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Integral Equations and Operator Theory

Reproducing Kernel Quaternionic Pontryagin Spaces

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Abstract. This paper studies various aspects of reproducing kernel spaces with a possibly indefinite metric when the field of scalar is replaced by the skew–field of quaternions. We first discuss in some details the positive case. A key fact which allows to consider the non–positive case is that Hermitian matrices with quaternionic entries have only real eigenvalues. This permits to extend the notion of functions with a finite number of negative squares to the present setting and we prove in particular that there is a one–to–one correspondence between such functions and reproducing kernel Pontryagin quaternionic spaces.

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1. Introduction

In this paper we present the theory of reproducing kernel quaternionic Pontryagin spaces. The paper is in part of a review nature and we first consider the case of quaternionic Hilbert space in some details. We use the survey of F. Zhang [47] for basic facts on the skew–field of quaternions. We also use [10, Chapter I] for the theory of topological vector spaces over a skew–field.

We note that a number of papers and chapters in books have been written on the theory of quaternionic Hilbert spaces and of reproducing kernel quaternionic Hilbert spaces (or more generally of Hilbert spaces of functions which take values in a Clifford algebra). We mention in particular [45], [33], [18], [14], [13], [11, Section 24 p. 184–198]. In [14] J. Cnops points out in particular important differences between the case of Clifford algebra valued functions and of complex–valued

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functions. For instance in a Hilbert module of functions it need not be true that all functions vanish at a given point w or there is a function such that $f(w) = 1$. This and other phenomena do not take place in the case of quaternionic valued functions since the quaternions form a skew–field.

The spectral theory of matrices with quaternionic entries is not an easy topic and we refer to the above mentioned paper of Zhang [47] for details and references. We also refer to the book [46]. Fortunately the case of hermitian matrices with quaternionic entries is well understood. As in the complex case, such a matrix has only real eigenvalues; see [15, p. 106]. This allows to extend to the quaternionic case the notion of functions with a finite number of negative squares (due to M.G Kreĭn and H. Langer in the case of $\mathbb{C}$; see [29]) and develop the corresponding theory of reproducing kernel quaternionic Pontryagin spaces. In the case of the complex numbers such a theory was developed by P. Sorjonen in [41] and L. Schwartz in [37] and by one of the authors and H. Dym in [3]. The case of quaternionic Pontryagin spaces does not seem to have been considered in previous works and this is the focus of the present paper. The main difficulty is to redo the various arguments (for vector spaces over the complex numbers) in the setting of quaternions, when commutativity does not hold anymore.

We now turn to the outline of the paper. The paper consists of thirteen sections, of which this introduction is the first. In the second section we review various facts on the quaternionic skew–field $\mathbb{H}$. In the third section we discuss matrices with entries in $\mathbb{H}$. In Section 4 we discuss the notion of hyperholomorphic functions. The notion of quaternionic vector spaces is considered in Section 5. In Section 6 we review some facts from the theory of linear operators in quaternionic Hilbert spaces. Sections 7 and 8 and 9 are respectively devoted to positive functions, quaternionic reproducing kernel Hilbert spaces and some facts on operators in quaternionic reproducing kernel Hilbert spaces. The last four sections deal with the non–positive case: In Section 10 we discuss quaternionic inner product spaces. In Section 11 we study the notion of functions with a finite number of negative squares. In Section 12 we study quaternionic Pontryagin spaces and in Section 13 we consider reproducing kernel quaternionic Pontryagin spaces.

Finally a word on notation. We denote by $\mathbb{C}^{m\times n}$ the set of matrices with m rows and n columns and with complex entries. Similarly, $\mathbb{H}^{m\times n}$ stands for the set of matrices with m rows and n columns and with quaternionic entries. In both cases we write m rather than $m \times 1$. The identity matrix in $\mathbb{H}^{n\times n}$ (and of course in $\mathbb{C}^{n\times n}$ and $\mathbb{R}^{n\times n}$) will be denoted by I_n. The conjugate of an element x (either in $\mathbb{C}$ or in $\mathbb{H}$) will be denoted by $\overline{x}$. Let $M = (m_{\ell j})_{\ell=1,\ldots m, j=1,\ldots n} \in \mathbb{H}^{m\times n}$. The matrix $\overline{M}$ is the element in $\mathbb{H}^{m\times n}$ with (ℓ, j) entry $\overline{m_{\ell,j}}$. The matrix M^t, called the transposed matrix, is the element of $\mathbb{H}^{n\times m}$ with (ℓ, j) entry $m_{j,\ell}$, and the adjoint of M, denoted by M^*, is equal to $M^* = \overline{M^t}$. A matrix $M \in \mathbb{H}^{n\times n}$ will be called unitary if $MM^* = M^*M = I_n$.

2. The skew–field $\mathbb{H}$

Consider the real four–dimensional vector space $\mathbb{R}^4$ with its standard (canonical) basis $\mathbf{e_0}$, $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$. The element $\mathbf{e_0}$ will be identified in the following with 1 and will be omitted. Let x, y be two elements from $\mathbb{R}^4$, that is, $x = \sum_{k=0}^{3} x_k \mathbf{e_k}$ and $y = \sum_{k=0}^{3} y_k \mathbf{e_k}$ with x_k, y_k $(k = 0, 1, 2, 3)$ real numbers. Then $\mathbb{R}^4$ becomes a **real linear space** if it is endowed with the usual (component–wise) operations of addition and of multiplication by (real) scalars. Moreover, the rules (which originate with R. Hamilton and form Caley's table)

$$\begin{aligned}\mathbf{e_1}\mathbf{e_2} &= -\mathbf{e_2}\mathbf{e_1} = \mathbf{e_3},\\ \mathbf{e_2}\mathbf{e_3} &= -\mathbf{e_3}\mathbf{e_2} = \mathbf{e_1},\\ \mathbf{e_3}\mathbf{e_1} &= -\mathbf{e_1}\mathbf{e_3} = \mathbf{e_2},\end{aligned}$$

and

$$\mathbf{e_1}^2 = \mathbf{e_2}^2 = \mathbf{e_1}^3 = -1,$$

make $\mathbb{R}^4$ into a skew–field.

The very last equality justifies the name **imaginary units** almost always used for $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$. We recommend to the reader to verify directly that the multiplication defined by the above table is indeed associative.

In this setting we recall Frobenius theorem, stating that $\mathbb{R}^n$ can be endowed with a field or skew–field structure only if $n = 1, 2$ or 4; see [43, p. 104] and [21, p. 94] for some historical remarks. The case of $n = 1$ and $n = 2$ corresponds to the fields of real and complex numbers respectively. The case $n = 4$ corresponds to the skew–field of quaternions $\mathbb{H}$.

We note the formula for the multiplication:

$$(xy)_0 = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3, \tag{2.1}$$

$$(xy)_1 = x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2, \tag{2.2}$$

$$(xy)_2 = x_0y_2 + x_2y_0 + x_3y_1 - x_1y_3, \tag{2.3}$$

$$(xy)_3 = x_0y_3 + x_3y_0 + x_1y_2 - x_2y_1. \tag{2.4}$$

For $x = \sum_{k=0}^{3} x_k \mathbf{e_k}$ we define the mapping (of quaternionic conjugation)

$$\textbf{bar} : \qquad x = \sum_{k=0}^{3} x_k \mathbf{e_k} \mapsto \overline{x} = x_0 - x_1\mathbf{e_1} - x_2\mathbf{e_2} - x_3\mathbf{e_3}.$$

The number $x_0 = \frac{x+\overline{x}}{2}$ is called the **real part**, or the **scalar part**, of the quaternion x.

We note that $\overline{xy} = \overline{y}\,\overline{x}$ and that

$$x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

There is one more way of constructing the set $\mathbb{H}$ relating the latter with complex numbers. To demonstrate that we note that for $x \in \mathbb{H}$ we can write:

$$x = (x_0 + x_1\mathbf{e_1}) + (x_2 + x_3\mathbf{e_1})\mathbf{e_2}.$$

We then identify $\mathbf{e_1}$ with the complex number i and $\mathbb{H}$ with $\mathbb{C}^2$; in this setting we use the notation $j := \mathbf{e_2}$. The set of elements of the form

$$q = z_1 + z_2 j, \quad z_1, z_2 \in \mathbb{C}, \quad j^2 = -1, \tag{2.5}$$

endowed both with an obvious component–wise addition and with the associative multiplication governed by the fundamental correlation between the imaginary units:

$$ij + ji = 0,$$

is then a realization of $\mathbb{H}$. In the representation (2.5) z_1 is called the **first complex component** and z_2 is called the **second complex component**.

We have for $z \in \mathbb{C}$ and $\overline{z}$, its complex conjugate, that

$$zj = j\overline{z}.$$

Let $p = z_1 + z_2 j$ and $q = w_1 + w_2 j$. We note that $pq = (pq)_1 + (pq)_2 j$ where

$$(pq)_1 = z_1 w_1 - z_2 \overline{w_2}, \tag{2.6}$$
$$(pq)_2 = z_2 \overline{w_1} + z_1 w_2. \tag{2.7}$$

The quaternionic conjugation can be defined by

$$\overline{z_1 + z_2 j} := \overline{z_1} - z_2 j.$$

Note that the classical conjugation in $\mathbb{C}^2$ is

$$(z_1, z_2) \mapsto (\overline{z_1}, \overline{z_2}),$$

while the one which appears here is

$$(z_1, z_2) \mapsto (\overline{z_1}, -z_2).$$

Lemma 2.1. *Let $q = z_1 + z_2 j \in \mathbb{H}$. Then,*

$$q\overline{q} = \overline{q}q = |z_1|^2 + |z_2|^2. \tag{2.8}$$

The inverse of the element $q = z_1 + z_2 j \neq 0$ is therefore given by

$$(z_1 + z_2 j)^{-1} = \frac{\overline{z_1} - z_2 j}{|z_1|^2 + |z_2|^2}.$$

Thus $\mathbb{H}$ is a skew–field generated by $\mathbb{C}^2$. Property (2.8) is crucial for many purposes, in particular it expresses a deep relationship between the Euclidian metric in $\mathbb{C}^2$ and the algebraic properties of $\mathbb{H}$. The next definition is quite natural; it determines the quaternionic modulus.

Definition 2.2. For $q = z_1 + z_2 j \in \mathbb{H}$ we set

$$|q| := \sqrt{|z_1|^2 + |z_2|^2}.$$

It is also easy to see that $|q|$ is a norm on the quaternions and in particular the triangle inequality holds:

Proposition 2.3. *Let p and q be in $\mathbb{H}$. Then*

$$|p+q| \leq |p| + |q|. \tag{2.9}$$

Assume $p \neq 0$. Equality holds if and only if $q = tp$ for some $t \in \mathbb{R}_+$.

Proof. Indeed, let $p = z_1 + z_2 j$ and $q = w_1 + w_2 j$. We have:

$$\begin{aligned}
|p+q| &= \sqrt{|z_1+w_1|^2 + |z_2+w_2|^2} \\
&= \sqrt{|z_1|^2 + |w_1|^2 + |z_2|^2 + |w_2|^2 + 2\mathrm{Re}\ (z_1\overline{w_1} + z_2\overline{w_2})} \\
&= \sqrt{|z_1|^2 + |z_2|^2 + |w_1|^2 + |w_2|^2 + 2\mathrm{Re}\ (z_1\overline{w_1} + z_2\overline{w_2})} \\
&\leq \sqrt{|z_1|^2 + |z_2|^2 + |w_1|^2 + |w_2|^2 + 2 \cdot \sqrt{|z_1|^2+|z_2|^2} \cdot \sqrt{|w_1|^2+|w_2|^2}} \\
&= \sqrt{|z_1|^2+|z_2|^2} + \sqrt{|w_1|^2+|w_2|^2}
\end{aligned}$$

since

$$\begin{aligned}
|\mathrm{Re}\ (z_1\overline{w_1} + z_2\overline{w_2})| &\leq |(z_1\overline{w_1} + z_2\overline{w_2})| \\
&\leq \sqrt{|z_1|^2+|z_2|^2} \cdot \sqrt{|w_1|^2+|w_2|^2}
\end{aligned}$$

by Cauchy–Schwartz inequality in $\mathbb{C}^2$. Equality holds if and only the vectors $(z_1, z_2)^t$ and $(w_1, w_2)^t$ are linearly independent over $\mathbb{C}$. Without loss of generality we may assume z_1 or z_2 different from 0. Then equality holds if and only if there is a positive number t such that $w_j = tz_j$ for $j = 1, 2$. This is equivalent to $q = tp$. □

Of course, inequality (2.9) in itself is trivially clear but we see it instructive to demonstrate a proof based on purely algebraic properties.

One may find all the above in many sources; see for instance [28, Appendices]. Note explicitly that it is easy to check that:

Proposition 2.4. *The sequence $q_n = z_{1n} + z_{2n} j$ is a Cauchy sequence in $\mathbb{H}$ if and only if both z_{1n} and z_{2n} are Cauchy sequences of complex numbers.*

Various realizations of the skew–field of quaternions will be reviewed in the next section. One construction is as follows (and follows the construction of the complex numbers as matrices): Let E_1 and E_2 be two matrices in $\mathbb{C}^{2\times 2}$ such that

$$E_1^2 = E_2^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad E_1E_2 + E_2E_1 = 0.$$

For instance one can take

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then the set of matrices of the form $(x_0I_2 + x_1E_1) + (x_2I_2 + x_3E_1)E_2$ where x_0, x_1, x_2 and x_3 are real numbers becomes isomorphic to $\mathbb{H}$ if we identify E_1 with the complex number i and set $E_2 = j$.

3. Matrix representations of quaternions and matrices with quaternionic entries

In the same way that one can see the skew–field of quaternions as $\mathbb{R}^4$ or $\mathbb{C}^2$ (with an appropriate structure) one can associate with a quaternion in a natural way two matrices, one in $\mathbb{R}^{4\times 4}$ and one in $\mathbb{C}^{2\times 2}$. We begin with the case of matrices in $\mathbb{R}^{4\times 4}$. In the next proposition $|x|^2$ denotes the Euclidian norm of x.

Proposition 3.1. *The map which to $x = (x_0, x_1, x_2, x_3)^t \in \mathbb{R}^4$ associates the matrix*

$$B(x) := \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \in \mathbb{R}^{4\times 4} \tag{3.1}$$

has the following properties:

$$B(x)B(y) = B(xy),$$

where the components of xy are given by (2.1)–(2.4),

$$\begin{aligned} B(x+y) &= B(x) + B(y), \\ B(\lambda x) &= \lambda B(x), \\ B(1) &= I_4, \\ B(\overline{x}) &= B(x)^t, \end{aligned} \tag{3.2}$$

where $\overline{x} = (x_0, -x_1, -x_2, -x_3)^t$ and where $B(x)^t$ is the transpose matrix of the matrix $B(x)$, and

$$\det B(x) = |x|^4$$

for any $x, y \in \mathbb{H}$ and $\lambda \in \mathbb{R}$.

These various properties are easily verified. It follows that

$$B(x)B(\overline{x}) = B(|x|^2) = |x|^2 \cdot I_4$$

and

$$B(x)^{-1} = B\left(\frac{\overline{x}}{|x|^2}\right).$$

Therefore:

Proposition 3.2. *The set*

$$\left\{B(x)\ ;\ x = (x_0, x_1, x_2, x_3)^t \in \mathbb{R}^4\right\}$$

endowed with the usual laws of multiplication and addition of matrices is a skew–field.

We set $\mathbf{e_0} = I_4$ and

$$\mathbf{e_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{e_3} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

These $\mathbf{e_j}$ satisfy the Caley table and thus the set of elements of the form

$$B(x) = \sum_{k=0}^{3} x_k \mathbf{e_k} \in \mathbb{H}$$

is a realization of the skew–field of quaternions. We identify x with $B(x)$.

For future use it is useful to note the following:

Lemma 3.3. *Let $x, y \in \mathbb{R}^4$ and let $z = xy \in \mathbb{R}^4$ be defined by (2.1)–(2.4). Then*

$$z = B(x)y.$$

The proof is a direct computation and will be omitted.

The counterpart of the above results for matrices with complex entries is:

Proposition 3.4. *The map which to $q = (z, w)^t \in \mathbb{C}^2$ associates the matrix*

$$\chi(q) := \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \tag{3.3}$$

has the following properties:

$$\chi(p)\chi(q) = \chi(pq)$$

where the components of pq are given by (2.6)–(2.7),

$$\chi(\overline{q}) = (\chi(q))^*,$$
$$\det \chi(q) = |q|^2,$$
$$\chi(1) = I_2.$$

The counterpart of Lemma 3.3 is:

Lemma 3.5. *Let $p, q \in \mathbb{C}^2$ and let $r = pq \in \mathbb{C}^2$ be defined by (2.6)–(2.7). Then*

$$r^t = q^t \chi(p).$$

We denote by $\mathbb{H}^{n\times m}$ the set of $n\times m$ matrices with entries in $\mathbb{H}$. When $m = 1$, we set $\mathbb{H}^{n\times 1} := \mathbb{H}^n$.

The map $x \mapsto B(x)$ extends to matrices in the following way: let

$$X = X_0 + X_1\mathbf{e_1} + X_2\mathbf{e_2} + X_3\mathbf{e_3} \in \mathbb{H}^{n\times m}$$

with the $X_\ell \in \mathbb{R}^{n\times m}$. We set

$$B(X) := \begin{pmatrix} X_0 & -X_1 & -X_2 & -X_3 \\ X_1 & X_0 & -X_3 & X_2 \\ X_2 & X_3 & X_0 & -X_1 \\ X_3 & -X_2 & X_1 & X_0 \end{pmatrix} \in \mathbb{R}^{4n\times 4m}.$$

We now collect the main properties of the map $X \mapsto B(X)$:

Proposition 3.6. *The map $X \mapsto B(X)$ has the following properties:*

$$B(XY) = B(X)B(Y)$$

where X and Y are of appropriate dimensions,

$$B(X^*) = B(X)^t$$

where X^* denotes the adjoint of the matrix X, and

$$B(X^{-1}) = (B(X))^{-1}.$$

when X is invertible.

The proof of Proposition 3.6 is a direct computation and is left to the reader. We just note the following fact, used in the proof of the second claim. Since the entries of X_ℓ are real we have for $\ell = 1, 2, 3$:

$$(X_\ell \mathbf{e}_\ell)^* = -\mathbf{e}_\ell X_\ell^* = -\mathbf{e}_\ell X_\ell^t = -X_\ell^t \mathbf{e}_\ell.$$

The following formula will be used in the sequel; see Theorem 9.9.

Lemma 3.7. *Let $p, q \in \mathbb{H}^n$ and $Q \in \mathbb{H}^{n\times n}$. Then,*

$$\text{Re } q^*Qp = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} B(q)^t B(Q) B(p) \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix representation

$$\chi(q) = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

of a quaternion $q = a + bj$ extends to matrices: if $Q = A + Bj$ belongs to $\mathbb{H}^{n\times m}$ with A and B in $\mathbb{C}^{n\times m}$ one defines $\chi(Q) \in \mathbb{C}^{2n\times 2m}$ by the formula

$$\chi(Q) := \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}.$$

Clearly,

$$\chi(Q) = \chi(P) \iff P = Q.$$

The following results are the counterparts of Proposition 3.6. They may be found in [47, p. 30] and were first proved in [30]. As in the case of Proposition 3.6 the results follow from direct computations.

Proposition 3.8. *The map $Q \mapsto \chi(Q)$ has the following properties:*

$$\chi(QR) = \chi(Q)\chi(R) \tag{3.4}$$

where Q and R are of appropriate dimensions,

$$\chi(Q^*) = (\chi(Q))^*, \tag{3.5}$$

and when Q is invertible,

$$\chi(Q^{-1}) = (\chi(Q))^{-1}. \tag{3.6}$$

The counterpart of Lemma 3.7 is:

Lemma 3.9. *Let $p, q \in \mathbb{H}^n$ and $Q \in \mathbb{H}^{n\times n}$. Then,*

$$\text{Re } q^*Qp = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \chi(q)^*\chi(Q)\chi(p) \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If $Q \in \mathbb{H}^{n\times m}$ we define its kernel to be

$$\ker Q := \{u \in \mathbb{H}^m \ ; \ Qu = 0\}$$

and its range to be

$$\text{ran } Q := \{Qu \ ; \ u \in \mathbb{H}^m\}.$$

Note that both the kernel and the range are right quaternionic vector spaces (the notion of quaternionic vector spaces is reviewed in Section 5). The dimension of ran Q will be called the rank of Q and denoted by rank Q.

Note also that one could define the dual notions

$$\left\{u \in \mathbb{H}^{1\times n} \ ; \ uQ = 0\right\}$$

and

$$\left\{uQ \ ; \ u \in \mathbb{H}^{1\times n}\right\}.$$

One then obtain left quaternionic vector spaces. The relationships between both notions follow from the fact that $uQ = 0 \iff Q^*u^* = 0$ and $u \in$ ran Q if and only if u^* belongs to the left range of Q^* (Q^* denotes the adjoint of Q, defined in the introduction and whose definition is recalled in the next paragraph).

The definition of the adjoint of a matrix is similar to the complex case: if $Q \in \mathbb{H}^{n\times m}$ its adjoint is the matrix $Q^* \in \mathbb{H}^{m\times n}$ with entries $\overline{q_{\ell j}}$. The matrix $Q \in \mathbb{H}^{n\times n}$ is said to be normal if it commutes with its adjoint and to be hermitian if $Q = Q^*$.

Quaternionic matrices which are normal have the following nice structure; see [47, Corollary 6.2 p. 41] and [12], [30].

Proposition 3.10. *The matrix $Q \in \mathbb{H}^{n\times n}$ is normal if and only if there exists a quaternionic unitary matrix U and complex numbers $c_1, \ldots, c_n$ such that*

$$Q = U^*\mathrm{diag}\ (c_1, c_2, \ldots, c_n)\ U.$$

As a corollary we have the following result, which plays a central role in the sequel.

Proposition 3.11. *The matrix $Q \in \mathbb{H}^{n\times n}$ is hermitian if and only if there exists a quaternionic unitary matrix U and real numbers $h_1, \ldots, h_n$ such that*

$$Q = U^*\mathrm{diag}\ (h_1, h_2, \ldots, h_n)\ U. \tag{3.7}$$

Definition 3.12. The signature of the hermitian matrix Q is $(\nu_+(Q), \nu_-(Q), \nu_0(Q))$ where $\nu_+(Q)$ (resp. $\nu_-(Q)$ and $\nu_0(Q)$) denotes the number of strictly positive h_j (resp. of strictly negative h_j and of $h_j = 0$) in the representation (3.7). The matrix Q is called positive (notation: $Q \geq 0$) if it is hermitian and if $c^*Qc \geq 0$ for all $c \in \mathbb{H}^n$. It is called strictly positive (notation: $Q > 0$) if it is positive and if $c^*Qc > 0$ for $c \neq 0$.

It follows directly from (3.7) that that $Q \geq 0$ if and only if $\nu_-(Q) = 0$ while $Q > 0$ if and only if $\nu_-(Q) = \nu_0(Q) = 0$, that is:

Proposition 3.13. *A matrix is positive if and only if it is hermitian and all its eigenvalues are positive. It is strictly positive if it is hermitian and if all its eigenvalues are strictly positive.*

We note that it is not enough to require that $c^*Qc \geq 0$ for all $c \in \mathbb{C}^n$ to insure positivity. Take for instance the matrix

$$A = \begin{pmatrix} 0 & \mathbf{e_2} \\ -\mathbf{e_2} & 0 \end{pmatrix}.$$

Then for every $c \in \mathbb{C}^2$ it holds that $c^*Ac = 0$ but

$$\begin{pmatrix} \mathbf{e_2} & 1 \end{pmatrix} A \begin{pmatrix} \mathbf{e_2} & 1 \end{pmatrix}^* = -2.$$

We note that for matrices for complex entries the assumption $Q = Q^*$ is superfluous and follows from the polarization identity

$$d^*Ac = \frac{1}{4}\sum_{k=0}^{3} i^k (c + i^k d)^* A (c + i^k d),$$

where $c, d \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$, together with the positivity condition $c^*Ac \geq 0$ for $c \in \mathbb{C}^n$.

The same holds in the case of matrices with entries in $\mathbb{H}$. Assume that $M \in \mathbb{H}^{n\times n}$ is such that $c^*Mc \geq 0$ for every $c \in \mathbb{H}^n$. We want to show that $M = M^*$. We could use the polarization identity in $\mathbb{H}$ (see [27, p. 108])) but we proceed as follows: let $c = a + bj$ where $a, b \in \mathbb{C}^n$. Then

$$\chi(c)^*\chi(M)\chi(c) \geq 0$$

and in particular

$$(1\ 0)\chi(c)^*\chi(M)\chi(c)(1\ 0)^* \geq 0,$$

that is,

$$(a\ b)^*\chi(M)\begin{pmatrix} a \\ b \end{pmatrix} \geq 0.$$

Since a and b are arbitrary it follows by the polarization identity that $\chi(M)$ is hermitian and so is M.

Proposition 3.11 is the counterpart of the spectral theorem for hermitian matrices in the present setting. It allows to give, as in the complex case, the following characterization and properties of positive matrices.

Corollary 3.14. *Let $Q \in \mathbb{H}^{n\times n}$. Then:*

- *Q is positive if and only if $Q = TT^*$ where $T \in \mathbb{H}^{n\times m}$ with $m = \operatorname{rank}_R Q$.*
- *Q is positive if and only if there is a hermitian positive matrix $M \in \mathbb{H}^{n\times n}$ such that $Q = M^2$.*

The proofs are direct consequences of Proposition 3.11 and will be omitted.

The following property of positive quaternionic hermitian matrices is proved as in the complex case, using the well–known formula

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \tag{3.8}$$
$$= \begin{pmatrix} I_p & 0 \\ Q_{21}Q_{11}^{-1} & I_q \end{pmatrix}\begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} - Q_{21}Q_{11}^{-1}Q_{12} \end{pmatrix}\begin{pmatrix} I_p & Q_{11}^{-1}Q_{12} \\ 0 & I_q \end{pmatrix},$$

where the $Q_{\ell k}$ are matrices (here with quaternionic entries) of appropriate dimensions. See [20, formula (0.4), p.3]. For a discussion of this formula in the present setting see [15, §8]. We recall that $Q_{22} - Q_{21}Q_{11}^{-1}Q_{12}$ is called the Schur complement of Q_{11} in Q.

Proposition 3.15. *A positive matrix $Q \in \mathbb{H}^{n\times n}$ can be factored as $Q = LL^*$ where $L \in \mathbb{H}^{n\times n}$ is a lower triangular matrix.*

Proof. We outline the proof for completeness. We first assume that $Q > 0$ (that is, that all the h_ℓ in (3.7) are strictly positive). Let $Q = (q_{\ell k})$. We first apply (3.8) with

$$\begin{aligned} Q_{11} &= q_{11}, \\ Q_{12} &= \begin{pmatrix} q_{12} \cdots q_{1n} \end{pmatrix}, \\ Q_{21} &= \begin{pmatrix} q_{21} \cdots q_{n1} \end{pmatrix}^t, \\ Q_{22} &= \begin{pmatrix} q_{22} & q_{23} & \cdots & q_{2n} \\ q_{32} & q_{33} & \cdots & q_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ q_{n2} & q_{n3} & \cdots & q_{nn} \end{pmatrix}. \end{aligned}$$

The matrix $Q_{22} - Q_{21}Q_{11}^{-1}Q_{12}$ is strictly positive since $Q > 0$ and we reiterate the procedure and apply formula (3.8) to this matrix where now Q_{11} is replaced by the 11 entry of the Schur complement.

When $Q \geq 0$ we replace Q by $Q(\epsilon) = Q + \epsilon I_n$. Then $Q(\epsilon) > 0$ for $\epsilon > 0$ and there are lower triangular matrices $L(\epsilon)$ such that $Q(\epsilon) = L(\epsilon)L(\epsilon)^*$. The entries of $L(\epsilon)$ are uniformly bounded for $\epsilon \in [0, 1]$ and thus one can take a subsequence ϵ_ℓ going to 0 and obtain a converging subsequence $L(\epsilon_\ell)$. The result follows. □

We compare now the above notions and facts with their real and complex counterparts. In the notation we remove the dependence of the signature on the matrix and e.g. write ν_+ for $\nu_+(Q)$.

Proposition 3.16. *Q is hermitian if and only if $B(Q)$ is self–transposed. The matrix Q has signature (ν_+, ν_-, ν_0) if and only if $B(Q)$ has signature $(4\nu_+, 4\nu_-, 4\nu_0)$. In particular Q is positive if and only if $B(Q)$ is positive.*

Proof. Indeed, write $Q = U^*\text{diag }(h_1, h_2, \ldots, h_n)\ U$ as in Proposition 3.11. The properties of the map $Q \mapsto B(Q)$ give:

$$B(Q) = \\ = B(U)^t \text{diag }(h_1, h_2, \ldots, h_n, h_1, h_2, \ldots, h_n, h_1, h_2, \ldots, h_n, h_1, h_2, \ldots, h_n) B(U)$$

and hence the result. □

Proposition 3.17. *Q is hermitian if and only if $\chi(Q)$ is hermitian. Q has signature (ν_+, ν_-, ν_0) if and only $\chi(Q)$ has signature $(2\nu_+, 2\nu_-, 2\nu_0)$. In particular Q is positive if and only if $\chi(Q)$ is positive.*

Proof. Assume that $\chi(Q) = (\chi(Q))^*$. By (3.5), $\chi(Q) = \chi(Q^*)$ and therefore $Q = Q^*$. Conversely, assume that $Q = Q^*$. Then $\chi(Q) = \chi(Q^*)$ and once more using (3.5) we get that $\chi(Q)$ is hermitian. From (3.7) and the properties of the map $Q \mapsto \chi(Q)$ mentioned in Proposition 3.8 we obtain

$$\chi(Q) = (\chi(U))^* \text{ diag }(h_1, h_2, \ldots, h_n, h_1, h_2, \ldots, h_n)\chi(U)$$

from which follow easily the other claims of the proposition. □

The following lemma is easily proved for matrices with complex entries, and indeed we reduce the proof to this case. The result itself will be used in the proof of Lemma 10.6.

Lemma 3.18. *Let $G \in \mathbb{H}^{n\times n}$ be hermitian and let $X \in \mathbb{H}^{n\times r}$. Assume that G has s negative eigenvalues. Then, X^*GX has at most s negative eigenvalues.*

Proof. By (3.4) and (3.5),

$$\chi(X^*GX) = (\chi(X))^* \cdot \chi(G) \cdot \chi(X)$$

and so by Proposition 3.17 the matrix $\chi(X^*GX)$ has at most $2s$ negative eigenvalues. That same proposition allows to conclude that X^*GX itself has at most s negative eigenvalues. □

The next lemma is used in the proof of Theorem 13.1. We first state the minimax principle for the eigenvalues of a hermitian matrix (see [23]).

Theorem 3.19. *Let $A \in \mathbb{C}^{n\times n}$ be hermitian and let $\lambda_1 \leq \lambda_2 \leq \cdots$ be its eigenvalues in increasing order. Then,*

$$\lambda_r = \min_{\mathcal{M}_r} \max_{\substack{c\in\mathcal{M}_r \\ \|c\|=1}} \{c^*Ac\}$$

where $\mathcal{M}_r$ runs through all r–dimensional subspaces of $\mathbb{C}^n$.

Lemma 3.20. *Let M and N be two hermitian matrices with quaternionic entries and assume $M \geq 0$. Then*

$$\nu_-(M+N) \leq \nu_-(M) + \nu_-(N).$$

Proof. We prove the first inequality. The result is true for hermitian matrices with complex entries. Indeed, from minimax principle the eigenvalues of a hermitian matrix we have

$$\lambda_r(\chi(M) + \chi(N)) \geq \lambda_r(\chi(N))$$

since $\chi(M)$ is a positive matrix. Thus,

$$\begin{aligned} \nu_-(\chi(M+N)) &= \nu_-(\chi(M) + \chi(N)) \\ &\leq \nu_-(\chi(M)) + \nu_-(\chi(N)). \end{aligned}$$

We conclude by using Proposition 3.17. □

As in the classical case a **signature matrix** will be a matrix $J \in \mathbb{H}^{n\times n}$ which is both hermitian and unitary. From Proposition 3.11 follows:

Proposition 3.21. *J is a quaternionic signature matrix if and only if it can be written as*

$$J = U^* \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} U \tag{3.9}$$

where U is unitary.

Another consequence of Proposition 3.11 is the following result.

Proposition 3.22. *Let $Q \in \mathbb{H}^{n\times n}$ of rank r be a hermitian matrix. Then there is an $r \times r$ signature matrix J and a matrix $M \in \mathbb{H}^{n\times r}$ such that*

$$Q = MJM^*. \tag{3.10}$$

For more information on matrices with quaternionic entries we refer to [22].

We mention the next definition and result for completeness. Since we will not use them in the sequel the proof is omitted.

Definition 3.23. Let $Q \in \mathbb{H}^{n\times n}$. $\mathcal{M}(Q)$ denotes the element of $\mathbb{C}^{2n\times 2n}$ defined by

$$\mathcal{M}(Q) = (\chi(q_{\ell k}))\,.$$

Proposition 3.24. *There is a permutation matrix P (depending only on n) such that*

$$\mathcal{M}(Q) = P^*\chi(Q)P.$$

4. Hyperholomorphic functions

Writing a quaternion as $x = \sum_{k=0}^{3} x_k\mathbf{e_k}$ and a quaternionic–valued function $f(x) = f(x_0, x_1, x_2, x_3)$ as

$$f = \sum_{k=0}^{3} f_k\mathbf{e_k},$$

the latter will be called **left–hyperholomorphic** if

$$Df{:=}\frac{\partial}{\partial x_0}f + \mathbf{e_1}\frac{\partial}{\partial x_1}f + \mathbf{e_2}\frac{\partial}{\partial x_2}f + \mathbf{e_3}\frac{\partial}{\partial x_3}f = 0. \tag{4.1}$$

The operator D is called the Cauchy-Riemann operator. Because of the non–commutativity of quaternionic multiplication it can act on the right also, and if we use the notation D_r them f will be called right–hyperholomorphic if it holds that

$$D_rf := \frac{\partial}{\partial x_0}f + \frac{\partial}{\partial x_1}f\mathbf{e_1} + \frac{\partial}{\partial x_2}f\mathbf{e_2} + \frac{\partial}{\partial x_3}f\mathbf{e_3} = 0.$$

Note that the set of left–hyperholomorphic functions has a natural structure of a quaternionic right–linear space, while right–hyperholomorphic functions form a left–linear space. See the next section for definitions.

Various representations of quaternions described in Sections 2 and 3 lead to the corresponding reformulations of equation 4.1. Using the matrix representation (3.1) one may define a (left–) hyperholomorphic $\mathbb{R}^4$–valued vector $f = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 \end{pmatrix}$ as a solution to the system

$$
\begin{aligned}
\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} &= 0, \\
\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_3} &= 0, \\
\frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_0} &= 0, \\
\frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} &= 0.
\end{aligned}
\tag{4.2}
$$

Sometimes (4.2) bears the name of the Fueter system.

Consider now the representation (2.5) which allows to identify $\mathbb{C}^2$ and $\mathbb{H}$, that is, the independent variable $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ is identified now with $z = (x_0 + ix_1, x_2 + ix_3) \in \mathbb{C}^2$. We use the canonical notation

$$
\begin{aligned}
\tfrac{\partial}{\partial \overline{z_1}} &:= \tfrac{1}{2}\left(\tfrac{\partial}{\partial x_0} + i\tfrac{\partial}{\partial x_1}\right); \quad \tfrac{\partial}{\partial z_1} := \tfrac{1}{2}\left(\tfrac{\partial}{\partial x_0} - i\tfrac{\partial}{\partial x_1}\right); \\
\tfrac{\partial}{\partial \overline{z_2}} &:= \tfrac{1}{2}\left(\tfrac{\partial}{\partial x_2} + i\tfrac{\partial}{\partial x_3}\right); \quad \tfrac{\partial}{\partial z_2} := \tfrac{1}{2}\left(\tfrac{\partial}{\partial x_2} - i\tfrac{\partial}{\partial x_3}\right),
\end{aligned}
$$

where of course we have written again i instead of $\mathbf{e_1}$ and will write j instead of $\mathbf{e_2}$ and we set $f = (f_0 + if_1) + (f_2 + if_3)j =: F_1 + F_2 j$. Condition (4.1) is equivalent to the following "hyperholomorphic Cauchy–Riemann conditions" for the mapping $(F_1, F_2)\,\Omega \subset \mathbb{C}^2 \to \mathbb{C}^2$:

$$
\begin{aligned}
\frac{\partial F_1}{\partial \overline{z_1}} &= \frac{\partial F_2}{\partial \overline{z_2}}, \\
\frac{\partial F_1}{\partial z_2} &= -\frac{\partial F_2}{\partial z_1}.
\end{aligned}
$$

In particular this means that given holomorphic functions $F_1(z_1)$ and an holomorphic $F_2(z_2)$ then $F_1 + jF_2$ is hyperholomorphic. To imbed all holomorphic mappings into the set of hyperholomorphic mappings, one may consider, for instance, the quaternionic operator

$$
\widetilde{D} f := \frac{\partial}{\partial x_0} f + \mathbf{e_1} \frac{\partial}{\partial x_1} f + \mathbf{e_2} \frac{\partial}{\partial x_2} f - \mathbf{e_3} \frac{\partial}{\partial x_3} f
$$

but some others as well. For more information on all the above the reader is referred to [39], [44], [31], [42], [40].

It is illustrative to verify that the quaternionic variable x itself **is not** left–hyperholomorphic.

Example 4.1. For every y in the open unit ball $\mathbb{S}$ of $\mathbb{H}$ the function

$$
x \mapsto K(x, y) = \frac{1 - \overline{x}y}{|1 - \overline{x}y|^4} \tag{4.3}
$$

is left–hyperholomophic in $\mathbb{S}$.

We will come back to this important example in the sequel. See Examples 7.3 and 8.2 and Corollary 9.5. The function $K(x, y)$ is called the Szëgo kernel of $\mathbb{S}$.

It is positive (in the sense of Definition 7.1 below) and it is the reproducing kernel of the Hardy space of the unit ball of $\mathbb{H}$. See [11].

5. Quaternionic vector spaces and quaternionic Hilbert spaces

For the definition of a right quaternionic vector space we refer to [9, A II.3, AII.95–118] . The reader needs only to take as a special case the skew–field of quaternions in the definitions there. The definitions follow the classical case of the complex numbers, with appropriate care due to the non–commutativity of the multiplication. There is also the related notion of left quaternionic vector space. Since we will consider here only the case of right quaternionic vector space, mostly we will only say quaternionic vector space.

Definition 5.1. Let $\mathcal{M}$ be a right quaternionic vector space. The elements $f_1, \ldots, f_k$ are called linearly independent if for $q_1, \ldots, q_k \in \mathbb{H}$ it holds that:

$$\sum_1^k f_k q_k = 0 \Longrightarrow q_1 = \cdots = q_k = 0.$$

The definition of a basis has to take into account the non–commutativity: a set $(e_\alpha)_{\alpha\in A}$ is a (right) basis of the quaternionic vector space $\mathcal{M}$ if every element $m \in \mathcal{M}$ can be written in a unique way as a finite linear combination

$$m = e_{\alpha_1} q_1 + \cdots + e_{\alpha_{i_m}} q_{i_m}$$

with coefficients $q_1, \ldots, q_{i_m} \in \mathbb{H}$.

Definition 5.2. Let $\mathcal{H}$ be a right quaternionic vector space. We denote by $\mathcal{H}_\mathbb{R}$ (resp. $\mathcal{H}_\mathbb{C}$) this same space when endowed with the structure of a real vector space (resp. of a complex vector space).

Proposition 5.3. *Let $\mathcal{H}$ be a finite–dimensional right quaternionic vector space, of dimension κ. Then it has dimension 2κ when considered as a vector space over the complex numbers and it has dimension 4κ when considered as a vector space over the real numbers. Moreover if $\{f_1, \ldots, , f_N\}$ is a basis of $\mathcal{H}$ then*

$$\{f_1, f_1\mathbf{e}_1, f_1\mathbf{e}_2, f_1\mathbf{e}_3, \cdots, f_N, f_N\mathbf{e}_1, f_N\mathbf{e}_2, f_N\mathbf{e}_3\}$$

is a basis of $\mathcal{H}_\mathbb{R}$, and

$$\{f_1, f_1\mathbf{e}_2, \cdots, f_N, f_N\mathbf{e}_2\}$$

is a basis of $\mathcal{H}_\mathbb{C}$.

Definition 5.4. Let $\mathcal{V}$ be a quaternionic right vector space. A norm on $\mathcal{V}$ is a map

$$\|\cdot\| : \mathcal{V} \to \mathbb{R}_+^0 := [0, \infty)$$

with the following properties: for $v, w \in \mathcal{V}$,

- $\|v\| = 0 \iff v = 0$.
- $\|v + w\| \leq \|v\| + \|w\|$.
- $\|vp\| = |p|\|v\|$, for any $p \in \mathbb{H}$.

All norms on a finite–dimensional quaternionic vector space are equivalent (see [17, Théorème 1.1 p. 24]).

Definition 5.5. A pre–Hilbert quaternionic space is a right vector $\mathbb{H}$–space $\mathcal{H}$ endowed with an $\mathbb{H}$–valued form $\langle \cdot, \cdot \rangle$ which is:

1. Hermitian, i.e.,
$$\langle f, g\rangle = \overline{\langle g, f\rangle}.$$
2. Positive:
$$\langle f, f\rangle \geq 0$$
and equality holds only for $f = 0$.
3. Linear, meaning that:
$$\langle fp, gq\rangle = \overline{q}\langle f, g\rangle p$$
for all $p, q \in \mathbb{H}$.

The previous definition appears already in [45, Section 1 p. 338]. See also [17, p. 441]. It is more restrictive than more recent ones for the case of general Clifford algebras, where usually it is not required that $\langle f, f\rangle$ is a real number; see [11, p. 35] for the definition. A similar definition holds for left quaternionic Hilbert space. Since we will consider only right quaternionic Hilbert spaces, mostly we will not mention the adjective **right** in the sequel.

We set $\|f\| := \sqrt{\langle f, f\rangle}$. It is a norm and it is usefull to note that
$$\langle fq, fq\rangle = |q|^2 \cdot \|f\|^2.$$

Lemma 5.6. *The Cauchy–Schwartz inequality*
$$|\langle f, g\rangle|^2 \leq \|f\|^2 \cdot \|g\|^2$$
holds in a quaternionic pre–Hilbert space.

Proof. Without loss of generality we assume that $g \neq 0$. Let $q = z_1 + z_2 j \in \mathbb{H}$ and $f, g \in \mathcal{H}$. We set $\langle f, g\rangle =: w_1 + w_2 j$. We have

$$\begin{aligned}
0 &\leq \langle f + gq, f + gq\rangle \\
&= \|f\|^2 + |q|^2 \cdot \|g\|^2 + \langle g, f\rangle q + \overline{q}\langle f, g\rangle \\
&= \|f\|^2 + (|z_1|^2 + |z_2|^2)\|g\|^2 + (\overline{z_1}w_1 + z_2\overline{w_2} + \overline{w_1}z_1 + w_2\overline{z_2}) \\
&= \left\| \begin{pmatrix} \overline{z_1} \\ z_2 \end{pmatrix} cdot\|g\| + \begin{pmatrix} \overline{w_1} \\ w_2 \end{pmatrix} \cdot \frac{1}{\|g\|} \right\|_{\mathbb{C}^2}^2 + \|f\|^2 - \frac{\left\| \begin{pmatrix} \overline{w_1} \\ w_2 \end{pmatrix} \right\|_{\mathbb{C}^2}^2}{\|g\|^2}.
\end{aligned}$$

Since the above holds for any choice of complex numbers z_1 and z_2 it follows that
$$0 \leq \|f\|^2 - \frac{\left\| \begin{pmatrix} \overline{w_1} \\ w_2 \end{pmatrix} \right\|_{\mathbb{C}^2}^2}{\|g\|^2}.$$
□

Remark 5.7. As in the classical case, the above proof uses only that $\langle f, f\rangle \geq 0$ and not the additional requirement that $\langle f, f\rangle = 0$ if and only if $f = 0$.

Two elements of a quaternionic pre–Hilbert space will be called orthogonal if $\langle f, g\rangle = 0$. If $\mathcal{M} \subset \mathcal{H}$ we set

$$\mathcal{M}^{\perp} := \left\{ h \in \mathcal{H} \;\middle|\; \langle h, m\rangle = 0,\ \ \forall m \in \mathcal{M} \right\}.$$

We note that the Gram–Schmidt orthonormalization process works as in the complex Hilbert space case since $\langle f, f\rangle$ is a strictly positive number for all $f \neq 0$. More precisely:

Theorem 5.8. *Let $\mathcal{H}$ be a quaternionic pre–Hilbert space and let $f_1, \ldots, f_k$ be linearly independent (see Definition* 5.1*). Then there exist $e_1, \ldots, e_k \in \mathcal{H}$ with the following properties:*

- *For all $\ell \leq k$*

  $$\text{l.s. } \{f_1, \ldots, , f_\ell\} = \text{l.s. } \{e_1, \ldots, , e_\ell\},$$

  where l.s. *stands for the set of linear combinations (with coefficients on the right).*
- *It holds that*

  $$\langle e_\ell, e_m\rangle = \delta_{\ell,m}.$$

The proof is as in the complex case and can also be derived from Proposition 3.15. One has to be careful with the place of scalars because of the lack of commutativity. One defines by induction

$$\begin{aligned} e_1 &= \frac{f_1}{\|f_1\|}, \\ e_2 &= \frac{f_2 - e_1\langle f_2, e_1\rangle}{\|f_2 - e_1\langle f_2, e_1\rangle\|}, \\ &\vdots \end{aligned}$$

The quaternionic pre–Hilbert space is called a quaternionic Hilbert space if $\|f\| := \sqrt{\langle f, f\rangle}$ defines a norm for which $\mathcal{H}$ is complete. very quaternionic pre–Hilbert space has a completion, as follows from [10, p. EVT I.6].

If $\mathcal{H}$ is a quaternionic Hilbert space and $\mathcal{M}$ is a closed subspace of it, then it holds that:

$$\mathcal{M} \oplus \mathcal{M}^{\perp} = \mathcal{H}.$$

See [10, Chapter I]). We note that in Bourbaki's treatise on topological vector spaces over skew–fields the setting is more general. On the other hand the special case of Hilbert spaces is treated in Bourbaki only for the real and complex scalars; see [10, Chapitre V]. In this context, we note that a quaternionic Hilbert space can be given a structure of complex Hilbert space (its so called **symplectic image** and a structure of real Hilbert space. See [33]). Sometimes (see e.g. Proposition 5.10) using the complex or real structure helps proving results for the quaternionic structure. In general we will prefer direct arguments.

Proposition 5.9. *Let $\mathcal{H}$ be a quaternionic Hilbert space with quaternionic inner product $\langle \cdot, \cdot \rangle$. Then:*

(**1**) *There exists a bilinear real form $\langle \cdot, \cdot \rangle_0$ such that*

$$\langle f, g \rangle = \langle f, g \rangle_0 + \mathbf{e_1} \langle f, g\mathbf{e_1} \rangle_0 + \mathbf{e_2} \langle f, g\mathbf{e_2} \rangle_0 + \mathbf{e_3} \langle f, g\mathbf{e_3} \rangle_0. \tag{5.1}$$

The form $\langle f, g \rangle_0$ endows $\mathcal{H}$ with the structure of a real Hilbert space.

(**2**) *There exists a sesquilinear hermitian form $\langle\langle \cdot, \cdot \rangle\rangle$ such that*

$$\langle f, g \rangle = \langle\langle f, g \rangle\rangle + \langle\langle f, gj \rangle\rangle j$$

and the form

$$\langle\langle f, g \rangle\rangle$$

endows $\mathcal{H}$ with the structure of a (right) complex Hilbert space.

Proof. We set

$$\langle \cdot, \cdot \rangle_0 = \text{Re } \langle \cdot, \cdot \rangle.$$

Clearly, $\langle \cdot, \cdot \rangle_0$ is a real bilinear form. We set

$$\langle f, g \rangle = \langle \cdot, \cdot \rangle_0 + a_1(f, g)\mathbf{e_1} + a_2(f, g)\mathbf{e_2} + a_3(f, g)\mathbf{e_3},$$

where the a_j are $\mathbb{R}$–valued. The next formulas (see e.g. [33, p.2]) show that $\langle f, g \rangle_0$ determines the inner product uniquely and that the a_j are bilinear real forms:

$$a_1(f, g) = \langle f, g\mathbf{e_1} \rangle_0, \tag{5.2}$$

$$a_2(f, g) = \langle f, g\mathbf{e_2} \rangle_0, \tag{5.3}$$

$$a_3(f, g) = \langle f, g\mathbf{e}_3 \rangle_0. \tag{5.4}$$

We prove the first equality. The others are proved in a similar way. Using the $\mathbb{H}$–linearity of the inner product we have

$$\begin{aligned} \langle f, g\mathbf{e_1} \rangle &= -\mathbf{e_1} \langle f, g \rangle \\ &= -\mathbf{e_1} \left(\langle f, g \rangle_0 + \mathbf{e_1} a_1(f, g) + \mathbf{e_2} a_2(f, g) + \mathbf{e_3} a_3(f, g) \right) \\ &= a_1(f, g) - \mathbf{e_1} \langle f, g \rangle_0 + \mathbf{e_2} a_3(f, g) - \mathbf{e_3} a_2(f, g), \end{aligned}$$

and hence we get (5.2). This ends the proof of (1). The claims of (2) are proved in much the same way. □

Following other authors we will not use the term **Hilbert modules** but rather **quaternionic pre–Hilbert space** or **quaternionic Hilbert space** since Hilbert modules has already a different meaning.

We conclude this section with the following result:

Proposition 5.10. *Let $\mathcal{H}$ be a quaternionic Hilbert space and let h_α be a uniformly bounded family of elements:* $\sup_\alpha \|h_\alpha\| < \infty$. *Then h_α possesses a weakly converging subsequence h_{α_n}.*

Proof. The result is true when one considers $\mathcal{H}$ endowed with its real Hilbert space structure; see e.g. [10, Théorème 4 p. EVT V.17]. In the notation of Proposition 5.9 there exists $h \in \mathcal{H}$ such that

$$\forall g \in \mathcal{H} \quad \lim_{n\to\infty} \langle h_{\alpha_n}, g\rangle_0 = \langle h, g\rangle_0.$$

Using (5.2)–(5.4) we have

$$\lim_{n\to\infty} \langle h_{\alpha_n}, g\rangle_\ell = \langle h, g\rangle_\ell, \quad \ell = 1, 2, 3,$$

and hence by (5.1) we obtain the weak convergence in the quaternionic inner product. □

The proofs of thee following two propositions are left to the reader.

Proposition 5.11. *Let $\mathcal{H}$ be a quaternionic Hilbert space. Then $f \perp g$ if and only if $\{f, f\mathbf{e_1}, f\mathbf{e_2}, f\mathbf{e_3}\}$ is orthogonal to $\{g, g\mathbf{e_1}, g\mathbf{e_2}, g\mathbf{e_3}\}$ in the real inner product.*

Proposition 5.12. *Let $\mathcal{H}$ be a quaternionic Hilbert space. Then $f \perp g$ if and only if $\{f, fj\}$ is orthogonal to $\{g, gj\}$ in the complex inner product.*

6. Linear operators in quaternionic Hilbert spaces

We here follow [10, Chapter I] and [11, Section 7, p. 35–43]. An operator A between two quaternionic right Hilbert spaces will be called (right) linear if

$$A(xp + yq) = A(x)p + A(y)q$$

for all x, y in the domain of A and $p, q \in \mathbb{H}$. The operator is called a **functional** if the range space is inside $\mathbb{H}$. It is called bounded (or continuous) if

$$\|A\| := \sup_{\|x\|\le 1} \|Ax\| < \infty.$$

In the above expression we denoted by the same symbol the norms in the (possibly different) quaternionic Hilbert spaces. The equivalence between continuity and boundedness is shown as in the case of complex numbers.

Of importance for the present work are the following:

- The Riesz representation theorem for continuous functionals holds; see [11, p. 24].
- The closed graph theorem holds; see [10, EVT 1.19].
- The open mapping theorem (see e.g. [34, Théorème 5.10 p. 96]) holds. See [10, Corollaire 1 p. EVT 1.19].
- The Hahn–Banach theorem (see [24]).

The Riesz representation theorem reads as follows:

Theorem 6.1. *Let $\mathcal{H}$ be a quaternionic right Hilbert space with quaternionic inner product $\langle\cdot,\cdot\rangle$, and let φ be a continuous right linear functional. Then there is a uniquely defined element $p_\varphi \in \mathcal{H}$ such that*

$$\varphi(x) = \langle x, p_\varphi\rangle, \quad \forall x \in \mathcal{H}.$$

As in the case of complex numbers the Riesz representation theorem allows to define the adjoint of an operator. We here focus on the case of bounded operators.

Proposition 6.2. *Let A be a bounded right linear operator from the quaternionic Hilbert space $(\mathcal{H}_1, \langle\cdot,\cdot\rangle_1)$ into the quaternionic Hilbert space $(\mathcal{H}_2, \langle\cdot,\cdot\rangle_2)$. Then there exists a uniquely defined bounded right linear operator*

$$A^* : \mathcal{H}_2 \to \mathcal{H}_1$$

such that for any $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$

$$\langle Ax, y\rangle_2 = \langle x, A^*y\rangle_1.$$

Finally, $\|A\| = \|A^*\|$.

Proof. The existence of A^*y follows form the Riesz representation theorem. We prove that $A^*(yq) = (A^*y)q$ for $y \in \mathcal{H}_2$ and $q \in \mathbb{H}$. We have:

$$\langle Ax, yq\rangle_2 = \langle x, A^*(yq)\rangle_1, \tag{6.1}$$

while on the other hand

$$\begin{aligned}\langle Ax, yq\rangle_2 &= q^*\langle Ax, y\rangle_2\\ &= q^*\langle x, A^*y\rangle_1\\ &= \langle x, (A^*y)q\rangle_1,\end{aligned} \tag{6.2}$$

and hence the result is true by comparing (6.1) and (6.2).

We now check that A^* is bounded and has the same norm as A. Let y be such that $A^*y \neq 0$. We have $\|A^*y\| \leq \sup_{\|x\|\leq 1} |\langle A^*y, x\rangle|$ by Cauchy–Schwartz inequality. The choice $x = \frac{A^*y}{\|A^*y\|}$ leads to

$$\|A^*y\|_1 = \sup_{\|x\|\leq 1} |\langle A^*y, x\rangle_1|.$$

Thus,

$$\|A^*y\|_1 = \sup_{\|x\|\leq 1} |\langle y, Ax\rangle|_2 \leq \|y\|_2 \cdot \|A\|$$

and so $\|A^*\| \leq \|A\|$. The equality holds by symmetry. □

The operator will be called self–adjoint if $A = A^*$. For another discussion of adjoints, in the more general setting of quaternionic Banach spaces see [39, §3.11].

We quote now the open mapping theorem in the setting of Hilbert quaternionic spaces. We will need this result in Section 12 to prove that all fundamental decompositions in a quaternionic Pontryagin space lead to equivalent topologies.

Theorem 6.3. *Let E and F two right quaternionic Hilbert spaces and let u be a one–to–one and onto, right–linear continuous map from E to F. Then u^{-1} is continuous.*

7. Positive functions

Definition 7.1. An $\mathbb{H}^{n\times n}$–valued function $K(z,w)$ defined for z,w in some set Ω is called positive if it is hermitian in the sense that

$$K(z,w)=K(w,z)^*$$

and if for every choice of integer $m\in\mathbb{N}$ and every choice of points $w_1,\ldots,w_m\in\Omega$ the $m\times m$ matrix with ℓj entry $K(w_\ell,w_j)$ is positive.

A first and trivial example is the case where $\Omega=\mathbb{H}$ and where $K(x,y)=x\overline{y}$. This example also serves to show that a product of positive functions (which a priori need not be hermitian at all) need not be positive even it is hermitian. The function $T(x,y)=K(x,y)^2=x\overline{y}x\overline{y}$ is not positive. We postpone the discussion after the proof of the next proposition. Another example is the real–valued function

$$\mathrm{Re}\,(\overline{x}y)=x_0y_0+x_1y_1+x_2y_2+x_3y_3.$$

Proposition 7.2. *Let $K(z,w)$ be a hermitian function from $\Omega\times\Omega$ into $\mathbb{H}^{n\times n}$. Then:*
(**1**) *It is positive if and only if the $\mathbb{R}^{4n\times 4n}$–valued function*

$$B(K(z,w))=\begin{pmatrix} K_0(z,w) & -K_1(z,w) & -K_2(z,w) & -K_3(z,w)\\ K_1(z,w) & K_0(z,w) & -K_3(z,w) & K_2(z,w)\\ K_2(z,w) & K_3(z,w) & K_0(z,w) & -K_1(z,w)\\ K_3(z,w) & -K_2(z,w) & K_1(z,w) & K_0(z,w)\end{pmatrix}$$

is positive on Ω in the usual sense, where we have set $K(z,w)=K_0(z,w)+K_1(z,w)\mathbf{e_1}+K_2(z,w)\mathbf{e_2}+K_3(z,w)\mathbf{e_3}$ with the K_j being $\mathbb{R}^{n\times n}$–valued.
(**2**) *It is positive if and only if the $\mathbb{C}^{2n\times 2n}$–valued function*

$$\chi(K(z,w))=\begin{pmatrix} A(z,w) & B(z,w)\\ -\overline{B(z,w)} & \overline{A(z,w)}\end{pmatrix} \tag{7.1}$$

is positive on Ω in the usual sense, where we have set $K(z,w)=A(z,w)+B(z,w)j$ with A and B being $\mathbb{C}^{n\times n}$–valued.

Proof. We present the complex case only. The real case is proved in a similar manner. Assume first that the function $K(z,w)$ is positive on Ω. Then, for any choice of points $w_1,\ldots,w_m\in\Omega$, the block matrix $M\in\mathbb{H}^{nm\times nm}$ with ℓk block equal to

$$A(w_\ell,w_k)+B(w_\ell,w_k)j$$

is positive. Set

$$A=(A(w_\ell,w_k))_{\ell,k\in\{1,\ldots,m\}}\,,\qquad B=(B(w_\ell,w_k))_{\ell,k\in\{1,\ldots,m\}}\,.$$

Then $M=A+Bj$ and by Proposition 3.17 the matrix

$$\chi(M)=\begin{pmatrix} A & B\\ -\overline{B} & \overline{A}\end{pmatrix}$$

is positive. Let P denote the permutation matrix which sends the lines

$$\{1, 2, \cdots, m, m+1, m+2, \cdots, 2m\}$$

to the lines

$$\{1, m+1, 2, m+2, \cdots, m, 2m, \cdots, 2m\}$$

Multiplying $\chi(M)$ by P on the left and on the right by P^* we get

$$P\chi(M)P^* = (\chi(K(w_\ell, w_k)))_{\ell,k\in\{1,\ldots,m\}} \tag{7.2}$$

and hence the function $\chi(K)$ is positive in the usual sense.

Equation (7.2) shows in fact that argument may be read backwards to prove the converse direction. □

Example 7.3. The function $\frac{1-\overline{x}y}{|1-\overline{x}y|^4}$ is positive in $\mathbb{S}$. Thus the function

$$\begin{pmatrix} \frac{1-(\overline{a}c+b\overline{d})}{(|1-(\overline{a}c+b\overline{d})|^2+|\overline{a}d-b\overline{c}|^2|)^2} & -\frac{(\overline{a}d-b\overline{c})}{(|1-(\overline{a}c+b\overline{d})|^2+|\overline{a}d-b\overline{c}|^2|)^2} \\ \frac{(\overline{d}a-c\overline{d})}{(|1-(\overline{a}c+b\overline{d})|^2+|\overline{a}d-b\overline{c}|^2|)^2} & \frac{1-(\overline{c}a+d\overline{b})}{(|1-(\overline{a}c+b\overline{d})|^2+|\overline{a}d-b\overline{c}|^2|)^2} \end{pmatrix} \tag{7.3}$$

where $x = a + bj$ and $y = c + dj$ is positive in the unit ball $\mathbb{B}_2$ of $\mathbb{C}^2$.

The positivity of the function $\frac{1-\overline{x}y}{|1-\overline{x}y|^4}$ follows from the reproducing kernel property (see [11]) and a direct proof seems quite difficult. Similarly a direct proof of the positivity of the kernel (7.3) seems also quite difficult.

Example 7.4. The function $K(x, y) = x\overline{y}$ is trivially positive. The corresponding functions A and B are given by (with $x = a + bj$ and $y = c + dj$)

$$A(x, y) = a\overline{c} + b\overline{d}, \quad B(x, y) = bc - ad.$$

We have

$$\begin{pmatrix} A(x,y) & B(x,y) \\ -\overline{B(x,y)} & \overline{A(x,y)} \end{pmatrix} = \begin{pmatrix} a\overline{c} + b\overline{d} & bc - ad \\ \overline{a}\overline{d} - \overline{b}\overline{c} & \overline{a}c + \overline{b}d \end{pmatrix} = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} c & d \\ -\overline{d} & \overline{c} \end{pmatrix}^*$$

which is a positive function.

The above equality is nothing but

$$\chi(x\overline{y}) = \chi(x)\chi(y)^*.$$

On the other hand:

Example 7.5. The function $K(x, y) = \overline{y}x$ is not positive.

Now

$$A(x, y) = a\overline{c} + \overline{b}d, \quad B(x, y) = \overline{c}b - d\overline{a}.$$

Thus

$$\begin{pmatrix} A(x,y) & B(x,y) \\ -\overline{B(x,y)} & \overline{A(x,y)} \end{pmatrix} = \begin{pmatrix} a\overline{c} + \overline{b}d & \overline{c}b - d\overline{a} \\ a\overline{d} - \overline{b}c & \overline{a}c + b\overline{d} \end{pmatrix}.$$

To show that this function is not positive we write

$$\begin{pmatrix} a\overline{c}+\overline{b}d & \overline{c}b-d\overline{a} \\ a\overline{d}-\overline{b}c & \overline{a}c+b\overline{d}\end{pmatrix}=\begin{pmatrix} a\overline{c}+\overline{b}d & 0 \\ 0 & \overline{a}c+b\overline{d}\end{pmatrix}+\frac{1}{2}\left\{\begin{pmatrix} b & \overline{a} \\ a & -\overline{b}\end{pmatrix}\begin{pmatrix} d & \overline{c} \\ c & -\overline{d}\end{pmatrix}^* - \right.$$
$$\left. -\begin{pmatrix} -b & \overline{a} \\ a & \overline{b}\end{pmatrix}\begin{pmatrix} -d & \overline{c} \\ c & \overline{d}\end{pmatrix}^*\right\}.$$

This formula expresses that this function has in fact **two** negative squares; see Section 11 and Proposition 11.4.

In connection with this example and to emphasize the non–commutativity we give the following example taken from matrix theory.

Example 7.6. Let $\Omega = \mathbb{C}^{n\times n}$. The function

$$K(A,B) = AB^*$$

is positive on Ω. The function $T(A,B) = B^*A$ is not positive on Ω for $n > 1$.

To see that the function $T(A,B)$ is not positive on Ω for $n > 1$ take $n = 2$ and $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}$ and $A_2 = \begin{pmatrix} a & b \\ 0 & 0\end{pmatrix}$. Then:

$$\begin{pmatrix} A_1^*A_1 & A_2^*A_1 \\ A_1^*A_2 & A_2^*A_2\end{pmatrix} = \begin{pmatrix} 1 & 0 & a^* & 0 \\ 0 & 0 & b^* & 0 \\ a & b & a^*a & a^*b \\ 0 & 0 & b^*a & b^*b\end{pmatrix}.$$

This matrix is not positive when $b \neq 0$.

We now return to the function $T(x,y) = x\overline{y}x\overline{y}$. If this function was positive so would be the function $\overline{y}x$. We just saw that this last function is not positive.

8. Reproducing kernel quaternionic Hilbert spaces

For completeness we mention the papers [26] and [32] but we will not need the full theory developed in those papers.

Definition 8.1. A quaternionic Hilbert space $\mathcal{H}$ of $\mathbb{H}^n$–valued functions defined on a set Ω is called a reproducing kernel quaternionic Hilbert space if there exists an $\mathbb{H}^{n\times n}$–valued function defined on $\Omega\times\Omega$ and with the following properties:

1. For every $w \in \Omega$ and $a \in \mathbb{H}^n$ the function
$$z \mapsto K(z,w)a$$
belongs to $\mathcal{H}$.
2. For every $f \in \mathcal{H}$ and w and a as above
$$\langle f, K(\cdot,w)a\rangle_{\mathcal{H}} = a^*f(w).$$

By Riesz' theorem an equivalent definition is that the functionals $f \mapsto a^*f(w)$ are all continuous.

We have in particular

$$\langle K(\cdot, w)b, K(\cdot, v)a\rangle_{\mathcal{H}} = a^*K(v, w)b.$$

Using this equality and computing the norm of an element of the form

$$\sum_{j=1}^{m} K(z, v_j)a_j$$

we see that

$$\sum_{\ell,j=1}^{m} a_\ell^* K(v_\ell, v_j)a_j \geq 0,$$

that is the kernel is positive as defined in Section 7.

The function $K(z, w)$ is called the reproducing kernel of the space. It is unique (if it exists). Two proofs are available; one relies on the fact that the representation of a continuous functional by Riesz theorem is unique. The other proof mimics the case of complex scalars, as in e.g. [20, p. 23].

Example 8.2. The Hardy space of the unit ball $\mathbb{S}$ of $\mathbb{H}$ consists of the closure of the functions left–hyperholomorphic in the unit ball of $\mathbb{R}^4$ and continuous up to the boundary, with respect to the norm associated to the quaternionic inner product

$$\langle f, g\rangle = \int_{\mathbb{S}} \overline{g(x)} f(x) d\lambda(x),$$

λ being the normalized Lebesgue measure on $\mathbb{S}$.

It is a the (right) quaternionic reproducing kernel Hilbert space with reproducing kernel $\frac{1-\overline{x}y}{|1-\overline{x}y|^4}$.

See [14], where it is called the Szëgo module of the unit ball of $\mathbb{R}^4$.

In Definition 8.1 one can make the weaker requirement that $\mathcal{H}$ is a quaternionic pre–Hilbert space. A consequence of Theorem 8.4 below is that a reproducing kernel quaternionic pre–Hilbert space has a unique completion as a quaternionic reproducing kernel Hilbert space. Before turning to this point we need a preliminary lemma.

Lemma 8.3. *A Cauchy sequence in a quaternionic pre–Hilbert space of* $\mathbb{H}^n$*–valued functions with reproducing kernel converges pointwise.*

Proof. Let (f_n) be a Cauchy sequence, let $w \in \Omega$ and let $p \in \mathbb{H}^n$. By the Cauchy–Schwartz inequality,

$$\begin{aligned} |p^*(f_n(w) - f_m(w))|^2 &= |\langle f_n(\cdot) - f_m(\cdot), K(\cdot, w)p\rangle|^2 \\ &\leq \|f_n - f_m\|^2 \cdot p^*K(w, w)p. \end{aligned}$$

Therefore $n \mapsto p^*f_n(w)$ is a Cauchy sequence and has a limit in $\mathbb{H}$. It follows readily that $n \mapsto f_n(w)$ has a limit in the metric of $\mathbb{H}^n$. □

Theorem 8.4. *Given an $\mathbb{H}^{n\times n}$–valued function $K(z,w)$ positive on a set Ω, there exists a uniquely defined reproducing kernel quaternionic Hilbert space of $\mathbb{H}^{n\times 1}$–valued functions defined on Ω and with reproducing kernel $K(z,w)$.*

Proof. The proof of course follows the classical one (see e.g. [5], [36]). We outline it for completeness.

Let $\overset{\circ}{\mathcal{H}}(K)$ denote the linear span of the functions of the form $z \mapsto K(z,w)p$ where w varies in Ω and p varies in $\mathbb{H}^n$, with the inner product

$$\langle K(\cdot,v)q, K(\cdot,w)p\rangle_{\overset{\circ}{\mathcal{H}}(K)} := p^*K(w,v)q. \tag{8.1}$$

It is readily seen as in the classical case that (8.1) indeed is well–defined and that the reproducing kernel property

$$p^*f(w) = \langle f(\cdot), K(\cdot,w)p\rangle_{\overset{\circ}{\mathcal{H}}(K)}$$

holds for all $f \in \overset{\circ}{\mathcal{H}}(K)$. The positivity of the function $K(z,w)$ implies that $\overset{\circ}{\mathcal{H}}(K)$ is a quaternionic pre–Hilbert space. That it admits a closure follows from general results; see for instance [10, p. EVT 1.6]. This closure is unique, up to an isomorphism which leaves invariant the pre–Hilbert space and a natural choice consists in the set of all the equivalence classes of Cauchy sequences, two such sequences being equivalent if their difference goes to 0 in norm. Here we are interested in the closure as a space of functions and not as a space of equivalence classes of Cauchy sequences.

Take two equivalence Cauchy sequences (f_n) and (g_n). By Lemma 8.3 the limits

$$\lim_{n\to\infty} f_n(w) \qquad \text{and} \qquad \lim_{n\to\infty} g_n(w)$$

exist and are equal. Thus for $\widetilde{(f_n)}$, an equivalence class of Cauchy sequences, we can define a function f on Ω by

$$f(w) := \lim_{n\to\infty} f_n(w)$$

where (f_n) is any sequence in the equivalence class. We set $\mathcal{H}(K)$ to be the set of these functions with the inner product

$$\langle f,g\rangle := \lim_{n\to\infty} \langle f_n, g_n\rangle_{\overset{\circ}{\mathcal{H}}(K)}.$$

In a way similar to the complex field, $\mathcal{H}(K)$ is a quaternionic Hilbert space with reproducing kernel K. The uniqueness of $\mathcal{H}(K)$ is proved as follows: let $(\mathcal{H}', \langle\cdot,\cdot\rangle_{\mathcal{H}'}$ be another quaternionic reproducing kernel Hilbert space of functions with reproducing kernel $K(z,w)$. By definition of the reproducing kernel the space $\overset{\circ}{\mathcal{H}}(K)$ is isometrically included in $\mathcal{H}'$ and so is $\mathcal{H}(K)$. Any $f \in \mathcal{H}' \ominus \mathcal{H}(K)$ is in particular orthogonal to the $K(\cdot,v)q$ and so $q^*f(v) \equiv 0$, that is $f \equiv 0$ and $\mathcal{H}(K) = \mathcal{H}'$. □

Theorem 8.5. *Let $K(z,w) : \Omega \times \Omega \to \mathbb{H}^{n\times n}$ be a positive function and let $\mathcal{H}(K)$ be the associated quaternionic reproducing kernel Hilbert space of $\mathbb{H}^n$–valued functions. Let $\Omega_1 \subset \Omega$. Then the restriction $K_1(z,w)$ of $K(z,w)$ to Ω_1 is still a positive function. The elements of $\mathcal{H}(K_1)$ are the restrictions of functions of $\mathcal{H}(K)$ to Ω_1 with the norm*

$$\|f_1\|_{\mathcal{H}(K_1)} = \inf \|f\|_{\mathcal{H}(K)},$$

where the infimum is over all functions $f \in \mathcal{H}(K)$ which coincide with f_1 on Ω_1.

Before we turn to the proof of the theorem let us present an example. Consider the function $K(x,y)$ defined by (4.3) and restrict it to the open unit disk, that is, both x and y are now complex numbers of modulus strictly less than 1. Then,

$$K(x,y) = K(z_1,w_1) = \frac{1}{(1-\overline{z_1}w_1)(1-z_1\overline{w_1})^2}.$$

We now have a $\mathbb{C}$–valued function positive in $\mathbb{D}$ and the corresponding reproducing kernel Hilbert space consists of functions of the form

$$f(z_1) = \sum_{n=0}^{\infty} \overline{z_1}^n f_n(z_1)$$

where the f_n belong to $\mathbf{B}_2$, the Bergman space of the disk, and are such that

$$\sum_0^{\infty} \|f_n\|^2_{\mathbf{B}_2} < \infty.$$

This last sum is then the square of the norm of the function f.

When only a finite number of f_n are different from 0 the function f is poly-analytic. See [7] for more on these functions. When $n = 0$ we get the Bergman space of the disk. We note that the Bergman space of the disk contains the Hardy space of the disk since the kernel

$$\frac{1}{(1-z\overline{w})^2} - \frac{1}{1-z\overline{w}}$$

is positive in $\mathbb{D}$ but the inclusion is contractive and not isometric.

Proof of Theorem 8.5. Consider the map i from $\mathcal{H}(K_1)$ into $\mathcal{H}(K)$ which to the function $z \mapsto K(z,w_1)c$ (with $z \in \Omega_1$ and $w_1 \in \Omega_1$) associates the function $z \mapsto K(z,w_1)c$ with $z \in \Omega$. Then, for $f \in \mathcal{H}(K)$ and $w_1 \in \Omega_1$ we have

$$\begin{aligned}\langle i^*f, K(\cdot,w_1)c\rangle_{\mathcal{H}(K_1)} &= \langle f, K(\cdot,w_1)c\rangle_{\mathcal{H}(K)} \\ &= c^*f(w_1).\end{aligned}$$

Thus i^*f is the restriction of a function of $\mathcal{H}(K)$ to Ω_1. □

9. Operators in quaternionic reproducing kernel Hilbert spaces and some applications

Definition 9.1. Let $L(z,w)$ be an $\mathbb{H}^{n\times n}$–valued function positive on a set Ω. A function $S:\Omega\rightarrow\mathbb{H}^{n\times n}$ is called a multiplier if the operator M_S of multiplication $f\mapsto Sf$ is bounded from $\mathcal{H}(L)$ into itself. It is called a Schur multiplier if $\|M_S\|\le 1$.

The next lemma is the key for the study of multipliers. The proof is as in the complex case and is reviewed for completeness.

Lemma 9.2. *In the notation of the previous definition, it holds that*

$$(M_S^*L(\cdot,w)c)(v)=L(v,w)S(w)^*c.$$

Proof. Let $d\in\mathbb{H}^n$. Then,

$$\begin{aligned}d^*((M_S^*(L(\cdot,w)c))(v)&=\langle M_S^*(L(\cdot,w)c),L(\cdot,v)d\rangle_{\mathcal{H}(L)}\\&=\langle L(\cdot,w)c,S(\cdot)L(\cdot,v)d\rangle_{\mathcal{H}(L)}\\&=\overline{\langle S(\cdot)L(\cdot,v)d,L(\cdot,w)c\rangle_{\mathcal{H}(L)}}\\&=\overline{(c^*S(w)^*L(w,v)d)}\\&=d^*L(v,w)S(w)^*c\end{aligned}$$

and hence the result. □

Proposition 9.3. *In the notation of the previous definition, S is a multiplier if and only if there exists a strictly positive number p such that the function*

$$L(z,w)-\frac{1}{p}S(z)L(z,w)S(w)^*$$

is positive in Ω. The smallest such p is equal to $\|M_S\|$.

The proof follows the complex case and relies on the formula

$$\left((I-\frac{1}{p}M_SM_S^*)(L(\cdot,w)c)\right)(v)=L(v,w)-\frac{1}{p}S(v)L(v,w)S(w)^*c.$$

The next result takes full advantage of the fact that $\langle f,f\rangle$ is a positive real number.

Proposition 9.4. *Let $K(z,w)$ be an $\mathbb{H}^{n\times n}$–valued function positive on a set Ω, and let $\mathcal{H}(K)$ be the associated quaternionic Hilbert space of $\mathbb{H}^{n\times 1}$–valued functions defined on Ω and with reproducing kernel $K(z,w)$. A function $f(z):\Omega\rightarrow\mathbb{H}^n$ is in $\mathcal{H}(K)$ if and only if there is a positive number $p>0$ such that the function*

$$K(z,w)-\frac{1}{p}f(z)f(w)^*$$

is positive in Ω,

Proof. Assume first that the given function is positive. As in the complex case the positivity of the kernel ensures that the formula

$$T(K(\cdot, w)c) = \frac{1}{p} f(z) f(w)^* c$$

which, a priori, defines a relation (that is, a linear subspace of $\mathcal{H}(K)\times\mathcal{H}(K)$, see [2] for more on linear relations) defines in fact a contraction. Furthermore $T^*f = f$ and so $f \in \mathcal{H}(K)$. The converse statement is proved by reading the arguments backwards. $\square$

As an application we have:

Corollary 9.5. *Let* $a \in \mathbb{H}$ *with* $|a| < 1$ *and let*

$$\varphi_a(x) = (x-a)(1-\overline{a}x)^{-1} = (1-x\overline{a})^{-1}(x-a).$$

The map which to f *associates the function* $C_a f$ *defined by*

$$C_a f(x) = (1-|a|^2)^{3/2} \frac{1-\overline{x}a}{|1-\overline{x}a|^4} f \circ \varphi_a(x)$$

is a contraction from the hyperholomorphic Hardy space of the unit sphere into itself.

Proof. We first note that

$$\begin{aligned} 1 - \overline{\varphi_a(x)}\varphi_a(y) &= 1 - (1-\overline{x}a)^{-1}(\overline{x}-\overline{a})(y-a)(1-\overline{a}y)^{-1} \\ &= (1-\overline{x}a)^{-1}\left\{(1-\overline{x}a)(1-\overline{a}y) - (\overline{x}-\overline{a})(x-a)\right\}(1-\overline{a}y)^{-1} \\ &= (1-\overline{x}a)^{-1}(1-\overline{x}y)(1-\overline{a}y)^{-1}(1-|a|^2). \end{aligned} \tag{9.1}$$

In particular

$$1 - |\varphi_a(x)|^2 = \frac{(1-|x|^2)(1-|a|^2)}{|1-\overline{x}a|^2}$$

and φ_a sends $\mathbb{S}$ into itself.

Let f be in the hyperholomorphic Hardy space and of norm equal to 1. Then

$$\frac{1-\overline{x}y}{|1-\overline{x}y|^4} - f(x)\overline{f(y)}$$

is positive in $\mathbb{S}$. (Note that the converse statement is false; the positivity of the kernel ensures only that $\|f\| \leq 1$). Replace in this kernel x by $\varphi_a(x)$ and y by $\varphi_a(y)$. We obtain that the kernel

$$\frac{1-\overline{\varphi_a(x)}\varphi_a(y)}{|1-\overline{\varphi_a(x)}\varphi_a(y)|^4} - f\circ\varphi_a(x)\overline{f\circ\varphi_a(y)}$$

is positive on $\mathbb{S}$. Using (9.1) we obtain that the kernel

$$\frac{(1-\overline{x}a)^{-1}(1-\overline{x}y)(1-\overline{a}y)^{-1}(1-|a|^2)}{|1-\overline{x}a|^{-4}|1-\overline{x}y|^4|1-\overline{a}y|^{-4}(1-|a|^2)^4} - f\circ\varphi_a(x)\overline{f\circ\varphi_a(y)}$$

is positive on $\mathbb{S}$. It follows that the kernel

$$\frac{1-\overline{x}y}{|1-\overline{x}y|^4}-\frac{1-\overline{x}a}{|1-\overline{x}a|^4}f\circ\varphi_a(x)\overline{f\circ\varphi_a(y)}\frac{1-\overline{a}y}{|1-\overline{a}y|^4}(1-|a|^2)^3$$

is positive on $\mathbb{S}$. Thus $\|C_af\|\leq 1$ and the result follows. □

We note that the above arguments take full advantage of the fact that $q\overline{q}$ is a real positive number for $q\in\mathbb{H}$. For the fact that the map C_a maps left–hyperholomorphic functions into left–hyperholomorphic functions we refer to [22, Lemma 3.103 p. 120].

For the sake of comparison we find it instructive to redo the similar argument for the Hardy space $\mathbf{H}_2$ of the disk and to show that the map which to $f\in\mathbf{H}_2$ associates the function

$$\frac{\sqrt{1-|a|^2}}{1-z\overline{a}}f\left(\frac{z-a}{1-z\overline{a}}\right)$$

is a contraction. Setting $b_a(z)=\frac{z-a}{1-z\overline{a}}$, we first recall that

$$\frac{1-b_a(z)\overline{b_a(w)}}{1-z\overline{w}}=\frac{1-|a|^2}{(1-z\overline{a})(1-a\overline{w})}. \tag{9.2}$$

Now from the positivity of the kernel

$$\frac{1}{1-z\overline{w}}-f(z)\overline{f(w)}$$

in $\mathbb{D}$ (that is, $\|f\|_{\mathbf{H}_2}\leq 1$) we obtain that the kernel

$$\frac{1}{1-b_a(z)\overline{b_a(w)}}-f\circ b_a(z)\overline{f\circ b_a(w)}$$

is positive in $\mathbb{D}$. Replacing $\frac{1}{1-b_a(z)\overline{b_a(w)}}$ by

$$\frac{(1-z\overline{a})(1-a\overline{w})}{(1-|a|^2)(1-z\overline{w})}$$

we obtain the result.

The same argument works in the reproducing kernel Hilbert space of functions analytic in the unit ball of $\mathbb{C}^N$ with reproducing kernel $\frac{1}{1-zw^*}$ where

$$z=(z_1,\ldots,z_N),\quad w=(w_1,\ldots,w_N)$$

and $zw^*=\sum_1^N z_\ell\overline{w_\ell}$. One replaces then b_a by an automorphism of the ball, namely

$$\frac{(1-|a|^2)^{1/2}}{1-za^*}(z-a)(I_N-a^*a)^{-1/2}.$$

Formula (9.2) is then still valid. See [35, Theorem 2.2.2 p. 26]. For related results, see [4].

For the case $N=1$ we recall that by Littewood's theorem (see [38, p. 16], [16]) the composition operator which to f associates the function $z\mapsto f\left(\frac{z-a}{1-z\overline{a}}\right)$ has norm less or equal to $\frac{1+|a|}{1-|a|}$.

Theorem 9.6. *Given an $\mathbb{H}^{n\times n}$–valued function positive on a set Ω, and let $\mathcal{H}(K)$ be the associated reproducing kernel quaternionic Hilbert space of $\mathbb{H}^{n\times 1}$–valued functions with reproducing kernel $K(z,w)$. Then, $\mathcal{H}(K)$ is finite dimensional (of dimension N, say) if and only if there exist N linearly independent elements $f_1,\ldots,f_N$ of $\mathcal{H}(K)$ such that*

$$K(z,w)=\sum_{j=1}^{N} f_j(z)f_j(w)^*. \tag{9.3}$$

Proof. The space is finite dimensional if and only if it has a finite dimensional basis. This basis may be supposed orthonormal. It is then easy to see that (9.3) gives a formula for the reproducing kernel. □

Definition 9.7. An $\mathbb{H}^{n\times n}$–valued function positive on a set Ω is said to be of finite rank if it is of the form (9.3), or, equivalently, if the associated quaternionic reproducing kernel Hilbert space $\mathcal{H}(K)$ is finite dimensional. The rank of the function is then defined to be the number N in (9.3), or, equivalently, the dimension of $\mathcal{H}(K)$.

In the previous theorems the domain where the functions are defined was not specified. The next result deals with the special case where $\Omega\subset\mathbb{H}$.

Theorem 9.8. *Given an $\mathbb{H}^{n\times n}$–valued function positive on an open set $\Omega\subset\mathbb{C}^2$, and let $\mathcal{H}(K)$ be the associated reproducing kernel quaternionic Hilbert space of $\mathbb{H}^{n\times 1}$–valued functions with reproducing kernel $K(z,w)$. Assume that for all $w\in\Omega$ the function $z\mapsto K(z,w)$ is left–hyperholomorphic. Then the entries of the elements of $\mathcal{H}(K)$ are also left–hyperholomophic.*

Proof. We consider the case of $\mathbb{H}$–valued functions. The case of matrix–valued functions is treated in a similar way. For $f\in\mathcal{H}(K)$, $p\in\Omega$ and $\epsilon\in\mathbb{R}$ small enough we have

$$\frac{K(p,q+\epsilon\mathbf{e_j})-K(p,q)}{\epsilon}=\overline{\frac{K(q+\epsilon\mathbf{e_j},p)-K(q,p)}{\epsilon}}$$

and hence

$$\frac{\partial K(p,q)}{\partial y_j}=\overline{\frac{\partial K(q,p)}{\partial x_j}}. \tag{9.4}$$

The family of functions $\frac{K(p,q+\epsilon\mathbf{e_j})-K(p,q)}{\epsilon}$ is uniformly bounded in norm and therefore has a weakly convergent subsequence, which is readily seen to converge to $\frac{\partial K(p,q)}{\partial y_j}$. Since

$$\frac{f(p+\epsilon\mathbf{e_j})-f(p)}{\epsilon}=\langle f(\cdot),\frac{K(\cdot,p+\epsilon\mathbf{e_j})-K(\cdot,p)}{\epsilon}\rangle_{\mathcal{H}(K)}$$

we have

$$\frac{\partial f}{\partial x_j}(p)=\langle f,\frac{\partial K(\cdot,p)}{\partial y_j}\rangle_{\mathcal{H}(K)}.$$

By the properties of the inner product,

$$\begin{aligned}
&\frac{\partial f}{\partial x_0}+\mathbf{e_1}\frac{\partial f}{\partial x_1}+\mathbf{e_2}\frac{\partial f}{\partial x_2}+\mathbf{e_3}\frac{\partial f}{\partial x_3}= \\
&= \quad \langle f,\frac{\partial K(\cdot,q)}{\partial x_0}\rangle_{\mathcal{H}(K)}+\mathbf{e_1}\langle f,\frac{\partial K(\cdot,q)}{\partial y_1}\rangle_{\mathcal{H}(K)}+\mathbf{e_2}\langle f,\frac{\partial K(\cdot,q)}{\partial y_2}\rangle_{\mathcal{H}(K)}+ \\
&\qquad +\mathbf{e_3}\langle f,\frac{\partial K(\cdot,q)}{\partial y_3}\rangle_{\mathcal{H}(K)} \\
&= \quad \langle f,\frac{\partial K(\cdot,q)}{\partial y_0}-\frac{\partial K(\cdot,q)}{\partial y_1}\mathbf{e_1}-\frac{\partial K(\cdot,q)}{\partial y_2}\mathbf{e_2}-\frac{\partial K(\cdot,q)}{\partial y_3}\mathbf{e_3}\rangle_{\mathcal{H}(K)}.
\end{aligned}$$

Using (9.4) we have

$$\begin{aligned}
&\frac{\partial K(\cdot,q)}{\partial y_0}-\frac{\partial K(\cdot,q)}{\partial y_1}\mathbf{e_1}-\frac{\partial K(\cdot,q)}{\partial y_2}\mathbf{e_2}-\frac{\partial K(\cdot,q)}{\partial y_3}\mathbf{e_3}= \\
&= \quad \overline{\frac{\partial K(q,\cdot)}{\partial x_0}+\mathbf{e_1}\frac{\partial K(q,\cdot)}{\partial x_1}+\mathbf{e_2}\frac{\partial K(q,\cdot)}{\partial x_2}+\mathbf{e_3}\frac{\partial K(q,\cdot)}{\partial x_3}} \\
&= \quad 0,
\end{aligned}$$

since for every $h\in\Omega$ the function $q\mapsto K(q,h)$ is left–hyperholomorphic. $\square$

Theorem 9.9. *Let K be an $\mathbb{H}^{n\times n}$–valued function positive on a set Ω and let $\mathcal{H}(K)$ be the associated reproducing kernel quaternionic Hilbert space. Let $\mathcal{H}(B(K))$ denote the reproducing kernel Hilbert space of $\mathbb{R}^{4n}$–valued functions with reproducing kernel $B(K)$ and let $\mathcal{H}(\chi(K))$ denote the reproducing kernel Hilbert space of $\mathbb{C}^{2n}$–valued functions with reproducing kernel $\chi(K)$ (defined by (7.1)). Then:*

(**1**) *The map*

$$\sum_1^m K(z,w_\ell)q_\ell\mapsto\sum_1^m B(K(z,w_\ell))B(q_\ell)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

extends to a unitary map from $\mathcal{H}_{\mathbb{C}}(K)$ onto $\mathcal{H}(B(K))$.

(**2**) *The map*

$$\sum_1^m K(z,w_\ell)q_\ell\mapsto\sum_1^m \chi(K(z,w_\ell))\chi(q_\ell)\begin{pmatrix}1\\0\end{pmatrix}$$

(where $q_\ell=a_\ell+b_\ell j\in\mathbb{H}^n$) extends to a unitary map from $\mathcal{H}_{\mathbb{R}}(K)$ onto $\mathcal{H}(\chi(K))$.

Proof. The proof of the first claim relies on formula (3.7).

Let $v,w\in\Omega$ and $p=a+bj$ and $q=c+dj$ be in $\mathbb{H}^n$. The first complex component of $q^*K(w,v)q$ is equal to

$$c^*K_1(w,v)a+d^t\overline{K_2(w,v)}a+d^t\overline{K_1(w,v)}a-c^*K_2(w,v)\overline{b}.$$

This expression is in turn equal to

$$\begin{pmatrix} c \\ -\overline{d} \end{pmatrix}^* \chi(K(w,v)) \begin{pmatrix} a \\ -\overline{b} \end{pmatrix}.$$

The real parts of these two expressions are in particular equal and therefore

$$\begin{aligned}\langle K(\cdot,v)p, K(\cdot,w)q\rangle_{\mathcal{H}(K)} &= \text{real part of } \; q^*K(w,v)q \\ &= \left\langle \chi(K(\cdot,v)) \begin{pmatrix} a \\ -\overline{b} \end{pmatrix}, \chi(K(\cdot,w)) \begin{pmatrix} c \\ -\overline{d} \end{pmatrix} \right\rangle_{\mathcal{H}(\chi(K))}.\end{aligned}$$

This expression extends on linear combinations with real coefficients and this allows to conclude the proof. □

We leave to the reader to check that the previous result is false if one considers the complex structures rather than the real structures of the spaces.

10. Quaternionic inner product spaces

We consider a (right) vector $\mathbb{H}$–space $\mathcal{P}$ endowed with a form $[\cdot,\cdot]$ which is hermitian and linear (as in Definition 5.5) but for which the positivity hypothesis is replaced by the following reality and non–degeneracy hypothesis:

- $[f, f]$ is a real number for all $f \in \mathcal{P}$.
- If $f \in \mathcal{P}$ is such that $[f, g] = 0$ for all $g \in \mathcal{P}$ then $f = 0$.

We will call such a space a (non–degenerate) quaternionic inner product space. As in the classical case, two elements f and g of $\mathcal{P}$ will be called orthogonal (notation: $f[\perp]g$) if $[f, g] = 0$. Two subspaces of $\mathcal{P}$ will be called orthogonal if every element of the first is orthogonal to every element of the second. For $\mathcal{M}$, a subset of $\mathcal{P}$, we set

$$\mathcal{M}^{[\perp]} = \{n \in \mathcal{P} \text{ such that } \; [n, m] = 0 \text{ for all } m \in \mathcal{M}\}.$$

Since we assume that $[f, f]$ is real the following definitions also make sense in the present setting: an element $f \in \mathcal{P}$ is positive (resp. negative or neutral) if $[f, f] \geq 0$ (resp. $[f, f] \leq 0$ or $[f, f] = 0$). A subspace $\mathcal{M} \subset \mathcal{P}$ will be called positive (resp. negative, neutral) if $[f, f] \geq 0$ (resp. $[f, f] \leq 0$ or $[f, f] = 0$) for all $f \in \mathcal{M}$. It will be called positive definite if $[f, f] > 0$ for $f \neq 0$ and similarly it will be called negative definite if $[f, f] < 0$ for such f. It will be called maximal positive (resp. negative, neutral) if it is maximal with respect to this property. We will say that $\mathcal{M}$ is maximal positive definite (resp. negative definite) if $[f, f] > 0$ (resp. $[f, f] < 0$) for all $f \neq 0 \in \mathcal{M}$.

From the definition of the inner products of $\mathcal{P}_{\mathbb{R}}$ and $\mathcal{P}_{\mathbb{C}}$ it is easy to prove that:

Proposition 10.1. *Let $\mathcal{P}$ be an inner product space and let $\mathcal{M}$ is a positive (resp. strictly positive) (resp. negative) subspace of $\mathcal{P}$. Then:*

(1) *$\mathcal{M}_{\mathbb{R}}$ is a positive (resp. strictly positive) (resp. negative) subspace of $\mathcal{P}_{\mathbb{R}}$.*

(2) *$\mathcal{M}_{\mathbb{C}}$ is a positive (resp. strictly positive) (resp. negative) subspace of $\mathcal{P}_{\mathbb{C}}$.*

Definition 10.2. The space $\mathcal{M}$ will be called projectively complete (or orthocomplemented) if

$$\mathcal{M} + \mathcal{M}^{[\perp]} = \mathcal{P}.$$

See [6, Definition7.8 p.44], [8, §9 p. 18]. The notion of orthogonal complement is more involved in a general inner product space than it is in a Hilbert space and not every subspace is projectively complete. The following proposition is what we need in the present work and is used several times in the sequel.

Proposition 10.3. *Every finite dimensional definite subspace of a quaternionic inner product space is projectively complete,*

Proof. We follow the proof of [8, Lemma 9.8 p. 20] and assume that the subspace $\mathcal{N}$ of the quaternionic inner product space $\mathcal{V}$ is positive definite and of dimension $N < \infty$. Let $n_1, \ldots, n_N$ be an orthonormal basis of $\mathcal{N}$ and for $f \in \mathcal{V}$ write:

$$f = \left(\sum_1^N n_j[f, n_j] \right) + \left(f - \sum_1^N n_j[f, n_j] \right).$$

We have $\sum_1^N n_j[f, f_j] \in \mathcal{N}$ and $(f - \sum_1^N n_j[f, f_j]) \in \mathcal{N}^{[\perp]}$ and this concludes the proof.

The case of negative definite spaces is treated in a similar way. □

Definition 10.4. Let $g_1, \ldots, g_n$ be n elements of a quaternionic inner product space $\mathcal{P}$. The Gram matrix associated to the g_j is the $n \times n$ hermitian matrix with (ℓ, j) entry equal to $[g_j, g_\ell]_{\mathcal{P}}$.

Let $g = (g_1 \, g_2 \, \cdots \, g_n) \in \mathcal{P}^n$. Then for $c, d \in \mathbb{H}^n$ we have

$$[gc, gd]_{\mathcal{P}} = d^* Gc.$$

Thus, with some abuse of notation one can interpret the Gram matrix as $G = [g^t, g]$, where the inner product is understood component–wise.

Definition 10.5. A quaternionic inner product space $\mathcal{H}$ with indefinite inner product $[\cdot, \cdot]$ is called a quaternionic antiHilbert space, or the antispace of a quaternionic Hilbert space, if $\mathcal{H}$ endowed with the inner product $-[\cdot, \cdot]$ is a quaternionic Hilbert space.

The following lemma played a central role in the arguments in [2]. It will also have an important role here.

Lemma 10.6. [2, Lemma 1.1.1′ p. 4]. *Let $g_1, \ldots, g_n$ be vectors in a quaternionic inner product space $\mathcal{P}$ and let $G = ([g_j, g_\ell])$ be their Gram matrix. Then the number of negative eigenvalues of G coincides with the maximum dimension of a subspace $\mathcal{N}$ of the span of the g_j which is the antispace of a quaternionic Hilbert space in the inner product of $\mathcal{P}$.*

Proof. We follow the proof of [2, Lemma 1.1.1′ p. 4]. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of G and let $\nu_-(G)$ denote the number of strictly negative eigenvalues of G. Write

$$G = U^* \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix} U$$

where $U \in \mathbb{H}^{n\times n}$ is unitary, D_- and D_+ are diagonal, with

$$D_- = \text{diag }(\lambda_1, \ldots, \lambda_{\nu_-(G)}).$$

The $\nu_-(G)$ vectors defined by $GU \begin{pmatrix} I_{\nu_-(G)} \\ 0 \end{pmatrix}$ have as Gram matrix D_- and therefore form an antispace of a quaternionic Hilbert space. Thus

$$\nu_-(G) \le \max \dim \mathcal{N}$$

where the maximum is taken over all such spaces.

Conversely, let $\mathcal{N} \subset \text{l.s. }\{g_1, \ldots, g_n\}$ be the antispace of a quaternionic Hilbert space, and let $n_1, \ldots, n_s$ be an orthonormal basis of $\mathcal{N}$. There exists a matrix $X \in \mathbb{H}^{n\times s}$ such that $(n_1\, n_2\, \cdots n_s) = (g_1\, g_2\, \cdots\, g_n)X$ and so

$$-I_s = X^*GX.$$

By Lemma 3.18 we have $\nu_-(G) \le s$, and this ends the proof. □

11. Number of negative squares

Thanks to Proposition 3.11 the next definition makes sense:

Definition 11.1. An $\mathbb{H}^{n\times n}$–valued function $K(z, w)$ defined on a set Ω has κ negative squares if it is hermitian and if for every choice of $m \in \mathbb{N}$ and of $z_1, \ldots, z_m \in \Omega$ the $m \times m$ block hermitian matrix with ℓj entry $K(z_\ell, z_j)$ has at most κ strictly negative eigenvalues, and exactly κ strictly negative eigenvalues for some choice of m, $z_1, \ldots, z_m$.

An apparently different definition is:

Definition 11.2. An $\mathbb{H}^{n\times n}$–valued function $K(z, w)$ defined on a set Ω has κ negative squares if it is hermitian and if for every choice of $m \in \mathbb{N}$, of $z_1, \ldots, z_m \in \Omega$ and of vectors $c_1, \ldots, c_m \in \mathbb{H}^n$ the $m\times m$ hermitian matrix with ℓj entry $c_\ell^* K(z_\ell, z_j)c_j$ has at most κ strictly negative eigenvalues, and exactly κ strictly negative eigenvalues for some choice of m, $z_1, \ldots, z_m$ and $c_1, \ldots, c_m \in \mathbb{H}^n$.

Proposition 11.3. *Definitions* 11.1 *and* 11.2 *are equivalent.*

Proof. Let $\overset{\circ}{\mathcal{P}}(K)$ denote the linear span of the functions of the form $z \mapsto K(z, w)p$ where w varies in Ω and p varies in $\mathbb{H}^n$. The (in general indefinite) inner product

$$[K(\cdot, v)q, K(\cdot, w)p]_{\overset{\circ}{\mathcal{P}}(K)} = p^*K(w, v)q.$$

is readily seen to be well–defined. By Lemma 10.6 the function K has κ negative squares in either way if and only if the dimension of a maximum antispace of a quaternionic Hilbert space in $\overset{\circ}{\mathcal{P}}(K)$ has dimension κ. $\square$

When the function K has a finite number of negative squares we will see in Section 13 that the space $\overset{\circ}{\mathcal{P}}(K)$ admits a closure which is a quaternionic reproducing kernel Pontryagin space (see the following section, Section 12, for the definition of a quaternionic Pontryagin space).

Proposition 11.4. *Let $K(z,w)$ be a $\mathbb{C}^{n\times n}$–valued function having κ negative squares on a set Ω. Then:*

(**1**) *The function $B(K(z,w))$ has 4κ negative squares.*
(**2**) *The function $\chi(K(z,w))$ has 2κ negative squares.*

As an illustration, see Example 7.5. The proof of the proposition follows exactly the proof of Proposition 7.2 and will be omitted.

The following result characterizes functions which have a finite number of negative squares. The proof is deferred to the last section of the paper.

Theorem 11.5. *An $\mathbb{H}^{n\times n}$–valued function $K(z,w)$ defined on a set Ω has at most κ negative squares if and only if it can be written as $K(z,w) = K_+(z,w) - K_-(z,w)$ where both K_+ and K_- are positive and where moreover K_- is of finite rank. It has exactly κ negative squares if moreover*

$$\mathcal{H}(K_+) \cap \mathcal{H}(K_-) = \{0\}.$$

12. Quaternionic Pontryagin spaces

A number of books have appeared on the theory of Kreĭn and Pontryagin spaces (and more generally, on spaces with an indefinite metric) and we refer in particular to [6], [8] and [25]. In this section we present a short introduction into the quaternionic setting.

Definition 12.1. A quaternionic (right) inner product space $(\mathcal{P}, [\cdot,\cdot])$ is called a quaternionic Pontryagin space if there exist two subspaces $\mathcal{P}_+$ and $\mathcal{P}_-$ of $\mathcal{P}$ with the following properties:

1. The space $\mathcal{P}_-$ is finite dimensional.
2. $\mathcal{P}_+$ and $\mathcal{P}_-$ are orthogonal:
$$[p_+, p_-] = 0 \quad \forall (p_+, p_-) \in \mathcal{P}_+ \times \mathcal{P}_-.$$
3. The space $\mathcal{P}_+$ endowed with the form $[\cdot,\cdot]$ and the space $\mathcal{P}_-$ endowed with the form $-[\cdot,\cdot]$ both are quaternionic Hilbert spaces.
4. It holds that
$$\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_-. \tag{12.1}$$

The last item means that every element in $\mathcal{P}$ admits a decomposition $p = p_+ + p_-$ where $(p_+, p_-) \in \mathcal{P}_+ \times \mathcal{P}_-$. This decomposition is unique. Indeed, if one can write $0 = p_+ + p_-$ $(p_+, p_-) \in \mathcal{P}_+ \times \mathcal{P}_-$ we have $p_- \in \mathcal{P}_+ \cup \mathcal{P}_-$ and hence $[p_-, p_-] = 0$ so that $p_- = 0$.

A decomposition as in Definition 12.1 is called a **fundamental decomposition**. Such a decomposition is in general not unique.

A more general concept would be that of quaternionic Kreĭn space, where the finite dimensionality requirement is dropped. Already in the case of the complex field $\mathbb{C}$ the situation is much more complicated; see [37], [19], [1]. We will not consider this case here.

We note that $\mathcal{P}$ endowed with the inner product

$$\langle f, g\rangle = [f_+, g_+] - [f_-, g_-] \tag{12.2}$$

is a quaternionic Hilbert space as defined in Section 5. We set $\|f\| = \langle f, f\rangle$.

Proposition 12.2. *The form* $[\cdot, \cdot]$ *is continuous with respect to the topology defined by* (12.2). *More precisely, it holds that*

$$|\langle f, g\rangle|^2 \leq \|f\|^2 \cdot \|g\|^2. \tag{12.3}$$

Proof. By the Cauchy-Schwarz inequality (5.6) we have

$$|[f_\pm, g_\pm]^2| \leq [f_\pm, f_\pm] \cdot [g_\pm, g_\pm].$$

Therefore, using the triangle inequality (2.9),

$$\begin{aligned} |\langle f, g\rangle|^2 &\leq (|[f_+, g_+]| + |[f_-, g_-]|)^2 \\ &\leq \left(\sqrt{[f_+, f_+]} \cdot \sqrt{[g_+, g_+]} + \sqrt{-[f_-, f_-]} \cdot \sqrt{-[g_-, g_-]}\right)^2 \end{aligned}$$

and using the Cauchy–Schwarz inequality in $\mathbb{R}^2$

$$\begin{aligned} &\leq ([f_+, f_+] - [f_-, f_-]) \cdot ([g_+, g_+] - [g_-, g_-]) \\ &= \|f\|^2 \cdot \|g\|^2 \end{aligned}$$

and hence the result. □

The decomposition (12.1) is not unique in general but the following fundamental result holds:

Theorem 12.3. *Let* $\mathcal{P}$ *be a quaternionic Pontryagin space. The norms defined by* (12.2) *corresponding to different fundamental decompositions are all equivalent.*

The same result (and almost the same proof) hold for quaternionic Kreĭn spaces. Before proving the theorem we note that:

Lemma 12.4. *Let* $\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_-$ *be a fundamental decomposition of* $\mathcal{P}$. *It holds that*

$$\begin{aligned} \mathcal{P}_+ &= \mathcal{P}_-^{[\perp]}, \\ \mathcal{P}_- &= \mathcal{P}_+^{[\perp]}. \end{aligned}$$

Proof. It is clear that $\mathcal{P}_- \subset \mathcal{P}_+^{[\perp]}$. Let now $h \in \mathcal{P}_+^{[\perp]} \backslash \mathcal{P}_-$ and let $h = h_+ + h_-$ be the decomposition of h along the fundamental decomposition with $(h_+, h_-) \in \mathcal{P}_+ \times \mathcal{P}_-$. Then,

$$0 = [h, p_+]$$

for all $p_+ \in \mathcal{P}_+$ since $h \in \mathcal{P}_+^{[\perp]}$. Thus

$$0 = [h_+, p_+]$$

and so $h_+ = 0$. Thus $h \in \mathcal{P}_- \cap \mathcal{P}_+$ and so $h = 0$ and hence the result. □

As a corollary we have that

$$\mathcal{P}_- = \mathcal{P}_-^{[\perp\perp]}, \quad \mathcal{P}_+ = \mathcal{P}_+^{[\perp\perp]}. \tag{12.4}$$

Proof of Theorem 12.3. We follow and somewhat simplify the proof of the classical result in [6, Theorem 7.19, p. 47]. Let $\mathcal{P} = \mathcal{N}_+ \oplus \mathcal{N}_-$ be another fundamental decomposition of $\mathcal{P}$.

STEP 1: *The space $\mathcal{N}_+^{[\perp\perp]}$ is closed in the topology defined by the first decomposition.*

This follows from the definition

$$\mathcal{N}_+^{[\perp\perp]} = \left\{ f \in \mathcal{P} \text{ such that } \; [f,g] = 0 \text{ for all } g \in \mathcal{N}_+^{[\perp]} \right\} \tag{12.5}$$

and from the continuity of the form $[\cdot,\cdot]$ (see Proposition 12.2 for the latter).

STEP 2: *The space $\mathcal{N}_+$ is closed in the topology defined by the first decomposition.*

The claim follows from the previous step and from (12.4).

STEP 3: *Let $n = n_+ + n_- \in \mathcal{P}$ with $n_\pm \in \mathcal{N}_\pm$. The maps $n \mapsto n_\pm$ are continuous.*

We use Proposition 10.3. Let $n_1, \ldots, n_N$ be an orthonormal basis of $\mathcal{N}_-$. Then

$$n_- = \left(\sum_1^N n_j [n, n_j] \right) \quad \text{and} \quad n_+ = \left(f - \sum_1^N n_j [f, n_j] \right).$$

The map $n \mapsto n_-$ is then clearly continuous and so is $n_+ = n - n_-$.

STEP 4: *The topology defined by the first decomposition restricted to $\mathcal{N}_+$ and the restriction of $[\cdot,\cdot]$ to $\mathcal{N}_+$ are equivalent.*

Let $n_+ \in \mathcal{N}_+$ and let $n_+ = p_+ + p_-$ be its decomposition along the first fundamental decomposition. Then,

$$[n_+, n_+] = [p_+, p_+] + [p_-, p_-] \le [p_+, p_+] - [p_-, p_-] = \langle n_+, n_+ \rangle$$

and therefore the identity map is a contraction from $\mathcal{N}_+$ endowed with its intrinsic norm onto $\mathcal{N}_+$ endowed with the norm defined by the first decomposition. Since $\mathcal{N}_+$ is a Hilbert space with either norm (by definition for the intrinsic norm and thanks to STEP 2 for the first), the open mapping theorem (see Theorem 6.3 and the discussion preceding it) the identity is bicontinuous.

In the preceding step one can replace $\mathcal{N}_+$ by $\mathcal{N}_-$ and $[\cdot,\cdot]$ by -$[\cdot,\cdot]$. The arguments are simpler since all norms on a finite dimensional quaternionic Hilbert space are equivalent; see [17, Théorème 1.1 p. 24].

STEP 5: *The two topologies are equivalent.* Let $n \in \mathcal{P}$ and let $n = n_+ + n_-$ be its decomposition with respect to the second fundamental decomposition. We denote by $\|\cdot\|_1$ the norm defined by the second decomposition. Using the preceding step and STEP 3 we have:

$$\begin{aligned} \|n\|_1 &\leq \|n_+\|_1 + \|n_-\|_1 \\ &\leq \|n_+\| + k\|n_-\| \quad (k > 0) \\ &\leq (1+k)\|n\| \end{aligned}$$

and once more the open mapping theorem allows to conclude. □

The different fundamental decompositions define thus equivalent topologies. We will endow the quaternionic Pontryagin space with any of these.

Proposition 12.5. *Let $\mathcal{P}$ be a quaternionic Pontryagin space and let $\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_-$ be a fundamental decomposition. Then $\mathcal{P}_+$ (resp. $\mathcal{P}_-$) is maximal strictly positive (resp. maximal strictly negative).*

Proof. Assume that there is a strictly positive subspace $\mathcal{L}$ such that $\mathcal{P}_+ \subset \mathcal{L}$ and let $h \in \mathcal{L} \setminus \mathcal{P}_+$. Let $h = h_+ + h_-$ be the decomposition of h along the fundamental decomposition with $(h_+, h_-) \in \mathcal{P}_+ \times \mathcal{P}_-$. Then, as in the argument in the proof of Lemma 12.4 one shows that $h_- = 0$ and so that $\mathcal{L} = \mathcal{P}_+$. □

As a corollary we have:

Proposition 12.6. *The dimensions of the spaces $\mathcal{P}_-$ (as vector spaces) are the same for all decompositions* (12.1).

Proof. Indeed all the spaces $\mathcal{P}_-$ are maximal strictly negative and by the finite dimensional hypothesis have the same dimension. □

One can be a bit more precise and prove the following result. See [6, Theorem 4.2, p. 24] for the complex case.

Proposition 12.7. *Let $\mathcal{L}$ be a strictly positive (resp. strictly negative) subspace of the quaternionic Pontryagin space $\mathcal{P}$. Let $\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_-$ be a fundamental decomposition and let P_+ and P_- be the associated orthogonal projections. Then, P_+ is a homeomorphism from $\mathcal{L}$ onto $P_+\mathcal{L}$ (resp. P_- is an homeomorphism from $\mathcal{L}$ onto $P_-\mathcal{L}$).*

Proof. We follow the proof in [6]. Let $h \in \mathcal{L}$ be such that $P_+ h = 0$. Then $h = (P_+ + P_-)h = P_- h \in \mathcal{P}_-$. So

$$[h, h] \geq 0$$

since $\mathcal{L}$ is a positive subspace, while

$$[h, h] \leq 0$$

since $h \in \mathcal{P}_-$. Hence $[h, h] = 0$ and $h = 0$. Thus P_+ is one–to–one on $\mathcal{P}_+$. Moreover, for $h = h_+ + h_- \in \mathcal{L}$ we have

$$[h_+, h_+] \geq -[h_-, h_-] \quad \text{since} \quad \mathcal{L} \quad \text{is a positive subspace.}$$

Thus,

$$\|P_+|_{\mathcal{L}} h\|^2 = \|h_+\|^2 \geq \frac{1}{2}\left(\|h_+\|^2 + \|h_-\|^2\right) = \frac{1}{2}\|h\|^2.$$

But P_+ is continuous. By the open mapping theorem $P_+|_{\mathcal{L}}$ is a homeomorphism. □

The following result is due to Pontryagin in the complex case and is of fundamental importance. We here follow the proof in Bognar's book (see [8, p. 185]), suitably adapted to the present case.

Theorem 12.8. *Any dense subspace of a quaternionic Pontryagin space $\mathcal{P}$ contains a maximal uniformly negative subspace.*

Proof. Let $\mathcal{L}$ denote the dense subspace. We fix $\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_-$ to be a fundamental decomposition of the quaternionic Pontryagin space $\mathcal{P}$. By Theorem 5.8 the Hilbert space $(\mathcal{P}_-, -[\cdot, \cdot])$ has an orthonormal basis $e_1, \ldots, , e_k$:

$$-[e_\ell, e_m] = \delta_{\ell m}.$$

As earlier we denote by $\langle \cdot, \cdot \rangle$ the positive inner product defined by the given fundamental decomposition and by $\| \cdot \|$ the associated norm. Let $\epsilon > 0$ to be specified later. There exist elements $g_1, \ldots, g_k \in \mathcal{L}$ such that $\|g_\ell - e_\ell\| \leq \epsilon$ for $\ell = 1, \ldots, k$. We claim that the space spanned by the g_j is strictly positive for small enough ϵ (it will follow in particular that the g_j are linearly independent). Indeed, let $y = \sum_1^k g_\ell q_\ell$ with $q_1, \ldots, q_k \in \mathbb{H}$ and set $x = \sum_1^k e_\ell q_\ell$. Then,

$$[x, x] = \langle x, x \rangle \tag{12.6}$$

and we have

$$\begin{aligned} \|y-x\| &\leq \sum_1^k \|(g_\ell - e_\ell)q_\ell\| \\ &= \sum_1^k \|(g_\ell - e_\ell)\| \cdot |q_\ell| \\ &\leq \epsilon\left(\sum_1^k |q_j|\right) \\ &\leq \epsilon\sqrt{k}\sqrt{\sum_1^k |q_\ell|^2} \\ &= \epsilon\sqrt{k}\|x\|. \end{aligned}$$

In particular, by the triangle inequality,

$$\|y\| \leq \|y-x\| + \|x\| \leq \|x\|(1+\epsilon\sqrt{k}).$$

Thus, in view of (12.3),

$$\begin{aligned} |[y,y]-[x,x]| &= |[y-x,x]+[y,y-x]| \\ &\leq \epsilon\sqrt{k}\|x\| + \epsilon\sqrt{k}\|x\|\|y\| \\ &\leq \epsilon\sqrt{k}\|x\|^2(2+\epsilon\sqrt{k}). \end{aligned}$$

Using (12.6) we see that $[y,y]>0$ for ϵ small enough and $x \neq 0$. □

Proposition 12.9. *Let $\mathcal{P}$ be a quaternionic Pontryagin space with decomposition* (12.1) *and negative index κ. A sequence x_n converges in $\mathcal{P}$ to $x \in \mathcal{P}$ if and only the following hold:*

1. *The sequence $[x_n, x_n]$ converges to $[x,x]$.*
2. *For z in a dense set $\mathcal{L}$ the sequence $[x_n, z]$ tends to $[x,z]$.*

Proof. The conditions are necessary since the inner product is continuous (see Proposition 12.2). We now show that they are also sufficient. By Theorem 12.8 the space $\mathcal{L}$ contains a maximal negative space, say $\mathcal{L}_-$, of dimension κ. Write

$$\mathcal{P} = \mathcal{L}_- + \mathcal{L}_-^{[\perp]}$$

and let $x_n = y_n + t_n$ and $x = y + t$ be the corresponding decompositions of the elements of the sequence x_n and of x. Then for all $z \in \mathcal{L}_-$ the limit

$$\lim_{n\to\infty} -[y_n, z]$$

exists. Since $\mathcal{L}_-$ is finite dimensional it follows that $\lim_{n\to\infty} y_n$ exists (and is equal to $y \in \mathcal{L}_-$). Therefore

$$\lim_{n\to\infty} [t_n, t_n]$$

exists, as well as

$$\lim_{n\to\infty}[t_n, z]$$

for $z \in \mathcal{L} \cap \mathcal{L}_-^{[\perp]}$. This implies that $\lim_{n\to\infty}\|t_n - t\| = 0$ since $\mathcal{L}^{[\perp]}$ is a Hilbert space. □

Proposition 5.9 still makes sense in the setting of quaternionic Pontryagin spaces.

Proposition 12.10. *Let $\mathcal{P}$ be a quaternionic Hilbert space and let*

$$[f,g] = [f,g]_0 + \mathbf{e_1}[f,g\mathbf{e_1}]_0 + \mathbf{e_2}[f,g\mathbf{e_2}]_0 + \mathbf{e_3}[f,g\mathbf{e_3}]_0.$$

where the $[f,g]_\ell$ ($\ell = 0,1,2,3$) are real valued. The form $[f,g]_0$ endows $\mathcal{P}$ with the structure of a real Pontryagin space $\mathcal{P}_\mathbb{R}$ and the form

$$[f,g]_0 + i[f,g]_1$$

endows $\mathcal{P}$ with the structure of a complex Pontryagin space $\mathcal{P}_\mathbb{C}$.

Theorem 12.11. *Let $\mathcal{P}$ be a quaternionic Pontryagin space of index κ. Then $\mathcal{P}_\mathbb{R}$ is a Pontryagin space of index 4κ and $\mathcal{P}_\mathbb{C}$ is a Pontryagin space of index 2κ.*

Proof. Let $\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_-$ be a fundamental decomposition of $\mathcal{P}$. Then by Proposition 5.3,

$$\dim\ (\mathcal{P}_-)_\mathbb{R} = 4\ \dim\ \mathcal{P}_-.$$

By Proposition 10.1 $(\mathcal{P}_-)_\mathbb{R}$ is a negative subspace and $(\mathcal{P}_+)_\mathbb{R}$ is a positive subspace, and they are still orthogonal in the real inner product. This concludes the proof. □

13. Reproducing kernel quaternionic Pontryagin spaces

As already mentioned the next result originates with work of P. Sorjonen [41] and L. Schwartz in [37] for the complex case.

Theorem 13.1. *Let Ω be a set. There is a one–to–one correspondence between quaternionic reproducing kernel Pontryagin spaces of $\mathbb{H}^n$–valued functions on Ω and $\mathbb{H}^{n\times n}$–valued functions which have a finite number of negative squares on Ω.*

Proof. We follow the proof of Theorem 8.4. For the classical case the reader is referred to [41], [37].

Assume that the function

$$K(z,w)\ :\ \Omega\times\Omega \Longrightarrow \mathbb{H}^{n\times n}$$

has κ negative squares. As in the proof of Theorem 8.4 let $\overset{\circ}{\mathcal{P}(K)}$ denote the linear span of the functions of the form $z \mapsto K(z,w)p$ where w varies in Ω and p varies in $\mathbb{H}^n$. The (now indefinite) inner product

$$[K(\cdot,v)q, K(\cdot,w)p]_{\overset{\circ}{\mathcal{P}(K)}} := p^*K(w,v)q.$$

is well–defined and, as earlier, the reproducing kernel property

$$p^* f(w) = [f(\cdot), K(\cdot, w)p]$$

holds for all $f \in \overset{\circ}{\mathcal{P}(K)}$. These two properties hold even if K does not have a finite number of negative squares. The sequel of the argument does take into account the finite number of negative squares. By Proposition 12.6 any maximal strictly negative subspace of $\overset{\circ}{\mathcal{P}(K)}$ has dimension κ. Let $\mathcal{N}_-$ be such a subspace. By Proposition 10.3 one can write

$$\overset{\circ}{\mathcal{P}(K)} = \mathcal{N}_- + \mathcal{N}_-^{[\perp]}.$$

The space $\mathcal{N}_-^{[\perp]}$ is a quaternionic pre–Hilbert space. Let $f_1, \ldots, f_\kappa$ be an orthonormal basis of $\mathcal{N}_-$. Then, $\mathcal{N}_-^{[\perp]}$ has reproducing kernel

$$K_{\mathcal{N}^{[\perp]}}(z, w) = K(z, w) - \sum_1^\kappa f_j(z) f(w)^*. \tag{13.1}$$

By Theorem 8.4, $\mathcal{N}_-^{[\perp]}$ has a unique completion as a reproducing kernel Hilbert space with reproducing kernel (13.1). We set $\mathcal{N}_+$ to be this completion and we set

$$\mathcal{P}(K) := \{f = f_- + f_+,\ f_+ \in \mathcal{N}_+,\ f_- \in \mathcal{N}_-,\}$$

with the inner product

$$[f, f] := [f_-, f_-] + [f_+, f_+]_{\mathcal{N}_+}.$$

This space is easily seen to be a quaternionic reproducing kernel Pontryagin space with reproducing kernel $K(z, w)$. We now prove its uniqueness and follow the arguments in [2, p. 10]. Let $(\mathcal{P}', [\cdot, \cdot]_{\mathcal{P}})$ be another quaternionic reproducing kernel Pontryagin space with reproducing kernel $K(z, w)$. Then $\overset{\circ}{\mathcal{P}(K)}$ and hence $\mathcal{N}_-$ and $\mathcal{N}_-^{[\perp]}$ are isometrically included in $\mathcal{P}'$. Thus $\mathcal{N}_-^{\perp}$ is dense in $\mathcal{P}' \ominus \mathcal{N}_-$ and so its closure is isometrically included in $\mathcal{P}'$. We deduce that $\mathcal{P}(K)$ is isometrically included in $\mathcal{P}'$ and equality follows as in the proof of uniqueness in Theorem 8.4. ☐

We conclude with the proof of Theorem 11.5. Assume first that the $\mathbb{H}^{n\times n}$–valued function $K(z, w)$ (with $z, w \in \Omega$) is the reproducing kernel of the reproducing kernel Pontryagin space $\mathcal{P}(K)$ and let

$$\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_-$$

be a fundamental decomposition. One defines functions $K_+(z, w)$ and $K_-(z, w)$ such that for every $w \in \Omega$ and $c \in \mathbb{H}^n$

$$K(z, w)c = K_+(z, w)c + K_-(z, w)c$$

is the decomposition of the function $z \mapsto K(z, w)c$ along this fundamental decomposition. The functions K_+ and K_- are positive in Ω and are the reproducing kernels of $\mathcal{P}_+$ and $\mathcal{P}_-$ respectively. Since $\mathcal{P}_-$ is finite dimensional the function K_- is of finite rank $\kappa = \dim\,\mathcal{P}_-$ (Definition 9.7).

Conversely, assume that $K(z,w) = K_+(z,w) - K_-(z,w)$ where $K_-(z,w)$ is of finite rank. It follows from Lemma 9.7 that K has a finite number of negative squares. There exists thus an associated quaternionic reproducing kernel Pontryagin space $\mathcal{P}(K)$ and the other claims are obtained by considering the decomposition of K as above from any fundamental decomposition of $\mathcal{P}(K)$.

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