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# On a question of G. Kuba

By

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**Abstract.** For any real number $x$ let $\{x\} = x - \lfloor x \rfloor$ be the fractionary part of $x$. In this note, we study the real algebraic numbers $b$ such that

$$\lim_{n\to\infty} \{b^n\} = 0.$$

**1. Introduction.** For any real number $x$ let $\{x\} = x - \lfloor x \rfloor$ be the fractionary part of $x$. In this paper, we study the real numbers $b$ such that

(1) $$\lim_{n\to\infty} \{b^n\} = 0.$$

One may immediately notice that there are two obvious families of real numbers $b$ satisfying condition (1); namely:

1) a topological family consisting of the numbers $b \in (0,1)$ and
2) a number-theoretical family consisting of the numbers $b \in \mathbb{Z}$.

A natural question to ask is whether there exist other real numbers $b$ except for the ones belonging to the families 1) or 2) above which satisfy condition (1). Our main result is that the answer to the above question is no if one looks only at numbers $b$ which are algebraic. That is, we have the following:

**Theorem.** *If $b$ is a real algebraic number satisfying condition* (1), *then either $b \in (0,1)$ or $b \in \mathbb{Z}$.*

In particular, our result gives a negative answer to a question raised by G. Kuba (see [2], page 162).

For any real number $x$ let $||x||$ be the distance from $x$ to the nearest integer to it. By a well-known theorem of Pisot and Vijayaraghavan (see Chapter VIII in [1]), we know that when $b > 1$ is an algebraic number, then $\lim_{n\to\infty} ||b^n|| = 0$ if and only if either $b$ is an integer or $b$ is a PV-number; that is, $b \notin \mathbb{Z}$ is an algebraic integer such that all its conjugates lie in the unit disc $\{z \mid |z| < 1\}$. From this theorem, it follows that if $b$ is a PV-number, then the cluster points of the set of values of $\{b^n\}_{n \geqq 0}$ are contained in the set $\{0,1\}$. What our Theorem shows is that this set of cluster points of $\{b^n\}_{n \geqq 0}$ can never consist of the

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point 0 alone. Notice that it can obviously consist of the point 1 alone (take for example $b$ to be the square of the golden section) or of both points 0 and 1 (take $b$ to be the golden section).

**2. Statements of Preliminary Results.** In this section, we state two lemmas whose proofs are given in the Appendix.

**Lemma 1.** *Let $d \geqq 2$ be a positive integer. Assume that $x_1, \ldots, x_d$ are all the roots of an irreducible polynomial $F \in \mathbb{Q}[X]$ of degree d. Assume that for some $n_0 \in \mathbb{N}$, some proper non-empty subset $I \subset \{1, \ldots, d\}$ and some complex numbers $\lambda_1, \ldots, \lambda_d$ one has*

$$\sum_{i \in I} \lambda_i x_i^n \in \mathbb{Q} \quad \textit{for all } n \geqq n_0. \tag{2}$$

*Then, $\lambda_i = 0$ for all $i \in I$.*

**Lemma 2.** *Let $d \geqq 1$ be a positive integer. Let $\chi = \{x_1, \ldots, x_d\}$ be a set of non-zero complex numbers which is closed under conjugation. Then, there exist infinitely many positive integers $n$ such that*

$$\sum_{i=1}^{d} x_i^n > 0. \tag{3}$$

**3. The Proof of the Theorem.** Assume that $b$ is a real algebraic number satisfying condition (1). Suppose that $b \notin (0,1) \cup \mathbb{Z}$. Notice that $b \notin (-1,0)$. Indeed, if $b \in (-1,0)$, then

$$\{b^{2n+1}\} = 1 + b^{2n+1} \longrightarrow 1,$$

which contradicts (1). Thus, $|b| > 1$. We first assume that $b > 1$ and we will come back to the negative $b$ case later.

Since the sequence $\{b^n\}_n$ converges to zero, it follows that the sequence $||b^n||_n$ converges to zero as well. By the Pisot-Vijayaraghavan Theorem, it follows that $b$ is either a positive integer or $b$ is a PV-number. Since we have excluded the integers, it follows that $b$ is a PV-number. In particular, all its conjugates lie in the unit disc $\{z | \, |z| < 1\}$.

For any $n \geqq 0$ let $k_n = \lfloor b^n \rfloor$ and $\varepsilon_n = \{b^n\}$. Then,

$$b^n = k_n + \varepsilon_n \quad \text{for all } n \geqq 0. \tag{4}$$

Let $d = [\mathbb{Q}(b) : \mathbb{Q}]$. Let $F \in \mathbb{Z}[X]$ be a non-zero polynomial of degree $d$ having $b$ as a root. Denote the roots of $F$ by $x_1, \ldots, x_d$ with the convention that $x_1 = b$. Write

$$F(X) = a_0 X^d + \cdots + a_d \in \mathbb{Z}[X].$$

Since

$$a_0 b^{n+d} + \cdots + a_d b^n = b^n F(b) = 0 \quad \text{for all } n \geqq 0,$$

it follows that

$$(a_0 k_{n+d} + \cdots + a_d k_n) + (a_0 \varepsilon_{n+d} + \cdots + a_d \varepsilon_n) = 0 \quad \text{for all } n \geqq 0.$$

Hence,

(5) $$a_0\varepsilon_{n+d}+\cdots+a_d\varepsilon_n\in\mathbb{Z}\quad\text{for all } n\geqq 0.$$

Since $\varepsilon_n\longrightarrow 0$, it follows that there exists $n_0\in\mathbb{N}$ such that

(6) $$|a_0\varepsilon_{n+d}+\cdots+a_d\varepsilon_n|<1\quad\text{for all } n\geqq n_0.$$

From formulae (5) and (6), it follows that

(7) $$a_0\varepsilon_{n+d}+\cdots+a_d\varepsilon_n=0\quad\text{for all } n\geqq n_0.$$

In particular, the sequence $(\varepsilon_n)_{n\geqq n_0}$ is a linearly recurrent sequence of order $d$. It's characteristic equation is exactly $F(X)=0$, whose roots are $x_1,\ldots,x_d$. Hence,

(8) $$\varepsilon_n=\sum_{i=1}^{d}\lambda_i x_i^n\quad\text{for all } n\geqq n_0.$$

Notice that $\lambda_1=0$. Indeed, since $|x_i|<1$ for all $i=2,\ldots,d$, it follows that

(9) $$\lim_{n\to\infty}|\sum_{i=2}^{d}\lambda_i x_i^n|=0.$$

If $\lambda_1\neq 0$, then since $x_1=b>1$, it would follow that the sequence

$$\varepsilon_n=|\sum_{i=1}^{d}\lambda_i x_i^n|\geqq|\lambda_1|b^n-|\sum_{i=2}^{d}\lambda_i x_i^n|$$

tends to infinity, which is impossible since $\varepsilon_n\in[0,1)$. Hence, $\lambda_1=0$ and

(10) $$\varepsilon_n=\sum_{i=2}^{d}\lambda_i x_i^n\quad\text{for all } n\geqq n_0.$$

Writing $\varepsilon_n=b^n-k_n$, we get

(11) $$x_1^n+\sum_{i=2}^{d}(-\lambda_i)x_i^n=b^n-\varepsilon_n=k_n\in\mathbb{Z}\subset\mathbb{Q}\quad\text{for all } n\geqq n_0.$$

However, since $x_1,\ldots,x_d$ are the roots of a polynomial with integer coefficients it follows, by the Newton-Euler formulae, that

(12) $$\sum_{i=1}^{d}x_i^n\in\mathbb{Q}.$$

Subtracting (11) from (12), we get

(13) $$\sum_{i=2}^{n}(\lambda_i+1)x_i^n\in\mathbb{Q}\quad\text{for all } n\geqq n_0.$$

By Lemma 1, it follows that $\lambda_i=-1$ for all $i=2,\ldots,n$. Hence,

(14) $$\varepsilon_n=-\sum_{i=2}^{d}x_i^n\quad\text{for all } n\geqq n_0.$$

Notice that the set $\{x_i|i=2,\ldots,d\}$ is closed under conjugation. By Lemma 2, we know that there exist infinitely many positive integers $n$ for which

$$\sum_{i=2}^{d}x_i^n>0.$$

By formula (14), such an integer $n \geqq n_0$ would imply that $\varepsilon_n < 0$, which is impossible since $\varepsilon_n \in [0, 1)$. Hence, there are no such PV-numbers $b$ in this case.

Assume now that $b < -1$. By the previous arguments, it follows that $b^2$ is an integer. Suppose that $b$ is not an integer. Write $b = -\sqrt{d}$ for some positive integer $d$ which is not a square and let $b_1 = \sqrt{d}$. Since $b^n + b_1^n \in \mathbb{Z}$ for all $n \geqq 0$ and since the sequence $\{b^n\}_n$ converges to zero, it follows that the sequence $\{b_1^n\}_n$ converges to 1. In particular, the sequence $||b_1^n||_n$ converges to zero. Since $b_1 > 1$, by the Pisot-Vijayaraghavan Theorem, it follows that $b_1$ is a PV-number, which is certainly not the case because $b_1$ and $-b_1 = b$ are conjugate.

The Theorem is therefore proved.

**4. Appendix: The proofs of the preliminary results.**

The Proof of Lemma 1. We may assume, without restricting generality, that $I = \{1, \ldots, l\}$ for some $l < d$ and that $\lambda_i \neq 0$ for all $i = 1, \ldots, l$. The assertion of Lemma 1 is obvious when $l = 1$. Indeed, in this case, since $\lambda_1 \neq 0$, it follows that

$$x_1 = \frac{\lambda_1 x_1^{n_0+1}}{\lambda_1 x_1^{n_0}} \in \mathbb{Q},$$

which contradicts the fact that $x_1$ is a root of an irreducible polynomial with rational coefficients of degree $d \geqq 2$. Assume now that $d > l \geqq 2$. Let

$$u_n = \sum_{i=1}^{l} \lambda_i x_i^n \quad \text{for all } n \geqq n_0. \tag{15}$$

Let $\sigma_1, \sigma_2, \ldots, \sigma_l$ denote the fundamental symmetric polynomials evaluated in the numbers $x_1, x_2, \ldots, x_l$; that is

$$\begin{aligned} \sigma_1 &= \sum_{i=1}^{l} x_i, \\ \sigma_2 &= \sum_{1 \leqq i < j \leqq l} x_i x_j, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \sigma_l &= \prod_{i=1}^{l} x_i. \end{aligned} \tag{16}$$

The sequence $(u_n)_{n \geqq n_0}$ is a linearly recurrent sequence satisfying the recurrence

$$u_{n+l} - \sigma_1 u_{n+l-1} + \cdots + (-1)^l \sigma_l u_n = 0 \quad \text{for all } n \geqq n_0. \tag{17}$$

We now look at the equations

$$-\sigma_1 u_{n+l-1} + \cdots + (-1)^l \sigma_l u_n = -u_{n+l} \quad \text{for } n = n_0, n_0 + 1, \ldots, n_0 + l - 1. \tag{18}$$

We regard equations (18) as a linear non-homogeneous system of equations in the unknowns $-\sigma_1, \ldots, (-1)^l \sigma_l$. Notice that it suffices to show that the system (18) is non-singular. Indeed, assume that the system (18) is non-singular. It follows, from Kramer's rule and from the fact that $u_n \in \mathbb{Q}$ for all $n \geqq n_0$, that $\sigma_i \in \mathbb{Q}$ for all $i = 1, \ldots, l$. In this case, the polynomial

$$F(X) = X^l - \sigma_1 X^{l-1} + \cdots + (-1)^l \sigma_l$$

is a non-zero polynomial in $\mathbb{Q}[X]$ having $x_1, \ldots, x_l$ as roots. This contradicts the fact that $x_1, \ldots, x_d$ are all the roots of an irreducible polynomial with rational coefficients.

In order to prove that system (18) is non-singular, notice that its determinant is equal to:

$$\begin{vmatrix} \lambda_1 x_1^{n_0} + \cdots + \lambda_l x_l^{n_0} & \lambda_1 x_1^{n_0+1} + \cdots + \lambda_l x_l^{n_0+1} & \ldots\ldots & \lambda_1 x_1^{n_0+l-1} + \cdots + \lambda_l x_l^{n_0+l-1} \\ \lambda_1 x_1^{n_0+1} + \cdots + \lambda_l x_l^{n_0+1} & \lambda_1 x_1^{n_0+2} + \cdots + \lambda_l x_l^{n_0+2} & \ldots\ldots & \lambda_1 x_1^{n_0+l} + \cdots + \lambda_l x_l^{n_0+l} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1 x_1^{n_0+l-1} + \cdots + \lambda_l x_l^{n_0+l-1} & \lambda_1 x_1^{n_0+l} + \cdots + \lambda_l x_l^{n_0+l} & \ldots\ldots & \lambda_1 x_1^{n_0+2l-2} + \cdots + \lambda_l x_l^{n_0+2l-2} \end{vmatrix}$$

$$= \begin{vmatrix} \lambda_1 x_1^{n_0} & \lambda_2 x_2^{n_0} & \ldots\ldots & \lambda_l x_l^{n_0} \\ \lambda_1 x_1^{n_0+1} & \lambda_2 x_2^{n_0+1} & \ldots\ldots & \lambda_l x_l^{n_0+1} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1 x_1^{n_0+d-1} & \lambda_2 x_2^{n_0+d-1} & \ldots\ldots & \lambda_l x_l^{n_0+d-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & x_1 & \ldots\ldots & x_1^{l-1} \\ 1 & x_2 & \ldots\ldots & x_2^{l-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & x_l & \ldots\ldots & x_l^{l-1} \end{vmatrix}$$

$$= \left(\prod_{i=1}^{l} \lambda_i\right) \left(\prod_{i=1}^{l} x_i\right)^{n_0} \prod_{1 \leqq i < j \leqq l} (x_i - x_j)^2 \neq 0.$$

The Proof of Lemma 2. Write

(19) $$u_n = \sum_{i=1}^{d} x_i^n.$$

Then,

$$u_{2n} = \sum_{i=1}^{d} x_i^{2n}.$$

If some of the $x_i$'s are real, then $x_i^{2n} > 0$. Thus, we may assume that all the $x_i$'s are complex non-real. Write $d = 2l$ and assume that $x_{l+i} = \overline{x_i}$ for $i = 1, \ldots, l$. Write

$$x_j = \rho_j e^{i\theta_j} \quad \text{for } j = 1, \ldots, l.$$

Then,

(20) $$u_n = 2 \sum_{i=1}^{l} \rho_i^n \cos(n\theta_i).$$

Let $V$ be the $\mathbb{Q}$-vector space generated by $\{1, \theta_1/\pi, \ldots, \theta_l/\pi\}$. We may assume that none of the numbers $\theta_i/\pi$ is rational. Indeed, suppose that $\theta_i = q_i \pi$ for some $q_i \in \mathbb{Q}$ and some $i = 1, \ldots, l$. Let $\Delta_i$ be the denominator of $q_i$. Then,

$$x_i^{2n\Delta_i} + x_{l+i}^{2n\Delta_i} = 2\rho_i^{2n\Delta_i} \cos(2n\Delta_i q_i \pi) = 2\rho_i^{2n\Delta_i} > 0.$$

Hence, we may replace the sequence $(u_n)_{n \geqq 0}$ by

$$v_n = 2 \sum_{j \neq i} \rho_j^{2n\Delta_i} \cos(2n\Delta_i \theta_j).$$

and continue by induction on $l$. Let $s+1$ be the dimension of $V$ over $\mathbb{Q}$. Clearly, $2 \leqq s+1 \leqq l+1$. Up to reindexing, we may assume that $1, \theta_1/\pi, \ldots, \theta_s/\pi$ is a basis of $V$ over $\mathbb{Q}$. Write

(21) $$\frac{\theta_i}{\pi} = \sum_{j=1}^{s} a_{ij} \left(\frac{\theta_j}{\pi}\right) + b_i \quad \text{for } i = s+1, \ldots, l,$$

where $a_{ij}$ and $b_i$ are rational for $j = 1, \ldots, s$ and $i = s+1, \ldots, l$. Let $B$ be the greatest common denominator of the $b_i$'s. We may replace $\theta_i$ by $2B\theta_i$. This means that we look only at the subsequence $(u_{2Bn})_{n \geqq 0}$. In this case,

$$\frac{2B\theta_i}{\pi} = \sum_{j=1}^{s} a_{ij}\left(\frac{2B\theta_j}{\pi}\right) + 2Bb_i \quad \text{for } i = s+1, \ldots, l. \tag{22}$$

Since the argument of a complex number is defined only modulo $2\pi$, it follows that we may assume that $b_i = 0$ for $i = s+1, \ldots, l$. Thus, formula (21) becomes:

$$\frac{\theta_i}{\pi} = \sum_{j=1}^{s} a_{ij}\left(\frac{\theta_j}{\pi}\right) \quad \text{for } i = s+1, \ldots, l. \tag{23}$$

Let

$$L = \max\left(1, \left\{\sum_{j=1}^{s} |a_{ij}|\right\}_{s+1 \leqq i \leqq l}\right). \tag{24}$$

Let $\Delta$ be the greatest common denominator of all the $a_{ij}$'s. Choose $\varepsilon \in \left(0, \frac{1}{4\Delta L}\right)$. Since $1, \theta_1/\pi, \ldots, \theta_s/\pi$ are linearly independent over $\mathbb{Q}$, it follows that $1, \frac{\theta_1}{2\pi\Delta}, \ldots, \frac{\theta_s}{2\pi\Delta}$ are also linearly independent over $\mathbb{Q}$. By a well-known theorem of Kronecker, it follows that there exist infinitely many $n$'s such that

$$\left(\left\{\frac{n\theta_1}{2\pi\Delta}\right\}, \ldots, \left\{\frac{n\theta_s}{2\pi\Delta}\right\}\right) \in (0, \varepsilon)^s. \tag{25}$$

Assume that $n$ is such that containment (25) holds. For $j = 1, \ldots, s$ write

$$\frac{n\theta_j}{2\pi\Delta} = k_j + \varepsilon_j \quad \text{where } k_j \in \mathbb{Z} \quad \text{and} \quad 0 < \varepsilon_j < \varepsilon < \frac{1}{4\Delta L}. \tag{26}$$

Hence,

$$n\theta_j = 2\pi\Delta k_j + 2\pi\Delta\varepsilon_j \quad \text{and} \quad 2\pi\Delta\varepsilon_j < \frac{\pi}{2L} \leqq \frac{\pi}{2} \tag{27}$$

when $j = 1, \ldots, s$. Moreover, notice that for $i = s+1, \ldots, l$ one has

$$n\theta_i = \sum_{j=1}^{s} a_{ij} n\theta_j = 2\pi\left(\sum_{j=1}^{s} a_{ij}\Delta k_j\right) + 2\pi\Delta\left(\sum_{j=1}^{s} a_{ij}\varepsilon_j\right). \tag{28}$$

Clearly, $\sum_{j=1}^{s} a_{ij}\Delta k_j \in \mathbb{Z}$. Moreover,

$$\left|2\pi\Delta\left(\sum_{j=1}^{s} a_{ij}\varepsilon_j\right)\right| < 2\pi\Delta\varepsilon\left(\sum_{j=1}^{s} |a_{ij}|\right) \leqq 2\pi\Delta\varepsilon L < \frac{\pi}{2}. \tag{29}$$

From formulae (27), (28) and inequality (29), it follows that $\cos(n\theta_i) > 0$ for all $i = 1, \ldots, l$. Hence, $u_n > 0$, whenever $n$ is such that containment (25) is satisfied.

Note (Added in the Proof): The referee observed that the main idea behind the proof of Lemma 2 is similar to a method developed by Turán in [3]. We thank the referee for pointing this out to us.

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