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Integral Equations and Operator Theory

Mean Oscillation and Hankel Operators on the Segal-Bargmann Space

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Abstract. For the *Segal-Bargmann space* of Gaussian square integrable entire functions on $\mathbb{C}^m$ we consider *Hankel operators* H_f with symbols in $f \in \mathcal{T}(\mathbb{C}^m)$. We completely characterize the functions in $\mathcal{T}(\mathbb{C}^m)$ for which the operators H_f and $H_{\bar{f}}$ are simultaneously bounded or compact in terms of the *mean oscillation* of f. The analogous description holds for the commutators $[M_f, P]$ where M_f denotes the "multiplication by f" and P is the *Toeplitz projection*. These results are already known in case of *bounded symmetric domains* Ω *in* $\mathbb{C}^m$ (see [BBCZ] or [C]). In the present paper we combine some techniques of [BBCZ] and [BC1]. Finally, we characterize the entire function $f \in \mathcal{H}(\mathbb{C}^m) \cap \mathcal{T}(\mathbb{C}^m)$ and the polynomials p in z and $\bar{z}$ for which the *Hankel operators* $H_{\bar{f}}$ and H_p are bounded (resp. compact).

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1. Introduction

Throughout this paper let $m \in \mathbb{N}$ be fixed. Let μ denote the Gaussian measure on the complex space $\mathbb{C}^m$ defined by $d\mu(z) = \pi^{-m} \exp(-\|z\|^2) dV(z)$, where V is the usual Lebesgue measure on $\mathbb{C}^m$. The *Segal-Bargmann space* $H^2(\mathbb{C}^m, \mu)$ is the closed subspace of $L^2(\mathbb{C}^m, \mu)$ of all square integrable holomorphic functions on $\mathbb{C}^m$. If P denotes the orthogonal projection from $L^2(\mathbb{C}^m, \mu)$ onto $H^2(\mathbb{C}^m, \mu)$ then for a function $f \in \mathcal{T}(\mathbb{C}^m)$ (for definition see section 2) the *Hankel operator*

$$H_f : \mathcal{D}(H_f) \subset H^2(\mathbb{C}^m, \mu) \longrightarrow H^2(\mathbb{C}^m, \mu)^{\perp}$$

is the densely defined (and in general unbounded) operator $H_f g = (I - P)M_f g$ for all $g \in \mathcal{D}(H_f)$ where M_f denotes the multiplication by f. Moreover, for $f \in \mathcal{T}(\mathbb{C}^m)$ the *commutator* of M_f and P given by $[M_f, P] := M_f P - P M_f$ is a densely defined

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operator on $L^2(\mathbb{C}^m,\mu)$. It is easy to verify that $[M_f, P]$ is bounded (resp. compact) if and only if both *Hankel operators* H_f and $H_{\bar f}$ are simultaneously bounded (resp. compact).

The authors of [BC1] prove that for bounded symbols $f \in L^\infty(\mathbb{C}^m)$ the *Hankel operator* H_f is compact if and only if $H_{\bar f}$ is compact (see also [S1]). Moreover, they determine the largest $*$-*algebra* Q in $L^\infty(\mathbb{C}^m)$ such that H_f and $H_{\bar f}$ are compact for symbols $f \in Q$. The functions in Q are characterized by a condition of *oscillation at infinity.*

In general, if we deal with unbounded symbols f in $\mathcal{T}(\mathbb{C}^m)$ also the question arises whether the *Hankel operator* H_f is bounded. Our main aim in this paper is to prove that

(A) *For $f \in \mathcal{T}(\mathbb{C}^m)$ the commutator $[M_f, P]$ is bounded if and only if the symbol f has bounded mean oscillation.*

We also completely characterize the compact commutators $[M_f, P]$ for symbols $f \in \mathcal{T}(\mathbb{C}^m)$ in terms of the *mean oscillation* of f.

(B) *The commutator $[M_f, P]$ is compact if and only if the symbol f has vanishing mean oscillation at infinity.*

The analogous results are already known for *Bergman spaces of bounded symmetric domains* Ω *in* $\mathbb{C}^m$ (see [BBCZ] and [C]) and it was a conjecture in [C] that both (A) and (B) above hold in the unbounded setting of the *Segal-Bargmann space.*

Finally, we determine the space of all entire functions in $\mathcal{T}(\mathbb{C}^m)$ as well as the space of all polynomials in z and $\bar z$ for which $[M_f, P]$ is bounded or compact.

2. Preliminaries

For $j = (j_1, \cdots, j_m) \in \mathbb{N}_0^m$ define $j! := j_1! \cdots j_m!$ and $|j| := j_1 + \cdots + j_m$. If $z \in \mathbb{C}^m$ then write $z^j := z_1^{j_1} \cdots z_m^{j_m}$. Throughout this paper $\langle \cdot, \cdot \rangle$ denotes the usual Euclidian scalar product and $\| \cdot \|$ the Euclidian norm in $\mathbb{C}^m$. For $R > 0$ and $a \in \mathbb{C}^m$ let $B(a, R)$ denote the ball in $\mathbb{C}^m$ with radius R centered in a. Further, we write $\langle \cdot, \cdot \rangle_2$ for the $L^2(\mathbb{C}^m,\mu)$-scalar product and $\| \cdot \|_2$ for the $L^2(\mathbb{C}^m,\mu)$-norm.

Because each *point evaluation* is a continous functional on $H^2(\mathbb{C}^m,\mu)$ the *Segal-Bargmann space* is a Hilbert space with *kernel function* $K(z,w) := \exp(\langle z, w\rangle)$ for $z, w \in \mathbb{C}^m$. We also use the *normalized kernel function* defined by

$$k_w(z) := K(z,w)\|K(\cdot,w)\|_2^{-1} = \exp\left(\langle z, w\rangle - \frac{1}{2}\|w\|^2\right), \qquad \forall\, z, w \in \mathbb{C}^m.$$

For $z, w \in \mathbb{C}^m$ let τ_z denote the z-*shift* on $\mathbb{C}^m$ given by $\tau_z(w) := z + w$. Define the linear space

$$\mathcal{T}(\mathbb{C}^m) := \{g \in L^2(\mathbb{C}^m,\mu) : g \circ \tau_x \in L^2(\mathbb{C}^m,\mu), \ \forall\, x \in \mathbb{C}^m\}.$$

It is easy to verify that a measurable function f on $\mathbb{C}^m$ belongs to $\mathcal{T}(\mathbb{C}^m)$ if and only if the functions $\lambda \mapsto f(\lambda)K(\lambda, x)$ belong to $L^2(\mathbb{C}^m,\mu)$ for every $x \in \mathbb{C}^m$.

Because the linear span of the set of all *kernel functions* $\{K(\cdot, x) : x \in \mathbb{C}^m\}$ is dense in the *Segal-Bargmann space*

$$\mathcal{D}(M_f) = \mathcal{D}(H_f) := \{h \in H^2(\mathbb{C}^m, \mu) : fh \in L^2(\mathbb{C}^m, \mu)\}$$

is a dense, linear subspace of $H^2(\mathbb{C}^m, \mu)$ whenever $f \in \mathcal{T}(\mathbb{C}^m)$. For $f \in \mathcal{T}(\mathbb{C}^m)$ define the *Berezin transform* $\tilde{f}$ of f by

$$\tilde{f}(\lambda) = \int_{\mathbb{C}^m} f \circ \tau_\lambda(u) d\mu(u) = \langle f k_\lambda, k_\lambda \rangle_2, \qquad \forall\ \lambda \in \mathbb{C}^m.$$

Clearly from this definition we have $\tilde{\bar{\tilde{f}}} = \bar{\tilde{f}}$ and $\tilde{f} \circ \tau_\lambda = \widetilde{f \circ \tau_\lambda}$.

Let $\mathcal{BC}(\mathbb{C}^m)$ be the space of all bounded continous functions on $\mathbb{C}^m$ and denote by $\mathcal{C}_0(\mathbb{C}^m)$ the subalgebra in $\mathcal{BC}(\mathbb{C}^m)$ of all continous functions vanishing at infinity. For $f \in \mathcal{BC}(\mathbb{C}^m)$ define the *oscillation of* f in $z \in \mathbb{C}^m$ by

$$\mathrm{Osc}_z(f) := \sup\{|f(z) - f(w)| : \|z - w\| < 1\}.$$

Then $z \mapsto \mathrm{Osc}_z(f)$ also is a continous function on $\mathbb{C}^m$. Now, we say f *is of bounded oscillation* [write $f \in \mathcal{BO}(\mathbb{C}^m)$] if $\mathrm{Osc}_z(f)$ is in $\mathcal{BC}(\mathbb{C}^m)$ as a function of z. We say the function f *is of vanishing oscillation* [write $f \in \mathcal{VO}(\mathbb{C}^m)$] if $\mathrm{Osc}_z(f) \to 0$ as $z \to \infty$. For $f \in \mathcal{T}(\mathbb{C}^m)$ the quantity

$$\mathrm{MO}(f, z) := \widetilde{|f|^2}(z) - |\tilde{f}(z)|^2$$

is a continous function on $\mathbb{C}^m$ and $\mathrm{MO}(f, \cdot)$ is called the *mean oscillation of* f. We say f *is of bounded mean oscillation on* $\mathbb{C}^m$ and write $f \in \mathcal{BMO}(\mathbb{C}^m)$ if

$$\|f\|_{\mathrm{BMO}} := \sup\{\mathrm{MO}(f, z)^{\frac{1}{2}} : z \in \mathbb{C}^m\} < \infty.$$

We say f *is of vanishing mean oscillation* and we write $f \in \mathcal{VMO}(\mathbb{C}^m)$ if

$$\lim_{z \to \infty} \mathrm{MO}(f, z) = 0.$$

For all $f, g \in \mathcal{T}(\mathbb{C}^m)$ and all $\lambda \in \mathbb{C}^m$ it is easy to verify that

$$0 \le \mathrm{MO}(g + h, \lambda)^2 \le 2\left[\mathrm{MO}(g, \lambda)^2 + \mathrm{MO}(h, \lambda)^2\right].$$

Thus $\mathcal{BMO}(\mathbb{C}^m)$ as well as $\mathcal{VMO}(\mathbb{C}^m)$ are linear spaces. For $S \subset \mathbb{C}^m$ and each $f \in \mathcal{T}(\mathbb{C}^m)$ we write

$$\|f\|_{\mathrm{BMO}(S)} := \sup\{\mathrm{MO}(f, z)^{\frac{1}{2}} : z \in S\}.$$

Let $\mathbb{P}[z, \bar{z}]$ be the space of complex polynomials on $\mathbb{C}^m$ in the complex variables z and $\bar{z}$. Each $p \in \mathbb{P}[z, \bar{z}]$ has the form

$$p(z, \bar{z}) = \sum_{l,j \in \mathbb{N}_0^m} a_{l,j} z^l \bar{z}^j, \qquad \text{where} \quad a_{l,j} \in \mathbb{C}. \tag{2.1}$$

For $p \in \mathbb{P}[z, \bar{z}]$ with (2.1) define the integer

$$\rho(p) := \max\{|l + j| : l, j \in \mathbb{N}_0^m,\ \ a_{l,j} \neq 0\} \in \mathbb{N}_0.$$

Lemma 2.1. *Let $R(z,\bar{z}) := z^l \bar{z}^j$ with $l, j \in \mathbb{N}_0^m$ be a monomial in z and $\bar{z}$ on $\mathbb{C}^m$. Then $\tilde{R}$ has the form $(*)$: $\tilde{R}(z,\bar{z}) = z^l\bar{z}^j + r(z,\bar{z})$, where $r \in \mathbb{P}[z,\bar{z}]$ with $\rho(r) < \rho(R) = |l+j|$.*

Proof. It follows from the definition of the *Berezin transform* that $\tilde{R}$ has the form

$$\tilde{R}(z,\bar{z}) = \prod_{k=1}^{m} \widetilde{R_k}(z_k, \overline{z_k}),$$

where $R_k : \mathbb{C} \to \mathbb{C}$ is defined by $R_k(z_k, \overline{z_k}) := z_k^{l_k} \overline{z_k}^{j_k}$ for $k = 1, \cdots, m$. Moreover,

$$\widetilde{R_k}(z_k, \overline{z_k}) = \frac{1}{\pi} \int_{\mathbb{C}} (z_k + w)^{l_k} \overline{(z_k + w)}^{j_k} \exp(-|w|^2) dV(w) = z_k^{l_k} \overline{z_k}^{j_k} + r_k(z_k, \overline{z_k})$$

with $\rho(r_k) < l_k + j_k$. From this the decomposition $(*)$ of $\tilde{R}$ follows. □

Corollary 2.2. *Let $p \in \mathbb{P}[z,\bar{z}]$ be as in* (2.1)*. Define $A(p) := \{(l,j) \in \mathbb{N}_0^{2m} : |(l,j)| = \rho(p)\}$ and*

$$Q_p(z,\bar{z}) := \sum_{(l,j)\in A(p)} a_{l,j} z^l \bar{z}^j.$$

Then it holds $\tilde{p}(z,\bar{z}) = Q_p(z,\bar{z}) + r(z,\bar{z})$ where $r \in \mathbb{P}[z,\bar{z}]$ with $\rho(r) < \rho(p)$.

Proof. This directly follows from Lemma 2.1 and the linearity of the *Berezin transform.* □

Corollary 2.3. *Let $p \in \mathbb{P}[z,\bar{z}] \subset \mathcal{T}(\mathbb{C}^m)$ be a non-constant polynomial. Then we have $MO(p,\cdot) \in \mathbb{P}[z,\bar{z}]$ and $\rho(MO(p,\cdot)) < \rho(|p|^2) - 1 = 2\rho(p) - 1$.*

Proof. Using Corollary 2.2 we conclude that $Q_{|p|^2} = Q_{\widetilde{|p|^2}} = Q_{|\tilde{p}|^2}$ and by the definition of $\mathrm{MO}(p,\cdot)$ it follows that

$$\rho(\mathrm{MO}(p,\cdot)) < \rho(|p|^2) = 2\rho(p).$$

Because of $\mathrm{MO}(p,\lambda) \geq 0$ for all $\lambda \in \mathbb{C}^m$ and $\rho(p) > 0$ we have $\rho(\mathrm{MO}(p,\cdot)) \neq 2\rho(p) - 1$ and Corollary 2.3 follows. □

Lemma 2.4. *Let $a, u \in \mathbb{C}^m$ and define $S_a \in \mathcal{T}(\mathbb{C}^m)$ by $S_a(u) := \langle u, a\rangle$. Then it follows that $\widetilde{S_a} = S_a$ and $MO(S_a, z) = \|a\|^2$ for all $z \in \mathbb{C}^m$.*

Proof. The function S_a is holomorphic and so we have $\widetilde{S_a} = S_a$. Define for $t \in \mathbb{R}$ the function $F : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F(t) := \int_{\mathbb{C}^m} \langle u, a\rangle \exp\big(\langle u, z\rangle + \langle ta+z, u\rangle\big) d\mu(u) = \langle ta+z, a\rangle \exp\big(\langle ta+z, z\rangle\big). \quad (2.2)$$

It follows that $(*)$ $F^{'}(0) = \exp\big(\|z\|^2\big)[\|a\|^2 + |\langle a, z\rangle|^2]$ and differentiation of (2.2) under the integral sign in $t = 0$ together with $(*)$ now shows that

$$\widetilde{|S_a|^2} = \|a\|^2 + |S_a|^2 = \|a\|^2 + |\tilde{S}_a|^2.$$ □

The inclusion $L^\infty(\mathbb{C}^m,\mu) \subset \mathcal{BMO}(\mathbb{C}^m)$ is valid but there are also unbounded functions in $\mathcal{BMO}(\mathbb{C}^m)$. Consider a linear polynomial $p = a_0 + \langle\cdot, b\rangle + \langle c, \cdot\rangle \in \mathbb{P}[z,\bar{z}]$ where $a_0 \in \mathbb{C}$ and $b, c \in \mathbb{C}^m$. Using Corollay 2.3 it follows that $\rho(\mathrm{MO}(p,\cdot)) < 1$ and so $\mathrm{MO}(p,\cdot)$ is constant. We conclude that $p \in \mathcal{BMO}(\mathbb{C}^m)$.

Lemma 2.5. *For $g \in \mathcal{T}(\mathbb{C}^m)$, $h \in \mathcal{BMO}(\mathbb{C}^m)$ and $\lambda \in \mathbb{C}^m$ we have*

(a) $MO(g,\lambda) = \|g\circ\tau_\lambda - \tilde{g}(\lambda)\|_2^2 = \widetilde{(|g-\tilde{g}(\lambda)|^2)}(\lambda)$,
(b) $MO(g,\lambda) \leq \|(I-P)(g\circ\tau_\lambda)\|_2^2 + \|(I-P)(\bar{g}\circ\tau_\lambda)\|_2^2$,
(c) $\|h\|_{BMO} \leq \sqrt{2}\max\{\|H_h\|, \|H_{\bar{h}}\|\}$.

Proof. (a) easy computation.

(b) The *Berezin symbol* of g can be written in the following form:

$$\tilde{g}(\lambda) = \langle g\circ\tau_\lambda, 1\rangle_2 = \overline{\langle \bar{g}\circ\tau_\lambda, K(\cdot,0)\rangle}_2 = \overline{P(\bar{g}\circ\tau_\lambda)(0)} = P(\overline{P(\bar{g}\circ\tau_\lambda)}).$$

This yields the inequality

$$\begin{aligned}\|P[g\circ\tau_\lambda] - \tilde{g}(\lambda)\|_2^2 &= \|P[g\circ\tau_\lambda] - P[\overline{P(\bar{g}\circ\tau_\lambda)}]\|_2^2 \\ &\leq \|g\circ\tau_\lambda - \overline{P(\bar{g}\circ\tau_\lambda)}\|_2^2 = \|(I-P)(\bar{g}\circ\tau_\lambda)\|_2^2. \qquad (2.3)\end{aligned}$$

From $\|g\circ\tau_\lambda\|_2^2 = \|(I-P)(g\circ\tau_\lambda)\|_2^2 + \|P(g\circ\tau_\lambda)\|_2^2$ and $\langle P(g\circ\tau_\lambda), \tilde{g}(\lambda)\rangle_2 = |\tilde{g}(\lambda)|^2$ it follows that

$$\|P[g\circ\tau_\lambda] - \tilde{g}(\lambda)\|_2^2 + \|(I-P)(g\circ\tau_\lambda)\|_2^2 = \|g\circ\tau_\lambda\|_2^2 - |\tilde{g}(\lambda)|^2 = \mathrm{MO}(g,\lambda).$$

This together with (2.3) imply (b).

(c) Follows from $\|(I-P)(h\circ\tau_\lambda)\|_2 = \|H_h k_\lambda\|_2 \leq \|H_h\|$ for all $\lambda \in \mathbb{C}^m$ together with standard estimates from (b). □

The following Theorem is an analog to Theorem F in [BBCZ] in the case of *bounded symmetric domains* Ω *in* $\mathbb{C}^m$. The *Bergman metric* is replaced by the *Euclidian metric on* $\mathbb{C}^m$.

Theorem 2.6. *For any smooth curve $\gamma : I := [0,1] \longrightarrow \mathbb{C}^m$ and any $f \in \mathcal{BMO}(\mathbb{C}^m)$ we have*

$$\left|\frac{d}{dt}\tilde{f}\circ\gamma(t)\right| \leq 2\|f\|_{BMO(\gamma(I))}\left\|\frac{d}{dt}\gamma(t)\right\|, \qquad \forall\, t \in I.$$

If $s = s(t)$ denotes the arclength of γ then $\frac{d}{dt}s(t) = \|\frac{d}{dt}\gamma(t)\|$.

Proof. Let $t \in I$. Then we differentiate under the integral sign in the definition of the *Berezin transform* $\tilde{f}$.

$$\begin{aligned}\frac{d}{dt}\tilde{f}\circ\gamma(t) &= \int_{\mathbb{C}^m} f(u)\frac{d}{dt}|k_{\gamma(t)}(u)|^2 d\mu(u) \qquad (2.4)\\ &= 2\int_{\mathbb{C}^m} f(u)\Re\left\{\left(\frac{d}{dt}k_{\gamma(t)}(u)\right)\overline{k_{\gamma(t)}(u)}\right\}d\mu(u) \\ &= 2\int_{\mathbb{C}^m}\Big(f(u) - \tilde{f}\circ\gamma(t)\Big)\Re\left[G_t(u)\right]d\mu(u)\end{aligned}$$

where $G_t(u) := \left[\frac{d}{dt}k_{\gamma(t)}(u) - \left\langle \frac{d}{dt}k_{\gamma(t)}, k_{\gamma(t)}\right\rangle_2 k_{\gamma(t)}(u)\right]\overline{k_{\gamma(t)}(u)}$. Here we have used

$$2\Re\left\langle \frac{d}{dt}k_{\gamma(t)}, k_{\gamma(t)}\right\rangle_2 = \frac{d}{dt}\left\langle k_{\gamma(t)}, k_{\gamma(t)}\right\rangle_2 = \frac{d}{dt}1 = 0.$$

For $u \in \mathbb{C}^m$ and $t \in I$ one easily computes

$$\frac{d}{dt}k_{\gamma(t)}(u) = \left[\left\langle u, \frac{d}{dt}\gamma(t)\right\rangle - \Re\left\langle \gamma(t), \frac{d}{dt}\gamma(t)\right\rangle\right]k_{\gamma(t)}(u)$$

and it follows that

$$\frac{d}{dt}k_{\gamma(t)}(u) - \left\langle \frac{d}{dt}k_{\gamma(t)}, k_{\gamma(t)}\right\rangle_2 k_{\gamma(t)}(u) = \left\langle u-\gamma(t), \frac{d}{dt}\gamma(t)\right\rangle k_{\gamma(t)}(u). \tag{2.5}$$

If we use the the equalities (2.4) and (2.5) as well as the *Cauchy-Schwarz inequality* we conclude that

$$\begin{aligned}&\left|\frac{d}{dt}\tilde{f}\circ\gamma(t)\right|\\ =&2\left|\int_{\mathbb{C}^m}\Big(f(u)-\tilde{f}\circ\gamma(t)\Big)\Re\left\{\left\langle u-\gamma(t), \frac{d}{dt}\gamma(t)\right\rangle|k_{\gamma(t)}(u)|^2\right\}d\mu(u)\right|\\ \leq&2\left[(|f-\tilde{f}\circ\gamma(t)|^2)\tilde{}\circ\gamma(t)\right]^{\frac{1}{2}}\left[(|\Gamma_t-\widetilde{\Gamma}_t\circ\gamma(t)|^2)\tilde{}\circ\gamma(t)\right]^{\frac{1}{2}}\end{aligned}$$

where $\Gamma_t \in \mathcal{T}(\mathbb{C}^m)$ is defined by $\Gamma_t(u) := \left\langle u, \frac{d}{dt}\gamma(t)\right\rangle$. An application of Lemma 2.5 (a) and Lemma 2.4 yields

$$\left|\frac{d}{dt}\tilde{f}\circ\gamma(t)\right| \leq 2\|f\|_{\mathrm{BMO}(\gamma(I))}\mathrm{MO}(\Gamma_t,\gamma(t))^{\frac{1}{2}} = 2\|f\|_{\mathrm{BMO}(\gamma(I))}\left\|\frac{d}{dt}\gamma(t)\right\|.$$

From this the desired result follows. □

Corollary 2.7. *For $f \in \mathcal{BMO}(\mathbb{C}^m)$ and $a, b \in \mathbb{C}^m$ we have the Lipschitz-inequality*

$$|\tilde{f}(a) - \tilde{f}(b)| \leq 2\|f\|_{BMO}\|a-b\|.$$

In particular, $\tilde{f} \in \mathcal{BO}(\mathbb{C}^m)$ and $\|Osc_z(\tilde{f})\|_\infty \leq 2\|f\|_{BMO}$.

Proof. Choose $\gamma_a^b : I := [0,1] \to \mathbb{C}^m$ with $\gamma_a^b(t) := a + t(b-a)$ and apply Theorem 2.6. □

Corollary 2.8. *Let $f \in \mathcal{VMO}(\mathbb{C}^m)$. For each $\varepsilon > 0$ there is a number $r > 0$ such that the inequality $(*)$: $|\tilde{f}(a) - \tilde{f}(b)| < \varepsilon\|a-b\|$ is valid for all $a, b \in A_r := \mathbb{C}^m \setminus B(0,r)$. In particular, $\tilde{f} \in \mathcal{VO}(\mathbb{C}^m)$.*

Proof. Fix $r_0 > 0$ and $a, b \in A_{r_0}$ with $a \neq b$. Define $z_1 := \frac{1}{2}(a+b)$ and $z_2 := \frac{1}{2}(a-b)$. Choose $z_3 \in \mathbb{C}^m$ with $z_3 \perp z_2$ and $\|z_3\| = \|z_2\|$ and consider the arcs $\gamma_1, \gamma_2 : I \to \mathbb{C}^m$ given by

$$\gamma_1(t) := z_1 + z_2\cos\pi t + z_3\sin\pi t, \qquad \gamma_2(t) := z_1 + z_2\cos\pi(1+t) + z_3\sin\pi(t+1).$$

We have $\gamma_1(0) = \gamma_2(1) = a$ and $\gamma_1(1) = \gamma_2(0) = b$. Because $a, b \in A_{r_0}$ it easy to check that either $\gamma_1(I) \subset A_{r_0}$ or $\gamma_2(I) \subset A_{r_0}$. Assume $\gamma_1(I) \subset A_{r_0}$ and apply Theorem 2.6

$$\begin{aligned}|\tilde{f}(a) - \tilde{f}(b)| &\leq \int_0^1 \left| \frac{d}{dt} \tilde{f} \circ \gamma_1(t) \right| dt \\ &\leq 2\|f\|_{\mathrm{BMO}(A_{r_0})} \int_0^1 \left\| \frac{d}{dt} \gamma_1(t) \right\| dt = \pi \|f\|_{\mathrm{BMO}(A_{r_0})} \|a - b\|.\end{aligned}$$

Finally choose $r_0 > 0$ such that $\|f\|_{\mathrm{BMO}(A_{r_0})} < \frac{\varepsilon}{\pi}$. □

3. The spaces $\mathcal{BMO}(\mathbb{C}^{\mathbf{m}})$ and $\mathcal{VMO}(\mathbb{C}^{\mathbf{m}})$

In this section we give a description of the space $\mathcal{BMO}(\mathbb{C}^m)$ [resp. $\mathcal{VMO}(\mathbb{C}^m)$]. We show in which sense they are related to $\mathcal{BO}(\mathbb{C}^m)$ [resp. $\mathcal{VO}(\mathbb{C}^m)$].

Theorem 3.1. *Let $f \in \mathcal{T}(\mathbb{C}^m)$.*

(a) *The Berezin transform $\widetilde{|f|^2}$ is a bounded continous function if and only if $M_f P$ is bounded. Moreover, there is a constant $C > 0$ such that*

$$(*): \quad \|\widetilde{|f|^2}\|_\infty \leq \|M_f P\|^2 \leq C \|\widetilde{|f|^2}\|_\infty$$

where $\|g\|_\infty := \sup\{|g(z)| : z \in \mathbb{C}^m\}$ for all $g \in \mathcal{BC}(\mathbb{C}^m)$.

(b) *The operator $M_f P$ is compact if and only if $\widetilde{|f|^2}(\lambda) \longrightarrow 0$ as $\lambda \to \infty$.*

Proof. (a) An analogous computation as in [BC1] Lemma 14 shows that there is a constant $C > 0$ such that

$$\|\widetilde{|f|^2}\|_\infty \leq \|PM_{|f|^2}P\| \leq C \|\widetilde{|f|^2}\|_\infty.$$

Using $\|PM_{|f|^2}P\| = \|(M_f P)^*(M_f P)\| = \|M_f P\|^2$ the inequality $(*)$ follows.

(b) Let $M_f P$ be compact. Then the operator $PM_{|f|^2}P = (M_f P)^*(M_f P)$ is compact and because $k_\lambda \to 0$ weakly in $H^2(\mathbb{C}^m, \mu)$ as $\lambda \to \infty$ it follows that

$$\widetilde{|f|^2}(\lambda) = \langle PM_{|f|^2}Pk_\lambda, k_\lambda \rangle_2 \leq \|PM_{|f|^2}Pk_\lambda\|_2 \longrightarrow 0, \qquad (\lambda \to \infty).$$

Let $\widetilde{|f|^2}(\lambda) \to 0$ as $\lambda \to \infty$ and let χ_R be the characteristic function of $B(0, R)$. It is easy to verify that $M_{f\chi_R}P$ is of *Hilbert-Schmidt* type. Hence, it is sufficient to show that

$$\|M_f P - M_{f\chi_R}P\| = \|M_{f(1-\chi_R)}P\| \longrightarrow 0, \qquad (R \to \infty).$$

According to (a) there is a constant $C > 0$ such that

$$\|M_{f(1-\chi_R)}P\|_2^2 \leq C \sup_{u \in \mathbb{C}^m} \int_{\|z\| \geq R} |f(z)|^2 |k_u(z)|^2 d\mu(z). \tag{3.1}$$

Let $\varepsilon > 0$, then choose $r > 0$ with $\widetilde{|f|^2}(z) < \frac{\epsilon}{C}$ for all $z \in \mathbb{C}^m \setminus B(0,r)$. It follows that

$$\sup_{\|u\|>r} \int_{\|z\|\geq R} |f(z)|^2 |k_u(z)|^2 d\mu(z) \leq \sup_{\|u\|>r} \widetilde{|f|^2}(u) < \frac{\varepsilon}{C}. \tag{3.2}$$

On $\overline{B(0,r)}$ the function $F_R : u \mapsto \int_{\|z\|\geq R} |f(z)|^2 |k_u(z)|^2 d\mu(z)$ converges monotonely to 0 as $R \to \infty$. Using *Dini's theorem* there is $R_0 > 0$ such that with $R > R_0$

$$\sup_{\|u\|\leq r} \int_{\|z\|\geq R} |f(z)|^2 |k_u(z)|^2 d\mu(z) < \frac{\varepsilon}{C}. \tag{3.3}$$

The inequalities (3.1), (3.2) and (3.3) prove that $\|M_{f(1-\chi_R)} P\|_2^2 < \varepsilon$ for each $R > R_0$. □

Definition 3.2. In the following we use the spaces $\mathcal{F}$ and $\mathcal{I}$ defined by

$$\mathcal{F} := \left\{ f \in \mathcal{T}(\mathbb{C}^m) : \widetilde{|f|^2} \in \mathcal{BC}(\mathbb{C}^m) \right\}, \qquad \mathcal{I} := \left\{ f \in \mathcal{T}(\mathbb{C}^m) : \widetilde{|f|^2} \in \mathcal{C}_0(\mathbb{C}^m) \right\}.$$

Corollary 3.3. *For $f \in \mathcal{F}$ the Hankel operator H_f is bounded and there is a constant $C > 0$ such that $\|H_f\|^2 \leq C \|\widetilde{|f|^2}\|_\infty$. Moreover, for $f \in \mathcal{I}$ the Hankel operator H_f is compact.*

Proof. This follows from Theorem 3.1 with $H_f = (I - P) M_f P$. □

Lemma 3.4. *Let $f \in \mathcal{BO}(\mathbb{C}^m)$ and fix $r \geq 0$. Then for all $z, w \in \mathbb{C}^m \setminus B(0,r)$ we have the inequality $|f(z) - f(w)| \leq C(f,r)\,(1 + \pi \|z - w\|)$ where $C(f,r) :=$ $\sup\{|Osc_z(f)| : \|z\| \geq r - 1\}$.*

Proof. Let $z, w \in \mathbb{C}^m \setminus B(0,r)$. Then choose $\gamma : I = [0,1] \to \mathbb{C}^m \setminus B(0,r)$ connecting z and w as in the proof of Corollary 2.8. Let $n \in \mathbb{N}$ be the greatest integer in $\pi\|z - w\|$ then divide $\gamma(I)$ into $n+1$ segments $[\gamma(t_i), \gamma(t_{i+1})]$ of equal length.

Because of $B(\gamma(t_i), 1) \subset \{z \in \mathbb{C}^m : \|z\| \geq r - 1\}$ and $\|\gamma(t_i) - \gamma(t_{i+1})\| < 1$ for $i = 0, \cdots, n$, it follows that

$$|f(z) - f(w)| \leq (1+n) C(f,r) \leq C(f,r)\,(1 + \pi \|z - w\|).$$

From this we obtain Lemma 3.4. □

Lemma 3.5. *We have $\mathcal{BO}(\mathbb{C}^m) \subset \mathcal{BMO}(\mathbb{C}^m)$ and the following statements are equivalent*

(a) $f \in \mathcal{BO}(\mathbb{C}^m)$,
(b) *there is a constant $C > 0$ with $|f(z) - f(w)| \leq C\,(1 + \|z - w\|)$ for all $z, w \in \mathbb{C}^m$,*
(c) *the function $z \mapsto \|f(z) - f \circ \tau_z\|_2$ is in $\mathcal{BC}(\mathbb{C}^m)$.*

Proof. The conclusion $(a) \Rightarrow (b)$ follows from Lemma 3.4 with $r = 0$. Suppose (b) holds and $z \in \mathbb{C}^m$. Then

$$\|f(z) - f \circ \tau_z\|_2^2 = \int_{\mathbb{C}^m} |f(z) - f(z+w)|^2 d\mu(w) \leq C^2 \int_{\mathbb{C}^m} [1 + \|w\|]^2 d\mu(w) < \infty.$$

Finally suppose (c) holds. It is easy to check that

$$\|f(z) - f \circ \tau_z\|_2^2 = \mathrm{MO}(f, z) + |f(z) - \tilde{f}(z)|^2. \tag{3.4}$$

Because the left hand side of the equality (3.4) is bounded we conclude that $f \in \mathcal{BMO}(\mathbb{C}^m)$ and

$$f - \tilde{f} \in \mathcal{BC}(\mathbb{C}^m) \subset \mathcal{BO}(\mathbb{C}^m).$$

It follows from Corollary 2.7 that $\tilde{f} \in \mathcal{BO}(\mathbb{C}^m)$ and so we obtain $f = (f - \tilde{f}) + \tilde{f} \in \mathcal{BO}(\mathbb{C}^m)$. □

Lemma 3.6. *We have $\mathcal{VO}(\mathbb{C}^m) \subset \mathcal{VMO}(\mathbb{C}^m)$ and the following statements are equivalent*

(a) $f \in \mathcal{VO}(\mathbb{C}^m)$,
(b) *for each $\varepsilon > 0$ there is $r > 0$ such that $|f(z) - f(w)| \leq \varepsilon(1 + \|z - w\|)$ for all $z, w \in \mathbb{C}^m \setminus B(0, r)$,*
(c) *the function $z \mapsto \|f(z) - f \circ \tau_z\|_2$ is in $\mathcal{C}_0(\mathbb{C}^m)$.*

Proof. The conclusion $(a) \Rightarrow (b)$ follows from Lemma 3.4 together with the convergence

$$\lim_{r \to 0} C(f, r) = 0.$$

Now, suppose (b) holds. Then fix $\varepsilon > 0$ and choose $R > 0$ such that for all $z \in \mathbb{C}^m$

$$\int_{\|w\|>R} |f(z) - f(z+w)|^2 d\mu(w) \leq C(f, 0)^2 \int_{\|w\|>R} [1 + \pi\|w\|]^2 d\mu(w) < \frac{\varepsilon}{2}. \tag{3.5}$$

Define $M := \int_{\mathbb{C}^m} [1 + \|w\|]^2 d\mu(w) > 0$ and choose a radius $r > 0$ such that for all $z, w \in \mathbb{C}^m \setminus B(0, r)$

$$|f(z) - f(w)|^2 \leq \frac{\varepsilon}{2M} (1 + \|z - w\|)^2. \tag{3.6}$$

If $\|z\| > r + R$ then we have $\|z + w\| > r$ for all $w \in B(0, R)$ and it follows with the inequalities (3.5) and (3.6) that

$$\begin{aligned}
&\|f(z) - f \circ \tau_z\|_2^2 \\
&= \int_{\|w\| \leq R} |f(z) - f(z+w)|^2 d\mu(w) + \int_{\|w\|>R} |f(z) - f(z+w)|^2 d\mu(w) \\
&\leq \frac{\varepsilon}{2M} \int_{\|w\| \leq R} [1 + \|w\|]^2 d\mu(w) + \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

and (c) follows.

Finally suppose (c) holds. Then the identity (3.4) shows that $f \in \mathcal{VMO}(\mathbb{C}^m)$ as well as $\tilde{f} - f \in \mathcal{C}_0(\mathbb{C}^m) \subset \mathcal{VO}(\mathbb{C}^m)$ and using Corollary 2.8 we conclude that $\tilde{f} \in \mathcal{VO}(\mathbb{C}^m)$. This together proves that $f = \tilde{f} - (\tilde{f} - f) \in \mathcal{VO}(\mathbb{C}^m)$. □

Corollary 3.7. *Using the notations above we have*

(*i*) $\mathcal{BMO}(\mathbb{C}^m) = \mathcal{BO}(\mathbb{C}^m) + \mathcal{F}$, (*ii*) $\mathcal{VMO}(\mathbb{C}^m) = \mathcal{VO}(\mathbb{C}^m) + \mathcal{I}$.

Moreover, the decompositions in (*i*) *and* (*ii*) *are given by* $f = \tilde{f} + (f - \tilde{f})$ *for* $f \in \mathcal{BMO}(\mathbb{C}^m)$ *[resp.* $f \in \mathcal{VMO}(\mathbb{C}^m)$*].*

Proof. (i) The inclusion "⊃" follows from Lemma 3.5 and $\mathcal{F} \subset \mathcal{BMO}(\mathbb{C}^m)$.

Let $f \in \mathcal{BMO}(\mathbb{C}^m)$. Then we conclude that $\tilde{f} \in \mathcal{BO}(\mathbb{C}^m)$ from Corollary 2.7 and it is enough to show that $f - \tilde{f} \in \mathcal{F}$

$$\begin{aligned}(|f-\tilde{f}|^2\tilde{)}(z) &= \|(f-\tilde{f})\circ\tau_z\|_2^2 \leq 2\left[\|f\circ\tau_z - \tilde{f}(z)\|_2^2 + \|\tilde{f}(z) - \tilde{f}\circ\tau_z\|_2^2\right] \\ &= 2\left[\mathrm{MO}(f,z) + \|\tilde{f}(z) - \tilde{f}\circ\tau_z\|_2^2\right]. \end{aligned} \tag{3.7}$$

Because of $f \in \mathcal{BMO}(\mathbb{C}^m)$ the function $\mathrm{MO}(f,\cdot)$ is bounded. Moreover, Lemma 3.5 together with $\tilde{f} \in \mathcal{BO}(\mathbb{C}^m)$ shows that also $z \mapsto \|\tilde{f}(z) - \tilde{f}\circ\tau_z\|_2^2$ is bounded and we conclude that $f - \tilde{f} \in \mathcal{F}$.

(ii) The inclusion "⊃" follows from Lemma 3.6 and $\mathcal{I} \subset \mathcal{VMO}(\mathbb{C}^m)$.

Let $f \in \mathcal{VMO}(\mathbb{C}^m)$. Then we conclude that $\tilde{f} \in \mathcal{VO}(\mathbb{C}^m)$ from Corollary 2.8 and it is enough to show that $f - \tilde{f} \in \mathcal{I}$. An application of Lemma 3.6 together with $\tilde{f} \in \mathcal{VO}(\mathbb{C}^m)$ yields

$$\|\tilde{f}(z) - \tilde{f}\circ\tau_z\|_2^2 \longrightarrow 0, \qquad (z \to \infty). \tag{3.8}$$

Finally, because of $f \in \mathcal{VMO}(\mathbb{C}^m)$ the inequalities (3.7) and (3.8) show that $f - \tilde{f} \in \mathcal{I}$. □

4. Bounded Hankel operators

We will prove (A) in section 1 (see Theorem 4.3). The main ingrediant for the proof is the decomposition $\mathcal{BMO}(\mathbb{C}^m) = \mathcal{BO}(\mathbb{C}^m) + \mathcal{F}$ of the space of all functions of *bounded mean oscillation* and the estimate in Theorem 4.1 between the norm of an *Hankel operator* and the *oscillation* of its symbol.

Theorem 4.1. *Let* $f \in \mathcal{BO}(\mathbb{C}^m)$ *then* H_f *is bounded with* $\|H_f\| \leq C\|Osc_z(f)\|_\infty$ *where* C *is a constant given by* $C := \frac{1}{\pi^m}\int_{\mathbb{C}^m}[\pi\|w\| + 1]\exp(-\frac{1}{2}\|w\|^2)dV(w)$.

Proof. For $f \in \mathcal{BMO}(\mathbb{C}^m)$ the operator $(I-P)M_fP$ is an integral operator on $H^2(\mathbb{C}^m, \mu)$ defined by

$$[(I-P)M_fPg](w) := \int_{\mathbb{C}^m}[f(w) - f(z)]\exp\big(\langle w, z\rangle\big)g(z)d\mu(z), \qquad \forall\, w \in \mathbb{C}^m.$$

Because of $f \in \mathcal{BO}(\mathbb{C}^m)$ Lemma 3.4 with $r = 0$ shows for all $z, w \in \mathbb{C}^m$ that

$$|f(z) - f(w)| \leq \|\mathrm{Osc}_z(f)\|_\infty(1 + \pi\|z - w\|). \tag{4.1}$$

Define $p(z) := \exp(\frac{1}{2}\|z\|^2)$. Then a translation by $w \in \mathbb{C}^m$ with C defined as above shows that

$$\int_{\mathbb{C}^m} [1 + \pi\|z - w\|] \exp\big(\Re\langle w, z\rangle\big) p(z) d\mu(z) = Cp(w). \tag{4.2}$$

After combining the inequalities (4.1) and (4.2) we conclude that

$$\int_{\mathbb{C}^m} |f(w) - f(z)| \exp\big(\Re\langle w, z\rangle\big) p(z) d\mu(z) \le C\|\mathrm{Osc}_z(f)\|_\infty p(w) \tag{4.3}$$

and an application of *Schur's lemma* (see [HS] or [S1]) together with the inequality (4.3) now show that $\|H_f\| = \|(I-P)M_f P\| \le C\|\mathrm{Osc}_z(f)\|_\infty$. □

Theorem 4.2. *Let $f \in \mathcal{BMO}(\mathbb{C}^m)$. Then the Hankel operator H_f is bounded and there is a constant $D > 0$, independent of f, such that $\|H_f\| \le D\|f\|_{BMO}$.*

Proof. For $f \in \mathcal{BMO}(\mathbb{C}^m)$ Corollary 3.7 shows that $\tilde{f} \in \mathcal{BO}(\mathbb{C}^m)$ and $f - \tilde{f} \in \mathcal{F}$. Using Corollary 2.7 and Theorem 4.1 we conclude that $H_{\tilde{f}}$ is bounded and there is $C > 0$ independent of f such that

$$\|H_{\tilde{f}}\| \le C\|\mathrm{Osc}_z(\tilde{f})\|_\infty \le 2C\|f\|_{\mathrm{BMO}}. \tag{4.4}$$

Now, using Corollary 2.7 again, it follows for all $z \in \mathbb{C}^m$ that

$$\begin{aligned}\|\tilde{f}(z) - \tilde{f}\circ\tau_z\|_2 &= \left[\int_{\mathbb{C}^m} |\tilde{f}(z) - \tilde{f}(z+w)|^2 d\mu(w)\right]^{\frac{1}{2}}\\ &\le 2\|f\|_{\mathrm{BMO}} \left[\int_{\mathbb{C}^m} \|w\|^2 d\mu(w)\right]^{\frac{1}{2}} = C_1\|f\|_{\mathrm{BMO}}\end{aligned}$$

where $C_1 := 2[\int_{\mathbb{C}^m} \|w\|^2 d\mu(w)]^{\frac{1}{2}}$. This together with (3.7) shows that

$$(|f - \tilde{f}|^2\tilde{)}(z) \le 2\left[\mathrm{MO}(f,z) + C_1^2\|f\|_{\mathrm{BMO}}^2\right] \le 2(1 + C_1^2)\|f\|_{\mathrm{BMO}}^2. \tag{4.5}$$

Using (4.5) and Corollary 3.3 there are constants $C_2, C_3 > 0$ such that

$$\|H_{f-\tilde{f}}\| \le C_2\|(|f - \tilde{f}|^2\tilde{)}\|_\infty^{\frac{1}{2}} \le C_3\|f\|_{\mathrm{BMO}}. \tag{4.6}$$

Finally, (4.4) together with (4.6) show $\|H_f\| \le \|H\|_{\tilde{f}} + \|H_{f-\tilde{f}}\| \le D\|f\|_{\mathrm{BMO}}$ where $D > 0$ is a constant independent of f. □

Theorem 4.3. *For $f \in \mathcal{T}(\mathbb{C}^m)$ the following are equivalent*

(a) *H_f and $H_{\bar{f}}$ are bounded operators,*

(b) *$f \in \mathcal{BMO}(\mathbb{C}^m) = \mathcal{BO}(\mathbb{C}^m) + \mathcal{F}$. In particular, we have $\tilde{f} \in \mathcal{BO}(\mathbb{C}^m)$ and $f - \tilde{f} \in \mathcal{F}$.*

Whenever (a) and (b) hold the quantities $\|[M_f, P]\|$, $\max\{\|H_f\|, \|H_{\bar{f}}\|\}$ and $\|f\|_{BMO}$ are equivalent.

Proof. Suppose (a) holds. Then Lemma 2.5, (c) shows that

$$\|f\|_{\mathrm{BMO}} \leq \sqrt{2} \max\left\{\|H_f\|, \|H_{\bar{f}}\|\right\}$$

and (b) follows.

Suppose (b) holds. Then we conlude $\bar{f} \in \mathcal{BMO}(\mathbb{C}^m)$ and using Theorem 4.2 we find $D_1, D_2 > 0$ with $\|H_f\| \leq D_1 \|f\|_{\mathrm{BMO}} < \infty$ and

$$\|H_{\bar{f}}\| \leq D_2 \|\bar{f}\|_{\mathrm{BMO}} = D_2 \|f\|_{\mathrm{BMO}} < \infty$$

and from this (a) follows. Moreover, $\|f\|_{\mathrm{BMO}}$ and $\max\left\{\|H_f\|, \|H_{\bar{f}}\|\right\}$ are equivalent.

Finally, the formulas

$$[M_f, P] = H_f - H_{\bar{f}}^*, \qquad (I-P)[M_f, P] = H_f, \qquad [M_f, P](I-P) = -H_{\bar{f}}^* \quad (4.7)$$

show that $\|[M_f, P]\|$ and $\max\left\{\|H_f\|, \|H_{\bar{f}}\|\right\}$ are equivalent. □

Corollary 4.4. *Let $f \in \mathcal{T}(\mathbb{C}^m)$ be an entire function on $\mathbb{C}^m$. Then the following are equivalent*

(a) *There is $a_0 \in \mathbb{C}$ and $b \in \mathbb{C}^m$ such that $f(z) = a_0 + \langle z, b\rangle$,*
(b) *the Hankel operator $H_{\bar{f}}$ is bounded.*

Proof. Suppose (a) holds. Then using Corollary 2.3 we conclude that

$$\rho(\mathrm{MO}(\bar{f}, \cdot)) = 0$$

and it follows that $\bar{f} \in \mathcal{BMO}(\mathbb{C}^m)$. Theorem 4.3 shows that $H_{\bar{f}}$ is bounded.

Suppose (b) holds, so $H_{\bar{f}}$ is bounded. Because of $H_f = 0$ Theorem 4.3 proves that f is in $\mathcal{BMO}(\mathbb{C}^m)$. Applying Corollary 3.7 we now obtain with $\tilde{f} = f$ that $f \in \mathcal{BO}(\mathbb{C}^m)$. It follows with Lemma 3.5 that there is a constant $C > 0$ such that

$$|f(z) - f(w)| \leq C(1 + \|z - w\|), \qquad \forall\, z, w \in \mathbb{C}^m. \quad (4.8)$$

Assume $f(z) = \sum_{j \in \mathbb{N}_0^m} b_j z^j$. Then the *Cauchy estimates* show for any $r > 0$ and $j \in \mathbb{N}_0^m$ that

$$|b_j| = \frac{|[D^j f](0)|}{j!} \leq \frac{1}{r^{|j|}} \sup\{|f(z)| : z \in P(0, \mathbf{r})\}. \quad (4.9)$$

Here, $P(0, \mathbf{r})$ is the polydisc in $\mathbb{C}^m$ with multiradius $\mathbf{r} := (r, \cdots, r)$ and center 0. It is easy to check that the inclusion $P(0, \mathbf{r}) \subset B(0, r\sqrt{m+1})$ holds and we obtain from (4.9) and (4.8)

$$|b_j| \leq \frac{1}{r^{|j|}} \sup\{|f(z)| : z \in B(0, r\sqrt{m+1})\} \leq \frac{1}{r^{|j|}} \left\{|f(0)| + C(1 + r\sqrt{m+1})\right\}.$$

Because $r > 0$ was arbitrary we conclude that $b_j = 0$ for $j \in \mathbb{N}_0^m$ such that $|j| > 1$ and (b) follows. □

Corollary 4.5. *For $p \in \mathbb{P}[z, \bar{z}]$ the statements* (a) *and* (b) *are equivalent:*

(a) *There is $a_0 \in \mathbb{C}$ and $c, d \in \mathbb{C}^m$ such that $p(z, \bar{z}) = a_0 + \langle z, c\rangle + \langle d, z\rangle$,*
(b) *the Hankel operators H_p and $H_{\bar{p}}$ are bounded.*

Proof. Suppose (a) holds. Then using Corollary 2.2 we conclude that

$$\rho(\mathrm{MO}(p,\cdot)) = 0$$

and it follows that $p \in \mathcal{BMO}(\mathbb{C}^m)$. Theorem 4.3 shows that H_p and $H_{\bar{p}}$ are bounded.

Suppose (b) holds, so H_p and $H_{\bar{p}}$ are bounded. Then Theorem 4.3 shows that $p \in \mathcal{BMO}(\mathbb{C}^m)$ and according to Corollary 3.7 we have $\tilde{p} \in \mathcal{BO}(\mathbb{C}^m)$. Using Corollary 2.2 it follows with the above defined set $A(p) := \{(l,j) \in \mathbb{N}_0^{2m} : |(l,j)| = \rho(p)\}$ that

$$\tilde{p}(z,\bar{z}) = Q_p(z,\bar{z}) + r(z,\bar{z}), \quad \text{where} \quad Q_p(z,\bar{z}) := \sum_{(l,j)\in A(p)} a_{l,j} z^l \bar{z}^j \tag{4.10}$$

and $\rho(r) < \rho(p)$. Choose $a \in \mathbb{C}^m$ with $Q_p(a,\bar{a}) \neq 0$. Because of $\tilde{p} \in \mathcal{BO}(\mathbb{C}^m)$ Lemma 3.5 shows that there is a constant $C > 0$ such that

$$|\tilde{p}(z,\bar{z})| \leq |\tilde{p}(0,0)| + C(1 + \|z\|).$$

Using (4.10) we obtain for all $t > 0$

$$t^{\rho(p)}|Q_p(a,\bar{a})| \leq |\tilde{p}(ta,t\bar{a})| + |r(ta,t\bar{a})| \leq |\tilde{p}(0,0)| + C\,[1 + t\|a\|] + |r(ta,t\bar{a})|.$$

Because of $\rho(r) < \rho(p)$ this leads to a contradiction for $\rho(p) > 1$. □

5. Compact Hankel operators

Finally, we prove (B) in section 1 about compact commutators $[M_f, P]$ with $f \in \mathcal{T}(\mathbb{C}^m)$. We use the decomposition $\mathcal{VMO}(\mathbb{C}^m) = \mathcal{VO}(\mathbb{C}^m) + \mathcal{I}$ which was proven in Corollary 3.7 and the fact that the *Hankel operator* $H_{\bar{f}}$ is compact for all $f \in \mathcal{VMO}(\mathbb{C}^m)$ (see Theorem 5.2). We show that there are no non-constant holomorphic symbols f such that $H_{\bar{f}}$ is compact.

Lemma 5.1. *For $r > 0$ consider a function $f : A_r := \mathbb{C}^m \setminus B(0,r) \to \mathbb{C}$ with*

$$|f(z) - f(w)| \leq C\|z - w\|, \qquad \forall\, z, w \in A_r,$$

where $C > 0$ is independent of f. Then there is $F : \mathbb{C}^m \to \mathbb{C}$ such that

$$(a)\ \ f(z) = F(z), \quad \forall\, (z \in A_r), \qquad\qquad (b)\ \ |F(z) - F(w)| \leq 2C\|z - w\|.$$

for all $z, w \in \mathbb{C}^m$.

Proof. If f is real-valued, then define $F(z) := \inf\{f(w) + C\|z - w\| : w \in A_r\}$.

We conclude that (a) holds from $f(z) \leq f(w) + C\|z - w\|$ for all $z, w \in A_r$.

Moreover, from

$$f(w) + C\|z_1 - w\| \leq f(w) + C\|z_2 - w\| + C\|z_1 - z_2\|, \qquad \forall\, z_1, z_2 \in \mathbb{C}^m, \quad w \in A_r$$

it follows that $|F(z_1) - F(z_2)| \leq C\|z_1 - z_2\|$. If f is complex-valued, then write $f = f_1 + if_2$, where f_1 and f_2 are real-valued. Choose F_1 and F_2 with

$$f_j(z) = F_j(z), \quad \forall\, z \in A_r, \qquad\qquad |F_j(z) - F_j(w)| \leq C\|z - w\|, \quad \forall\, z, w \in \mathbb{C}^m$$

for $j = 1, 2$. Then (a) and (b) in Lemma 5.1 immediately follow with $F := F_1 + iF_2$ and the triangle inequality. □

Theorem 5.2. *Let $f \in \mathcal{VMO}(\mathbb{C}^m)$. Then the Hankel operator $H_{\tilde{f}}$ is compact.*

Proof. Let $\varepsilon > 0$. Applying Corollary 2.8 there is a number $r > 0$ such that for the *Berezin transform* $\tilde{f}$ and all $z, w \in A_r := \mathbb{C}^m \setminus B(0, r)$ the inequality $|\tilde{f}(z) - \tilde{f}(w)| < \varepsilon \|z - w\|$ holds. Due to Lemma 5.1 there is a function $F : \mathbb{C}^m \to \mathbb{C}$ such that

(i) $F(z) = \tilde{f}(z) \quad \forall\, z \in A_r$, $\qquad (ii)$ $|F(z) - F(w)| < 2\varepsilon \|z - w\| \quad \forall\, z, w \in \mathbb{C}^m$.

Using Theorem 4.1 and (ii) we conclude that H_F is bounded and there is a constant $C > 0$ such that $\|H_F\| < 2\varepsilon C$. The function $\tilde{f} - F$ has compact support and so $H_{\tilde{f}-F}$ is compact. Because $\varepsilon > 0$ was arbitrary and with $\|H_{\tilde{f}} - H_{\tilde{f}-F}\| = \|H_F\| \leq 2\varepsilon C$ we conclude that $H_{\tilde{f}}$ is compact. □

Theorem 5.3. *For $f \in \mathcal{T}(\mathbb{C}^m)$ the following are equivalent*

(a) *The commutator $[M_f, P]$ is compact,*
(b) *H_f and $H_{\bar{f}}$ are compact operators,*
(c) *$f \in \mathcal{VMO}(\mathbb{C}^m) = \mathcal{VO}(\mathbb{C}^m) + \mathcal{I}$. In particular, $\tilde{f} \in \mathcal{VO}(\mathbb{C}^m)$ and $f - \tilde{f} \in \mathcal{I}$.*

Proof. The equivalence $(a) \Leftrightarrow (b)$ follows from the equations in (4.7). Suppose (b) holds. Then using Lemma 2.5, (b) we conclude that

$$|\mathrm{MO}(f, z)| \leq \|H_{f \circ \tau_z} 1\|^2 + \|H_{\bar{f} \circ \tau_z}\|^2 = \|H_f k_z\|^2 + \|H_{\bar{f}} k_z\|^2 \longrightarrow 0, \qquad (z \to \infty)$$

because $k_z \to 0$ weakly in $H^2(\mathbb{C}^m, \mu)$ as $z \to \infty$. The second part of (c) follows from Corollary 3.7.

Suppose (c) holds. Then $f = \tilde{f} + (f - \tilde{f})$ where $\tilde{f} \in \mathcal{VO}(\mathbb{C}^m)$ and $f - \tilde{f} \in \mathcal{I}$. Due to Corollary 3.3 the *Hankel operator* $H_{f-\tilde{f}}$ is compact. Because $f \in \mathcal{VMO}(\mathbb{C}^m)$ we conclude from Theorem 5.2 that $H_{\tilde{f}}$ is compact and so $H_f = H_{\tilde{f}} - H_{f-\tilde{f}}$ is compact.

For a function $f \in \mathcal{VMO}(\mathbb{C}^m)$ we also have $\bar{f} \in \mathcal{VMO}(\mathbb{C}^m)$ and the same argument shows that $H_{\bar{f}}$ is compact. □

Example. Let $f \in \mathcal{T}(\mathbb{C}^m)$ be an entire function such that $H_{\bar{f}}$ is compact. Then by Corollary 4.4 we have $f(z) = a_0 + \langle \cdot, b \rangle$ where $b \in \mathbb{C}^m$. It follows that $H_{\bar{f}} = H_{\langle b, \cdot \rangle}$ and using Lemma 2.4 we obtain with $\widetilde{\langle b, \cdot \rangle} = \langle b, \cdot \rangle$

$$\mathrm{MO}(\langle b, \cdot \rangle, \lambda) = \widetilde{|\langle b, \cdot \rangle|^2}(\lambda) - |\langle b, \lambda \rangle|^2 = \|b\|^2.$$

Applying Theorem 5.3 we conclude that $b = 0$ and so $f \equiv a_0$ is constant.

Remark 5.4. A similar argument shows that for $p \in \mathbb{P}[z, \bar{z}]$ the *Hankel operator* H_p is compact if and only if p is holomorphic. In this case we obtain $H_p = 0$.

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