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Integral Equations and Operator Theory

Double Operator Integrals in a Hilbert Space

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Abstract. Double operator integrals are a convenient tool in many problems arising in the theory of self-adjoint operators, especially in the perturbation theory. They allow to give a precise meaning to operations with functions of two ordered operator-valued non-commuting arguments. In a different language, the theory of double operator integrals turns into the problem of scalar-valued multipliers for operator-valued kernels of integral operators.

The paper gives a short survey of the main ideas, technical tools and results of the theory. Proofs are given only in the rare occasions, usually they are replaced by references to the original papers. Various applications are discussed.

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C O N T E N T S

1. Introduction

1.1.

Formally, Double Operator Integrals are objects of the type

$$\mathbf{Q} = \int_{\Lambda} \int_{M} \phi(\lambda, \mu) dE(\lambda) \mathbf{T} dF(\mu). \tag{1.1}$$

So far, this is only a formal expression. In (1.1) $(\Lambda, E(\cdot))$ and $(M, F(\cdot))$ are two spaces with spectral measure. The values of the measure $E(\cdot)$ are orthogonal projections in a Hilbert space $\mathfrak{H}$, and similar for the measure $F(\cdot)$ in a Hilbert space $\mathfrak{G}$. Both spaces $\mathfrak{H}, \mathfrak{G}$ are always supposed separable. The scalar-valued function $\phi(\lambda, \mu)$ (*the symbol* of the integral) is defined on $\Lambda \times M$. Finally, $\mathbf{T}$ is a linear bounded operator acting from $\mathfrak{G}$ to $\mathfrak{H}$ (notation $\mathbf{T} \in \mathcal{B}(\mathfrak{G}, \mathfrak{H})$). It is clear that under any reasonable definition the result $\mathbf{Q}$ of integration is also an operator acting from $\mathfrak{G}$ to $\mathfrak{H}$. Hence, the integral (1.1) defines a linear mapping

$$\mathcal{J}_{\phi}^{E,F} : \mathbf{T} \mapsto \mathbf{Q}. \tag{1.2}$$

Following I. Gohberg and M.G. Krein [22], we use the term *transformer* for linear mappings acting on operators. So, the mapping $\mathcal{J}_{\phi}^{E,F}$ is a special case of a transformer. Often we use for it a shortened notation $\mathcal{J}_{\phi}$. In particular, we do this when the spectral measures E, F are fixed. Sometimes we write

$$\mathbf{Q}_{\phi} := \mathcal{J}_{\phi}^{E,F} \mathbf{T}. \tag{1.3}$$

If E, F are the spectral measures of self-adjoint operators $\mathbf{A}, \mathbf{B}$ ($E = E^{\mathbf{A}},\ F = F^{\mathbf{B}}$), then instead of (1.3) we write

$$\mathbf{Q}_{\phi} := \mathcal{J}_{\phi}^{\mathbf{A},\mathbf{B}} \mathbf{T}. \tag{1.4}$$

In a simplest situation, double operator integrals (1.1), and also integrals of higher multiplicity, first appeared in 1956, in the paper [18] by Y.L. Daletskii and S.G. Krein. Their purpose was differentiation of the functions $h(\mathbf{A}(t))$ where h is a smooth scalar-valued function on $\mathbb{R}$ (say, with compact support) and $\mathbf{A}(t)$ is a smooth function whose values are bounded self-adjoint operators in a Hilbert space $\mathfrak{H}$. The starting point was the representation

$$h(\mathbf{A}(t)) = \int_{\mathbb{R}} h(\lambda) dE^{\mathbf{A}(t)}(\lambda) = -\int_{\mathbb{R}} h'(\lambda) E^{\mathbf{A}(t)}(\lambda) d\lambda.$$

Based upon this representation and beginning from the case $\dim \mathfrak{H} < \infty$, the authors of [18] came to the equality, now known as the Daletskii – Krein differentiation formula; see equation (1.12) of this Introduction. It was justified under rather restrictive assumptions on h, and the authors did not consider the double and multiple operator integrals as a subject deserving a special study.

The authors of the present paper started their work on double operator integrals in 1964. Our interest in the topic was motivated by the work of M.Sh. Birman on the stationary approach to the scattering theory, see his papers [2], [3]. This required an analysis of many natural questions to which no answers were known at that time. In particular, it was necessary to be able to show that for a wide class of non-smooth symbols ϕ the assumption $\mathbf{T} \in \mathfrak{S}_1$ (where $\mathfrak{S}_1$ stands for the trace class) implies $\mathbf{Q}_\phi \in \mathfrak{S}_1$. In this connection, M.G. Krein attracted Birman's attention to the paper [18], and influenced by his remarks we started our work on the subject.

It was understood rather soon that indeed, the double operator integrals provide an appropriate tool for study of questions of this nature. But it was also realized that one could not answer them without developing a comprehensive theory of such integrals. This was done in a series of authors' papers starting from [5]. As a result, varied important questions, including the above one, got their adequate answers. Among contributions of other mathematicians to this topic the most substantial results are due to V. Peller [30], [32] and S. Rotfeld [34]. Section 10 contains further historical remarks and comments on the literature on the subject.

The most important applications of the double operator integrals concern Perturbation Theory. With their help, an integral representation of the operator $h(\mathbf{B}) - h(\mathbf{A})$ can be given. It yields useful estimates of the norm of the latter operator in various operator ideals. Further, this techniques allows to justify the Daletskii – Krein differentiation formula in a more general situation. It is important that if $\mathbf{A}'(t)$ belongs to an operator ideal, then the derivative in (1.12) exists in the norm of the same ideal. This fact plays a basic role in applications of double integrals to the Spectral Shift Function of I.M. Lifshits – M.G. Krein, see section 9.

In most cases the theory of operator integrals deals with the symbols continuous in at least one of the variables. However, there are also some useful results for discontinuous symbols. They are closely related to the theory of the so-called triangle transformer for which $\Lambda = M = \mathbb{R}$ and $\sigma(\lambda, \mu) = \theta(\lambda - \mu)$, where θ is the Heaviside function. For $F = E$ this transformer is one of the main technical tools in the theory of Volterra operators, see the book [22].

It turns out that the main estimates for this transformer extend, upon a different technical basis, to the case when the spectral measures are different. A consequence of this fact is a general result on integration of functions of bounded variation (in one variable), Theorem 7.3, which in its turn leads to an important Theorem 8.6.

There exists a realization of operator integrals, which on the first sight has nothing in common with the expression (1.1). Namely, let $\mathbf{T}$ be an integral operator acting between two L^2-spaces, with the kernel $T(\lambda, \mu)$. Given a scalar-valued, bounded function $\phi(\lambda, \mu)$, consider the integral operator $\mathbf{Q}$ with the kernel $T(\lambda, \mu)\phi(\lambda, \mu)$. Some useful properties of the transformation $\mathbf{T} \mapsto \mathbf{Q}$ were studied in [22], section II.5.

It turns out that this transformation can be written as a double integral (1.1) with the symbol ϕ, if one chooses the spectral measures E, F in an appropriate way. What is more, this realization is exhaustive, i.e. any transformer (1.1) can be realized as the above multiplier transformation. However, the "usual" scalar-valued kernels are not sufficient for this purpose, and one needs integral operators with operator-valued kernels. We discuss this material in sections 3.2 and 4.1.

This point of view allows to consider the pseudo-differential operators on $\mathbb{R}^d$ as a special case of a transformer of the type (1.4) applied to the identity operator $\mathbf{I}$, see section 6.

The authors' papers on the theory and applications of operator integrals were originally published in Russian mathematical journals of minor importance; not all of them were translated into English, and the complete exposition was never written. This paper is an attempt of such exposition. This is a survey where all the basic problems and main applications are discussed in detail. As a rule, the proofs are absent. Instead, we give the relevant references.

The remaining part of Introduction is an informal description of the contents of the paper.

1.2. Preliminary remarks

The first problem in the theory of operator integrals is to give their rigorous definition for as broad as possible class of symbols. It turns out that there is no universal such definition: the proper definition, and hence also the class of admissible symbols, depend on the space of operators we wish to deal with. In this respect the space $\mathfrak{S}_2$ of Hilbert – Schmidt operators plays a special role: here the integral (1.1) can be well defined for an arbitrary bounded and measurable symbol (measurability with respect to an appropriate measure σ on $\Lambda \times M$). The measure σ is determined by the given spectral measures E and F; the operator $\mathbf{Q}_\phi$ is also Hilbert – Schmidt and moreover,

$$\|\mathbf{Q}_\phi\|_{\mathfrak{S}_2} \le (\sigma)\text{-}\sup|\phi|\,\|\mathbf{T}\|_{\mathfrak{S}_2}. \tag{1.5}$$

All this, including the construction of the measure σ, will be explained in section 3.

For other spaces of operators the situation is more complex. The most important case is when the integral (1.1) can be well defined for any bounded operator $\mathbf{T}$ and the resulting operator $\mathbf{Q}_\phi$ is also bounded. Then the transformer $\mathcal{J}_\phi^{E,F}$ acts in the space $\mathcal{B}(\mathfrak{G},\mathfrak{H})$ and by Closed Graph Theorem is bounded. Theorem 4.1 gives a full description of the class $\mathfrak{M} = \mathfrak{M}(E,F)$ of all admissible symbols of this type. If $\phi \in \mathfrak{M}$, then the transformer $\mathcal{J}_\phi^{E,F}$ is also bounded in the space $\mathfrak{S}_1$ and in the space $\mathfrak{S}_\infty$ of all compact operators. It is possible to consider the action of the integral (1.1) between other spaces of operators, and the spaces for $\mathbf{T}$ and for $\mathbf{Q}$ may differ from each other. It is worth mentioning that the exhaustive description of the class of admissible symbols for the most of cases is not known. However, there are many sufficient conditions which allow one to apply the general results of the theory of operator integrals.

If a space $\mathfrak{S}$ of operators is chosen, then the symbols ϕ, such that the transformer $\mathcal{J}_\phi = \mathcal{J}_\phi^{E,F}$ is bounded in $\mathfrak{S}$, form a commutative algebra of functions on $\Lambda \times M$, with complex conjugation as the involution. Namely, it turns out that

$$\mathcal{J}_{\phi_1+\phi_2} = \mathcal{J}_{\phi_1} + \mathcal{J}_{\phi_2};\ \mathcal{J}_{\phi_1\phi_2} = \mathcal{J}_{\phi_1}\mathcal{J}_{\phi_2};\ \mathcal{J}_{\bar\phi} = \mathcal{J}_\phi^*. \tag{1.6}$$

Moreover, if $\mathfrak{S}$ is a Banach space, then the above algebra is a Banach algebra with respect to an appropriate norm. This points out on the possibility to develop a sort of operational calculus for the integrals (1.1). Our main goal is the detailed study of the transformer (1.2) in various classes of operators, obtaining estimates for the corresponding norm of $\mathcal{J}_\phi$, and so on.

1.3. Functions of non-commuting operators

Suppose that $\mathfrak{G} = \mathfrak{H}$ and in (1.1) $\Lambda = M = \mathbb{R}$, $E = E^{\mathbf{A}}$, and $F = F^{\mathbf{B}}$ where $\mathbf{A}, \mathbf{B}$ are self-adjoint operators. Then it is natural to regard $\mathbf{Q}_\phi$ as the function ϕ of the pair $\mathbf{A}, \mathbf{B}$, separated by the operator $\mathbf{T}$. Here the "argument" $\mathbf{A}$ stands on the left side of $\mathbf{T}$, and $\mathbf{B}$ stands on the right side of $\mathbf{T}$. The operators $\mathbf{A}$ and $\mathbf{B}$ are not assumed commuting. Even if they do commute, this does not affect the general picture: indeed, the presence of the operator $\mathbf{T}$ prevents any possible gains which might come from the commutation of $\mathbf{A}$ and $\mathbf{B}$. Of course, this situation changes if one makes some assumptions on the properties of the commutators $[\mathbf{A}, \mathbf{B}]$, $[\mathbf{A}, \mathbf{T}]$ and $[\mathbf{B}, \mathbf{T}]$. However, such assumptions are not natural for the general theory of double operator integrals.

It is quite clear, how to define the operator $\mathbf{Q}_\phi$ for the case when $\phi(\lambda, \mu) = \alpha(\lambda)\beta(\mu)$ where α and β are bounded functions. Indeed, then by Spectral Theorem

$$\alpha(\mathbf{A})\mathbf{T}\beta(\mathbf{B}) = \int \alpha(\lambda)dE(\lambda)\,\mathbf{T}\int \beta(\mu)dF(\mu).$$

Formally, this can be re-written as

$$\mathbf{Q}_\phi = \alpha(\mathbf{A})\mathbf{T}\beta(\mathbf{B}) = \int_{\mathbb{R}}\int_{\mathbb{R}} \alpha(\lambda)\beta(\mu)dE(\lambda)\mathbf{T}dF(\mu). \tag{1.7}$$

Actually, one integrates only over the spectra, that is over $\sigma(\mathbf{A}) \times \sigma(\mathbf{B})$. Moreover, for the operator $\mathbf{Q}_\phi$ given by (1.7) we have

$$\|\mathbf{Q}_\phi\| \le \sup_{\lambda,\mu} |\alpha(\lambda)||\beta(\mu)|\|\mathbf{T}\|. \tag{1.8}$$

More exactly,

$$\|\mathbf{Q}_\phi\| \le \|\alpha\|_{\mathsf{L}^\infty(\Lambda;E)}\|\beta\|_{\mathsf{L}^\infty(M;F)}\|\mathbf{T}\|$$

where, say, $\mathsf{L}^\infty(\Lambda; E)$ denotes the space L^∞ on Λ with respect to the spectral measure E.

The equality (1.7) can serve as the definition of the integral (1.1) for the functions $\phi(\lambda, \mu) = \alpha(\lambda)\beta(\mu)$. Clearly, this definition extends to the finite linear combinations of such monoms, in particular to the case when ϕ is a polynomial in λ, μ and the operators $\mathbf{A}, \mathbf{B}$ are bounded. However, the estimate similar to (1.8), i.e.

$$\|\mathbf{Q}_\phi\| \le \sup_{\lambda,\mu} |\phi(\lambda,\mu)| \|\mathbf{T}\|$$

is no longer valid. This is one of the main difficulties we encounter when developing the theory of operator integrals. The estimate (1.5) shows that this difficulty can be overcome if one is interested in estimates in the Hilbert – Schmidt norm, rather than in the usual operator norm.

A similar situation arises if we are dealing with two families of mutually commuting self-adjoint operators, unitary, or normal operators. Then $E(\cdot)$ and $F(\cdot)$ are the joint spectral measures for these families, and depending on the situation we take $\Lambda = \mathbb{R}^d,\ \mathbb{T}^d$, or $\mathbb{C}^d$, and similar for M. In sections 6.2, 6.3 we will see that other choice of Λ or M can also appear in a natural way.

1.4. On applications to the perturbation theory

Here we briefly discuss one of possible applications of double operator integrals. Let $\mathbf{A}$ and $\mathbf{B}$ be two self-adjoint operators in a Hilbert space $\mathfrak{H}$, for simplicity we assume them bounded. Let $h(\lambda)$ be a function defined on an interval which contains the spectra of both operators. One of the central problems of the perturbation theory is study of the operator $h(\mathbf{B}) - h(\mathbf{A})$, depending on the properties of $\mathbf{T} := \mathbf{B} - \mathbf{A}$. In particular, it would be useful to have an explicit representation of $h(\mathbf{B}) - h(\mathbf{A})$ in terms of $\mathbf{T}$. To achieve this goal, let us try to give meaning to the "crazy" formula

$$h(\mathbf{B}) - h(\mathbf{A}) = \frac{h(\mathbf{B}) - h(\mathbf{A})}{\mathbf{B} - \mathbf{A}}\mathbf{T}. \tag{1.9}$$

Rather surprisingly, this can be done in the language of operator integrals, with the help of the algebraic properties (1.6) of the transformers $\mathcal{J}_\phi = \mathcal{J}_\phi^{\mathbf{A},\mathbf{B}}$. The proper realization of the meaningless equality (1.9) is the formula

$$h(\mathbf{B}) - h(\mathbf{A}) = \int_{\sigma(\mathbf{A})}\int_{\sigma(\mathbf{B})} \phi_h(\lambda,\mu) dE^{\mathbf{A}}(\lambda)\mathbf{T}dF^{\mathbf{B}}(\mu). \tag{1.10}$$

where

$$\phi_h(\lambda,\mu) := \frac{h(\mu) - h(\lambda)}{\mu - \lambda}.$$

The formula (1.10) will be justified in section 8.1.

An integral representation similar to (1.10) is valid also for *quasi-commutators* $\mathbf{J}h(\mathbf{B}) - h(\mathbf{A})\mathbf{J}$ where $\mathbf{J}$ is one more bounded operator. Namely,

$$\mathbf{J}h(\mathbf{B}) - h(\mathbf{A})\mathbf{J} \tag{1.11}$$
$$= \int_{\sigma(\mathbf{A})}\int_{\sigma(\mathbf{B})} \phi_h(\lambda,\mu) dE^{\mathbf{A}}(\lambda)(\mathbf{JB} - \mathbf{AJ})dF^{\mathbf{B}}(\mu).$$

Clearly, (1.10) is a particular case of (1.11). Let us point out a minor difference between these two equalities: if in the right-hand side of (1.11) we change places of the spectral measures, then the left-hand side becomes $h(\mathbf{B})\mathbf{J} - \mathbf{J}h(\mathbf{A})$; for (1.10) this change is irrelevant. It is also irrelevant for the case when $\mathbf{A} = \mathbf{B}$, that is for commutators. Note also that for the equality (1.11) the assumption $\mathfrak{G} = \mathfrak{H}$ is unnecessary, and the operator $\mathbf{J}$ acts from $\mathfrak{G}$ to $\mathfrak{H}$.

Both formulas (1.10) and (1.11) can be treated as a way to linearize a non-linear problem. Let us explain this for (1.10). The transformer $\mathcal{J}_{\phi_h}$ is a linear mapping which acts on any operator $\mathbf{T}$ of an appropriate class, rather than on the operator $\mathbf{B} - \mathbf{A}$ only. Thus the *non-linear* problem on the representation of the operator $h(\mathbf{B}) - h(\mathbf{A})$ in terms of the unperturbed operator $\mathbf{A}$ and the perturbation $\mathbf{B} - \mathbf{A}$ is embedded into the *linear* problem on the properties of the linear transformer $\mathcal{J}_{\phi_h}$. It is noteworthy that such linearization often gives very precise results for the original problem.

The representation (1.10) for the operator $h(\mathbf{B}) - h(\mathbf{A})$ leads to useful estimates for its norm in various spaces of operators. Further, consider the operator-valued function $\mathbf{A}(t) = \mathbf{A} + t\mathbf{T}$ (so that $\mathbf{A}(0) = \mathbf{A},\ \mathbf{A}(1) = \mathbf{B}$). By (1.10),

$$\frac{h(\mathbf{A}(t)) - h(\mathbf{A}(0))}{t} = \int_{\sigma(\mathbf{A})}\int_{\sigma(\mathbf{A}(t))} \phi_h(\lambda,\mu)dE^{\mathbf{A}}(\lambda)\mathbf{T}dE^{\mathbf{A}(t)}(\mu).$$

Formally passing to the limit as $t \to 0$, we come to the equality

$$\left.\frac{dh(\mathbf{A}(t))}{dt}\right|_{t=0} = \int_{\sigma(\mathbf{A})}\int_{\sigma(\mathbf{A})} \phi_h(\lambda,\mu)dE^{\mathbf{A}}(\lambda)\mathbf{T}dE^{\mathbf{A}}(\mu) \tag{1.12}$$

known as *the Daletskii – S. Krein formula.* The limiting procedure here needs a justification which can be given under certain assumptions on the function h and the operator $\mathbf{T}$. We discuss this problem in section 8.3.

It is also possible to calculate the further derivatives and to write the Taylor formula for the operator-valued functions. However, this requires integrals of multiplicity greater than two. Their theory is developed up to a lesser extent and we do not include it in our exposition.

1.5. Double operator integrals as multiplier transformation

The double operator integral admits a useful equivalent interpretation as a multiplier transformation for kernels of the integral operators. Here we give some preliminary explanations, restricting ourselves to the Hilbert – Schmidt operators acting on scalar-valued functions.

Let (Λ,ρ) and (M,τ) be two separable measure spaces and $\Gamma = \Lambda \times M$, $\sigma = \rho \times \tau$. Consider the Hilbert spaces $\mathfrak{G} = \mathsf{L}^2(M,\tau)$ and $\mathfrak{H} = \mathsf{L}^2(\Lambda,\rho)$. Any operator $\mathbf{T} \in \mathfrak{S}_2(\mathfrak{G},\mathfrak{H})$ can be realized as an integral operator with a kernel $T(\lambda,\mu)$ from $\mathsf{L}^2(\Gamma,\sigma)$:

$$v(\lambda) = (\mathbf{T}u)(\lambda) = \int_M T(\lambda,\mu)u(\mu)d\tau(\mu). \tag{1.13}$$

Moreover,

$$\|\mathbf{T}\|^2_{\mathfrak{S}_2} = \|T\|^2_{\mathsf{L}^2(\Gamma,\sigma)} = \int_\Gamma |T(\lambda,\mu)|^2 d\rho(\lambda) d\tau(\mu). \tag{1.14}$$

Let now a function $\phi \in \mathsf{L}^\infty(\Gamma,\sigma)$ be given. Consider the mapping $\mathcal{M}_\phi$ which transforms the operator (1.13) into the integral operator whose kernel is the product $\phi(\lambda,\mu)T(\lambda,\mu)$:

$$(\mathcal{M}_\phi \mathbf{T} u)(\lambda) = \int_M \phi(\lambda,\mu) T(\lambda,\mu) u(\mu) d\tau(\mu). \tag{1.15}$$

It is clear that the mapping $\mathcal{M}_\phi$ is linear, meets the properties (1.6) and is bounded in $\mathfrak{S}_2(\mathfrak{G},\mathfrak{H})$:

$$\|\mathcal{M}_\phi \mathbf{T}\|_{\mathfrak{S}_2} \le \|\phi\|_{\mathsf{L}^\infty(\Gamma,\sigma)} \|\mathbf{T}\|_{\mathfrak{S}_2}, \tag{1.16}$$

cf. (1.5). This leads to the conclusion that there should be a connection between the multiplier transformation $\mathcal{M}_\phi$ and the transformer $\mathcal{J}_\phi$ generated by the double operator integral. It turns out that indeed this is the case.

In order to show this, let us consider the spectral measure $E(\cdot)$ in $\mathfrak{H}$, formed by the operators of multiplication by the characteristic functions χ_δ of the measurable subsets $\delta \subset \Lambda$, i.e. $(E(\delta)u)(\lambda) = \chi_\delta(\lambda)u(\lambda)$ for any $u \in \mathfrak{H}$. Let also $F(\cdot)$ be the similar spectral measure in the space $\mathfrak{G}$. It is not difficult to verify that

$$\mathcal{M}_\phi = \mathcal{J}_\phi^{E,F}. \tag{1.17}$$

Moreover, let E, F be arbitrary spectral measures. It turns out that any transformer $\mathcal{J}_\phi^{E,F}$ which is bounded in the space $\mathfrak{S}_2$ can be realized as the multiplication transformation $\mathcal{M}_\phi$ in an appropriate class of kernels. However, the scalar-valued kernels are not sufficient for achieving this goal. One has to consider the kernels whose values themselves are operator-valued functions, and the appropriate language is the one of von Neumann's direct integrals of Hilbert spaces. This material is discussed in section 3.2 and its generalization for the spaces of operators different from $\mathfrak{S}_2$ – in section 5. See also section 6 for applications of this scheme to pseudodifferential operators.

2. Auxiliary material

The most of the material we need can be found in the textbook [13].

2.1. Reminder on spectral measures

Below E (in more detailed notation, (Λ, E)) is a spectral measure in a separable Hilbert space $\mathfrak{H}$, defined on a σ-algebra of subsets of a given set Λ. This σ-algebra is not reflected in our notations, and all the subsets encountered are assumed measurable, i.e. they belong to this σ-algebra. The values $E(\delta)$ of the spectral measure E are mutually commuting orthoprojections in $\mathfrak{H}$. For each element $\mathfrak{h} \in \mathfrak{H}$ the function $\rho_{\mathfrak{h}}(\delta) = (E(\delta)\mathfrak{h}, \mathfrak{h})$ is a finite scalar measure.

For two scalar measures ν_1, ν_2 defined on the same σ-algebra the relation $\nu_1 \prec \nu_2$ means that given a measurable set δ, the equality $\nu_2(\delta) = 0$ implies $\nu_1(\delta) = 0$. The measures ν_1, ν_2 are called equivalent ($\nu_1 \sim \nu_2$), if $\nu_1 \prec \nu_2$ and $\nu_2 \prec \nu_1$. The class of all measures equivalent to a given measure ν is called its *type*.

It is convenient to single out a scalar measure on Λ, say ρ, whose type coincides with the type of the spectral measure E. This means that $\rho_{\mathfrak{h}} \prec \rho$ for each $\mathfrak{h} \in \mathfrak{H}$ and there exists an element $\mathfrak{h}_0 \in \mathfrak{H}$ such that $\rho \sim \rho_{\mathfrak{h}_0}$. Any such $\mathfrak{h}_0$ is called element of maximal type with respect to the spectral measure E. The elements of maximal type (and measures ρ) do always exist provided that $\mathfrak{H}$ is separable.

2.2. Integration with respect to a spectral measure

Let $\alpha(\lambda)$ be a measurable and (E)-a.e. finite function on Λ, then the integral $I_\alpha := \int_\Lambda \alpha(\lambda) dE(\lambda)$ is well defined; this is an operator in $\mathfrak{H}$ which is bounded if and only if $\alpha \in L^\infty(\Lambda) := L^\infty(\Lambda, \rho)$. As a rule, this is the only case we are interested in in this paper. If $\Lambda = \mathbb{R}$ and E is a spectral measure of a self-adjoint operator $\mathbf{A}$ in $\mathfrak{H}$, then by definition $I_\alpha = \alpha(\mathbf{A})$.

The mapping $\alpha \mapsto I_\alpha$ satisfies the following properties:

$$I_{\alpha_1+\alpha_2} = I_{\alpha_1} + I_{\alpha_2}, \ I_{\alpha_1\alpha_2} = I_{\alpha_1} I_{\alpha_2}, \ I_{\bar{\alpha}} = I_\alpha^*; \tag{2.1}$$

$$\|I_\alpha\| = \|\alpha\|_{L^\infty}. \tag{2.2}$$

This shows that $\alpha \mapsto I_\alpha$ is an isometric isomorphism of the Banach C^*-algebra $L^\infty(\Lambda)$ onto a commutative and involutive sub-algebra of the algebra $\mathcal{B} = \mathcal{B}(\mathfrak{H})$.

It is useful to add that for any $\mathfrak{h} \in \mathfrak{H}$

$$I_\alpha \mathfrak{h} = \int_\Lambda \alpha(\lambda) d(E(\lambda)\mathfrak{h}); \qquad \|I_\alpha \mathfrak{h}\|^2 = \int_\Lambda |\alpha(\lambda)|^2 d\rho_{\mathfrak{h}}(\lambda); \tag{2.3}$$

in the first integral we integrate with respect to the vector-valued measure $E(\cdot)\mathfrak{h}$.

2.3. Direct integral of Hilbert spaces

Each Hilbert space $\mathfrak{H}$ with a given spectral measure (Λ, E) can be decomposed into the direct integral of Hilbert spaces:

$$\mathfrak{H} = \int_\Lambda \oplus H(\lambda) d\rho(\lambda) \tag{2.4}$$

where ρ is a chosen scalar measure whose type coincides with the type of E. The meaning of the equality (2.4) is that there is a unitary operator which identifies each element $\mathfrak{h} \in \mathfrak{H}$ with a function $\mathfrak{h}(\lambda) = \mathfrak{h}_{\Lambda,E}(\lambda)$ with values in $H(\lambda)$ (in writing, $\mathfrak{h} \sim \mathfrak{h}(\lambda)$). Each function $\mathfrak{h}(\lambda)$ is measurable, in an appropriate sense. As a matter of fact, the term "function" is here not quite accurate, since $\mathfrak{h}(\lambda)$ takes its values in different spaces $H(\lambda)$ for different $\lambda \in \Lambda$. See e.g. [13], Section 7.1 for more detail.

Unitarity means that

$$(\mathfrak{h}_1, \mathfrak{h}_2)_{\mathfrak{H}} = \int_{\Lambda} (\mathfrak{h}_1(\lambda), \mathfrak{h}_2(\lambda))_{H(\lambda)} d\rho(\lambda), \ \forall \mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{H}.$$

The decomposition (2.4) diagonalizes each operator I_α, i.e.

$$\mathfrak{h} \sim \mathfrak{h}(\lambda) \Longrightarrow I_\alpha \mathfrak{h} \sim \alpha(\lambda)\mathfrak{h}(\lambda). \tag{2.5}$$

3. Double operator integrals on $\mathfrak{S}_2$

3.1. Basic definition

Let $(\Lambda, E), (M, F)$ be two spectral measures in the spaces $\mathfrak{H}$, $\mathfrak{G}$. It is convenient to fix scalar measures ρ on Λ and τ on M whose types coincide with the types of the spectral measures E, F respectively.

The Hilbert-Schmidt class $\mathfrak{S}_2 = \mathfrak{S}_2(\mathfrak{G}, \mathfrak{H})$ is a Hilbert space, with respect to the scalar product

$$\langle \mathbf{T}, \mathbf{S} \rangle = tr(\mathbf{T}\mathbf{S}^*) = tr(\mathbf{S}^*\mathbf{T}). \tag{3.1}$$

We shall construct a certain spectral measure on $\mathfrak{S}_2$, the *tensor product* of measures (Λ, E) and (M, F), and define the transformer $\mathcal{J}_\phi$ as integral with respect to this spectral measure.

Consider the mappings

$$\begin{cases} \mathcal{E}(\delta) : \mathbf{T} \mapsto E(\delta)\mathbf{T} & \text{for } \delta \subset \Lambda, \ \mathbf{T} \in \mathfrak{S}_2; \\ \mathcal{F}(\partial) : \mathbf{T} \mapsto \mathbf{T}F(\partial) & \text{for } \partial \subset M, \ \mathbf{T} \in \mathfrak{S}_2. \end{cases} \tag{3.2}$$

Each operator $\mathcal{E}(\delta)$ is an orthogonal projection in $\mathfrak{S}_2$, the mapping $\delta \mapsto \mathcal{E}(\delta)$ is σ-additive, and $\mathcal{E}(\Lambda) = \mathcal{I}$ (the identity transformer on $\mathfrak{S}_2$). So we see that $\mathcal{E}$ is a spectral measure in $\mathfrak{S}_2$, and the same for $\mathcal{F}$. The type of $\mathcal{E}$ (of $\mathcal{F}$) coincides with that of E (of F), so that both types are defined by the above measures ρ, τ. It follows directly from the definition that for any bounded measurable functions $\alpha(\lambda), \beta(\mu)$ we have

$$\int_{\Lambda} \alpha(\lambda) d(\mathcal{E}(\lambda)\mathbf{T}) = \int_{\Lambda} \alpha(\lambda) dE(\lambda) \cdot \mathbf{T},$$
$$\int_{M} \beta(\mu) d(\mathcal{F}(\mu)\mathbf{T}) = \mathbf{T} \cdot \int_{M} \beta(\mu) dF(\mu).$$

The measures $\mathcal{E}$ and $\mathcal{F}$ commute, since one corresponds to the multiplication from the left and another from the right.

The mapping

$$\mathcal{G}(\delta \times \partial) = \mathcal{E}(\delta)\mathcal{F}(\partial) : \mathbf{T} \mapsto E(\delta)\mathbf{T}F(\partial) \tag{3.3}$$

is an additive projection-valued function on the set of all "measurable rectangles" $\delta \times \partial \subset \Lambda \times M$ (orthogonal projections in $\mathfrak{S}_2$). It turns out (see [16]) that this function is σ-additive. The σ-additive projection-valued function $\mathcal{G}(\Delta)$ extends, in a unique way, from the set of measurable rectangles $\Delta = \delta \times \partial$ to the minimal

σ-algebra $\mathfrak{A}_0$ of subsets in $\Lambda \times M$, generated by such rectangles, and the extension is σ-additive, so it is a spectral measure in $\mathfrak{S}_2$. We denote it by the same symbol $\mathcal{G}$. It is convenient to add to $\mathfrak{A}_0$ all the subsets $e' \subset e$ of sets $e \in \mathfrak{A}_0$ of $\mathcal{G}$-measure zero, putting $\mathcal{G}(e') = 0$. The resulting family $\mathfrak{A}$ is also a σ-algebra, and the spectral measure $\mathcal{G}$ on $\mathfrak{A}$ is *N-full*, cf. [13], section I.3.7. The type of $\mathcal{G}$ coincides with the type of the scalar measure $\rho \times \tau$ on $\Lambda \times M$.

Now we take by definition

$$\mathcal{J}_\phi = \int_{\Lambda \times M} \phi(\lambda, \mu) d\mathcal{G}(\lambda, \mu), \tag{3.4}$$

or

$$\mathcal{J}_\phi \mathbf{T} = \int_{\Lambda \times M} \phi(\lambda, \mu) d(\mathcal{G}(\lambda, \mu)\mathbf{T}). \tag{3.5}$$

So, for bounded ϕ this is a bounded transformer in $\mathfrak{S}_2$. The relations (2.1), (2.2) turn into

$$\mathcal{J}_{\phi_1+\phi_2} = \mathcal{J}_{\phi_1} + \mathcal{J}_{\phi_2}, \ \mathcal{J}_{\phi_1\phi_2} = \mathcal{J}_{\phi_1}\mathcal{J}_{\phi_2}; \tag{3.6}$$

$$\mathcal{J}_{\bar{\phi}} = \mathcal{J}_\phi^*; \tag{3.7}$$

$$\|\mathcal{J}_\phi\| = \|\phi\|_{L^\infty(\Lambda \times M)}. \tag{3.8}$$

If $\phi(\lambda, \mu) = \alpha(\lambda)$, then $\mathcal{J}_\phi = \int_\Lambda \alpha(\lambda) d\mathcal{E}(\lambda)$, or $\mathcal{J}_\phi \mathbf{T} = \int_\Lambda \alpha(\lambda) dE(\lambda) \cdot \mathbf{T}$. The similar formula is valid for $\mathcal{J}_\phi$ with $\phi(\lambda, \mu) = \beta(\mu)$. Using this observation and (3.6), we see that

$$\int_{\Lambda \times M} \alpha(\lambda)\beta(\mu) d(\mathcal{G}(\lambda, \mu)\mathbf{T}) = \int_\Lambda \alpha(\lambda) dE(\lambda) \cdot \mathbf{T} \cdot \int_M \beta(\mu) dF(\mu).$$

This shows that our definition is compatible with the "naive" description suggested in section 1.

3.2. Integrals on $\mathfrak{S}_2$ as multipliers

In section 1.4 we already discussed the possibility to interpret the double operator integral as a multiplier transformation for the kernels of integral operators. We did this for the simplest situation when $\mathfrak{H} = \mathsf{L}^2(\Lambda, \rho)$, $\mathfrak{G} = \mathsf{L}^2(M, \tau)$ and the spectral measures E, F are formed by the operators of multiplication by the characteristic functions of the measurable subsets in Λ and M respectively. In order to cover the general case, we need the apparatus of direct integrals of Hilbert spaces.

Consider the decomposition (2.4) of the space $\mathfrak{H}$ and the similar decomposition

$$\mathfrak{G} = \int_M \oplus G(\mu) d\tau(\mu) \tag{3.9}$$

of the space $\mathfrak{G}$, which corresponds to the spectral measure (M, F). There is a class of integral operators which is closely connected with the pair of decompositions (2.4), (3.9). Namely, consider the measure space $(\Lambda \times M, \rho \times \tau)$. Let $T(\lambda, \mu)$ be a measurable function (kernel) on $\Lambda \times M$, whose values are linear operators acting from $G(\mu)$ to $H(\lambda)$. Suppose that $T(\lambda,\mu) \in \mathfrak{S}_2(G(\mu), H(\lambda))$ a.e. and

$$\|T(\lambda,\mu)\|_{\mathfrak{S}_2} \in L^2(\Lambda \times M, \rho \times \tau).$$

Consider the operator $\mathbf{T}$ acting according to the rule

$$\mathfrak{h}(\lambda) = \int_M T(\lambda,\mu)\mathfrak{g}(\mu)d\tau(\mu). \tag{3.10}$$

Then $\mathbf{T} \in \mathfrak{S}_2(\mathfrak{G}, \mathfrak{H})$ and, moreover,

$$\|\mathbf{T}\|^2_{\mathfrak{S}_2} = \int_{\Lambda\times M} \|T(\lambda,\mu)\|^2_{\mathfrak{S}_2(G(\mu),H(\lambda))} d\rho(\lambda)d\tau(\mu).$$

Conversely, each operator $\mathbf{T} \in \mathfrak{S}_2(\mathfrak{G}, \mathfrak{H})$ can be represented, in a unique way, as the integral operator (3.10) with an appropriate operator-valued kernel $T(\lambda,\mu)$. This scheme allows one to write

$$\mathfrak{S}_2(\mathfrak{G}, \mathfrak{H}) = \int_{\Lambda\times M} \oplus \mathfrak{S}_2(G(\mu), H(\lambda))d\rho(\lambda)d\tau(\mu). \tag{3.11}$$

It turns out that it realizes the decomposition of this space, corresponding to the spectral measure $(\Lambda\times M, \mathcal{G})$. We have to show that for any measurable subset $\Delta \subset \Lambda\times M$ and any operator $\mathbf{T} \in \mathfrak{S}_2$ the kernel of operator $\mathcal{G}(\Delta)\mathbf{T}$ is $\chi_\Delta(\lambda,\mu)T(\lambda,\mu)$. It is sufficient to consider operators $\mathbf{T} = (\cdot,\omega)\vartheta$ of rank one, since they span the space $\mathfrak{S}_2$. The kernel of such $\mathbf{T}$ is $T(\lambda,\mu) = \vartheta(\lambda)\overline{\omega}(\mu)$ where $\vartheta(\lambda) = \vartheta_{\Lambda,E}(\lambda)$ and $\omega(\mu) = \omega_{M,F}(\mu)$.

Suppose that $\Delta = \delta\times\partial$, then $\mathcal{G}(\Delta)\mathbf{T} = (\cdot, F(\partial)\omega)E(\delta)\vartheta$, and the corresponding kernel is $\chi_\delta(\lambda)\vartheta(\lambda)\chi_\partial(\mu)\overline{\omega}(\mu) = \chi_\Delta(\lambda,\mu)T(\lambda,\mu)$. This extends to arbitrary measurable $\Delta \subset \Lambda \times M$, q.e.d.

Applying the general formula (2.5) to this new situation, we find for any $\phi \in L^\infty(\Lambda \times M)$:

$$\mathbf{T} \sim T(\lambda,\mu) \Longrightarrow \mathcal{J}^{E,F}_\phi \mathbf{T} \sim \phi(\lambda,\mu)T(\lambda,\mu). \tag{3.12}$$

In other words, the transformer $\mathcal{J}^{E,F}_\phi$ is realized as the multiplier transformation, cf. (1.15).

4. Integrals on $\mathfrak{S}_1$ and $\mathcal{B}$

4.1. Class $\mathfrak{M}$

After the transformer $\mathcal{J}_\phi$ is well defined on the class $\mathfrak{S}_2$, the next important task is its extension to the space $\mathcal{B} = \mathcal{B}(\mathfrak{G}, \mathfrak{H})$ of all bounded operators. This is not always possible: we need some additional assumptions on the symbol ϕ. The scheme we use below, has many classical analogs. It exploits the duality arguments.

Let $\mathfrak{S}_1$ stand for the trace class of operators, then

$$\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \mathcal{B}. \tag{4.1}$$

Moreover, the space $\mathcal{B}$ is adjoint to $\mathfrak{S}_1$, with respect to the duality form given by trace, cf. (3.1):

$$\langle \mathbf{T}, \mathbf{S} \rangle = tr(\mathbf{T}\mathbf{S}^*), \qquad \mathbf{T} \in \mathfrak{S}_1,\ \mathbf{S} \in \mathcal{B}. \tag{4.2}$$

Any transformer $\mathcal{J}_\phi$ with the L^∞-symbol maps $\mathfrak{S}_1$ into $\mathfrak{S}_2$. Suppose that for a given function ϕ its image lies in $\mathfrak{S}_1$ and, moreover, $\mathcal{J}_\phi$ is bounded as a transformer on $\mathfrak{S}_1$. It is possible (and not difficult) to show that then the transformer $\mathcal{J}_{\overline{\phi}}$ is also bounded in $\mathfrak{S}_1$ and has the same norm. The adjoint transformer $\mathcal{J}_{\overline{\phi}}{}^*$ acts in the space $\mathcal{B}$. The equality (3.7) shows that it is natural to define

$$\mathcal{J}_\phi \mathbf{T} = (\mathcal{J}_{\overline{\phi}}\big|_{\mathfrak{S}_1})^* \mathbf{T}, \qquad \forall \mathbf{T} \in \mathcal{B}. \tag{4.3}$$

The properties (3.6) of the transformers $\mathcal{J}_\phi$ extend to the whole of $\mathcal{B}$.

If $\mathbf{T} \in \mathfrak{S}_\infty$ (the space of all compact operators), then also $\mathcal{J}_\phi \mathbf{T} \in \mathfrak{S}_\infty$. Indeed, it is sufficient to show this for the dense in $\mathfrak{S}_\infty$ set $\mathfrak{K}$ of finite rank operators. But if $\mathbf{T} \in \mathfrak{K}$, then $\mathcal{J}_\phi \mathbf{T} \in \mathfrak{S}_1 \subset \mathfrak{S}_\infty$. So, the defined in (4.3) transformer $\mathcal{J}_\phi$ acts from $\mathfrak{S}_\infty$ to $\mathfrak{S}_\infty$ and, moreover,

$$\|\mathcal{J}_\phi\|_{\mathcal{B}\to\mathcal{B}} = \|\mathcal{J}_\phi\|_{\mathfrak{S}_1\to\mathfrak{S}_1} = \|\mathcal{J}_\phi\|_{\mathfrak{S}_\infty\to\mathfrak{S}_\infty}. \tag{4.4}$$

By interpolation, we obtain

$$\|\mathcal{J}_\phi\|_{\mathcal{B}\to\mathcal{B}} \geq \|\mathcal{J}_\phi\|_{\mathfrak{S}_2\to\mathfrak{S}_2} = \|\phi\|_{L^\infty}. \tag{4.5}$$

There is another, more direct way to define the transformer $\mathcal{J}_\phi$ on the space $\mathfrak{S}_\infty$. Namely, let $\phi \in L^\infty$. Then $\mathcal{J}_\phi$ is well defined on the class $\mathfrak{S}_2$ which is dense in $\mathfrak{S}_\infty$. If for a given ϕ the estimate holds

$$\|\mathcal{J}_\phi \mathbf{T}\|_{\mathcal{B}} \leq C\|\mathbf{T}\|_{\mathcal{B}}, \qquad \forall \mathbf{T} \in \mathfrak{S}_2,$$

then $\mathcal{J}_\phi$ extends to the whole of $\mathfrak{S}_\infty$ by continuity.

The two definitions are equivalent. Indeed, the space $\mathfrak{S}_1$ is adjoint to $\mathfrak{S}_\infty$ with respect to the same duality form (4.2); this time we should take $\mathbf{T} \in \mathfrak{S}_\infty,\ \mathbf{S} \in \mathfrak{S}_1$. Therefore, the adjoint transformer $(\mathcal{J}_\phi\big|_{\mathfrak{S}_\infty})^*$ is bounded in $\mathfrak{S}_1$, and it is easy to see that this transformer is nothing but $\mathcal{J}_{\overline{\phi}}\big|_{\mathfrak{S}_1}$. Now, the adjoint to the latter is the transformer $\mathcal{J}_\phi$ on $\mathcal{B}$ and its restriction to $\mathfrak{S}_\infty$ coincides with the original transformer.

Denote by $\mathfrak{M}_{\mathcal{B}}$ the set of all functions ϕ on $\Lambda \times M$, such that the transformer $\mathcal{J}_\phi$ is bounded on $\mathcal{B}$. This is a normed algebra of function, with respect to the norm

$$\|\phi\|_{\mathfrak{M}_{\mathcal{B}}} = \|\mathcal{J}_\phi\|_{\mathcal{B}\to\mathcal{B}}.$$

The mapping $\phi \mapsto \overline{\phi}$ is an involution in $\mathfrak{M}_{\mathcal{B}}$. It easily follows from (4.5) that the algebra $\mathfrak{M}_{\mathcal{B}}$ is complete and hence, is a Banach C^*-algebra. The Banach algebras $\mathfrak{M}_{\mathfrak{S}_1}$ and $\mathfrak{M}_{\mathfrak{S}_\infty}$ are introduced in the same way. It follows from the above reasoning that

$$\mathfrak{M} := \mathfrak{M}_{\mathcal{B}} = \mathfrak{M}_{\mathfrak{S}_1} = \mathfrak{M}_{\mathfrak{S}_\infty},$$

including equality of the corresponding norms.

The class $\mathfrak{M}$ depends on the choice of the spectral measures E, F. We shall use the detailed notations $\mathfrak{M}(E,F)$ when it is useful to reflect this dependence explicitly.

Now we show how to interpret the transformers $\mathcal{J}_\phi$ in $\mathcal{B}$ as multipliers. If $\mathbf{T} \in \mathfrak{S}_2$, we shall denote by $T = T(\lambda,\mu)$ its operator-valued kernel with respect to the pair of decompositions (2.4), (3.9) of the spaces $\mathfrak{H}$ and $\mathfrak{G}$. Denote by $\mathcal{K}(\mathfrak{S}_1)$ the linear space of kernels of all trace class operators. Being equipped by the trace norm of the corresponding operator, it becomes a Banach space. For any bounded operator $\mathbf{S}$, the linear functional $\varphi_{\mathbf{S}}(\mathbf{T}) = tr(\mathbf{TS}^*)$ can be considered as a linear functional on $\mathcal{K}(\mathfrak{S}_1)$. It is natural to call this functional the *(generalized) kernel* of the operator $\mathbf{S}$. This is in parallel with the generally accepted definition of a distribution.

Suppose now that $\phi \in \mathfrak{M}$ and $\mathbf{S} \in \mathcal{B}$. Then we interpret the generalized kernel of the operator $\mathcal{J}_\phi\mathbf{S}$ as the product $\phi(\lambda,\mu)S(\lambda,\mu)$. Of course, for each particular operator $\mathbf{S} \notin \mathfrak{S}_2$ this definition needs an accurate realization.

4.2. Criterion of $\phi \in \mathfrak{M}$

The above construction does not give any analytical description of the algebra $\mathfrak{M}$. Such description is given by the next result.

Theorem 4.1. *Let $\phi \in L^\infty(\Lambda, M)$. Then the following three statements are equivalent:*

(i) $\phi \in \mathfrak{M} = \mathfrak{M}(E,F)$.

(ii) *For any $\mathfrak{g} \in \mathfrak{G}$, $\mathfrak{h} \in \mathfrak{H}$ the integral operator*

$$\mathbf{K}_{\mathfrak{g},\mathfrak{h}} : L^2(M;\tau_{\mathfrak{g}}) \to L^2(\Lambda;\rho_{\mathfrak{h}}), \qquad (\mathbf{K}_{\mathfrak{g},\mathfrak{h}}u)(\lambda) = \int_M \phi(\lambda,\mu)u(\mu)d\tau_{\mathfrak{g}}(\mu)$$

belongs to $\mathfrak{S}_1$, and

$$\sup_{\|\mathfrak{g}\|=\|\mathfrak{h}\|=1} \|\mathbf{K}_{\mathfrak{g},\mathfrak{h}}\|_{\mathfrak{S}_1} =: C < \infty.$$

Moreover, $\|\phi\|_{\mathfrak{M}} = C$.

(iii) *The symbol ϕ admits the factorization*

$$\phi(\lambda,\mu) = \int_{\mathcal{T}} \alpha(\lambda,t)\beta(\mu,t)d\eta(t) \tag{4.6}$$

(where $(\mathcal{T},\eta)$ is an auxiliary measure space) such that

$$\begin{cases} A^2 := (E)\text{-}\sup\limits_{\lambda} \int_{\mathcal{T}} |\alpha(\lambda,t)|^2 d\eta(t) < \infty; \\ B^2 := (F)\text{-}\sup\limits_{\mu} \int_{\mathcal{T}} |\beta(\mu,t)|^2 d\eta(t) < \infty. \end{cases} \tag{4.7}$$

For any such factorization

$$\|\phi\|_{\mathfrak{M}} \le AB, \tag{4.8}$$

and there exists a factorization such that

$$cAB \le \|\phi\|_{\mathfrak{M}}, \ c > 0. \tag{4.9}$$

The constant c does not depend on spectral measures E, F.

The result is due the authors, [6], [9], except for the necessity for $\phi \in \mathfrak{M}$ of the condition *(iii)*. It was established by Peller, [30]. Note that the constant factor c in (4.9) can be estimated from below by an expression involving the Grothendick constant.

Outline the proof of sufficiency of *(iii)*. Suppose that ϕ admits the factorization (4.6) with the estimates (4.7). Consider the operator-valued functions

$$\mathbf{A}(t) = \int_\Lambda \alpha(\lambda, t) dE(\lambda), \qquad \mathbf{B}(t) = \int_M \beta(\lambda, t) dF(\mu).$$

The operators $\mathbf{A}(t),\ t \in \mathcal{T}$ mutually commute and the same is true for $\mathbf{B}(t)$. Formally, the factorization (4.6) leads to the equality

$$\begin{aligned}\mathbf{Q} = \mathcal{J}_\phi \mathbf{T} &= \int_\Lambda \int_M \int_{\mathcal{T}} \alpha(\lambda, t)\beta(\mu, t) d\eta(t)\, dE(\lambda)\mathbf{T} dF(\mu) \\ &= \int_{\mathcal{T}} \mathbf{A}(t)\mathbf{T}\mathbf{B}(t) d\eta(t).\end{aligned}$$

The expression in the right-hand side has no immediate meaning, since for any given $t \in \mathcal{T}$ the operators $\mathbf{A}(t),\ \mathbf{B}(t)$ can be unbounded. However, for each $\mathfrak{h} \in \mathfrak{H}$ and each $\mathfrak{g} \in \mathfrak{G}$ the vector-valued functions $\mathbf{A}(t)\mathfrak{h},\ \mathbf{B}(t)\mathfrak{g}$ are well defined for almost all $t \in \mathcal{T}$. This follows from the estimates

$$\begin{aligned}\int_{\mathcal{T}} \|\mathbf{A}(t)\mathfrak{h}\|^2 d\eta(t) = \int_{\mathcal{T}} \int_\Lambda |\alpha(\lambda, t)|^2 d(E(\lambda)\mathfrak{h}, \mathfrak{h}) d\eta(t) \le A^2 \|\mathfrak{h}\|^2, \\ \int_{\mathcal{T}} \|\mathbf{B}(t)\mathfrak{g}\|^2 d\eta(t) \le B^2 \|\mathfrak{g}\|^2.\end{aligned}$$

Now, consider the sesqui-linear form

$$\Omega(\mathfrak{g}, \mathfrak{h}) = \int_{\mathcal{T}} (\mathbf{T}\mathbf{B}(t)\mathfrak{g}, \mathbf{A}(t)^* \mathfrak{h}) dt, \qquad \mathfrak{h} \in \mathfrak{H},\ \mathfrak{g} \in \mathfrak{G}.$$

This form is bounded:

$$|\Omega(\mathfrak{g}, \mathfrak{h})| \le \|\mathbf{T}\| \int_{\mathcal{T}} \|\mathbf{A}(t)\mathfrak{h}\| \|\mathbf{B}(t)\mathfrak{g}\| d\eta(t) \le AB \|\mathbf{T}\| \|\mathfrak{g}\| \|\mathfrak{h}\|.$$

Let $\mathbf{Q} \in \mathcal{B}(\mathfrak{G}, GH)$ be the operator associated with the sesqui-linear form Ω, i.e. $\Omega(\mathfrak{g}, \mathfrak{h}) = (\mathbf{Q}\mathfrak{g}, \mathfrak{h})$ for any $\mathfrak{h} \in \mathfrak{H}$, $\mathfrak{g} \in \mathfrak{G}$. An elementary additional argument allows one to identify $\mathbf{Q}$ as $\mathcal{J}_\phi \mathbf{T}$, and (4.8) follows.

Note that the statement *(ii)* can be easily proved using the representation of $\mathcal{J}_\phi$ as a multiplier transform.

5. Transformers $\mathcal{J}_\phi$ on other classes

5.1.

Now we discuss the transformer $\mathcal{J}_\phi$ on the classes of operators, other than $\mathfrak{S}_2$, $\mathfrak{S}_1$ and $\mathcal{B}$. "Classes" here means symmetric ideals of compact operators, complete with respect to a certain norm or quasinorm. Respectively, they are called symmetrically normed, or quasinormed ideals. Each such ideal $\mathfrak{S}$ is described in terms of the behaviour of the singular numbers $s_n(\mathbf{T}) = \lambda_n(\sqrt{\mathbf{T}^*\mathbf{T}})$ of operators $\mathbf{T} \in \mathfrak{S}$. The main source in the theory of symmetric ideals is the book [21]. Many important results for the quasinormed case are due to Rotfeld, [33], [34]. See also [11], [13] for a short account on these results.

The most popular ideals are the Schatten classes

$$\mathfrak{S}_p = \{\mathbf{T} \in \mathfrak{S}_\infty : \{s_n(\mathbf{T})\} \in l_p\}, \qquad 0 < p < \infty. \tag{5.1}$$

For $1 \le p < \infty$, these are complete normed spaces, with respect to the norm

$$\|\mathbf{T}\|_{\mathfrak{S}_p} := \|\{s_n(\mathbf{T})\}\|_{l_p}.$$

Another important spaces are "weak $\mathfrak{S}_p$-ideals"

$$\mathfrak{S}_{p,w} = \{\mathbf{T} \in \mathfrak{S}_\infty : s_n(\mathbf{T}) = O(n^{-1/p})\}, \qquad 0 < p < \infty. \tag{5.2}$$

The natural functional $[\mathbf{T}]_p = \sup_n (n^{1/p} s_n(\mathbf{T}))$ generates the metric topology on $\mathfrak{S}_{p,w}$ but does not satisfy the triangle inequality. The spaces $\mathfrak{S}_{p,w}$ are complete in this topology. For $1 < p < \infty$, a norm in $\mathfrak{S}_{p,w}$ can be introduced, which is equivalent to $[\mathbf{T}]_p$. The expression for this norm is rather complicated and we do not present it here. The spaces $\mathfrak{S}_p,\ p < 1$ and $\mathfrak{S}_{p,w},\ p \le 1$ are not normalizable. The spaces $\mathfrak{S}_{p,w}$ (for all p) are non-separable; the closure in $\mathfrak{S}_{p,w}$ of the set $\mathfrak{K}$ is the separable ideal

$$\mathfrak{S}^\circ_{p,w} = \{\mathbf{T} \in \mathfrak{S}_\infty : s_n(\mathbf{T}) = o(n^{-1/p})\}. \tag{5.3}$$

We also mention the spaces

$$\mathfrak{S}_{p,1} = \{\mathbf{T} \in \mathfrak{S}_\infty : \sum_n n^{p^{-1}-1} s_n(\mathbf{T}) < \infty\}, \qquad 0 < p < \infty. \tag{5.4}$$

These are complete separable quasinormed spaces, with a quasinorm equivalent to the sum in (5.4). If $1 \le p < \infty$, this sum itself satisfies the triangle inequality and is standardly taken as the norm in $\mathfrak{S}_{p,1}$. Evidently, $\mathfrak{S}_{1,1} = \mathfrak{S}_1$.

For $1 < p < \infty$, there are duality relations (the duality form is given by (3.1))

$$\mathfrak{S}^*_p = \mathfrak{S}_{p'};\ (\mathfrak{S}^\circ_{p,w})^* = \mathfrak{S}_{p',1};\ \mathfrak{S}^*_{p,1} = \mathfrak{S}_{p',w}, \qquad 1/p' = 1 - 1/p. \tag{5.5}$$

For $p \ge 1$, the spaces $\mathfrak{S}_{p,w}$, $\mathfrak{S}^\circ_{p,w}$ and $\mathfrak{S}_{p,1}$ are special cases of the spaces $\mathfrak{S}_\Pi$, $\mathfrak{S}^\circ_\Pi$ and $\mathfrak{S}_\pi$, introduced in [21], sections III.14 and III.15. See this book for the additional information on these spaces, including the definition of the norm in $\mathfrak{S}_{p,w}$.

For brevity, we shall call the symmetrically normed ideals appearing in (5.5) "nice ideals". It is convenient to treat also the algebra $\mathcal{B}$ as a nice ideal.

There are many other useful symmetric ideals (both normed and quasi-normed) but we do not discuss them in this paper.

By interpolation, the inclusion $\phi \in \mathfrak{M}$ implies boundedness of $\mathcal{J}_\phi$ in any nice ideal $\mathfrak{S}$. However, it says nothing about properties of $\mathcal{J}_\phi$ as acting, say, on $\mathfrak{S}_p$ with $p < 1$, or on $\mathfrak{S}_{p,w}$ with $p \le 1$. Also, it may happen that $\phi \notin \mathfrak{M}$ but $\mathcal{J}_\phi$ can be well defined as a bounded transformer acting in some ideal $\mathfrak{S}$, for example in $\mathfrak{S}_p$ with some $p \in (1, \infty)$. To define $\mathcal{J}_\phi$ on such ideals, the extension by continuity from the set of all finite rank operators and the duality arguments are used.

Given a symmetrically normed ideal $\mathfrak{S}$, the set of symbols ϕ, such that the transformer $\mathcal{J}_\phi$ is bounded on $\mathfrak{S}$, form a commutative Banach algebra of functions on $\Lambda \times M$, with complex conjugation as the involution. We denote this algebra as $\mathfrak{M}_{\mathfrak{S}}$. It follows from the duality arguments and interpolation that for $1 < p < \infty$

$$\mathfrak{M}_{\mathfrak{S}_p} = \mathfrak{M}_{\mathfrak{S}_{p'}}; \qquad \mathfrak{M}_{\mathfrak{S}^\circ_{p,w}} = \mathfrak{M}_{\mathfrak{S}_{p',1}} = \mathfrak{M}_{\mathfrak{S}_{p,w}} \tag{5.6}$$

and for all nice ideals the following topological embeddings hold:

$$\mathfrak{M} \subset \mathfrak{M}_{\mathfrak{S}} \subset \mathfrak{M}_{\mathfrak{S}_2} = L^\infty(\Lambda \times M). \tag{5.7}$$

These algebras depend on the spectral measures E, F but this is not reflected in the notations. For any symmetrically quasi-normed ideal $\mathfrak{S}$, the similarly defined set $\mathfrak{M}_{\mathfrak{S}}$ is a commutative quasi-Banach algebra.

No exhaustive description of the algebras $\mathfrak{M}_{\mathfrak{S}}$ in the analytic terms is known so far, except for the cases $\mathfrak{S} = \mathfrak{S}_2$ (see section 3) and $\mathfrak{S} = \mathcal{B}, \mathfrak{S}_1, \mathfrak{S}_\infty$ (see Theorem 4.1). A part of this Theorem, except for *(iii)*, extends to the ideals $\mathfrak{S}_p$, $\mathfrak{S}_{p,w}$ with $p < 1$, see [34].

5.2. Analytic tests for $\phi \in \mathfrak{M}$ and $\phi \in \mathfrak{M}_{\mathfrak{S}}$

Such tests can be obtained based upon Theorem 4.1 and interpolation between the results for $\mathfrak{S} = \mathfrak{S}_2$ and $\mathfrak{S} = \mathfrak{S}_1$. We do not know how to interpolate between L^∞ and the class described by the factorization (4.6). However, the specific tests for $\phi \in \mathfrak{M}$ admit such interpolation. The following quite useful remark is implied by the first inclusion in (5.7).

Remark 5.1. If $\phi \in \mathfrak{M}$, then $\phi \in \mathfrak{M}_{\mathfrak{S}}$ for any nice ideal $\mathfrak{S}$, and

$$\|\phi\|_{\mathfrak{M}_{\mathfrak{S}}} \le \|\phi\|_{\mathfrak{M}}.$$

Below we present some results without proof. It is always supposed that the spectral measure E is Borelian. The spectral measure (M, F) can be arbitrary.

Theorem 5.2. *Let $\Lambda = \mathbb{R}^d$ and suppose that for some numbers $m_1 \ge 0$ and m_2 such that $m_1 < d < m_2$, we have*

$$(\tau)\text{-}\sup_{\mu} \int_{\mathbb{R}^d} (|\xi|^{m_1} + |\xi|^{m_2}) |\widehat{\phi}(\xi, \mu)|^2 d\xi = K^2 < \infty \tag{5.8}$$

where $\widehat{\phi}(\xi,\mu)$ stands for the Fourier transform of ϕ with respect to the variable λ. Then $\phi \in \mathfrak{M}$ and $\|\phi\|_{\mathfrak{M}} \le CK$ where the constant $C = C(d, m_1, m_2)$ does not depend on E, F.

The roles of E, F can be inverted.

If we take here $m_1 = 0,\ m_2 =: 2m > d$, this estimate turns into

$$\|\phi\|_{\mathfrak{M}} \le C(d,m)\,(\tau)\text{-}\sup_{\mu} \|\phi(\cdot,\mu)\|_{H^m(\mathbb{R}^d)}, \qquad 2m > d. \tag{5.9}$$

The general result of Theorem 5.2 is more flexible than the estimate (5.9) since it does not require $\phi(\cdot,\mu) \in L^2(\mathbb{R}^d)$. So, the decay of ϕ as $|\lambda| \to \infty$ can be slower.

Theorem 5.2 applies to any Borelian spectral measure E on $\mathbb{R}^d$ and the constant C in the estimate does not depend on E. Usually this is convenient, but on the other hand this can be considered as a weak point: indeed, dependence on the properties of a given spectral measure would be important in many questions.

In part, this defect is corrected in the next result which can be easily derived from Theorem 5.2. This result reflects dependence of the estimate on the (closed) support of E.

Theorem 5.3. *Let Λ be a domain in $\mathbb{R}^d$ with uniformly Lipschitz boundary, or a compact d-dimensional smooth Riemannian manifold. Then*

$$\|\phi\|_{\mathfrak{M}} \le C(\Lambda, m)\,(\tau)\text{-}\sup_{\mu} \|\phi(\cdot,\mu)\|_{H^m(\Lambda)}, \qquad 2m > d. \tag{5.10}$$

We might suppose Λ to be a non-compact manifold, but then some additional assumptions on the geometry of Λ at infinity would be necessary.

According to Remark 5.1, all the conditions presented automatically guarantee boundedness of $\mathcal{I}_\phi$ in any nice ideal. The next result gives a test for the boundedness of $\mathcal{I}_\phi$ only in some of such ideals. The symbol $\mathsf{W}_p^\alpha(\Lambda)$ means the Sobolev space, possibly of the fractional order α of smoothness.

Theorem 5.4. *Let Λ be as in Theorem 5.3, or $\Lambda = \mathbb{R}^d$. Suppose that*

$$N_{p,\alpha}(\phi) := (M)\text{-}\sup_{\mu} \|\phi(\cdot,\mu)\|_{\mathsf{W}_p^\alpha(\Lambda)} < \infty$$

for some $p > 2$ and α such that $p\alpha > d$, then $\phi \in \mathfrak{M}_{\mathfrak{S}}$ for $\mathfrak{S} = \mathfrak{S}_r$ and $\mathfrak{S} = \mathfrak{S}_{r,w}$ with $|r^{-1} - 1/2| < \alpha/d$. The estimates

$$\|\phi\|_{\mathfrak{S}_r} \le C(r,p,\alpha) N_{p,\alpha}, \qquad \|\phi\|_{\mathfrak{S}_{r,w}} \le C'(r,p,\alpha) N_{p,\alpha}$$

are satisfied.

6. Pseudodifferential operators as double operator integrals

6.1. General pseudodifferential operators

Let $\mathfrak{G} = \mathfrak{H} = L^2(\mathbb{R}^d)$, $\Lambda = \mathbb{R}^d$, and E be the joint spectral measure of the family of operators of multiplication by $x_1, \dots x_d$. Let F be the similar measure in Fourier representation. In order to distinguish in notations between the spectral measures

E and F, we denote by Ξ^d the d-dimensional Euclidean space in the variable ξ. Denoting the Fourier transform by Φ, we have

$$E(\delta) : u(x) \mapsto \chi_\delta(x)u(x), \qquad F(\partial) : u(x) \mapsto \Phi^*_{\xi\to x}(\chi_\partial(\xi)\hat{u}(\xi)). \tag{6.1}$$

It is natural to take the Lebesgue measure as both ρ and τ. Then the representation of $\mathfrak{H}$ as the direct integral with respect to the spectral measure E is trivial. It is realized by the equality $\mathfrak{H} = L^2(\mathbb{R}^d)$; for any $x \in \mathbb{R}^d$ we have $H(x) = \mathbb{C}^1$, and for each $u \in \mathfrak{H}$ its representative is the function $u(x)$ itself. For the spectral measure F, the picture is similar but the representative of a function $u \in \mathfrak{H}$ is its Fourier transform $\hat{u}(\xi)$. This can be written as $\mathfrak{H} = \Phi^* L^2(\Xi^d)$.

Let now a function $\phi(x, \xi)$ on $\mathbb{R}^d \times \Xi^d$ be given. Suppose that $\phi \in \mathfrak{M}(E, F)$, then the operator $\mathcal{J}_\phi\mathbf{T}$ is well defined as a bounded operator in $\mathfrak{H} = L^2(\mathbb{R}^d)$, as soon as $\mathbf{T}$ is bounded.

Let in particular $\mathbf{T} = \mathbf{I}$, the identity operator in $L^2(\mathbb{R}^d)$. With respect to these two decompositions of $\mathfrak{H}$, the operator $\mathbf{I}$ is given by the inverse Fourier transform: $u = \Phi^*\hat{u}$. This means that the generalized kernel of $\mathbf{I}$ is $(2\pi)^{-d/2}e^{ix\xi}$. According to the general scheme of operator integrals as multipliers, the operator $\mathbf{Q} := \mathcal{J}_\phi\mathbf{I}$ is defined by the kernel $Q(x, \xi) = (2\pi)^{-d/2}\phi(x, \xi)e^{ix\xi}$. In other words,

$$\mathbf{Q}u = \Phi^*_{\xi\to x}\big(\phi(x, \xi)\hat{u}(\xi)\big) = Op(\phi), \tag{6.2}$$

where $Op(\phi)$ stands for the pseudodifferential operators with the symbol ϕ. So, we did realize the general pseudodifferential operator as a double operator integral, applied to the operator $\mathbf{T} = \mathbf{I}$.

Now, Theorem 5.2 and its particular case, the inequality (5.9) give convenient tests for boundedness of a pseudodifferential operator in $L^2(\mathbb{R}^d)$. In particular, we get for $2m > d$:

$$\|Op(\phi)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le C\, ess\, sup_\xi \|\phi(\cdot, \xi)\|_{H^m(\mathbb{R}^d)}; \tag{6.3}$$

$$\|Op(\phi)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le C\, ess\, sup_x \|\phi(x, \cdot)\|_{H^m(\Xi^d)}, \tag{6.4}$$

$$C = C(d, m).$$

This shows that for the boundedness one needs only finite order smoothness of the symbol. Moreover, the smoothness conditions can be formulated alternatively either in terms of x, or in terms of ξ . The latter possibility is of a special interest, since often the symbols of pseudodifferential operators are smooth in ξ.

6.2. Pseudodifferential operators with homogeneous symbols of zero order

Such symbols are non-smooth at $\xi = 0$. However, in this case the above scheme can be refined in such a way that only the smoothness of ϕ on the sphere $|\xi| = 1$ is involved.

Let Θ^{d-1} stand for the unit sphere in Ξ^d. For $\xi \ne 0$, denote $\theta(\xi) = \xi/|\xi| \in \Theta^{d-1}$. Consider a new projection-valued function F_0 on Θ^{d-1}. Namely, let F be the spectral measure on Ξ^d, introduced in (6.1). Given a measurable subset $\partial \subset \Theta^{d-1}$, we denote by $cone(\partial)$ the conic subset in Ξ^d, such that $cone(\partial)\cap\Theta^{d-1}=\partial$. Then we define

$$F_0(\partial)=F(cone(\partial)).$$

It is clear that F_0 is a spectral measure in $L^2(\mathbb{R}^d)$.

Let now the symbol ϕ be homogeneous in ξ of zero order, i.e.

$$\phi(x,\xi)=\sigma(x,\theta(\xi)). \tag{6.5}$$

Then the transformer $\mathcal{J}_\phi$ can be written in a more convenient way:

$$\mathcal{J}_\phi\mathbf{T}=\int_{\mathbb{R}^d\times\Xi^d}\phi(x,\xi)dE(x)\mathbf{T}dF(\xi)=\int_{\mathbb{R}^d\times\Theta^{d-1}}\sigma(x,\theta)dE(x)\mathbf{T}dF_0(\theta).$$

In particular,

$$Op(\phi)=\mathcal{J}_\phi\mathbf{I}=\int_{\mathbb{R}^d\times\Theta^{d-1}}\sigma(x,\theta)dE(x)\,\mathbf{I}\,dF_0(\theta).$$

This representation allows us to apply the result of Theorem 5.3 with $M=\Theta^{d-1}$ (and the roles of Λ, M interchanged). We come to the following result: let ϕ be given by (6.5) and $2m>d-1$, then

$$\|Op(\phi)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}\le C(d,m)ess\,sup_x\|\sigma(x,\cdot)\|_{H^m(\Theta^{d-1})}. \tag{6.6}$$

For the class of pseudodifferential operators considered, this relaxes the smoothness condition on the symbol, cf. the estimate (6.4). Its counterpart, the estimate (6.3), can not be improved in a similar way.

6.3. Pseudodifferential operators of negative order

Let us consider the operators

$$(\mathbf{Q}u)(x)=(2\pi)^{-d/2}b(x)\int_{\mathbb{R}^d}\zeta(\xi)\phi(x,\xi)e^{ix\cdot\xi}\widehat{(cu)}(\xi)d\xi. \tag{6.7}$$

Here $\phi(x,\xi)=|\xi|^{-\kappa}\sigma(x,\theta_\xi)$ is a homogeneous in ξ function of the order $-\kappa<0$, $\zeta(\xi)$ is a smooth cut-off function which is 0 at $\xi=0$ and is 1 for large enough $|\xi|$; we need not this regularizing factor if $\kappa<d$. In the coordinate representation, $\mathbf{Q}$ is typically an integral operator with the "weakly polar" kernel K, $K(x,y)=O(|x-y|^{\kappa-d})$ as $|x-y|\to 0$. If the weight functions b,c decay fast enough at infinity, then such operator is compact in $L^2(\mathbb{R}^d)$ and its singular numbers are of the order $s_n(\mathbf{Q})=O(n^{-\kappa/d})$. Our goal is to make this argument rigorous and obtain for $s_n(\mathbf{Q})$ the qualified estimates.

The main idea (for $\kappa<d$): let first $\phi(x,\xi)=|\xi|^{-\kappa}$, and let $\mathbf{Q}_0$ be the corresponding operator (6.7), with $\zeta\equiv 1$. The operator $\mathbf{Q}_0$ can be re-written in the "coordinate representation":

$$(\mathbf{Q}u)(x)=C(d,\kappa)b(x)\int_{\mathbb{R}^d}\frac{c(y)u(y)}{|x-y|^{d-\kappa}}dx.$$

To this operator Theorem 10.3 from [11] applies. For the "symmetric case" $b=c$ it gives

$$s_n(\mathbf{Q}_0)\le C\|b\|^2_{\mathsf{L}^{2d/\kappa}}n^{-\kappa/d}.$$

It can be re-written as $[\mathbf{Q}_0]_{d/\kappa,w} \le \big(C\|b\|_{\mathsf{L}^{2d/\kappa}}\big)^{\kappa/d}$. It remains to insert the additional multiplier $\sigma(x,\theta_\xi)$ which is homogeneous of order 0 and therefore, can be considered as a function on the unit sphere Θ^{d-1}. This allows us to apply Theorem 5.4. Here we need estimates in the class $\mathfrak{S}_{d/\kappa,w}$.

Below we present a particular case of the main result:

$$ns_n^{d/\kappa}(\mathbf{Q}) \le C\|b\|^2_{L^{2d/\kappa}}\|\sigma\|_{\mathfrak{M}_{\mathfrak{S}_{d/\kappa,w}}}.$$

A certain specific condition on σ can be derived from here with the help of Theorem 5.4. It allows symbols σ of a rather low smoothness. The complete symbol can be not smooth at all, due to the presence of the weight function $b(x)$, with no smoothness conditions imposed on it. The result applies, in particular, to the case when $b = \chi_\Omega$ where Ω is a bounded domain in $\mathbb{R}^d$. This corresponds to the case of operators acting in $\mathsf{L}^2(\Omega)$.

Similar estimates, with some changes in the formulation, apply also to $\kappa \ge d$. Here we have to use results for the non-normalizable classes $\mathfrak{M}_{\mathfrak{S}_{p,w}}$ with $p \le 1$.

These estimates were used in [12] for calculating spectral asymptotics for the operators of the type discussed, and also for more general class of operators with anisotropically-homogeneous symbols.

7. Integrals with discontinuous symbols. The triangle transformer

7.1. Discontinuities on the diagonal

Starting with this section, we often deal with the transformers $\mathcal{J}_\phi$ whose symbols are discontinuous. This requires some precautions even in the framework of $\mathfrak{S}_2$-theory, since the symbol ϕ has to be well defined $\mathcal{G}$-everywhere (cf. section 3), including the set of its discontinuity points. In this subsection we discuss this question for the most important case when $\Lambda = M = \mathbb{R}$, both spectral measures E, F are Borelian, and the set of discontinuities is a subset of the diagonal

$$\boldsymbol{diag} = \{(\lambda,\lambda) : \lambda \in \mathbb{R}\} \subset \mathbb{R}^2. \tag{7.1}$$

First of all, consider the restriction of the spectral measure $\mathcal{G}$ to the diagonal. Given a set $\varsigma \subset \mathbb{R}$, let ς° be its natural image on the diagonal,

$$\varsigma^\circ = \{(\lambda,\lambda) : \lambda \in \varsigma\} \subset \boldsymbol{diag}. \tag{7.2}$$

Let $\mathcal{E},\ \mathcal{F}$ be the spectral measures in $\mathfrak{S}_2$, defined in (3.2). It is not difficult to show that

$$\mathcal{G}(\varsigma^\circ) = \sum_{\lambda\in\varsigma}\mathcal{E}(\{\lambda\})\mathcal{F}(\{\lambda\}). \tag{7.3}$$

Due to the separability of Hilbert spaces $\mathfrak{G}$ and $\mathfrak{H}$, the number of non-zero terms in the right-hand side of (7.3) is no more than countable. If $\mathbf{A}$ and $\mathbf{B}$ are self-adjoint operators and $E = E^{\mathbf{A}},\ F = F^{\mathbf{B}}$, then the points $\{\lambda \in \mathbb{R} : \mathcal{G}(\{\lambda,\lambda\}) \ne 0\}$ are nothing but the common eigenvalues of $\mathbf{A}$ and $\mathbf{B}$.

We see that $\mathcal{G}\big|_{\boldsymbol{diag}}$ is an atomic measure. According to the construction in section 3, the measure $\mathcal{G}$ is N-full. Therefore, *any function on* $\boldsymbol{diag}$ *is* $\mathcal{G}$*-measurable.*

Let $\mathsf{M} = \mathsf{M}(\mathbb{R})$ stand for the Banach space of all bounded functions on $\mathbb{R}$, with the standard norm $\|\omega\|_{\mathsf{M}} = \sup_{\lambda\in\mathbb{R}} |\omega(\lambda)|$. With any function $\omega \in \mathsf{M}$ we associate a function $\omega^\circ := \omega(\lambda)\delta_{\lambda,\mu}$ on $\mathbb{R}^2$. In other words,

$$\omega^\circ(\lambda, \lambda) = \omega(\lambda), \qquad \omega^\circ(\lambda, \mu) = 0,\ \mu \neq \lambda. \tag{7.4}$$

We take by definition, at first for $\mathbf{T} \in \mathfrak{S}_2$:

$$\mathbf{Q}_{\omega,\boldsymbol{diag}} = \mathcal{J}_{\omega,\boldsymbol{diag}}\mathbf{T} = \int_{\boldsymbol{diag}} \omega d(\mathcal{G}\mathbf{T}) := \int_{\mathbb{R}^2} \omega^\circ(\lambda, \mu) d(\mathcal{G}(\lambda, \mu)\mathbf{T}).$$

In order to simplify our further notations, denote $E_\lambda := E(\{\lambda\}),\ F_\lambda := F(\{\lambda\})$. It immediately follows from the definition of $\mathbf{Q}_{\omega,\boldsymbol{diag}}$ and the equality (7.3) that

$$\mathbf{Q}_{\omega,\boldsymbol{diag}} = \sum_{\lambda\in\mathbb{R}} \omega(\lambda) E_\lambda \mathbf{T} F_\lambda. \tag{7.5}$$

Let now $\mathbf{T} \in \mathcal{B}$. Estimating the quadratic form of the operator in the right-hand side of (7.5), we find for $\mathfrak{g} \in \mathfrak{G}, \mathfrak{h} \in \mathfrak{H}$:

$$|(\mathbf{Q}_{\omega,\boldsymbol{diag}}\mathfrak{g}, \mathfrak{h})| \le \|\omega\|_{\mathsf{M}} \|\mathbf{T}\| \sum_{\lambda\in\mathbb{R}} \|F_\lambda \mathfrak{g}\| \|E_\lambda \mathfrak{h}\| \le \|\omega\|_{\mathsf{M}} \|\mathbf{T}\| \|\mathfrak{g}\| \|\mathfrak{h}\|$$

which means that

$$\|\mathbf{Q}_{\omega,\boldsymbol{diag}}\| \le \|\omega\|_{\mathsf{M}} \|\mathbf{T}\|.$$

Using duality and interpolation, we derive from here that

$$\|\mathbf{Q}_{\omega,\boldsymbol{diag}}\|_{\mathfrak{S}} \le \|\omega\|_{\mathsf{M}} \|\mathbf{T}\|_{\mathfrak{S}}$$

for any nice symmetrically normed ideal $\mathfrak{S}$. As we know, for $\mathfrak{S} \neq \mathfrak{S}_2$ this estimate has no analogs for integrals over the whole of $\mathbb{R}^2$. The estimates obtained show that the equality (7.5) can be taken as the definition of the transformer $\mathcal{J}_{\omega,\boldsymbol{diag}}$ on the class $\mathcal{B}$ and on any nice ideal $\mathfrak{S}$. It is clear that this definition is compatible with the general definition (4.3), for the symbol $\phi = \omega^\circ$. The above inequality for $\|\mathbf{Q}_{\omega,\boldsymbol{diag}}\|$ shows that for arbitrary spectral measures E, F

$$\omega \in \mathsf{M} \Longrightarrow \{\omega^\circ \in \mathfrak{M} = \mathfrak{M}(E, F),\ \|\omega^\circ\|_{\mathfrak{M}} \le \|\omega\|_{\mathsf{M}}\}. \tag{7.6}$$

According to Theorem 4.1, any function $\phi \in \mathfrak{M}$ admits a factorization of the type (4.6) – (4.7). It is not difficult to present such a factorization for $\phi = \omega^\circ$: we take $\mathcal{T} = \mathbb{R},\ \eta = \delta$ (δ-measure concentrated at 0), $\alpha(\lambda, t) = \omega^\circ(\lambda, \lambda - t)$, and $\beta(\mu, t) = \delta_{\mu,t}$. The estimates (4.7) hold with $A = \|\omega\|_{\mathsf{M}},\ B = 1$.

The next statement is implied by the equality (7.5).

Proposition 7.1. *Let* $\mathbf{T} \in \mathcal{B}$. *The condition*

$$E_\lambda \mathbf{T} F_\lambda = 0, \qquad \forall \lambda \in \mathbb{R} \tag{7.7}$$

is necessary and sufficient in order to have

$$\int_{\boldsymbol{diag}} \omega d(\mathcal{G}\mathbf{T}) = 0 \tag{7.8}$$

for any bounded function ω.

Proposition 7.1, though quite elementary, is useful when one considers integrals (1.1) with the symbols ϕ discontinuous on $\boldsymbol{diag}$. Namely, the condition (7.7) guarantees independence of the resulting operator $\mathbf{Q} = \mathcal{J}_\phi^{E,F}\mathbf{T}$ on the way ϕ is defined on $\boldsymbol{diag}$. If the condition (7.7) is violated, then the operator $\mathbf{Q}$ is not well defined, unless we somehow extend ϕ to the set $\{\{\lambda,\lambda\} \in \boldsymbol{diag} : \mathcal{G}(\{\lambda,\lambda\}) \neq 0\}$. Still, the property of ϕ to be $\mathcal{G}$-measurable does not depend on the mode of such extension.

7.2. Symbols with the derivative of bounded total variation

The simplest discontinuous symbol is $\phi(\lambda,\mu) = \theta(\lambda-\mu)$, where

$$\theta(t) = 0 \ (t \leq 0); \qquad \theta(t) = 1 \ (t > 0). \tag{7.9}$$

So, θ is the left continuous realization of Heaviside function. We call the corresponding transformer $\mathcal{J}_\theta^{E,F}$ *the triangle transformer.* This term is generally accepted in the case $F = E$, when the transformer $\mathcal{J}_\theta^E$ plays the central role in the theory of Volterra operators, see [22]. We preserve the same term in the general case, though for $F \neq E$ the transformer $\mathcal{J}_\theta^{E,F}$ is no more related to the Volterra operators.

Since the symbol $\theta(\lambda-\mu)$ is defined everywhere on $\mathbb{R}\times\mathbb{R}$ and takes two values 0 and 1, the transformer $\mathcal{J}_\theta^{E,F}$ defines an orthogonal projector in the Hilbert space $\mathfrak{S}_2$. Its properties in other symmetric ideals, including $\mathfrak{S}_p$ with $p \neq 2$, look problematic, since all the tests presented in Section 5 require continuity of the symbol at least in one variable. However, this obstacle can be overcome, and the following result takes place.

Theorem 7.2. *Let E, F be arbitrary Borelian spectral measures on $\mathbb{R}$, possibly in two different Hilbert spaces. The triangle transformer $\mathcal{J}_\theta^{E,F}$, originally defined on the class $\mathfrak{K}$ of all finite rank operators, extends to all the spaces $\mathfrak{S}_p$, $1 < p < \infty$ as a bounded operator, and*

$$\|\mathcal{J}_\theta^{E,F}\|_{\mathfrak{S}_p\to\mathfrak{S}_p} \leq C(p)$$

where $C(p) = C(p') \to \infty$ as $p \to 1, \infty$. The result remains valid for the transformer $\mathcal{J}_{\widetilde{\theta}}^{E,F}$ where $\widetilde{\theta}(t) = \theta(t+)$.

Note that the function $\widetilde{\theta}$ is nothing but the right continuous realization of Heaviside function.

For the case $F = E$ this result is due to Gohberg and M.G.Krein [22], section III.6. Their proof uses an operator identity for Volterra operators. This identity allows to derive the result for a given $p \geq 4$ from the one for $p/2$. Since the result for $p = 2$ (with $C(2) = 1$) is known, this gives a basis for induction. For $p = 2^r$, $r \in \mathbb{N}$

this proof gives $C(2^r) = \cot(2^{-r-1}\pi)$. As it was shown by Gohberg and Krupnik [23], this value is sharp.

The result for other $p > 2$ then follows by interpolation and it extends to $p \in (1,2)$ by duality. For $p > 2, p \neq 2^r$ the sharp value of the constant $C(p)$ is unknown so far. For the classes $\mathcal{B}$, $\mathfrak{S}_1$ and $\mathfrak{S}_\infty$ the result fails which is clear from the behaviour of $C(p)$.

The consequence of Gohberg - M.G. Krein's result for Volterra operators claims that if such an operator $\mathbf{Q}$ has the imaginary part from $\mathfrak{S}_p$, $1 < p < \infty$, then also $\mathbf{Q} \in \mathfrak{S}_p$. This result is known as Matsaev's Theorem [27], see also [22], Theorem III.6.2. Matsaev proved it (before Gohberg and M.G. Krein and for $p < 2$ only) with the help of the theory of entire functions.

The original argument used by Gohberg and M.G.Krein does not extend to the transformer $\mathcal{J}_\theta^{E,F}$ with $F \neq E$, since the theory of Volterra operators does not apply. The proof of Theorem 7.2 makes use of the identity

$$\mathbf{Q}^*\mathbf{Q} = \mathcal{J}_\theta^E\big(\mathbf{Q}^*\mathbf{T} - \mathbf{T}^*\mathbf{Q}\big) + \mathbf{T}^*\mathbf{Q}, \qquad \mathbf{Q} = \mathcal{J}_\theta^{E,F} \tag{7.10}$$

which turns out to be a consequence of the functional equation

$$\theta(u+v)\big(\theta(u) + \theta(v) - 1\big) = \theta(u)\theta(v), \qquad \forall u, v \in \mathbb{R}.$$

The identity (7.10) is an analog of the relation for Volterra operators which Gohberg and M.G.Krein used in their proof. As soon as (7.10) is established, the rest part of Gohberg - M.G.Krein's reasoning goes through and leads to Theorem 7.2 in its full generality. It gives the same values of the constants $C(p)$.

The following statement, which also was proved in [37], is a consequence of Theorem 7.2. Below V stands for the space of all functions of bounded total variation on $\mathcal{R}$. Further, we let

$$\mathsf{V}_l\,(\mathsf{V}_r) = \{f \in \mathsf{V} : f \text{ is left (right) continuous}\}.$$

Theorem 7.3. *Let E, F be arbitrary Borelian spectral measures on $\mathbb{R}$ and $\phi(\lambda,\mu)$ be a Borelian function on $\mathbb{R}^2$, such that for (F)-almost all $\mu \in \mathbb{R}$ the function $\phi(\cdot,\mu)$ lies in V_l. Suppose also that the norms $\|\phi(\cdot,\mu)\|_{\mathsf{V}}$ are (F)-essentially bounded. Then $\phi \in \mathfrak{M}_{\mathfrak{S}_p}$ for any $1 < p < \infty$, and*

$$\|\phi\|_{\mathfrak{M}_{\mathfrak{S}_p}} \le 2C(p)\,(F)\text{-}\sup_\mu \|\phi(\cdot,\mu)\|_{\mathsf{V}} \tag{7.11}$$

where $C(p)$ is the same constant as in Theorem 7.2. The result remains valid if the condition $\phi(\cdot,\mu) \in \mathsf{V}_l$ is replaced by $\phi(\cdot,\mu) \in \mathsf{V}_r$.

By interpolation, the similar statement is valid in the classes $\mathfrak{M}_{\mathfrak{S}_{p,w}}$.

8. Applications to Operator Theory

8.1. Transformers Z_h

Let $\mathbf{A}, \mathbf{B}$ be two self-adjoint operators acting in the Hilbert spaces $\mathfrak{G}, \mathfrak{H}$ respectively. Let $h(\lambda)$ be a uniformly Lipschitz function on $\mathbb{R}$, then the function

$$\phi_h(\lambda, \mu) = \frac{h(\mu) - h(\lambda)}{\mu - \lambda}$$

is well defined (and continuous) outside the diagonal and bounded. Suppose that it is somehow extended to ***diag*** and the extended function is bounded on $\mathbb{R}^2$. Note that this function is always $\mathcal{G}$-measurable, cf. the end of section 7.1. If at some point $\lambda \in \mathbb{R}$ the function h is differentiable, the natural choice of extension is $\phi_h(\lambda, \lambda) = h'(\lambda)$. Otherwise, the value of $\phi_h(\lambda, \lambda)$ can be chosen arbitrarily. This choice is indifferent if the operators $\mathbf{A}, \mathbf{B}$ have no common eigenvalues, since in this case $\mathcal{G}(\boldsymbol{diag}) = 0$.

Below we suppose that some extension of ϕ_h to the whole of $\mathbb{R}^2$ is chosen and fixed. Then the transformer

$$Z_h^{\mathbf{A},\mathbf{B}} := \mathcal{J}_{\phi_h}^{\mathbf{A},\mathbf{B}} = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{h(\mu) - h(\lambda)}{\mu - \lambda} dE^{\mathbf{A}}(\lambda)(\cdot) dF^{\mathbf{B}}(\mu) \tag{8.1}$$

is well defined, at least on the class $\mathfrak{S}_2$. This transformer naturally arises in many problems of Perturbation Theory; this was already discussed in the section 1.3. Now we return to this material and treat it in a more systematic way. We shall write $Z_h^{\mathbf{A}}$ instead of $Z_h^{\mathbf{A},\mathbf{A}}$. We do not reflect the choice of extension in the notations, since the formulas presented in Theorems 8.1, 8.2 hold true independently of it. Moreover, for any symmetrically normed ideal $\mathfrak{S}$ the membership $\phi_h \in \mathfrak{M}_{\mathfrak{S}}$ does not depend on this choice. This follows from the material of section 7.1.

Theorem 8.1. *Let $\mathfrak{G} = \mathfrak{H}$ and $\mathbf{A}, \mathbf{B}$ be self-adjoint operators in $\mathfrak{H}$ with the same domain, and suppose that $\mathbf{B} - \mathbf{A} \in \mathfrak{S}$ where $\mathfrak{S}$ is a nice ideal. Suppose also that the function $h(\lambda)$ is such that $\phi_h \in \mathfrak{M}_{\mathfrak{S}}$. Then, independently on the way ϕ_h is defined on the diagonal,*

$$h(\mathbf{B}) - h(\mathbf{A}) = Z_h^{\mathbf{A},\mathbf{B}}(\mathbf{B} - \mathbf{A}). \tag{8.2}$$

The result is not very difficult but remarkable for several reasons.

1) It allows the operators $\mathbf{A}, \mathbf{B}$ to be unbounded.

2) The assumption on h is formulated in general terms not involving analytic properties of the function. This is important, since the necessary and sufficient conditions on h ensuring $\phi_h \in \mathfrak{M}_{\mathfrak{S}}$ are unknown so far, with the only exclusion $\mathfrak{S} = \mathfrak{S}_2$ when such condition is $h \in Lip1$.

Theorem 8.1 extends to the quasi-commutators $\mathbf{JB} - \mathbf{AJ}$. Here $\mathbf{J}$ is a linear bounded operator acting from $\mathfrak{G}$ to $\mathfrak{H}$. The operators $\mathbf{A}, \mathbf{B}$ are not supposed bounded, and $\mathbf{JB} - \mathbf{AJ}$ is understood as the operator generated by the sesqui-linear form $(\mathbf{JB}\mathfrak{g}, \mathfrak{h}) - (\mathbf{J}\mathfrak{g}, \mathbf{A}\mathfrak{h})$ where $\mathfrak{h} \in Dom\mathbf{A}$, $\mathfrak{g} \in Dom\mathbf{B}$.

Theorem 8.2. *Let* $\mathbf{A}, \mathbf{B}$ *be self-adjoint operators in the spaces* $\mathfrak{G}$ *and* $\mathfrak{H}$ *respectively and let* $\mathbf{J} \in \mathcal{B}(\mathfrak{G}, \mathfrak{H})$. *Suppose that* $\mathbf{JB} - \mathbf{AJ} \in \mathfrak{S}$ *where* $\mathfrak{S}$ *is a nice ideal, and that* $\phi_h \in \mathfrak{M}_{\mathfrak{S}}$. *Then, independently on the way* ϕ_h *is defined on the diagonal,*

$$\mathbf{J}h(\mathbf{B}) - h(\mathbf{A})\mathbf{J} = Z_h^{\mathbf{A},\mathbf{B}}(\mathbf{JB} - \mathbf{AJ}).$$

Theorem 8.2 turns into Theorem 8.1 if we take $\mathfrak{G} = \mathfrak{H}$ and $\mathbf{J} = \mathbf{I}$. Both theorems were proved in [9] which contains also other results of the similar nature.

Proof of Theorem 8.2 for the particular case $\mathbf{A}, \mathbf{B} \in \mathcal{B}$, $\mathfrak{S} = \mathcal{B}$. We use the equalities $\mathbf{AJ} = \mathcal{J}_\lambda \mathbf{J}$, $\mathbf{JB} = \mathcal{J}_\mu \mathbf{J}$. Using the properties of the algebra $\mathfrak{M}$, we find

$$\mathcal{J}_{\phi_h}(\mathbf{JB} - \mathbf{AJ}) = \mathcal{J}_{\phi_h}(\mathcal{J}_\mu \mathbf{J} - \mathcal{J}_\lambda \mathbf{J})$$
$$= \mathcal{J}_{\phi_h(\mu-\lambda)}\mathbf{J} = \mathcal{J}_{h(\mu)-h(\lambda)}\mathbf{J} = \mathbf{J}h(\mathbf{B}) - h(\mathbf{A})\mathbf{J}. \qquad Q.E.D.$$

The proof for the general case is a bit more complex. The above argument does not go through even if $\mathbf{A}, \mathbf{B} \in \mathcal{B}$, $\mathbf{JB} - \mathbf{JA} \in \mathfrak{S}_p$ for some $p \in (1, \infty)$ and $\phi_h \in \mathfrak{M}_{\mathfrak{S}_p}$ but $\phi_h \notin \mathfrak{M}$. To obtain the desired result, we construct a special family of finite rank operators $\mathbf{J}_k$ strongly converging to $\mathbf{J}$. Then we apply Theorem 8.2 to $\mathbf{J}_k$ and pass to the limit as $k \to \infty$. The realization of this scheme requires a careful construction of $\mathbf{J}_k$.

8.2. Tests for $\phi_h \in \mathfrak{M}_{\mathfrak{S}}$

For practical usage of Theorems 8.1, 8.2 one needs tools for checking the inclusion $\phi_h \in \mathfrak{M}_{\mathfrak{S}}$ for a given ideal $\mathfrak{S}$. Applying to ϕ_h general results is not very productive since the tests obtained turn out to be too rough. Indeed, smoothness of ϕ_h is more or less the same as that of h'. However, this naive argument does not take into account that for each μ the only "dangerous" point for the function $\phi_h(\cdot, \mu)$ is $\lambda = \mu$.

The simplest particular case of Theorem 8.1 says that

$$|h(\mu) - h(\lambda)| \le L|\mu - \lambda| \Longrightarrow \|h(\mathbf{B}) - h(\mathbf{A})\|_{\mathfrak{S}_2} \le L\|\mathbf{B} - \mathbf{A}\|_{\mathfrak{S}_2}.$$

It would be interesting to find an elementary proof of this estimate.

Here is the test for $\phi_h \in \mathfrak{M}$ found by Peller, [32]; the similar result for bounded operators was established by him earlier in [30]. The condition is very precise but still not necessary. Recall that a function $h(x)$ on $\mathbb{R}$ belongs to the Besov space $\mathcal{B}^1_{\infty,1}(\mathbb{R})$ if

$$r_0(h) := \int_0^\infty \big(\sup_{x\in\mathbb{R}} |h(x+t) - 2h(x) + h(x-t)|\big) \frac{dt}{t^2} < \infty. \tag{8.3}$$

Any function $h \in \mathcal{B}^1_{\infty,1}(\mathbb{R})$ has the bounded continuous derivative, and we denote

$$r(h) := r_0(h) + \sup_t |h'(t)|.$$

Naturally, for a function $h \in \mathcal{B}^1_{\infty,1}(\mathbb{R})$ we set $\phi_h(\lambda, \lambda) = h'(\lambda)$.

Theorem 8.3. *Let $h \in \mathcal{B}^1_{\infty,1}(\mathbb{R})$. Then $\phi_h \in \mathfrak{M}$ for any spectral measures E, F, and*

$$\|\phi_h\|_{\mathfrak{M}} \le Cr(h),$$

where C is an absolute constant. In particular, for any nice ideal $\mathfrak{S}$ the estimate is satisfied

$$\|\mathbf{J}h(\mathbf{B}) - h(\mathbf{A})\mathbf{J}\|_{\mathfrak{S}} \le Cr(h)\|\mathbf{JB} - \mathbf{AJ}\|_{\mathfrak{S}}. \tag{8.4}$$

The condition $r_0(h) < \infty$ takes care of both the smoothness of h and its decay at infinity. The following sufficient condition is rougher but sometimes more convenient, since the smoothness conditions and those at infinity are separated.

Theorem 8.4. *Suppose that $h' \in Lip(\epsilon) \cap L^p$ for some $\epsilon > 0$ and $p < \infty$. Then $\phi_h \in \mathfrak{M}$ for any spectral measures E, F.*

In particular, for any self-adjoint operators $\mathbf{A} \in \mathcal{B}(\mathfrak{H})$, $\mathbf{B} \in \mathcal{B}(\mathfrak{G})$ with spectra lying in a finite segment $[c, d]$, and for any $\mathbf{J} \in \mathcal{B}(\mathfrak{G}, \mathfrak{H})$ we have the estimate in an arbitrary nice ideal $\mathfrak{S}$:

$$\|\mathbf{J}h(\mathbf{B}) - h(\mathbf{A})\mathbf{J}\|_{\mathfrak{S}} \le C(\epsilon, d - c)\|\mathbf{JB} - \mathbf{AJ}\|_{\mathfrak{S}}$$

under the single condition $h' \in Lip(\epsilon)$.

Let us mention also the similar result for unitary operators.

Theorem 8.5. *Suppose that h is a differentiable function on the unit circle, such that $h' \in Lip(\epsilon)$ with some $\epsilon > 0$. Then for any unitary operators $\mathbf{U}$ in $\mathfrak{H}$, $\mathbf{V}$ in $\mathfrak{G}$ and any operator $\mathbf{J} \in \mathcal{B}(\mathfrak{G}, \mathfrak{H})$ we have*

$$\|\mathbf{J}h(\mathbf{V}) - h(\mathbf{U})\mathbf{J}\|_{\mathfrak{S}} \le C(\epsilon)\|\mathbf{JV} - \mathbf{UJ}\|_{\mathfrak{S}}.$$

where $\mathfrak{S}$ is an arbitrary nice ideal.

The next result easily follows from Theorem 7.3.

Theorem 8.6. *Suppose that h admits the integral representation*

$$h(x) = h(0) + \int_0^x \eta(s)ds, \ \forall x \in \mathbb{R}, \qquad \eta \in \mathsf{V}_l. \tag{8.5}$$

Then

$$\|\phi_h\|_{\mathfrak{M}_{\mathfrak{S}_p}} \le 2C(p)\|\eta\|_{\mathsf{V}}, \qquad 1 < p < \infty.$$

In particular,

$$\|h(\mathbf{B}) - h(\mathbf{A})\|_{\mathfrak{S}_p} \le 2C(p)\|\mathbf{B} - \mathbf{A}\|_{\mathfrak{S}_p}, \qquad 1 < p < \infty. \tag{8.6}$$

Recall that V_l stands for the space of left continuous functions of bounded total variation on $\mathbb{R}$, cf. section 7.3. It follows from the representation (8.6) that the derivative $h'(x) = \eta(x)$ exists at any point x where η is continuous. In the points of discontinuity of η the similar equality holds for the derivative from the left:

$$h'(x - 0) = \eta(x). \tag{8.7}$$

Proof of Theorem 8.6. It is sufficient to consider the case when the function η is monotone (non-decreasing). Then an elementary calculation shows that for each $\mu \in \mathbb{R}$ the function $\phi(\cdot,\mu)$ is also non-decreasing, and $\|\phi(\cdot,\mu)\|_{\mathsf{V}} \le \|\eta\|_{\mathsf{V}}$. The function $\phi(\cdot,\mu)$ is continuous at each point $\lambda_0 \in \mathbb{R}$, with the only possible exception of $\lambda_0 = \mu$. Left continuity at this point is nothing but the equality (8.7). So, Theorem 7.3 applies and we are done.

The estimate (8.6) is usually referred to as a Theorem by Davies who had proved it in [19]. His proof is directly based upon Matsaev's theorem, see section 7.2. In this connection we would like to notice that Theorem 7.3, whose immediate corollary is Theorem 8.6, was published in [37] some twenty years earlier.

8.3. Differentiation of functions of self-adjoint operators

Let $\mathbf{A}, \mathbf{T}$ be self-adjoint operators in $\mathfrak{H}$, $\mathbf{T}$ being bounded. Consider the operator-valued function $\mathbf{A}(t) = \mathbf{A} + t\mathbf{T}$, $Dom\mathbf{A}(t) = Dom\mathbf{A}$. Given a function $h(\lambda)$ such that $\phi_h \in \mathfrak{M}$, we derive from Theorem 8.1 that

$$h(\mathbf{A}(s)) - h(\mathbf{A}(t)) = (s-t)\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_h(\lambda,\mu)dE^{\mathbf{A}(t)}(\lambda)\mathbf{T}dE^{\mathbf{A}(s)}(\mu).$$

Dividing both parts by $s-t$ and formally passing to the limit as $s \to t$, we arrive at the *Daletskii – S.Krein formula*, [18]

$$\frac{dh(\mathbf{A}(t))}{dt} = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_h(\lambda,\mu)dE^{\mathbf{A}(t)}(\lambda)\mathbf{T}dE^{\mathbf{A}(t)}(\mu) = Z_h^{\mathbf{A}(t)}\mathbf{T}. \tag{8.8}$$

This formula needs a careful justification, since here we are dealing with the limit of an integral with respect to a varying spectral measure. In [18] (where a different approach had been applied), (8.8) was justified for the bounded operators and $h \in C^2$, which is rough. Better results were obtained in [6], [9], and then in [30], [32]. The results similar to Theorem 8.1 are unknown so far. We have to assume that $\phi_h \in \mathfrak{M}$ (independently of the symmetrically normed ideal $\mathfrak{S}$ the operator $\mathbf{T}$ belongs to), and even this is not enough: at least one of the functions $\alpha(\lambda,t), \beta(\mu,t)$ appearing in the factorization (4.6) of ϕ_h has to possess some additional properties. It turns out that these properties are satisfied if $h \in \mathcal{B}^1_{\infty,1}(\mathbb{R})$. This leads to the following result.

Theorem 8.7. *Let $h \in \mathcal{B}^1_{\infty,1}(\mathbb{R})$ and $\mathbf{T} \in \mathfrak{S}$ where $\mathfrak{S}$ is a nice ideal. Then at each point $t \in \mathbb{R}$ the derivative $dh(\mathbf{A}(t))/dt$ does exist in the topology of $\mathfrak{S}$ and is given by the equality* (8.8). *Besides, the estimate is valid*

$$\left\|\frac{dh(\mathbf{A}(t))}{dt}\right\|_{\mathfrak{S}} \le Cr(h)\|\mathbf{T}\|_{\mathfrak{S}}, \qquad \forall t \in \mathbb{R} \tag{8.9}$$

where C is an absolute constant.

The only exclusive case when a stronger result is known, is that of $\mathfrak{S} = \mathfrak{S}_2$; see [28]. In $\mathfrak{S}_2$ the formula (8.8) holds under the natural condition that the function

h has continuous and bounded derivative everywhere, and instead of (8.9) a better estimate is valid:

$$\|\frac{dh(\mathbf{A}(t)}{dt}\|_{\mathfrak{S}_2} \le \sup_t |h'(t)| \|\mathbf{T}\|_{\mathfrak{S}_2}.$$

8.4. Fractional powers of self-adjoint operators

Let $0 < \gamma < 1$, then for $t > 0$

$$h(\lambda) := \lambda^\gamma = c_\gamma \lambda \int_0^\infty t^{\gamma-1}(t+\lambda)^{-1} dt, \qquad c_\gamma = \pi^{-1} \sin \pi\gamma.$$

This implies that

$$\phi_h(\lambda, \mu) = c_\gamma \int_0^\infty \frac{t^\gamma}{(t+\lambda)(t+\mu)} dt = \int_0^\infty \alpha(\lambda, t)\alpha(\mu, t) dt \tag{8.10}$$

where $\alpha(\lambda, t) = \sqrt{c_\gamma}\, t^{\gamma/2}(t+\lambda)^{-1}$. We have

$$\int_0^\infty |\alpha(\lambda, t)|^2 dt = c_\gamma \int_0^\infty t^\gamma (t+\lambda)^{-2} dt = \gamma \lambda^{\gamma-1}. \tag{8.11}$$

Let now $\mathbf{A}, \mathbf{B}$ be two positive definite self-adjoint operators in $\mathfrak{H}$ (i.e. $\mathbf{A}, \mathbf{B} \ge \epsilon \mathbf{I}$, $\epsilon > 0$) with the same domain and let $\mathbf{T} := \mathbf{B} - \mathbf{A} \in \mathcal{B}$. The equalities (8.2) and (8.10) yield that

$$\begin{aligned}\mathbf{B}^\gamma - \mathbf{A}^\gamma &= \int_0^\infty \int_0^\infty \frac{\mu^\gamma - \lambda^\gamma}{\mu - \lambda} dE^{\mathbf{A}}(\lambda)\mathbf{T} dF^{\mathbf{B}}(\mu) \\ &= c_\gamma \int_0^\infty t^\gamma (\mathbf{A}+t)^{-1} \mathbf{T} (\mathbf{B}+t)^{-1} dt.\end{aligned}$$

It follows from (8.11) and (4.8) that

$$\|\mathbf{B}^\gamma - \mathbf{A}^\gamma\|_{\mathfrak{S}} \le \gamma \epsilon^{\gamma-1} \|\mathbf{B} - \mathbf{A}\|_{\mathfrak{S}}, \tag{8.12}$$

so that the estimate blows up as $\epsilon \to 0$.

Put $\delta = (1-\gamma)/2$ and along with ϕ_h consider the function

$$\widetilde{\phi_h}(\lambda, \mu) = \lambda^\delta \phi_h(\lambda, \mu) \mu^\delta = \int_0^\infty \widetilde{\alpha}(\lambda, t)\widetilde{\alpha}(\mu, t) dt$$

where $\widetilde{\alpha}(\lambda, t) = \lambda^\delta \alpha(\lambda, t)$. By (8.11), $\int_0^\infty |\widetilde{\alpha}(\lambda, t)|^2 dt = \gamma$ for all $\lambda > 0$. Since

$$\mathcal{J}_{\widetilde{\phi_h}}^{\mathbf{A},\mathbf{B}} \mathbf{T} = \mathbf{A}^\delta \big(Z_h^{\mathbf{A},\mathbf{B}} \mathbf{T} \big) \mathbf{B}^\delta, \tag{8.13}$$

we obtain a useful inequality

$$\|\mathbf{A}^\delta (\mathbf{B}^\gamma - \mathbf{A}^\gamma) \mathbf{B}^\delta\|_{\mathfrak{S}} \le \gamma \|\mathbf{B} - \mathbf{A}\|_{\mathfrak{S}}, \qquad 2\delta = 1 - \gamma. \tag{8.14}$$

In contrast with (8.12), the inequality (8.14) is satisfied for any positive (not necessarily positive definite) self-adjoint operators $\mathbf{A}, \mathbf{B}$.

Suppose again that $\mathbf{A}, \mathbf{B}$ are positive definite, then $\mathbf{A}(t) = \mathbf{A} + t\mathbf{T}$ is also positive definite for any $t \in [0,1]$. If $\mathbf{T}$ belongs to a nice ideal $\mathfrak{S}$, then the function $\mathbf{A}^\gamma(t)$ is differentiable in $\mathfrak{S}$ and

$$\frac{d\mathbf{A}^\gamma(t)}{dt} = Z_h^{\mathbf{A}(t)}\mathbf{T} = \mathcal{J}_{\widetilde{\phi}_h}^{\mathbf{A}(t)}\big(\mathbf{A}^{-\delta}(t)\mathbf{T}\mathbf{A}^{-\delta}(t)\big), \qquad 0 \le t \le 1. \tag{8.15}$$

Here the second equality comes from (8.13)

Now, assuming that $\mathbf{B} \ge \mathbf{A}$, we outline the proof of a useful inequality (see [4])

$$\|\mathbf{B}^\gamma - \mathbf{A}^\gamma\|_{\mathfrak{S}} \le \|(\mathbf{B}-\mathbf{A})^\gamma\|_{\mathfrak{S}}, \qquad 0 < \gamma < 1. \tag{8.16}$$

It follows from (8.15) that

$$\Big\|\frac{d\mathbf{A}^\gamma(t))}{dt}\Big\|_{\mathfrak{S}} \le \gamma\|\mathbf{A}^{-\delta}(t)\mathbf{T}\mathbf{A}^{-\delta}(t)\|_{\mathfrak{S}} = \|(\mathbf{A}^{-\delta}(t)\mathbf{T}^\delta)\mathbf{T}^\gamma(\mathbf{T}^\delta\mathbf{A}^{-\delta}(t))\|_{\mathfrak{S}}.$$

Since $\mathbf{A}(t) = \mathbf{A} + t\mathbf{T} \ge t\mathbf{T}$, we derive from the Heinz inequality (see e.g. [13], section 10.4) that for all $t \in (0,1)$

$$\|\mathbf{A}(t)^{-\delta}\mathbf{T}^\delta\| \le t^{-\delta}, \qquad \|\mathbf{T}^\delta\mathbf{A}(t)^{-\delta}\| \le t^{-\delta}.$$

Therefore,

$$\Big\|\frac{d\mathbf{A}^\gamma(t)}{dt}\Big\|_{\mathfrak{S}} \le \gamma t^{-2\delta}\|\mathbf{T}^\gamma\|_{\mathfrak{S}} = \gamma t^{\gamma-1}\|\mathbf{T}^\gamma\|_{\mathfrak{S}}.$$

Integrating this inequality in t over the segment $[0,1]$, we obtain (8.16). Since this inequality does not involve the lower bound of the operator $\mathbf{A}$, it extends to arbitrary non-negative operators $\mathbf{A}$ and $\mathbf{B} \ge \mathbf{A}$.

9. Applications to the theory of Spectral Shift Function

9.1. Calculation of the trace of an operator integral

Let $\phi \in \mathfrak{M}$ and

$$\mathbf{Q} = \mathcal{J}_\phi\mathbf{T} = \int_\Lambda\int_M \phi(\lambda,\mu)dE(\lambda)\mathbf{T}dF(\mu), \qquad \mathbf{T} \in \mathfrak{S}_1. \tag{9.1}$$

Then $\mathbf{Q} \in \mathfrak{S}_1$ and it is natural to try to calculate $tr\mathbf{Q}$. We shall discuss this problem for the case when $\mathfrak{G} = \mathfrak{H}$, $\Lambda = M$ and $E = F$.

As we know from Theorem 4.1, the assumption $\phi \in \mathfrak{M}$ is equivalent to existence of a factorization

$$\phi(\lambda,\mu) = \int_{\mathcal{T}} \alpha(\lambda,t)\beta(\mu,t)dt, \tag{9.2}$$

see (4.6), such that the conditions (4.7) are satisfied. According to (4.8), we have the estimate

$$\|\mathbf{Q}\|_{\mathfrak{S}_1} \le AB\|\mathbf{T}\|_{\mathfrak{S}_1}. \tag{9.3}$$

It is important that the factorization (9.2) determines the values of the symbol $\phi(\lambda,\mu)$ for E-almost all $\lambda,\mu \in \Lambda$. More precisely, this means that there exists a subset $\Lambda_0 \subset \Lambda$ such that $E(\Lambda \setminus \Lambda_0) = 0$ and the equality (9.2) holds for *all* $\lambda, \mu \in \Lambda_0$. Moreover, by Cauchy's inequality

$$|\phi(\lambda, \mu)| \leq AB, \qquad \forall \lambda, \mu \in \Lambda_0.$$

This shows in particular that the function $\phi(\lambda, \lambda)$ is well defined as an element of $\mathsf{L}^\infty(\Lambda; E)$.

It follows from (9.1), at least formally, that in the case $E = F$

$$tr\mathbf{Q} = \int_\Lambda \int_\Lambda \phi(\lambda, \mu) tr(dE(\lambda)\mathbf{T}dE(\mu)). \tag{9.4}$$

The orthogonality of the spectral measure $E(\cdot)$ gives rise to the conclusion that $dE(\mu)dE(\lambda) = \delta_{\lambda,\mu}dE(\lambda)$. By the known properties of the trace,

$$tr(dE(\lambda)\mathbf{T}dE(\mu)) = tr(dE(\mu)dE(\lambda)\mathbf{T}) = \delta_{\lambda,\mu} tr(dE(\lambda)\mathbf{T}).$$

Hence, we find from (9.4):

$$tr\mathbf{Q} = \int_\Lambda \phi(\lambda, \lambda) tr(\mathbf{T}dE(\lambda)). \tag{9.5}$$

It is convenient to re-write the equality (9.5) as follows. Let us introduce the scalar complex-valued measure on Λ, namely

$$\mathfrak{m}_{\mathbf{T},E}(\delta) = tr(\mathbf{T}E(\delta)), \qquad \delta \subset \Lambda.$$

Then (9.5) takes the form

$$tr\mathbf{Q} = \int_\Lambda \phi(\lambda, \lambda) d\mathfrak{m}_{\mathbf{T},E}(\lambda) = \int_\Lambda \phi(\lambda, \lambda) \mathfrak{m}_{\mathbf{T},E}(d\lambda). \tag{9.6}$$

The arguments which led us to the equality (9.6), were of a rather heuristic nature. However, it is not difficult to justify it. To achieve this goal, one starts with operators $\mathbf{T}$ of rank one. They span $\mathfrak{S}_1$, and the result extends to the whole of this space with the help of the estimate (9.3). The equality (9.6) is an analogue of the classical formula for the trace of an integral operator in L^2.

9.2. Two representations for $h(\mathbf{B}) - h(\mathbf{A})$

One such representation is given by the formula (8.2). Recall that $\mathbf{A}, \mathbf{B}$ are self-adjoint operators in $\mathfrak{H}$ with the common domain. We assume that $h \in \mathcal{B}^1_{\infty,1}(\mathbb{R})$, cf. (8.3). As we know, this assumption implies the existence of continuous and bounded derivative h' and guarantees that the function

$$\phi_h(\lambda, \mu) = \frac{h(\mu) - h(\lambda)}{\mu - \lambda}, \ \mu \neq \lambda; \qquad \phi_h(\lambda, \lambda) = h'(\lambda) \tag{9.7}$$

belongs to the algebra $\mathfrak{M} = \mathfrak{M}(E^{\mathbf{A}}, F^{\mathbf{B}}$.

If $\mathbf{T} := \mathbf{B} - \mathbf{A} \in \mathfrak{S}_1$, then by Theorem 8.1.

$$h(\mathbf{B}) - h(\mathbf{A}) = Z_h^{\mathbf{A},\mathbf{B}}\mathbf{T}. \tag{9.8}$$

Our next task is to find the trace of $h(\mathbf{B}) - h(\mathbf{A})$. Unfortunately, the formula (9.6) does not apply, since the spectral measures $E(\cdot), F(\cdot)$ in (9.8) differ from each other. For this reason, we now derive for $h(\mathbf{B}) - h(\mathbf{A})$ another representation. This new representation is more complex than (9.8) but is free from the above mentioned defect.

Let us consider the operator-valued function $\mathbf{A}(t) = \mathbf{A} + t\mathbf{T},\ t \in \mathbb{R}$. Then $\mathbf{A}(0) = \mathbf{A}$ and $\mathbf{A}(1) = \mathbf{B}$. By Theorem 8.7, the assumption $h \in \mathcal{B}^1_{\infty,1}(\mathbb{R})$ is sufficient for the Daletskii – S.Krein formula (8.8) to be valid. Hence, we have

$$\frac{dh(\mathbf{A}(t))}{dt} = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_h(\lambda,\mu) dE^{\mathbf{A}(t)}(\lambda)\mathbf{T}dE^{\mathbf{A}(t)}(\mu), \qquad \forall t \in \mathbb{R}. \tag{9.9}$$

It is important that under the condition $\mathbf{T} \in \mathfrak{S}_1$ the derivative in (9.9) exists in the sense of convergence in $\mathfrak{S}_1$. By integrating in $t \in [0,1]$ we derive from (9.9):

$$h(\mathbf{B}) - h(\mathbf{A}) = \int_0^1 dt \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_h(\lambda,\mu) dE^{\mathbf{A}(t)}(\lambda)\mathbf{T}dE^{\mathbf{A}(t)}(\mu). \tag{9.10}$$

Just this is the second representation of $h(\mathbf{B}) - h(\mathbf{A})$ which was our goal.

Now we use (9.10) for calculating the trace of $h(\mathbf{B}) - h(\mathbf{A})$. It is more convenient to start with finding the trace of derivative $dh(\mathbf{A}(t))/dt$. Here the formula (9.6) does apply, and we get

$$tr\frac{dh(\mathbf{A}(t))}{dt} = \int_{\mathbb{R}} h'(\lambda)\mathfrak{m}_{\mathbf{T},E^{\mathbf{A}(t)}}(d\lambda), \qquad \forall t \in \mathbb{R}. \tag{9.11}$$

Finally, we derive from (9.11) by integration over $0 \le t \le 1$ that

$$tr\big(h(\mathbf{B}) - h(\mathbf{A})\big) = \int_{\mathbb{R}} h'(\lambda)d\Xi(\lambda) \tag{9.12}$$

where

$$\Xi(\delta) = \int_0^1 tr(E^{\mathbf{A}(t)}(\delta)\mathbf{T})dt. \tag{9.13}$$

It is not difficult to justify all the steps of this calculation. The function $\Xi(\cdot)$ defined by (9.13) is a real-valued σ-additive measure (a charge). If the operator $\mathbf{T}$ is non-negative, then Ξ is also non-negative. For the total variation of Ξ we have the estimate

$$\big|\Xi\big| \le \|\mathbf{T}\|_{\mathfrak{S}_1}. \tag{9.14}$$

It is also clear that

$$\Xi(\mathbb{R}) = tr\mathbf{T}. \tag{9.15}$$

9.3. Spectral Shift Function

As a matter of fact, *the measure* (9.13) *is absolutely continuous with respect to the Lebesgue measure*, i.e. there exists a real-valued function $\xi = \xi(\cdot;\mathbf{B},BA) =\in \mathsf{L}^1(\mathbb{R})$ such that

$$\Xi(\delta) = \int_\delta \xi(\lambda)d\lambda. \tag{9.16}$$

The function ξ appearing in (9.16) is *the spectral shift function*, introduced by a physicist I.M. Lifshits in [25] for the case of finite rank perturbations. The consistent mathematical theory of the spectral shift function was developed by M.G. Krein in [24]. In particular, M.G. Krein established the formula

$$tr\big(h(\mathbf{B}) - h(\mathbf{A})\big) = \int_{\mathbb{R}} h'(\lambda)\xi(\lambda)d\lambda \tag{9.17}$$

for a wide class of functions h, under the assumption $\mathbf{B} - \mathbf{A} \in \mathfrak{S}_1$. The spectral shift function proved very efficient tool in many problems of Mathematics and Theoretical Physics.

Two important questions arise in connection with the formula (9.17): what are the properties of the function $\xi(\lambda)$ and what is the class of functions for which the equality (9.17) is satisfied. We do not aim in giving a survey of properties of the spectral shift function and restrict ourselves to a few remarks.

The techniques of double operator integrals does not provide us with the tools to show absolute continuity of the measure Ξ. This probably is a penalty for the inclusion of a non-linear problem in the framework of a linear one, cf. the discussion in section 1.3, below the formulas (1.10) and (1.11). In the representation (9.13) the connection between the operator-valued function $\mathbf{A}(t)$ and the operator $\mathbf{T} = \mathbf{B} - \mathbf{A}$ is lost. It is also remarkable that the measures $tr(E^{\mathbf{A}(t)}(\delta)\mathbf{T})$ appearing in the integrand of (9.13) are not necessarily absolutely continuous, but the integration moves off the singular component.

The approach of M.G. Krein uses Complex Function Theory, it is based upon a thorough analysis of the function

$$\Delta(z) = \det\big(\mathbf{I} + \mathbf{T}(\mathbf{A} - z\mathbf{I})^{-1}\big)$$

which is analytic outside the spectrum of $\mathbf{A}$. Namely, it turns out that

$$\xi(\lambda) = \pi^{-1} \lim_{\varepsilon \to 0} \arg \Delta(\lambda + i\varepsilon) \text{ a.e.} \tag{9.18}$$

In this representation the connection between $\mathbf{A}, \mathbf{B}$ and $\mathbf{T}$ is taken into account in a more explicit way than in (9.13). The relations (9.14) – (9.16) mean that

$$\int_{\mathbb{R}} |\xi(\lambda)| d\lambda \le \|\mathbf{T}\|, \qquad \int_{\mathbb{R}} \xi(\lambda) d\lambda = tr\mathbf{T}.$$

Further, suppose that $rank\mathbf{T} < \infty$ and the signature of $\mathbf{T}$ is (n_+, n_-). Then $-n_- \le \xi(\lambda) \le n_+$ for a.e. λ. These inequalities easily follow from the representation (9.18) but not from (9.13).

In this respect, one may ask the question: why at all the approach based on the theory of double integrals is useful? Another natural question concerns the additional information which can be extracted from the representation (9.13) or, in more detailed writing,

$$\int_{\delta} \xi(\lambda) d\lambda = \int_0^1 tr\big(E^{\mathbf{A}(t)}(\delta)\mathbf{T}\big) dt. \tag{9.19}$$

One of general facts of this type is the two-sided inequality (proved in [10])

$$tr\big(E^{\mathbf{B}}(-\infty,\lambda)\mathbf{T}\big) \le \int_{-\infty}^{\lambda} \xi(\mu;\mathbf{B},\mathbf{A})d\mu \le tr\big(E^{\mathbf{A}}(-\infty,\lambda)\mathbf{T}\big).$$

There are also many other useful applications of the formula (9.19).

Another advantage of the representation (9.19) is substantial expansion of the class of admissible functions h. In the M.G. Krein's approach, h has to belong to Wiener's class W_1, that is

$$h \in \mathsf{C}^1_{loc}, \qquad h'(\lambda) = \int_{\mathbb{R}} e^{-\lambda t} d\sigma(t), \qquad |\sigma|(\mathbb{R}) < \infty.$$

This class is much narrower than $B^1_{\infty,1}$.

10. Remarks on the literature

Double, and also multiple operator integrals first appeared in the paper [18] where the Daletskii – S.Krein formula (8.8) was derived, under some restrictive assumptions about the function h.

The consecutive theory of double operator integrals was worked out in the series of authors' publications [5] – [9] and [37]. In particular, the $\mathfrak{S}_2$-theory of such integrals was developed in [5], [6]. In the same papers, the Condition *(ii)* of Theorem 4.1 as a criterion of $\phi \in \mathfrak{M}_{\mathfrak{S}_1}$ was established, and the definition (4.3) of the transformer $\mathcal{J}_\phi$ on the class $\mathcal{B}$ was suggested. Theorem 8.5 for $\mathbf{J} = \mathbf{I}$ and $\mathfrak{S} = \mathfrak{S}_p,\ 1 \le p \le \infty$ was also obtained there.

Realization of an integral as a multiplier transform was found in [7]. This lead to the possibility to realize the pseudodifferential operators as double operator integrals. The material of sections 6.1 and 6.2 of the text presented is borrowed from this paper.

The element-wise multiplication of matrices ("Schur multiplication") was introduced by I. Schur as far back as in 1911, see [35] and also section 4 of the recent survey paper [20]. Among many other results, a discrete analog of the factorization (4.6)–(4.7) and the estimate (4.8) were found there. The multiplier transformation (1.15) can be viewed as an analog of Schur multiplication for the (generalized) kernels of integral operators. It is worth mentioning that many properties of the discrete Schur multiplication are not valid in the "continuous" case. Unfortunately, the authors were unaware of the Schur's results when starting the work on double operator integrals.

One of topics studied in [7] were the analytical tests for $\phi \in \mathfrak{M}$ and $\phi \in \mathfrak{M}_p$, $1 < p < \infty$ Theorems 5.3 and 5.4 were proved there. Theorem 5.2 was obtained later, in [8].

In the paper [9] many technical tools of the theory were worked out. This includes the sufficiency part of the statement *(iii)* of Theorem 4.1; its necessity was established later by V. Peller in [30]. The study of the transformers Z_h started in [5] and was continued in [9]. Theorem 8.1 was proved there. Besides, in [9] the problem of continuity of the transformers $\mathcal{J}_\phi^{E,F}$ with respect to the varying spectral measure was studied in detail, and the Daletskii – S.Krein formula (8.8) was justified under rather mild assumptions about the function h. More advanced results in this direction are due to V. Peller, [30] – [32], and to J. Arazy, T. Barton and Y. Friedman, [1].

In [10] the results of [9] were applied to the theory of the spectral shift function. The formula (9.13) was obtained there, its another proof was suggested by B. Simon [36]. For the systematic exposition of the theory of the spectral shift function see e.g. the book [38] and the paper [17].

Theorem 7.3 was obtained in [37]. As it was already mentioned in section 8.2, Theorem 8.6 is its elementary consequence. In the paper [14] the authors revisited this material and found some new results, in particular concerning commutators and quasi-commutators with unbounded operator $\mathbf{J}$, cf. (8.4).

Applications to the fractional powers of self-adjoint operators were discussed in [4] and [15].

In the book [26] V.P. Maslov developed the theory of functions of several "ordered" non-commuting operator-valued variables. As it was mentioned in section 1.3, a double operator integral can also be considered as a special case of such function. In this connection we would like to mention that the material of this paper has almost no intersections with the book [26]. This concerns both the technical means of the theory and the nature of applications.

As it was already mentioned in the Introduction, recently a new interest in operator integrals arose on another technical basis, see [29], [28] and references therein.

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