal{L}\left(\mathbb{X}_p^s(]0,a[)\right)$ and, moreover, the estimate

$$\|\mathcal{K}_a\| \le C_\theta\big(1 + \big\|k_a|\mathbb{TX}_p^{s_0}(]-a,a[)\big\|\big)^{1-\theta}\big(1 + \big\|k_a|\mathbb{TX}_p^{s_1}(]-a,a[)\big\|\big)^{\theta},$$

holds for some positive constant C_θ also depending on the parameter θ. Alternatively, a similar result can be obtained by applying methods of real interpolation. □

We conclude this section by noting that the transmission property, formulated above as a condition on the kernel function, e.g. in the Bessel potential space setting

$$k_a \in \mathbb{TH}_p^s(]-a,a[), \tag{3.3}$$

has an equivalent description in the form of conditions on a Fourier symbol of the finite interval convolution operator.

In the case $-1 + 1/p < s < 1/p$ the Fourier transform $\widehat{k}$ of the extension by zero $k = \ell_0 k_a$ to the real axis $\mathbb{R}$ falls into the Wiener algebra and vanishes at infinity $\widehat{k}(\xi) \to 0$ at ∞. Moreover, $e^{\pm ia\xi}\widehat{k}(\xi)$ have holomorphic and uniformly bounded extensions in the corresponding complex half-planes $\pm\Im m\,\xi > 0$, respectively. This implies $\widehat{k}(\xi) = \mathcal{O}(e^{\mp ia\xi}) = \mathcal{O}(e^{-a|\Im m\,\xi|})$ as $\Im m\,\xi \to \pm\infty$.

For $p = 2$, $s \ge 1/2$, the transmission property (3.3) is equivalent to the existence of an extension $\ell k_a \in \mathbb{L}_1(\mathbb{R})$ such that

$$\widehat{\ell k_a} \in \mathbb{L}_2\left(\mathbb{R}, \lambda^{s-1}\right) \qquad \text{if} \qquad n - \frac{1}{2} < s < n + \frac{1}{2}, \quad n = 1, 2, \ldots$$

and, for the remaining values of s, $s = n + 1/2$, $n = 0, 1, 2, \ldots$, we can find ℓk_a such that

$$\inf_{\nu > s}\left\|\lambda^{\nu-1}\,\widehat{\ell k_a}\,|\mathbb{L}_2(\mathbb{R})\right\| < \infty.$$

The case $p \neq 2$ is much more complicated because we have to deal with the p-multiplier space $\mathcal{M}_p(\mathbb{R})$ instead of $\mathcal{M}_2(\mathbb{R}) = \mathbb{L}_\infty(\mathbb{R})$, cf. Remark 3.2.

4. Fredholm Property and Invertibility

As in [12], we consider the Banach subalgebra of the Wiener algebra

$$\mathbb{WX}_p^s(\mathbb{R}) \;=\; \left\{ \text{const} + \phi : \mathcal{F}^{-1}\phi \in \mathbb{TX}_p^s(\mathbb{R}) \right\}. \tag{4.1}$$

Lemma 4.1 ([12]). *The singular integral operator $S_{\mathbb{R}}$, defined by*

$$S_{\mathbb{R}}\varphi(x) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{\varphi(y)}{y-x}dy,$$

is bounded in $\mathbb{WX}_p^s(\mathbb{R})$.

Due to Lemma 4.1 the projector

$$P_{\mathbb{R}} = \frac{1}{2}(I + S_{\mathbb{R}})$$

is bounded in the algebra $\mathbb{WX}_p^s(\mathbb{R})$, which makes $\mathbb{WX}_p^s(\mathbb{R})$ a decomposing Banach algebra [10].

Theorem 4.2. *Let $1 < p < \infty$, $-1 + 1/p < s < \infty$ and $k_a \in \mathbb{TX}_p^s(]-a, a[)$.*

i. *The convolution operator $\mathcal{K}_a$, presented in* (1.7)*, is a Fredholm operator in the space $\mathbb{X}_p^s(]0, a[)$ with zero Fredholm index.*
ii. *If $\mathcal{K}_a$ is left or right invertible, then it is invertible.*

Proof. Due to the fact that we are working with an integrable kernel on a finite interval, the convolution operator can be written as the identity operator plus a compact operator. Therefore, this is a Fredholm operator and it has zero Fredholm index in all spaces where it is bounded. The latter is obtained, e.g., by application of the Krasnosel'skij theorem on interpolation of compact operators (see [20] and [32, §§ 1.10.1, 1.16.4]), from which follows that if an operator is compact in $\mathbb{L}_p$ and bounded in any $\mathbb{X}_p^s$, then it is compact in $\mathbb{X}_p^s$. This statements implies also the assertion ii of the theorem. □

Remark 4.3. For the non-critical space orders and for Φ in the Wiener algebra, the foregoing theorem can also be derived from Theorem 2.1 and Proposition 2.3 which establish a direct connection between finite interval convolution operator and a convolution operator on the half-line. This approach is interesting and useful because it provides the possibility to obtain explicit invertibility conditions for finite interval convolution operators and, moreover, to write down explicitly the inverse operator (cf. Theorem 4.5). Although, for this we need to carry out the factorization of the matrix symbol. An alternative proof proceeds as follows.

i) Being Φ an invertible element of the algebra $[\mathbb{WX}_p^s(\mathbb{R})]^{2\times 2}$ having determinant minus one, it follows that a factorization

$$\Phi = \Phi_- \ \operatorname{diag}\left[\left(\frac{\lambda_-}{\lambda_+}\right)^{\kappa_1}, \left(\frac{\lambda_-}{\lambda_+}\right)^{\kappa_2}\right] \Phi_+$$

(in $[\mathbb{WX}^s_p(\mathbb{R})]^{2\times 2}$), with $\kappa_1 \geq \kappa_2$, must have partial indices such that $\kappa_1 + \kappa_2 = 0$. Therefore, taking into account Theorem 2.1 and Proposition 2.3, we obtain

$$\begin{aligned} \operatorname{Ind} \mathcal{K}_a &= \dim \operatorname{Ker} \mathcal{K}_a - \dim \operatorname{Coker} \mathcal{K}_a \\ &= -\kappa_2 - \kappa_1 = 0. \end{aligned} \tag{4.2}$$

ii) If $\dim \operatorname{Ker} \mathcal{K}_a = 0$ or $\dim \operatorname{Coker} \mathcal{K}_a = 0$, from (4.2) it immediately follows that $\mathcal{K}_a$ is an invertible operator.

Remark 4.4. Due to the operator relations presented in Theorem 2.1 and Proposition 2.3, we have that our initial convolution operator defined in (1.7) has the same Fredholm indices as $W_{\Phi,\mathbb{R}_+}$. Thus, the question of the existence of a (canonical) factorization of Φ in a decomposing algebra is independent of the particular extension k that we may take for k_a.

Theorem 4.5. *Let $k_a \in \mathbb{TX}^s_p(]-a,a[)$ for some non-critical space order and with Φ (cf. (2.8)) in the Wiener algebra. The convolution operator $\mathcal{K}_a$, defined in (1.7), is invertible if and only if the matrix-valued function Φ admits a canonical factorization [10] in $[\mathbb{WX}^s_p(\mathbb{R})]^{2\times 2}$.*

Moreover, assuming that this is the case and that a canonical factorization of Φ is given by

$$\Phi = \Phi_- \Phi_+ = \begin{bmatrix} 1 + \Phi^-_{11} & \Phi^-_{12} \\ \Phi^-_{21} & 1 + \Phi^-_{22} \end{bmatrix} \begin{bmatrix} 1 + \Phi^+_{11} & \Phi^+_{12} \\ \Phi^+_{21} & -1 + \Phi^+_{22} \end{bmatrix},$$

with $\Phi^{\pm}_{ij}(\infty) = 0$ (for $i, j = 1, 2$), then the inverse of $\mathcal{K}_a$ reads

$$\mathcal{K}_a^{-1} \varphi(x) = \varphi(x) + \int_0^a \gamma(x,y)\varphi(y)dy, \tag{4.3}$$

where

$$\begin{aligned} \gamma(x,y) &= \omega_1(x-y) + \omega_2(x-y-a) \\ &\quad + \int_{\max(x,y)-a}^{\min(x,y)} [\omega_2(x-z-a)\omega_1(z-y) - \omega_1(x-z-a)\omega_2(z-y)]\, dz, \end{aligned} \tag{4.4}$$

with $\widehat{\omega}_1 = \Phi^-_{22} + \tau_{-a}\Phi^-_{12}$ and $\widehat{\omega}_2 = \Phi^-_{21} + \tau_{-a}\Phi^-_{11}$.

Proof. We know, from Theorem 2.1 and Proposition 2.3, that $\mathcal{K}_a$ and $W_{\Phi,\mathbb{R}_+}$ are invertible only at the same time. Therefore, the first statement follows if we take into account the definition of $[\mathbb{WX}^s_p(\mathbb{R})]^{2\times 2}$ (cf. (4.1) and (3.1)).

For the second part, we will take advantage of the Gohberg-Sementsul formula [17, 28]. For this purpose, we first observe that the existence of γ is granted by the canonical factorization of Φ and that from (4.4) we obtain the representations

$$\begin{aligned} \gamma(x,0) &= \gamma(a, a-x) = \omega_2(x-a), \\ \gamma(0,y) &= \gamma(a-y, a) = \omega_1(-y). \end{aligned}$$

Thus, for $x, y \in [0,a]$, it follows

$$\begin{aligned} \gamma(x,0) + \int_0^a k_a(x-z)\gamma(z,0)dz &= -k_a(x) \\ \gamma(0,y) + \int_0^a k_a(z-y)\gamma(0,z)dz &= -k_a(-y), \end{aligned}$$

which shows that the conditions of the Gohberg-Sementsul identity are satisfied and therefore we arrive at formula (4.3), see [13, 17, 28]. Additionally, due to $\Phi_\pm^{-1} \in [\mathbb{WX}_p^s(\mathbb{R})]^{2\times 2}$ and the boundedness of the projector $P_\mathbb{R}$, it remains to note that

$$T = \mathcal{F}^{-1}\Phi_+^{-1} \cdot \mathcal{F}\,\ell_0 r_{\mathbb{R}\to\mathbb{R}_+}\,\mathcal{F}^{-1}\Phi_-^{-1}\cdot\mathcal{F}$$

is a bounded operator on $[\mathbb{L}_p(\mathbb{R})]^2$ which preserves the smoothness order. □

Theorem 4.6. *Let $k_a \in \mathbb{TX}_p^s(]-a,a[)$, for some non-critical space order, Φ being in the Wiener algebra and consider a generalized factorization of Φ in $[\mathbb{WX}_p^s(\mathbb{R})]^{2\times 2}$,*

$$\begin{aligned} \Phi &= \Phi_- \; D \; \Phi_+ \\ &= \begin{bmatrix} 1+\Phi_{11}^- & \Phi_{12}^- \\ \Phi_{21}^- & 1+\Phi_{22}^- \end{bmatrix} \begin{bmatrix} \left(\frac{\lambda_-}{\lambda_+}\right)^{\kappa_1} & 0 \\ 0 & \left(\frac{\lambda_-}{\lambda_+}\right)^{\kappa_2} \end{bmatrix} \begin{bmatrix} 1+\Phi_{11}^+ & \Phi_{12}^+ \\ \Phi_{21}^+ & -1+\Phi_{22}^+ \end{bmatrix}, \end{aligned} \tag{4.5}$$

with $\Phi_{ij}^\pm(\infty) = 0$ (for $i,j = 1,2$) and partial indices $\kappa_1 \geq \kappa_2$.

Then the convolution operator $\mathcal{K}_a$, defined in (1.7)*, is generalized invertible and a generalized inverse of it is given in the form*

$$\mathcal{K}_a^- \;=\; P\Big[\left(W_{\Phi,\mathbb{R}}^-\right)_{11} + W_1\left(W_{\Phi,\mathbb{R}}^-\right)_{21}\Big]_{|P\mathbb{X}_p^s(\mathbb{R}_+)} \tag{4.6}$$

where P is the projector introduced in the proof of Theorem 2.1*,*

$$W_1 = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\lambda_+^{-s'}\tau_a\cdot\mathcal{F}\ell_0$$

and the elements $\left(W_{\Phi,\mathbb{R}}^-\right)_{jk}$ are defined by the generalized factorization (4.5) *due to the representation*

$$\begin{bmatrix} \left(W_{\Phi,\mathbb{R}}^-\right)_{11} & \left(W_{\Phi,\mathbb{R}}^-\right)_{12} \\ \left(W_{\Phi,\mathbb{R}}^-\right)_{21} & \left(W_{\Phi,\mathbb{R}}^-\right)_{22} \end{bmatrix} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Phi_+^{-1}P_\mathbb{R}D^{-1}P_\mathbb{R}\Phi_-^{-1}\cdot\mathcal{F}\ell. \tag{4.7}$$

Proof. Using the methods exposed in the proof of Theorem 2.1 and the complemented projectors P and Q defined there, one can write the equivalence after extension relation between $\widetilde{\mathcal{K}_a} = PW_{|P\mathbb{X}_p^s(\mathbb{R}_+)}$ and $T = PW + Q$ in the form

$$\begin{bmatrix} \widetilde{\mathcal{K}_a} & 0 \\ 0 & I_{Q\mathbb{X}_p^s(\mathbb{R}_+)} \end{bmatrix} = \begin{bmatrix} I_{P\mathbb{X}_p^s(\mathbb{R}_+)} & -PW_{|Q\mathbb{X}_p^s(\mathbb{R}_+)} \\ 0 & I_{Q\mathbb{X}_p^s(\mathbb{R}_+)} \end{bmatrix} (PW+Q)\,. \tag{4.8}$$

In addition, for $PW + Q$ one has the identity (2.7) and, according to relations (2.6) and (2.9), we obtain

$$\begin{bmatrix} W_2 & 0 \\ W & W_1 \end{bmatrix} = \begin{bmatrix} r_{\mathbb{R}\to\mathbb{R}_+}\Lambda_-^{s'}\ell & 0 \\ 0 & I \end{bmatrix} W_{G_-,\mathbb{R}_+} W_{\Phi,\mathbb{R}_+} W_{G_+,\mathbb{R}_+} \begin{bmatrix} I & 0 \\ 0 & r_{\mathbb{R}\to\mathbb{R}_+}\Lambda_+^{-s'}\ell_0 \end{bmatrix}$$

where $I = I_{\mathbb{X}_p^s(\mathbb{R}_+)}$. Therefore, noting that (4.7) is a generalized inverse to $W_{\Phi,\mathbb{R}}$, we obtain the following generalized inverse of the operator in the left hand-side of (2.7):

$$\begin{bmatrix} (PW+Q)^- & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ W_2\,(W-I) & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & r_{\mathbb{R}\to\mathbb{R}_+}\Lambda_+^{s'}\ell_0 \end{bmatrix} W_{G_+^{-1},\mathbb{R}_+} \begin{bmatrix} \left(W_{\Phi,\mathbb{R}}^-\right)_{11} & \left(W_{\Phi,\mathbb{R}}^-\right)_{12} \\ \left(W_{\Phi,\mathbb{R}}^-\right)_{21} & \left(W_{\Phi,\mathbb{R}}^-\right)_{22} \end{bmatrix} W_{G_-^{-1},\mathbb{R}_+} \begin{bmatrix} r_{\mathbb{R}\to\mathbb{R}_+}\Lambda_-^{-s'}\ell_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} W_2 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & W_1 \\ 0 & I \end{bmatrix}. \quad (4.9)$$

In addition, from (4.8), generalized invertibility of $PW+Q$ also leads to generalized invertibility of $\widetilde{\mathcal{K}_a}$. Thus, combining this with (4.9), the desired formula (4.6) is proved. □

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References

[1] H. Bart and V.E. Tsekanovskii: Matricial coupling and equivalence after extension. T. Ando (ed.) et al., *Operator Theory and Complex Analysis*, Birkhäuser, Basel, Oper. Theory Adv. Appl. **59**, 143–160.

[2] M.A. Bastos and A.F. dos Santos: Convolution operators on a finite interval with periodic kernel – Fredholm property and invertibility, *Integr. Equ. Oper. Theory* **16** (1993), 186–223.

[3] M.A. Bastos, A.F. dos Santos and R. Duduchava: Finite interval convolution operators on the Bessel potential spaces H_p^s, *Math. Nachr.* **173** (1995), 49–63.

[4] A. Böttcher, Y.I. Karlovich and I.M. Spitkovskii: *Convolution Operators and Factorization of Almost Periodic Matrix Functions*, Oper. Theory Adv. Appl. **131**, Birkhäuser, Basel, 2002.

[5] L. Boutet de Monvel: Boundary problems for pseudo-differential operators, *Acta Math.* **126** (1971), 11–51.

[6] M.C. Câmara, A.B. Lebre and F.-O. Speck: Meromorphic factorization, partial index estimates and elastodynamic diffraction problems, *Math. Nachr.* **157** (1992), 291–317.

[7] L.P. Castro: A relation between convolution type operators on intervals in Sobolev spaces, *Appl. Anal.* **74** (2000), 393–412.

[8] L.P. Castro and F.-O. Speck: On the inversion of higher order Wiener-Hopf operators, *J. Integral Equations Appl.* **8** (1996), 269–285.

[9] L.P. Castro and F.-O. Speck: Relations between convolution type operators on intervals and on the half-line, *Integr. Equ. Oper. Theory* **37** (2000), 169–207.

[10] K. Clancey and I. Gohberg: *Factorization of Matrix Functions and Singular Integral Operators*, Oper. Theory Adv. Appl. **3**, Birkhäuser Verlag, Basel, 1981.

[11] R. Duduchava: *Integral Equations in Convolution with Discontinuous Presymbols, Singular Integral Equations with Fixed Singularities, and their Applications to Some Problems of Mechanics*, Teubner Verlagsgesellschaft, Leipzig, 1979.

[12] R. Duduchava: Wiener-Hopf equations with the transmission property, *Integr. Equ. Oper. Theory* **15** (1992), 412–426.

[13] I. Feldman, I. Gohberg and N. Krupnik: Convolution equations on finite intervals and factorization of matrix functions, *Integr. Equ. Oper. Theory* **36** (2000), 201–211.

[14] M.P. Ganin: On a Fredholm integral equation whose kernel depends on the difference of the arguments (in Russian), *Izv. Vysš. Učebn. Zaved. Matematika* **2** (1963), 31–43.

[15] I.C. Gohberg and I.A. Fel'dman: *Convolution Equations and Projection Methods for Their Solution*, Translations of Mathematical Monographs **41**, American Mathematical Society, Providence, R.I., 1974.

[16] I.Ts. Gokhberg and M.G. Kreĭn: Systems of integral equations on a half line with kernels depending on the difference of arguments, *Am. Math. Soc., Transl.*, II. Ser. **14** (1960), 217–287.

[17] I.C. Gohberg and A.A. Sementsul: On the inversion of finite Toeplitz matrices and their continuous analogues (in Russian), *Mat. issled.* **7** (1972), 201–223.

[18] G. Grubb and L. Hörmander: The transmission property, *Math. Scand.* **67** (1990), 273–289.

[19] Yu.I. Karlovich and I.M. Spitkovskij: On the Noetherian property of certain singular integral operators with matrix coefficients of class SAP and systems of convolution equations on a finite interval connected with them, *Sov. Math., Dokl.* **27** (1983), 358–363.

[20] M. Krasnosel'skij: On a theorem of M. Riesz, *Sov. Math., Dokl.* **1** (1960), 229–231.

[21] M.G. Kreĭn: Integral equations on a half-line with kernel depending upon the difference of the arguments, *Am. Math. Soc., Transl.*, II. Ser. **22** (1962), 163–288.

[22] A.B. Kuijper: A note on first kind convolution equations on a finite interval, *Integr. Equ. Oper. Theory* **14** (1991), 146–152.

[23] G. Monegato and L. Scuderi: Polynomial collocation methods for 1D integral equations with nonsmooth solutions, M. Bonnet (ed.) et al., *Mathematical Aspects of Boundary Element Methods*, Chapman Hall, Res. Notes Math. **414** (2000), 227–239.

[24] A. Moura Santos, F.-O. Speck and F.S. Teixeira: Minimal normalization of Wiener-Hopf operators in spaces of Bessel potentials, *J. Math. Anal. Appl.* **225** (1998), 501–531.

[25] V.Yu. Novokshenov: Convolution equations on a finite segment and factorization of elliptic matrices, *Math. Notes* **27** (1980), 449–455.

[26] B.V. Pal'cev: The asymptotics of the spectrum and eigenfunctions of convolution operators on a finite interval with a kernel with a homogeneous Fourier transform, *Sov. Math., Dokl.* **15** (1974), 1243–1247.

[27] S. Rempel and B. Schulze: *Index Theory of Elliptic Boundary Problems*, Mathematische Lehrbücher und Monographien II, Mathematische Monographien **55**, Akademie-Verlag, Berlin, 1982.

[28] L.A. Sakhnovich: *Integral Equations with Difference Kernels on Finite Intervals*, Oper. Theory Adv. Appl. **84**, Birkhäuser, Basel, 1996.

[29] F.-O. Speck: *General Wiener-Hopf Factorization Methods*, Research Notes in Mathematics **119**, Pitman, London, 1985.

[30] G. Talenti: Sulle equazioni integrali di Wiener-Hopf (in Italian), *Boll. Unione Mat. Ital.*, IV. Ser. **7**, Suppl. al Fasc. 1 (1973), 18–118.

[31] H. Triebel: *Theory of Function Spaces II*, Monographs in Mathematics **84**, Birkhäuser Verlag, Basel, 1992.

[32] H. Triebel: *Interpolation Theory, Function Spaces, Differential Operators*, Barth, Leipzig, 1995.

[33] N. Wiener and E. Hopf: Über eine Klasse Singulärer Integralgleichungen (in German), *Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl.* **30/32** (1931), 696–706.

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