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Bodies of constant brightness

By

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Abstract. We give a necessary condition for a convex body in $\mathbb{R}^n$ to have minimal volume in the class of all convex sets with prescribed constant brightness.

Such a condition permits us to prove that in the class of all convex bodies of revolution with given constant brightness the unique element of minimal volume is a body described by Blaschke as an example of a non-spherical solid of constant brightness.

1. Introduction. In this paper we deal with the class $\mathfrak{B}_B$ of convex bodies in $\mathbb{R}^n$ having prescribed constant brightness. The simplest example of a body of this kind is a ball. Constant brightness means that the $(n-1)$-dimensional volume of the orthogonal projection onto a hyperplane does not change under rotations of the body.

The first example of a non-spherical three-dimensional convex body of constant brightness was given by Blaschke in 1916 ([1], pp. 151–154). This body, which we call the Blaschke-Firey body, is a body of revolution, whose principal section reminds one of a Reuleaux triangle.

Inspired by the work of Blaschke, Firey ([4], 1965) introduced a special addition for convex bodies, called the Blaschke addition. This new operation involves the area measure of the bodies and is based on the existence and uniqueness theorems for the Minkowski problem (see [7] Sections 7.1 and 7.2).

Modulo the group of translations, under the Blaschke addition the class $\mathfrak{B}_B$ can be seen as a compact convex subset of the whole class of convex bodies with the classical Hausdorff topology.

The volume, as a continuous function, has a maximum and a minimum in $\mathfrak{B}_B$. Since each body in $\mathfrak{B}_B$ has the same surface area, we deduce from the isoperimetric inequality that the ball with prescribed brightness is the unique body of maximum volume.

As far as the minimum of the volume is concerned, only the solution in the two-dimensional case is available; in 1914 Lebesgue ([5]) proved the minimality property of the Reuleaux triangle.

According to the Kneser-Süss inequality (see [7], Theorem 7.1.3), a certain power of the volume is a concave function with respect to the Blaschke addition. It follows from the Kneser-Süss inequality and its equality condition that a convex body of minimal volume in $\mathfrak{B}_B$ must be indecomposable in the Blaschke sense. This argument leads to the necessary condition proved in [2].

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In Section 3 we strengthen such a condition and we eliminate some elements of $\mathfrak{B}_B$ from the list of possible minimizers. It turns out that in the class of bodies of revolution all the possible minimizers can be represented in a suitable parametric form. By using such a representation, in Section 4, we show that the unique (up to a rigid motion) rotationally symmetric minimizer in $\mathfrak{B}_B$ is just the Blaschke-Firey body.

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2. Preliminaries. Let us start with recalling some basic results from the Brunn-Minkowski theory we shall use. For more details we refer to [7].

We denote by $\mathscr{K}^n$ the class of all compact convex sets of $\mathbb{R}^n$ with nonempty interior. For every $K \in \mathscr{K}^n$ the *support function* of K is defined by:

$$h_K(z) = \sup_{x \in K} \langle x, z \rangle \quad \text{for } z \in S^{n-1},$$

where $S^{n-1} = \{z \in \mathbb{R}^n : \|z\| = 1\}$ and $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ denote the usual scalar product and the induced norm.

The support function of a translate of K by a vector $v \in \mathbb{R}^n$ satisfies $h_{K+v}(z) = h_K(z) + \langle z, v \rangle$, for every $z \in S^{n-1}$.

It can be seen that h_K is a Lipschitz function; this implies the existence of the spherical gradient $\nabla_S h_K$ almost everywhere on S^{n-1}.

If h_K is differentiable at z, both $h_K(z)$ and $\nabla_S h_K(z)$ uniquely determine the point of K lying on the support plane to K with exterior normal vector z. This point is the reverse image of z through the Gauss map, thus the reverse Gauss map is well defined almost everywhere on S^{n-1}.

For every Borel subset ω of S^{n-1} we define the *area measure* of $K \in \mathscr{K}^n$, $S_K(\omega)$, as the $(n-1)$-dimensional Hausdorff measure of the reverse image of ω through the Gauss map.

Such a measure is a finite non-negative Borel measure, having support with affine dimension n and satisfying the relation:

$$\int_{S^{n-1}} z \, dS_K(z) = 0.$$

Conversely, if a measure μ, defined on S^{n-1}, verifies all the above conditions then, by the Minkowski existence and uniqueness theorem, there exists a unique (up to translation) convex body K such that $S_K = \mu$.

If B denotes the unit ball of $\mathbb{R}^n$, we see that S_B coincides with $\mathscr{H}^{n-1}$, the $(n-1)$-dimensional Hausdorff measure.

The *Blaschke sum* of two convex bodies K and L is the body whose area measure is $S_K + S_L$. We shall write $r \cdot K \# s \cdot L$, $r, s \geqq 0$, to denote the convex body having area measure equal to $rS_K + sS_L$. Such a body is only determined up to a translation. Therefore, when we use the Blaschke addition, it is meant that we consider the equivalence classes with respect to the group of translations.

An interesting aspect of the Blaschke addition is its link with the *brightness function* of convex bodies. We recall that by the brightness of K in the direction v we mean the $(n-1)$-dimensional Hausdorff measure of the orthogonal projection of K onto $v^\perp$. Let us denote it by $V(K|_{v^\perp})$.

By the projection formula the brightness function at v can be expressed as follows:

$$V(K|_{v^\perp}) = \frac{1}{2} \int_{S^{n-1}} |\langle v, z\rangle| \, dS_K(z) \ .$$

This formula relates the brightness of K to S_K. In particular it shows that the class $\mathfrak{B}_K$ of all convex bodies having the same brightness function as K can be seen as a convex subset of $\mathscr{K}^n$ endowed with the Blaschke addition.

Furthermore, since the subspace spanned by the functions $z \mapsto |\langle z, v\rangle|$, $v \in S^{n-1}$, is dense in the space of all real even continuous functions on S^{n-1}, we have that two centered convex bodies of dimension n having the same brightness function must coincide.

The projection formula even shows that $V(K|_{v^\perp})$ is a mixed volume. Since we use just a little part of the results on mixed volumes, we shall introduce some of them in the shortest (even if unnatural) way.

For K, $L \in \mathscr{K}^n$, let us define the mixed volume $V_1(K, L)$ by means of

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(z) \, dS_K(z).$$

The Minkowski inequality asserts that

$$V_1(K, L) \geqq V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}},$$

where equality holds if and only if K and L are homothetic. Setting $L = K$ we deduce that $V_1(K, K) = V(K)$. Thus the volume can be expressed in terms of both the support function and the area measure of the body.

Among all the consequences of the last inequality we are interested in the Kneser-Süss inequality:

$$V(r \cdot K \# (1-r) \cdot L)^{\frac{n-1}{n}} \geqq r V(K)^{\frac{n-1}{n}} + (1-r) V(L)^{\frac{n-1}{n}} \ , \forall r \in [0,1] \ ,$$

where equality holds if and only if K and L are homothetic. Hence a positive power of the volume is a concave function on $\mathscr{K}^n$. Since $\mathfrak{B}_K$ is a convex set, it admits a unique body of maximum volume (up to translation); furthermore we realize that it is $\frac{1}{2} \cdot (K \# - K)$, the only centrally symmetric element of $\mathfrak{B}_K$.

On the other hand, the bodies of minimum volume in $\mathfrak{B}_K$ must be indecomposable with respect to the Blaschke addition. The class of all elements which are indecomposable in $\mathfrak{B}_K$ is denoted by $\mathfrak{I}_K$. A characterization of the elements of $\mathfrak{I}_K$ in terms of the support of their area measures can be found in [2].

In the next section we shall give a further necessary condition for a body of constant brightness to be of minimal volume.

3. Minimizers in $\mathfrak{B}_B$. Now we deal with the problem of finding the bodies of minimal volume in the class of all convex bodies of prescribed constant brightness in $\mathbb{R}^n$.

As we have seen, every minimizer in $\mathfrak{B}_B$ is an element of $\mathfrak{I}_B$. For our aim the following characterization proved in [2] is needed.

Theorem 1 (see [2], Theorem A). *A convex body K belongs to $\mathfrak{I}_B$ if and only if there exists a Borel subset E_K of S^{n-1} such that*

(i) *E_K is antisymmetric, i.e. $z \in E_K$ if and only if $-z \notin E_K$, $\forall z \in S^{n-1}$;*
(ii) *$S_K(\omega) = 2 \, \mathscr{H}^{n-1}(E_K \cap \omega)$, for every Borel subset ω* of S^{n-1}.

Consequently, the volume of a body $K \in \mathfrak{I}_B$ is given by $\frac{2}{n} \int_{E_K} h_K(z)\, dz$, where E_K is an antisymmetric subset of S^{n-1} and dz denotes integration with respect to $\mathscr{H}^{n-1}$. In order to minimize the volume we expect to integrate h_K over the subset of S^{n-1} where its values are as small as possible.

More precisely we assert:

Theorem 2. *Let K be a minimizer in $\mathfrak{B}_B$. Then there exists an open set $E \subset S^{n-1}$ and a vector $v \in \mathbb{R}^n$ such that*

(i) $S_K(\omega) = 2\, \mathscr{H}^{n-1}(E \cap \omega)$, *for every Borel subset ω of S^{n-1};*

(ii) $h_{K-v}(z) < h_{K-v}(-z)$ *if and only if $z \in E$.*

In order to prove this theorem we need some preliminary lemmas.

Lemma 1. *Let $K \in \mathfrak{I}_B$. Then*

$$\mathscr{H}^{n-1}(\{z \in S^{n-1} : h_{K-v}(z) = h_{K-v}(-z)\}) = 0\,,$$

for every $v \in \mathbb{R}^n$.

Proof. Set $F^0_{K-v} = \{z \in S^{n-1} : h_{K-v}(z) = h_{K-v}(-z)\}$ and suppose $\mathscr{H}^{n-1}(F^0_K) > 0$. We know that there exists an antisymmetric subset $E_K \subset S^{n-1}$, such that

$$S_K(\omega) = 2\, \mathscr{H}^{n-1}(\omega \cap E_K)\ , \quad \text{for every Borel subset } \omega \text{ of } S^{n-1}.$$

Since F^0_K is symmetric, $F^0_K \cap E_K$ has positive measure. According to the Lebesgue-Besicovitch differentiation theorem (see [3], Theorem 1, p. 43), applied to the indicator function of $F^0_K \cap E_K$ and the restriction of $\mathscr{H}^{n-1}$ to ∂S^{n-1}, there exist $z_0 \in S^{n-1}$ and $r > 0$ such that

$$\text{(3.1)} \qquad \frac{\mathscr{H}^{n-1}(B_{(z_0,\rho)} \cap F^0_K \cap E_K)}{\mathscr{H}^{n-1}(B_{(z_0,\rho)})} \geqq \frac{2}{3}\ , \ \forall \rho \in (0,r],$$

where $B_{(z,r)}$ is the set of all points of S^{n-1} whose spherical distance from z is less than r.

We denote by $\gamma : S^{n-1} \to \partial K$ the reverse of the Gauss map, which is a well defined map almost everywhere on S^{n-1}.

From the Lipschitz property of h_K, it follows $\nabla_S h_K(u) = -\nabla_S h_K(-u)$ a.e. in F^0_K (see [3], Corollary 1, p. 84); moreover, if h_K is differentiable at u, then $h_K(u)$ and $\nabla_S h_K(u)$ uniquely determine $\gamma(u)$. Hence we obtain

$$\text{(3.2)} \qquad \gamma(u) = -\gamma(-u), \ \text{a.e in } F^0_K.$$

Consider now the set $I = \gamma(B_{(z_0,r)}) \cap -\gamma(B_{(-z_0,r)})$. We can write

$$\begin{aligned}\mathscr{H}^{n-1}(I) &\overset{(3.2)}{\geqq} \mathscr{H}^{n-1}(\gamma(B_{(z_0,r)} \cap F^0_K)) = S_K(B_{(z_0,r)} \cap F^0_K) \\ &= 2\,\mathscr{H}^{n-1}(B_{(z_0,r)} \cap F^0_K \cap E_K) \overset{(3.1)}{\geqq} \frac{4}{3}\, \mathscr{H}^{n-1}(B_{(z_0,r)}).\end{aligned}$$

On the other hand we have:

$$\begin{aligned}\mathscr{H}^{n-1}(I) &\leqq \mathscr{H}^{n-1}(\gamma(B_{(-z_0,r)})) = S_K(B_{(-z_0,r)}) = 2\,\mathscr{H}^{n-1}(B_{(-z_0,r)} \cap E_K) \\ &= 2\,\mathscr{H}^{n-1}(B_{(z_0,r)}) - 2\,\mathscr{H}^{n-1}(B_{(z_0,r)} \cap E_K) \overset{(3.1)}{\leqq} \frac{2}{3}\mathscr{H}^{n-1}(B_{(z_0,r)}).\end{aligned}$$

This contradiction concludes the proof of Lemma 1. □

Lemma 2. *For every $K \in \mathfrak{I}_B$ there exists a vector $v \in \mathbb{R}^n$ such that*

$$\int_{F_{K-v}} z\, dz = 0,$$

where $F_{K-v} = \{z \in S^{n-1} : \ h_{K-v}(z) < h_{K-v}(-z)\}$.

Proof. From the previous lemma we deduce that F_{K-v} is not empty for every $v \in \mathbb{R}^n$.

Let us define the map $f : \mathbb{R}^n \to \kappa_{n-1}B$, where κ_{n-1} is the $(n-1)$-dimensional volume of the unit ball in $\mathbb{R}^{n-1}$, by

$$f(x) = \int_{F_{K-x}} z\, dz.$$

We have to prove that $f^{-1}(0)$ is not empty.

Let $\{x_m\}$ be a sequence in $\mathbb{R}^n$ converging to x. Outside of the set $\{z \in S^{n-1} : h_{K-x}(z) = h_{K-x}(-z)\}$, the indicator functions of the sets F_{K-x_m} converge to the indicator function of F_{K-x}. The previous lemma and Lebesgue's bounded convergence theorem then yield the continuity of f.

Moreover, there exists a constant $\lambda > 0$ such that

$$\langle f(x), x\rangle > 0\,, \ \forall x \in \partial\lambda B = \{x \in \mathbb{R}^n : \ \|x\| = \lambda\}. \tag{3.3}$$

To see this, let rB be a ball containing K; from the inequality $h_K(z) + h_K(-z) \geqq 0$ we deduce

$$\{z \in S^{n-1} : \langle z, x\rangle > r\} = \{z \in S^{n-1} : h_{rB-x}(z) < 0\} \subset F_{K-x}\,, \ \forall x \in \mathbb{R}^n\,.$$

So F_{K-x} approaches the hemisphere $\{z \in S^{n-1} : \langle z, x\rangle > 0\}$ as $\|x\|$ tends to infinity. This implies (3.3).

From (3.3) and the Poincaré-Bohl theorem (Theorem 2.1.5 in [6]) it follows that the degree of f at 0 relative to $\operatorname{int}(\lambda B)$ is 1 as the degree of the identity map. Therefore (Theorem 2.1.1 in [6]), $f^{-1}(0)$ is not empty. $\square$

We notice that the vector of Lemma 2 is unique. This can be shown without difficulty, but the proof is not carried out, since it is not needed.

Proof of Theorem 2. We take $E = F_{K-v}$ and v the vector of Lemma 2. We have to prove that E_K and F_{K-v} coincide up to a negligible set.

Let L be the convex body whose area measure is given by

$$S_L(\omega) = 2\,\mathscr{H}^{n-1}(\omega \cap F_{K-v})\,, \ \text{for every Borel subset } \omega \text{ of } S^{n-1}.$$

Since F_{K-v} is, up to a negligible set, an antisymmetric subset of S^{n-1}, Theorem 1 implies that $L \in \mathfrak{I}_B$; besides we can write

$$V(K) = \frac{2}{n}\int_{E_K} h_{K-v}(z)\, dz \geqq \frac{2}{n}\int_{F_{K-v}} h_{K-v}(z)\, dz = V_1(L, K).$$

From the Minkowski inequality we deduce $V(K) \geqq V(L)$; since K is a minimizer, K and L must be homothetic, that is E_K and F_{K-v} coincide up to a negligible set. $\square$

4. Bodies of revolution. In this section we shall see how Theorem 2 permits us to find the unique (up to rigid motion) body of minimum volume in the class $\mathfrak{R}_B$ of all bodies of revolution in $\mathfrak{B}_B$.

Let us try to give Theorem 2 a geometrical meaning. Henceforth we suppose $v = 0$. Consider the convex hull MK of the union of K and $-K$, that is

$$MK = \operatorname{conv}(K \cup (-K)) \ .$$

Clearly MK is a centrally symmetric convex body. We state that it coincides with SK, the closure of the convex hull of the singular points of both K and its reflection. Obviously $SK \subset MK$, thus it is enough to prove that SK contains K.

Since $-F_K$ is an open subset of S^{n-1}, Lemma 4.6.2 in [7] implies that the reverse Gauss map carries every point of $-F_K$ in at least one singular point of K. Then for every direction $u \in -F_K$ we have $h_K(u) \leqq h_{SK}(u)$; on the other hand, for $u \in F_K$, we have $h_K(u) < h_{-K}(u) \leqq h_{SK}(u)$. Lemma 1 implies $h_K \leqq h_{SK}$ almost everywhere on S^{n-1}, and thus $K \subset SK$.

Such a requirement is clear in the two-dimensional case. Indeed $\mathfrak{B}_B$ is the class of convex sets of constant width, $\mathfrak{I}_B$ contains the class of Reuleaux polygons, while it is not hard to prove that conditions (i) and (ii) of Theorem 2 are only satisfied by the regular Reuleaux polygons.

Let us turn to the rotational case. It is evident how Theorem 1 and Theorem 2 can be rephrased in this particular case.

We call n-dimensional *Blaschke-Firey body* the unique (up to rigid motion) element K of $\mathfrak{R}_B$ whose area measure is given by

$$S_K(\omega) = 2\mathscr{H}^{n-1}(\omega \cap E_K)\,, \quad \text{for every Borel subset } \omega \text{ of } S^{n-1},$$

where

$$E_K = \left\{ z \in S^{n-1} : \langle z, u \rangle > \frac{1 - \sqrt[n-1]{\frac{1}{4}}}{\langle z, u \rangle} \right\},$$

for some $u \in S^{n-1}$. Simple calculations show that $\int\limits_{S^{n-1}} z\, dS_K(z) = 0$.

In spite of the seemingly artificial definition this appears to be the most natural extension of both the Reuleaux triangle and the body constructed in [1] to higher dimension. The n-dimensional Blaschke-Firey body satisfies conditions (i) and (ii) of Theorem 2 and MK is the revolution of a hexagon.

We state the following:

Theorem 3. *The n-dimensional Blaschke-Firey body is the unique (up to rigid motion) element of minimal volume in the class* $\mathfrak{R}_B \subset \mathscr{K}^n$.

Proof. Let K be an element of minimum volume in $\mathfrak{R}_B \subset \mathscr{K}^n$. Consider a plane through its axis of revolution and fix on it an orthogonal system $(O; x, y)$ so that the origin is just the symmetry center of MK (that is v in Lemma 2), and the y-axis is the axis of revolution of K.

We call K', MK' and F'_K the intersections of the relevant sets with the xy plane. As an open subset of S^1, F'_K is the union of at most countably many connected open arcs. Parametrize S^1 by the angle formed with the y-axis and let (φ_1, φ_2) be a connected component of F'_K. Suppose that $0 < \varphi_1 < \varphi_2 \leqq \frac{\pi}{2}$. The reverse Gauss map carries (φ_1, φ_2) onto an arc γ joining two points, P_1 and P_2, of $\partial K'$. Notice that the reverse Gauss map is well defined everywhere on S^1 since K' is strictly convex.

Rotating in $\mathbb{R}^n$ the arc γ around the y-axis, gives a C^1-regular and strictly convex hypersurface Γ, whose surface area measure is twice $\mathscr{H}^{n-1}$. We shall prove that Γ is analytic.

Let $(x(t), y(t))$, $t \in (\varphi_1, \varphi_2)$, be the parametrization of the arc γ induced by the reverse Gauss map. Assuming Γ of class C^2, we have $y'(t) = -\tan(t)x'(t)$ and we deduce a condition on its gaussian curvature which corresponds to the following differential equation

$$\frac{\cos t \sin^{n-2} t}{x'(t)x^{n-2}(t)} = \frac{1}{2}.$$

Solving this system of differential equations in (φ_1, φ_2) with initial conditions $x(\varphi_2) = x_2$, $y(\varphi_2) = y_2$ yields

$$\tilde{\gamma}(t) = \left((2\sin^{n-1} t - a)^{\frac{1}{n-1}},\ y_2 + \int_t^{\varphi_2} \frac{2\sin^{n-1}\tau}{(2\sin^{n-1}\tau - a)^{\frac{n-2}{n-1}}}\, d\tau \right), \tag{4.1}$$

where a is determined by $a = 2\sin^{n-1}\varphi_2 - x_2^{n-1}$ and where x_i, y_i are the coordinates of P_i, $i = 1, 2$.

Form the surface $\tilde{\Gamma}$ of revolution with $\tilde{\gamma}$ as meridian. The projections of $\tilde{\Gamma}$ and Γ on $y^{\perp}$ have the same $(n-1)$-dimensional volumes and then $2\sin^{n-1}\varphi_1 - a = x_1^{n-1}$. Therefore, $\operatorname{conv}\Gamma$ and $\operatorname{conv}\tilde{\Gamma}$ have the same area measures, hence by Minkowski's uniqueness theorem they are translates of each other. This implies $\gamma = \tilde{\gamma}$.

Consider now the upper part of $K \setminus \operatorname{conv}\Gamma$. Estimating its brightness in the direction y by means of the projection formula, we find

$$2\kappa_{n-1}\sin^{n-1}\varphi_1 \geqq \kappa_{n-1}x_1^{n-1},$$

hence the constant a in (4.1) is non-negative and the function $x(t)$ is concave in (φ_1, φ_2).

Conditions (i) and (ii) of Theorem 2 imply that the point Q, intersection of the tangent lines at P_1 and P_2, belongs to $-K'$. Furthermore the segments P_1Q and P_2Q are in the boundary of MK'.

Let x_Q be the x-coordinate of Q. We have

$$\begin{aligned} y_1 - y_2 &= (x_Q - x_1)\tan\varphi_1 + (x_2 - x_Q)\tan\varphi_2 \\ &= \big(x(t) - x_Q\big)\tan t \Big|_{t=\varphi_1}^{t=\varphi_2} \\ &= \int_{\varphi_1}^{\varphi_2} \frac{x(t) - x_Q}{\cos^2 t}\, dt + \int_{\varphi_1}^{\varphi_2} x'(t)\tan t\, dt \\ &= \int_{\varphi_1}^{\varphi_2} \frac{x(t) - x_Q}{\cos^2 t}\, dt - \int_{\varphi_1}^{\varphi_2} y'(t)\, dt. \end{aligned}$$

Therefore

$$\int_{\varphi_1}^{\varphi_2} \frac{x(t)}{\cos^2 t}\, dt = \int_{\varphi_1}^{\varphi_2} \frac{x_Q}{\cos^2 t}\, dt. \tag{4.2}$$

Since $x(t)$ is a concave increasing function in $[\varphi_1, \varphi_2]$, we can write

$$\begin{aligned} \int_{\varphi_1}^{\varphi_2} \frac{x(t)}{\cos^2 t}\, dt &= \frac{1}{2}\int_{\varphi_1}^{\varphi_2} \left[\frac{x(t)}{\cos^2 t} + \frac{x(\varphi_1 + \varphi_2 - t)}{\cos^2(\varphi_1 + \varphi_2 - t)}\right] dt \\ &= \frac{1}{2}\int_{\varphi_1}^{\varphi_2} \left[\frac{x(t)}{\cos^2 t} + \frac{x(\varphi_1 + \varphi_2 - t)}{\cos^2 t} + x(\varphi_1 + \varphi_2 - t) \right. \end{aligned}$$

$$\times\left(\frac{1}{\cos^2(\varphi_1+\varphi_2-t)}-\frac{1}{\cos^2 t}\right)\Bigg]\,dt$$

$$\geqq \frac{1}{2}\int_{\varphi_1}^{\varphi_2}\frac{x_1+x_2}{\cos^2 t}\,dt+\frac{1}{2}\int_{\varphi_1}^{\frac{\varphi_1+\varphi_2}{2}}[x(\varphi_1+\varphi_2-t)-x(t)]$$

$$\times\left(\frac{1}{\cos^2(\varphi_1+\varphi_2-t)}-\frac{1}{\cos^2 t}\right)dt \geqq \frac{1}{2}\int_{\varphi_1}^{\varphi_2}\frac{x_1+x_2}{\cos^2 t}\,dt\,.$$

Therefore, from (4.2), we get

$$x_Q \geqq \frac{x_1+x_2}{2}\,. \tag{4.3}$$

Let R be the projection on the x-axis of the singular points of MK'. The set R divides $[0,1]$ in at most countably many intervals. Inequality (4.3) implies that every interval is larger than any other on its right-hand side. Hence R has no cluster points in the interior of $[-1,1]$.

Thus in the half-plane $\{x>0\}$ there are two singular points of MK' at a minimal distance from the axis of revolution. Let Q_1 be the one with positive y-coordinate and call r_1 its x-coordinate. We can assume $Q_1 \in K'$.

The point Q_1 is the endpoint of an analytic arc of $\partial K'$ which cuts orthogonally the y-axis. Then such an arc is a circular arc of radius $2^{\frac{1}{n-1}}$, and we deduce $r_1 = \sqrt[n-1]{2}\sin\alpha_1$, where $(-\alpha_1,\alpha_1)$ is a connected component of F'_K.

Now we set $r_0=\alpha_0=0$ and we label the x-coordinates of the singular points of MK' in the increasing order. Let Q_i be the point of $\partial MK'$ with r_i as x-coordinate and positive y-coordinate. Let α_i denote the angle formed by the y-axis and the outward normal to MK' at the edge $Q_{i-1}Q_i$.

The arc of $\partial(-K')$ joining Q_0 to Q_2 admits the parametrization (4.1) with $\varphi_1=\alpha_1$, $\varphi_2=\alpha_2$ and $x_1=0$. This implies $a=r_1^{n-1}$ and $x_2=r_2=\sqrt[n-1]{2\sin^{n-1}\alpha_2-r_1^{n-1}}$.

The arc of $\partial(K')$ joining Q_1 to Q_3 admits the parametrization (4.1) with $\varphi_1=\alpha_2$, $\varphi_2=\alpha_3$ and $x_1=r_1$. This implies $a=2\sin^{n-1}\alpha_2-r_1^{n-1}$ and then $a=r_2^{n-1}$.

Repeating the same argument, it is easy to verify by induction that the arc of $\partial K'$ or of $\partial(-K')$ joining Q_{i-1} and Q_{i+1} is represented by

$$\gamma(t)=\left((2\sin^{n-1}t-r_i^{n-1})^{\frac{1}{n-1}},\ y(Q_{i+1})+\int_t^{\alpha_{i+1}}\frac{2\sin^{n-1}\tau}{(2\sin^{n-1}\tau-r_i^{n-1})^{\frac{n-2}{n-1}}}\,d\tau\right), \tag{4.4}$$

$$t\in(\alpha_i,\alpha_{i+1}),\ i\geqq 1.$$

Rewriting (4.2) yields

$$\int_{\alpha_i}^{\alpha_{i+1}}\frac{(2\sin^{n-1}\tau-r_i^{n-1})^{\frac{1}{n-1}}}{\cos^2\tau}\,d\tau=\int_{\alpha_i}^{\alpha_{i+1}}\frac{r_i}{\cos^2\tau}\,d\tau\ ,\quad i\geqq 1\,. \tag{4.5}$$

From (4.4) we deduce

$$r_{i-1}^{n-1}+r_i^{n-1}=2\sin^{n-1}\alpha_i\ ,\quad i\geqq 1\,. \tag{4.6}$$

Equalities (4.5) and (4.6) show that K is uniquely determined (up to rigid motion) by r_1.

Next we shall write the volume of K as a function of the sequence $\{r_i\}$ and compare it with the volume of the Blaschke-Firey body, which corresponds to the pair $\{0,1\}$.

For every $x \in [0,1]$ let us denote with $\ell_K(x)$ the length of the chord of K' at a distance x from the y-axis and parallel to it. The inverse function of $x(t)$ in (4.4) is given by

$$t = \arcsin\left(\frac{x^{n-1} + r_i^{n-1}}{2}\right)^{\frac{1}{n-1}}, \quad x \in (r_{i-1}, r_{i+1}).$$

Recalling that $\frac{d}{dx}(y(t(x))) = -\tan t(x)$, we obtain the explicit formula

$$\ell_K'(x) = -\frac{\left(\frac{x^{n-1} + r_i^{n-1}}{2}\right)^{\frac{1}{n-1}}}{\sqrt{1-\left(\frac{x^{n-1} + r_i^{n-1}}{2}\right)^{\frac{2}{n-1}}}} - \frac{\left(\frac{x^{n-1} + r_{i+1}^{n-1}}{2}\right)^{\frac{1}{n-1}}}{\sqrt{1-\left(\frac{x^{n-1} + r_{i+1}^{n-1}}{2}\right)^{\frac{2}{n-1}}}}, \quad \forall x \in (r_i, r_{i+1}). \tag{4.7}$$

Taking into account the identity

$$V(K) = (n-1)\kappa_{n-1}\int_0^1 x^{n-2}\ell_K(x)\,dx = -\kappa_{n-1}\int_0^1 x^{n-1}\ell_K'(x)\,dx$$

yields

$$V(K) = \sum_{i=0,1,\dots} \kappa_{n-1}\int_{r_i}^{r_{i+1}} \frac{x^{n-1}\left(\frac{x^{n-1} + r_i^{n-1}}{2}\right)^{\frac{1}{n-1}}}{\sqrt{1-\left(\frac{x^{n-1} + r_i^{n-1}}{2}\right)^{\frac{2}{n-1}}}} + \frac{x^{n-1}\left(\frac{x^{n-1} + r_{i+1}^{n-1}}{2}\right)^{\frac{1}{n-1}}}{\sqrt{1-\left(\frac{x^{n-1} + r_{i+1}^{n-1}}{2}\right)^{\frac{2}{n-1}}}}\,dx. \tag{4.8}$$

In order to prove the minimality of the Blaschke-Firey body it is sufficient to show that the right-hand side of (4.8) decreases eliminating r_1 from the sequence $\{r_i\}$. Notice that the sequence $\{0, r_2, r_3, \dots\}$ does not necessarily correspond to a convex body. In case $\{r_i\}$ is not finite the conclusion will then follow by an asymptotic argument.

Writing $f(x,y) = \frac{\left(\frac{x^{n-1} + y^{n-1}}{2}\right)^{\frac{1}{n-1}}}{\sqrt{1-\left(\frac{x^{n-1} + y^{n-1}}{2}\right)^{\frac{2}{n-1}}}}$, what we have to prove is that

$$\int_0^{r_{i+1}} x^{n-1} f(x, r_i)\,dx - \int_0^{r_i} x^{n-1} f(x, r_{i+1})\,dx - \int_{r_i}^{r_{i+1}} x^{n-1} f(x,0)\,dx > 0, \quad \forall i \geqq 1. \tag{4.9}$$

The left-hand side of (4.9), as a function of r_{i+1}, is zero in r_i and has positive derivative in $\left[r_i, \sqrt[n-1]{2 - r_i^{n-1}}\right)$. Indeed by differentiating (4.9) we get

$$\begin{aligned}
&r_{i+1}^{n-1} f(r_i, r_{i+1}) - \int_0^{r_i} x^{n-1}\frac{\partial f}{\partial r_{i+1}}(x, r_{i+1})\,dx - r_{i+1}^{n-1} f(r_{i+1}, 0)\\
&\quad = r_{i+1}^{n-1}[f(r_i, r_{i+1}) - f(0, r_{i+1})] - \int_0^{r_i} x r_{i+1}^{n-2}\frac{\partial f}{\partial x}(x, r_{i+1})\,dx\\
&\quad = r_{i+1}^{n-1}[f(r_i, r_{i+1}) - f(0, r_{i+1})] - r_{i+1}^{n-2}\left[r_i f(r_i, r_{i+1}) - \int_0^{r_i} f(x, r_{i+1})\,dx\right]\\
&\quad = r_{i+1}^{n-2}\left[\int_0^{r_i} f(x, r_{i+1})\,dx + (r_{i+1} - r_i) f(r_i, r_{i+1}) - r_{i+1} f(0, r_{i+1})\right]
\end{aligned}$$

which is clearly positive, since $f(x, r_{i+1})$ is a positive and strictly increasing function with respect to x.

This concludes the proof. □

As a final remark, if β_n is the volume of the Blaschke-Firey body in $\mathbb{R}^n$, by using (4.8), it is possible to see that the ratio $\frac{\kappa_n}{\beta_n}$ tends to 1 as n goes to infinity. Nevertheless the diameter of the n-dimensional Blaschke-Firey body tends to infinity with n.

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