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Ventcel's boundary conditions and analytic semigroups

By

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Abstract. In this paper we prove that first-order degenerate ordinary differential operators generate analytic semigroups in spaces of continuous functions if Ventcel's boundary conditions are assumed. Some applications to different types of degeneration are also discussed.

1. Introduction. In this paper we prove the analyticity for the semigroups generated by some degenerate ordinary differential operators in spaces of continuous functions on a real interval I. We deal with first-order degeneracy, in the sense that we allow either the coefficient of the second order term to vanish at the endpoints like the distance from the boundary of I, or the coefficient of the first order term to behave as the inverse of the above distance.

We assume Ventcel's boundary conditions, i.e., if $(L, (D(L))$ is our operator, we impose that $\lim_{x \to \partial I} Lu(x) = 0$ for every $u \in D(L)$. Observe that for the evolution problem $u_t = Au$, $u(0) = u_0$, Ventcel's conditions are equivalent to the classical homogeneous Dirichlet conditions provided that the initial datum u_0 vanishes at the boundary.

The study of one-dimensional degenerate evolution problems under Ventcel's boundary conditions started with the work of Feller (see [11] and [12]) and is motivated by some diffusion problems. It turned out to be useful in many fields both in pure and in applied mathematics. Applications to models in genetics can be found in [8] and connections with approximation processes in [4] and in [1]. The subsequent work of Clément and Timmermans (see [6]) characterized exactly when a second order differential operator generates a C_0-semigroup in spaces of continuous functions under the above boundary conditions.

The problem of the regularity of the semigroup has been left open for a long time. Recently the analyticity has been proved for operators like $a(x)D^2 + b(x)D$ on $C([0,1])$ if $a(x) > 0$ on $]0,1[$, $a(0) = a(1) = 0$, $\sqrt{a} \in C^1([0,1])$ and $b/\sqrt{a}$ bounded (see [10]). These assumptions force a to have at least double zeros at $0, 1$ and exclude first order degeneration which is the most natural and important in the applications (see however [9] for a special case of simple degeneration in weighted L^p-spaces).

The analyticity of semigroups generated by self-adjoint second order degenerate operators like $D[a(x)D]$ has been proved in [5] in the L^p setting, $1 < p < +\infty$, for general a. The case of first-order degeneracy in spaces of continuous functions has been studied in [14] which is the starting point of the present investigation.

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Our operators have the form

$$A = m(x)\left[D^2 + \frac{b(x)}{x}D\right] \quad \text{or} \quad B = a(x)\left[x^\alpha D^2 + b(x)x^\sigma D + c(x)\right]$$

on the half-line $[0, +\infty[$, and

$$A_1 = m(x)[x(1-x)D^2 + b(x)D] \quad \text{or} \quad A_2 = m(x)\left[D^2 + \frac{b(x)}{x(1-x)}D\right]$$

on the interval $[0,1]$. We shall suppose m, a, b, c continuous and bounded with $\inf_{x\in I} m(x) > 0$ and $\inf_{x\in I} a(x) > 0$.

By Clément and Timmermans' result, the possibility of assuming Ventcel's conditions depends on the values of the function b at the boundary of I. In particular, for the operator A we have to require $b(0) < 1$ and, under this condition, it generates a C_0-semigroup. Similar conditions hold for the operators A_1, A_2 and B. The operator A (as well as A_1 and A_2) has been considered in [14] where it is proved that it generates an analytic semigroup under Ventcel's conditions but only in the case where $b(0) \leqq -1$ while different boundary conditions have been assumed if $b(0) > -1$ (see also [2]). In Sections 2 and 3 we extend the results of [14] for the operator A to the case $b(0) < 1$ using similar but slightly modified techniques, showing that it generates analytic semigroups on different spaces of continuous functions on $[0, +\infty]$. These changes allow us to drop the assumption of Hölder continuity of the function b at the endpoints and to assume only Dini continuity. The analyticity of the semigroups generated by the operators A_1 and A_2 is deduced from that of the semigroup generated by A via a localization procedure described in Proposition 2.4 which can be of independent interest. Applications to more general degree of degeneracy are described in Section 4.

Observe that it is not true, in general, that the semigroup generated by an operator $(L, (D(L))$ is always analytic when Ventcel's boundary conditions are imposed. For example, consider the space $C([-\infty, +\infty])$ of all continuous functions on $\mathbb{R}$ having finite limits at $\pm\infty$ and the operator $(L, (D(L))$ given by

$$L = D^2 + xD, \quad D(L) = \left\{u \in C([-\infty, +\infty]) \cap C^2(\mathbb{R}) : \lim_{x\to\pm\infty} Lu(x) = 0\right\}.$$

It is easy to see that Clément-Timmermans condition holds whence $(L, D(L))$ generates a C_0-semigroup on $C([-\infty, +\infty])$. In fact this semigroup is just the Ornstein-Uhlenbeck semigroup considered in [7] and the same computations as in [7, Lemma 3.3] show that the semigroup is not analytic. By a change of variable one can produce similar counterexamples on a bounded interval, e.g., putting $x = \tan s$ one obtains that the operator

$$(\cos^4 s)D^2 + \sin s \cos s(1 + 2\cos^2 s)D$$

generates a C_0-semigroup on $C([-\pi/2, \pi/2])$ which is not analytic. Note that the degeneration is of order 4 at the endpoints and that the sufficient condition of [10] for the analyticity in presence of high-order zeros does not hold.

N o t a t i o n. We shall study our differential operators on a space $E([a,b])$ of continuous functions defined on a real (bounded or not) interval.

If $-\infty < a < b < +\infty$, the space $E([a,b])$ is defined by putting $E([a,b]) = C([a,b])$.

If one of the endpoints is infinite, say $b = +\infty$, we prescribe for $u \in E([a, b])$ one of the following conditions at $+\infty$:

1) $\lim\limits_{x \to +\infty} u(x) = 0$;
2) there exists $\lim\limits_{x \to +\infty} u(x) \in \mathbb{C}$;
3) u is bounded on a neighborhood of $+\infty$;
4) u is bounded and uniformly continuous on a neighborhood of $+\infty$

and we define the corresponding spaces

$$E_i([a, +\infty]) = \{u \in C([a, +\infty[) : u \text{ satisfies } (i)\}, \quad i = 1, \dots, 4,$$

endowed with the sup norm $\|\cdot\|_{E_i([a,b])}$.

Similar definitions hold if $a = -\infty$. We shall drop the index i and write $E([a, b])$ if it is not necessary to distinguish between the different boundary conditions at infinity.

Occasionally we shall write $C([0, +\infty])$ for $E_2([0, +\infty])$ and $C_b([0, +\infty[)$ for $E_3([0, +\infty])$.

With $C^k(]0, +\infty])$ we denote the set of all functions $u \in C^k(]0, +\infty[)$ such that the limits $\lim\limits_{x \to +\infty} u^{(i)}(x)$ exist finite for $i = 0, \dots, k$.

The symbol **1** denotes the constant function of value 1.

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2. First-order degeneracy on $[0, +\infty[$. In this section we study the operator

$$A = m(x)\left[D^2 + \frac{b(x)}{x} D\right]$$

and we shall assume the following hypotheses on the coefficients m and b:

(i) m uniformly continuous and bounded on $[0, +\infty[$ with $\inf\limits_{x \geqq 0} m(x) > 0$;

(ii) b continuous on $[0, +\infty[$, $b(0) < 1$ and $b(x)/x$ bounded on a neighborhood of $+\infty$; in the case of $E_4([0, +\infty])$ we also assume $b(x)/x$ uniformly continuous on a neighborhood of $+\infty$;

(iii) $\dfrac{b(x) - b(0)}{x}$ summable on a neighborhood of 0.

As remarked in the introduction, the hypothesis $b(0) < 1$ is necessary and sufficient to impose Ventcel's condition at $x = 0$. Accordingly, we define

$$D(A) = \{u \in E([0, +\infty]) \cap C^2(]0, +\infty[) : u', u'' \in E([1, +\infty]), \lim_{x \to 0} Au(x) = 0\}. \tag{2.1}$$

Clearly $D(A)$ is dense in $E([0, +\infty])$ and $(A, D(A))$ is a linear operator on $E([0, +\infty])$.

Lemma 2.1. *The operator $(A, D(A))$ is closed and dissipative.*

Proof. Let $\delta > 0$ and C be such that $\left|\dfrac{b(x)}{x}\right| \leqq C$ for $x \geqq \delta$. Then, for every $\varepsilon > 0$, we can find a constant C_ε such that, for $x \geqq \delta$,

$$\left|\frac{b(x)}{x} u'(x)\right| \leqq C|u'(x)| \leqq \varepsilon \sup_{x \geqq \delta} |u''(x)| + C_\varepsilon \sup_{x \geqq \delta} |u(x)|.$$

Since $u''(x) = \dfrac{1}{m(x)} Au(x) - \dfrac{b(x)}{x} u'(x)$ we obtain

$$\sup_{x \geqq \delta} |u''(x)| \leqq C \sup_{x \geqq \delta} |Au(x)| + \varepsilon \sup_{x \geqq \delta} |u''(x)| + C_\varepsilon \sup_{x \geqq \delta} |u(x)|.$$

If we take $\varepsilon < 1$, we get for a suitable constant K (depending on δ)

$$\sup_{x \geqq \delta} |u''(x)| \leqq K\Big(\sup_{x \geqq \delta} |Au(x)| + \sup_{x \geqq \delta} |u(x)|\Big). \tag{2.2}$$

Consider now a sequence (u_n) in $D(A)$ converging to u in E and such that $Au_n \to v$. By (2.2) we obtain the uniform convergence of u_n' and u_n'' on the interval $[\delta, +\infty[$; since δ is arbitrary we deduce $u \in C^2(]0, +\infty[)$ and $Au = v$. Moreover $v(0) = \lim\limits_n Au_n(0) = 0$. It follows that $u \in D(A)$ and $Au = v$, that is A is closed.

In order to prove the dissipativity of A, we fix $\lambda > 0$ and show that the following inequalities hold:

$$\inf_{x \geqq 0} \big(\lambda u(x) - Au(x)\big) \leqq \lambda \inf_{x \geqq 0} u(x), \quad \lambda \sup_{x \geqq 0} u(x) \leqq \sup_{x \geqq 0} \big(\lambda u(x) - Au(x)\big). \tag{2.3}$$

Assume that $u(x_0) = \sup\limits_{x \geqq 0} u(x)$ for some $0 \leqq x_0 < +\infty$; then $Au(x_0) \leqq 0$ (indeed, this follows by $Au(0) = 0$ if $x_0 = 0$ and by $u'(x_0) = 0$ and $u''(x_0) \leqq 0$ if $x_0 > 0$), and hence the second inequality in (2.3) holds.

Now, suppose that $\sup\limits_{x \geqq 0} u(x) = \limsup\limits_{x \to +\infty} u(x)$ and that the supremum is not attained at any point.

If u' is definitively positive, we have $\sup\limits_{x \geqq 0} u(x) = \lim\limits_{x \to +\infty} u(x)$. Suppose by contradiction that $\liminf\limits_{x \to +\infty} Au(x) > 0$; then there exist $\delta > 0$ and $M > 0$ such that $\dfrac{Au(x)}{m(x)} \geqq \delta$ for every $x \geqq M$. Let $c > 0$ be such that $\left|\dfrac{b(x)}{x}\right| \leqq c$ for $x \geqq M$ and observe that $u'(x) \leqq \delta/(2c)$ implies $u''(x) \geqq \delta/2$ for $x \geqq M$. Since $u \in E([0, +\infty])$ we can find $x_1 > M$ and $x_2 > x_1$ such that $u'(x_1) > \delta/(2c)$ and $u'(x_2) < \delta/(2c)$. Hence the number

$$x_3 = \inf\left\{x > x_1 : u'(x) \leqq \frac{\delta}{2c}\right\}$$

is well defined and satisfies $x_3 > x_1$, $u'(x_3) = \delta/(2c)$ and $u''(x_3) \geqq \delta/2$, contradicting the definition of x_3.

This shows that $\liminf\limits_{x \to +\infty} Au(x) \leqq 0$ and consequently we can find a sequence (x_n) tending to $+\infty$ such that $\lim\limits_{n \to +\infty} Au(x_n) \leqq 0$. Then

$$\begin{aligned}\sup_{x \geqq 0} \lambda u(x) &= \lim_{n \to +\infty} \lambda u(x_n) \\ &\leqq \lim_{n \to +\infty} \big(\lambda u(x_n) - Au(x_n)\big) \leqq \sup_{x \geqq 0} \big(\lambda u(x) - Au(x)\big).\end{aligned}$$

Suppose now that u' is not definitively positive; we can find an increasing sequence (y_n) tending to $+\infty$ and such that

$$u(y_n) > \sup_{x \geqq 0} u(x) - \frac{1}{n}.$$

For every n, we consider a point $z_n > y_n$ satisfying $u(z_n) > u(y_n)$ and a point $w_n > z_n$ such that $u'(w_n) < 0$. Denote by x_n the maximum of u in the interval $[y_n, w_n]$. The point x_n is in the interior of $[y_n, w_n]$ and satisfies $u'(x_n) = 0$, $Au(x_n) \leqq 0$; moreover, the sequence $(u(x_n))$ converges to the supremum of u and thus we can proceed as before to obtain

$$\sup_{x \geqq 0} \lambda u(x) \leqq \sup_{x \geqq 0} \big(\lambda u(x) - Au(x)\big).$$

The first inequality in (2.3) follows by the second one applied to $-u$. □

Remark 2.2. Inequalities (2.3) show that for every $\lambda > 0$ the operator $(\lambda - A)^{-1}$ is positive whenever it exists. In the case of E_1 and E_2 the proof of dissipativity of A is easier since $\lim_{x \to +\infty} Au(x) = 0$ for every $u \in D(A)$ (see [6] in the case of E_1).

In order to prove that $(A, D(A))$ generates an analytic semigroup we consider first the case of constant coefficients, i.e., we assume $m = \mathbf{1}$ and $b(x) = b < 1$. The following theorem is stated in [14, Lemma 2.6 and Proposition 2.7] in the case $b \leqq -1$. Nevertheless, the same proof still holds in our case as one can easily check; moreover no new problems can arise at $+\infty$ since A is regular on $[\delta, +\infty[$ for $\delta > 0$. For these reasons we omit the details.

Theorem 2.3. *Let* $A = D^2 + \dfrac{b}{x} D$ *with* $b < 1$ *and* $D(A)$ *defined by* (2.1). *Then* A *generates an analytic semigroup of angle* $\pi/2$ *on* $E([0, +\infty])$. *The semigroup is positive and contractive.*

The following proposition allows us to localize near the endpoints the problem of analyticity under separated boundary conditions (see also [2]).

Proposition 2.4. *Let* $L = p(x)\big[D^2 + q(x)D\big]$ *be a dissipative second order differential operator on* $E([a,b])$, $-\infty \leqq a < b \leqq +\infty$, *such that* $D(L)$ *contains all* C^2*-functions compactly supported in* $]a,b[$.

Assume that p *is continuous and strictly positive in* $]a,b[$, $q \in C([a,b])$ *and that there exist differential operators* $L_0 : D(L_0) \to E([a,c])$, $L_2 : D(L_2) \to E([d,b])$ *with* $a < c \leqq +\infty$, $-\infty \leqq d < b$ *and cut-off functions* ϕ_0, ϕ_2 *with the following properties:*

1. L_0 *and* L_2 *generate analytic semigroups of angle* Θ *on* $E([a,c])$ *and* $E([d,b])$ *respectively.*
2. ϕ_0 *is a* C^∞*-function supported in* $[a,b[$, *equal to* 1 *in a neighborhood of* a *and to* 0 *in a neighborhood of* c, *such that for every* $u \in E([a,c]) \cap C^2(]a,c[)$, $u\phi_0 \in D(L_0)$ *implies* $u\phi_0 \in D(L)$ *and* $L(u\phi_0) = L_0(u\phi_0)$.
3. ϕ_2 *is a* C^∞*-function supported in* $]a,b]$, *equal to* 1 *in a neighborhood of* b *and to* 0 *in a neighborhood of* d, *such that for every* $u \in E([d,b]) \cap C^2(]d,b[)$, $u\phi_2 \in D(L_2)$ *implies* $u\phi_2 \in D(L)$ *and* $L(u\phi_2) = L_2(u\phi_2)$.

Then L *generates an analytic semigroup of angle* Θ.

Proof. We may assume that $0 \leqq \phi_i \leqq 1$ for $i = 0, 2$ and that the supports of ϕ_0, ϕ_2 are disjoint. Let ϕ_1 be another positive cut-off function such that $\phi_0^2 + \phi_1^2 + \phi_2^2 = 1$. Let $c_1, d_1 \in]a,b[$ be such that the support of ϕ_1 is contained in the interval $]c_1, d_1[$ and observe that $c_1 < c$, $d_1 > d$. Consider the operator $L_1 : D(L_1) \to C([c_1, d_1])$ defined as L with Neumann boundary conditions at the endpoints c_1 and d_1; L_1 generates an analytic semigroup of angle $\pi/2$.

By assumption 1., we can find a constant C and $R \geqq 0$ such that for every $|\lambda| \geqq R$ satisfying $|\arg \lambda| \leqq \Theta_1 < \Theta + \pi/2$, we have

$$\|(\lambda - L_i)^{-1}\| \leqq C/|\lambda|, \qquad i = 0, 1, 2 \tag{2.4}$$

(we consider the operatornorm defined by the space where L_i acts).

Consider the operator $S(\lambda) = \sum_{i=0}^{2} \phi_i(\lambda - L_i)^{-1}\phi_i : E([a,b]) \to D(L)$ which satisfies $\|S(\lambda)\| \leqq C_1/|\lambda|$. We have $L\phi_i(\lambda - L_i)^{-1}\phi_i = L_i\phi_i(\lambda - L_i)^{-1}\phi_i$ and consequently

$$(\lambda - L)S(\lambda) = I - \sum_{i=0}^{2}[L_i, \phi_i](\lambda - L_i)^{-1}\phi_i,$$

where $[L_i, \phi_i] = L_i\phi_i - \phi_i L_i$ is a first-order differential operator defined on $D(L_i)$ and supported on a compact subset $[\alpha_i, \beta_i]$ of $]a, b[$. It follows

$$\|[L_i, \phi_i]u\|_{E([a,b])} \leqq K\Big[\|u\|_{C([\alpha_i,\beta_i])} + \big(\|u\|_{C([\alpha_i,\beta_i])}\|u''\|_{C([\alpha_i,\beta_i])}\big)^{\frac{1}{2}}\Big] \tag{2.5}$$

for all $u \in D(L_i)$, with K a suitable positive constant. Since L_i is non-degenerate on $[\alpha_i, \beta_i]$ we have also

$$\|u''\|_{C([\alpha_i,\beta_i])} \leqq K_1\big[\|u\|_{C([\alpha_i,\beta_i])} + \|L_i u\|_{C([\alpha_i,\beta_i])}\big], \quad u \in D(L_i). \tag{2.6}$$

The estimate

$$\|[L_i, \phi_i](\lambda - L_i)^{-1}\phi_i f\|_{E([a,b])} \leqq \frac{K_2}{\sqrt{|\lambda|}}\|f\|_{E([a,b])}, \quad f \in E([a,b])$$

then follows from (2.4), (2.5) and (2.6) putting $u = (\lambda - L_i)^{-1}f$. Therefore if we take $|\lambda| \geqq R_1$ for a suitable R_1, we have $\|(\lambda - L)S(\lambda) - I\| < 1/2$.

It follows that for $|\lambda| \geqq R_1$ and $|\arg \lambda| \leqq \Theta_1$ the operator $B = (\lambda - L)S(\lambda)$ is invertible with $\|B^{-1}\| < 2$. Consequently, the operator $R(\lambda) = S(\lambda)B^{-1}$ is a right inverse of $\lambda - L$ satisfying

$$\|R(\lambda)\| \leqq \frac{2C_1}{|\lambda|}. \tag{2.7}$$

The above discussion shows that, for $|\lambda| \geqq R_1$ and $|\arg \lambda| \leqq \Theta_1$, the operator $R(\lambda)$ coincides with $(\lambda - L)^{-1}$ whenever $\lambda - L$ is injective, in particular for $\lambda > 0$, since L is dissipative.

Remembering that if λ belongs to the resolvent of L and $|\omega - \lambda| < \|(\lambda - L)^{-1}\|^{-1}$ then ω belongs to the resolvent of L, is not difficult to deduce, using (2.7) and a simple argument based on connectness, that $\rho(L) \supset \{\lambda \in \mathbb{C} : |\arg \lambda| \leqq \Theta_1 \text{ and } |\lambda| > R_1\}$. This last fact, together with (2.7), conclude the proof. □

Remark 2.5. Observe that the above proof is still valid if we only require that $\lambda_n - L$ is injective for a sequence $\lambda_n \to +\infty$.

Theorem 2.6. *The operator $(A, D(A))$ generates an analytic semigroup of angle $\pi/2$ in $E([0, +\infty])$. The semigroup is positive and contractive.*

Proof. Since A degenerates only at $x = 0$ we can use Proposition 2.4 and give the proof in the case where $E([0, +\infty]) = C([0, +\infty])$.

Let $\delta > 0$ and consider a C^∞-function $0 \leqq \phi_\delta \leqq 1$ such that ϕ_δ is equal to 1 in a neighborhood of 0 and to 0 for $x \geqq \delta$.

Let $c(x) = \dfrac{b(x) - b(0)}{x}$ and define the functions

$$c_\delta(x) = c(x)\phi_\delta(x), \quad p_\delta(x) = \int_0^x c_\delta(t)\,dt.$$

Moreover, consider the function m_δ defined by $m_\delta(x) = m(x)$ for $x \leqq \delta$ and $m_\delta(x) = m_\delta(\delta)$ for $x \geqq \delta$. If we put

$$v_\delta(x) = e^{p_\delta(x)}u(x) - \int_0^x u(t)e^{p_\delta(t)}c_\delta(t)\,dt,$$

we get

$$v_\delta'(x) = e^{p_\delta(x)}u'(x), \quad v_\delta(0) = u(0).$$

Hence we obtain

$$u(x) = v_\delta(x)e^{-p_\delta(x)} + \int_0^x v_\delta(t)e^{-p_\delta(t)}c_\delta(t)\,dt$$

and $A_\delta u = B_\delta v_\delta$, where

$$A_\delta u(x) = m_\delta(x)\left[u''(x) + \left(\frac{b(0)}{x} + c_\delta(x)\right)u'(x)\right],$$

$$B_\delta v_\delta(x) = m_\delta(x)e^{-p_\delta(x)}\left[v_\delta''(x) + \frac{b(0)}{x}v_\delta'(x)\right].$$

Defining $D(A_\delta) = D(A)$ and

$$D(B_\delta) = \left\{v \in C([0, +\infty]) \cap C^2(]0, +\infty]) : \lim_{x\to 0} B_\delta v(x) = 0\right\}$$

we obtain $u \in D(A) = D(A_\delta)$ if and only if $v_\delta \in D(B_\delta)$.

Finally, consider the operator

$$B = m(0)\left[D^2 + \frac{b(0)}{x}D\right], \quad D(B) = D(B_\delta);$$

by Theorem 2.3, the operator B generates an analytic semigroup of angle $\pi/2$ and then, given $\Theta < \pi$ we can find M and R such that $\|(B-\lambda)^{-1}\| \leqq M/|\lambda|$ if $|\lambda| \geqq R$ and $|\arg\lambda| \leqq \Theta$.

We observe that, for every $v \in D(B)$,

$$\|Bv - B_\delta v\| \leqq K_\delta\|Bv\|,$$

with $K_\delta \to 0$ as $\delta \to 0$.

Moreover, we can write for $|\lambda| \geqq R$ and $|\arg\lambda| \leqq \Theta$

$$\lambda - B_\delta = (I + (B - B_\delta)(\lambda - B)^{-1})(\lambda - B)$$

with

$$\begin{aligned}\|(B - B_\delta)(\lambda - B)^{-1}u\| &\leqq K_\delta\|B(\lambda - B)^{-1}u\|\\ &\leqq K_\delta\big(\|u\| + \lambda\|(\lambda - B)^{-1}u\|\big)\\ &\leqq K_\delta(1 + M)\|u\| \leqq \frac{1}{2}\|u\|\end{aligned}$$

for every $\delta \leqq \delta_0$, δ_0 small enough.

It follows that, for $|\lambda| \geqq R$, $|\arg \lambda| \leqq \Theta$ and $\delta \leqq \delta_0$, the operator $\lambda - B_\delta$ is invertible and $\|(\lambda - B_\delta)^{-1}\| \leqq 2M/|\lambda|$ with M independent of δ. These facts will be essential to show that A_δ generates an analytic semigroup for sufficiently small δ.

To this aim we solve the equation $A_\delta u - \lambda u = f$. We have

$$\begin{aligned} A_\delta u(x) - \lambda u(x) &= B_\delta v_\delta(x) - \lambda v_\delta(x) e^{-p_\delta(x)} - \lambda \int_0^x v_\delta(t) e^{-p_\delta(t)} c_\delta(t)\, dt \\ &= B_\delta v_\delta(x) - \lambda v_\delta(x) - \lambda\left(\left(e^{-p_\delta(x)} - 1\right) v_\delta(x) + \int_0^x v_\delta(t) e^{-p_\delta(t)} c_\delta(t)\, dt\right). \end{aligned} \tag{2.8}$$

Consider the operator

$$S_\delta v(x) = \left(e^{-p_\delta(x)} - 1\right) v(x) + \int_0^x v(t) e^{-p_\delta(t)} c_\delta(t)\, dt.$$

We have $A_\delta u - \lambda u = f$ if and only if $B_\delta v_\delta - \lambda v_\delta - \lambda S_\delta v_\delta = f$, i.e., if and only if $(B_\delta - \lambda)(I - \lambda(B_\delta - \lambda)^{-1} S_\delta) v_\delta = f$.

The operator $B_\delta - \lambda$ is invertible and $\|\lambda(B_\delta - \lambda)^{-1}\| \leqq M$. In order to estimate $\|S_\delta\|$, we observe that, if $x > \delta$,

$$S_\delta v(x) = \left(e^{-p_\delta(\delta)} - 1\right) v(x) + \int_0^\delta v(t) e^{-p_\delta(t)} c_\delta(t)\, dt;$$

hence, for δ small enough we obtain $\|S_\delta\| \leqq 1/(2M)$ and consequently $\|\lambda(B_\delta - \lambda)^{-1} S_\delta\| \leqq 1/2$. This yields

$$\|\left(I - \lambda(B_\delta - \lambda)^{-1} S_\delta\right)^{-1}\| \leqq 2$$

and by (2.8) we obtain $\|u\| \leqq M_1/|\lambda| \|f\|$.

It follows that A_δ generates an analytic semigroup of angle $\pi/2$; since A degenerates only at $x = 0$, we can apply Proposition 2.4 with $(L_0, D(L_0)) = (A_\delta, D(A_\delta))$, $D(L_2) = \{u \in C^2([1, +\infty]) : u'(1) = 0\}$ and $L_2 u = Lu$ for $u \in D(L_2)$ to deduce that the same holds for A. Positivity and contractivity follow by Lemma 2.1. □

3. First-order degeneracy on [0,1]. We consider here the differential operators on $C([0,1])$

$$A_1 = m(x)[x(1-x)D^2 + b(x)D], \quad A_2 = m(x)\left[D^2 + \frac{b(x)}{x(1-x)} D\right].$$

We assume m, b continuous on $[0,1]$ with m strictly positive and $\dfrac{b(x) - b(0)}{x}$, $\dfrac{b(x) - b(1)}{1 - x}$ summable on neighborhoods of $0, 1$ respectively. In order to impose Ventcel's conditions at $x = 0, 1$ we assume also that the function b satisfies $b(0) < 1$, $b(1) > -1$. Accordingly, we define the domains of A_i, $i = 1, 2$ in the following way:

$$D(A_i) = \left\{u \in C([0,1]) \cap C^2(]0,1[) : \lim_{x \to 0,1} A_i u(x) = 0\right\}.$$

A straightforward argument shows that $(A_i, D(A_i))$ is dissipative for $i = 1, 2$. Moreover, the change of variable $x = (1 - \cos \pi t)/2$ transforms $(A_1, D(A_1))$ into $(A_2, D(A_2))$ so that every result for A_2 still holds for A_1.

Theorem 3.1. *For $i = 1, 2$ the operator $(A_i, D(A_i))$ generates an analytic semigroup of angle $\pi/2$. The semigroup is positive and contractive.*

Proof. It is sufficient to give the proof for A_2.

Let m be extended on $]-\infty, +\infty[$ by $m(x) = m(1)$ for $x \geqq 1$, $m(x) = m(0)$ for $x \leqq 0$ and let ψ_0, ψ_2 be cut-off functions such that $\psi_0 \equiv 1$, $\psi_2 \equiv 0$ in a neighborhood of 0, $\psi_0 \equiv 0$, $\psi_2 \equiv 1$ in a neighborhood of 1.

Consider the operators

$$L_0 = m(x)\Big[D^2 + \frac{b(x)\psi_0(x)}{x(1-x)}D\Big], \quad L_2 = m(x)\Big[D^2 + \frac{b(x)\psi_2(x)}{x(1-x)}D\Big]$$

with domains

$$D(L_0) = \big\{u \in C([0, +\infty]) \cap C^2(]0, +\infty]) : \lim_{x\to 0} L_0 u(x) = 0\big\},$$

$$D(L_2) = \big\{u \in C([-\infty, 1]) \cap C^2([-\infty, 1[) : \lim_{x\to 1} L_2 u(x) = 0\big\}.$$

By Theorem 2.6, L_0 and L_2 generate analytic semigroups of angle $\pi/2$. Finally, apply Proposition 2.4 with cut-off functions ϕ_0 and ϕ_2 such that $\phi_0 \equiv 1$ in a neighborhood of 0, $\operatorname{supp} \phi_0 \subset \{x : \psi_0(x) = 1\}$ and similarly for ϕ_2 near $x = 1$. □

We investigate now the asymptotic behaviour of the semigroups $(T_1(t))_{t \geqq 0}$ and $(T_2(t))_{t \geqq 0}$ generated by A_1 and A_2. In the following lemma, which provides the compactness of the above semigroups, we suppose $D(A_2)$ endowed with the graph norm.

Lemma 3.2. *The inclusion $D(A_2) \hookrightarrow C([0,1])$ is compact.*

Proof. If $b(0) \leqq -1$ and $b(1) \geqq 1$ this fact is proved in [14, Theorem 4.1]. Suppose $-1 < b(0) < 1$ and consider the interval $[0, 1/2]$. The function

$$\gamma(x) = \exp\Bigg(\int_{1/2}^{x} \frac{b(t)}{t(1-t)}\,dt\Bigg) \tag{3.1}$$

is equivalent to $Cx^{b(0)}$ as $x \to 0$ for some $C > 0$ and therefore it is integrable near $x = 0$.

We can write

$$A_2 u(x) = \frac{m(x)}{\gamma(x)}\frac{d}{dx}\big[\gamma(x)u'(x)\big].$$

Then we obtain

$$\gamma(x)u'(x) - u'(1/2) = \int_{1/2}^{x} \frac{A_2 u(t)\gamma(t)}{m(t)}\,dt$$

and hence, with $K = \int_0^1 \gamma(t)m(t)^{-1}\,dt$,

$$|\gamma(x)u'(x)| \leqq |u'(1/2)| + K\|A_2 u\|. \tag{3.2}$$

Since A_2 is regular in $]0,1[$ we can estimate $u'(1/2)$ with the graph norm of u so obtaining

$$\sup_{x \leqq 1/2} |u'(x)| \leqq \frac{C}{\gamma(x)}\big[\|u\| + \|A_2 u\|\big].$$

By $\gamma(x)^{-1} \in L^1(0, 1/2)$ we get the compactness of the inclusion of $D(A_2)$ into $C([0, 1/2])$. A similar argument in $[1/2, 1]$ yields the thesis. □

Theorem 3.3. *The semigroups generated by A_1 and A_2 are compact.*

Proof. Since A_1 reduces to A_2 by a change of variable it is sufficient to prove the theorem for A_2. However this is immediate since the semigroup is analytic and A_2 has compact resolvent by the above lemma. □

In order to state the following result we introduce the function $\psi(x) = \int_0^x \gamma(t)^{-1}\,dt$ and observe that the conditions $b(0) < 1$ and $b(1) > -1$ imply that $\psi \in D(A_i)$ and $A_i\psi = 0$ for $i = 1, 2$.

Proposition 3.4. *The semigroups $(T_1(t))_{t \geqq 0}$ and $(T_2(t))_{t \geqq 0}$ converge in norm as $t \to +\infty$ to the projection*

$$Pu(x) = u(0) + \frac{u(1) - u(0)}{\psi(1)}\psi(x).$$

Proof. Let us consider only $(T_2(t))_{t \geqq 0}$. Since $(T_2(t))_{t \geqq 0}$ is compact, positive and contractive, by [15, B-IV, Theorem 2.5] it converges in norm, as $t \to +\infty$, to a projection P such that $\operatorname{Im}(P) = \operatorname{Ker}(A_2)$. The functions $\mathbf{1}$ and ψ generate $\operatorname{Ker}(A_2)$ so that

$$Pu = c_1\mathbf{1} + c_2\psi$$

for suitable constants c_1 and c_2. Put $U(x,t) = T_2(t)u(x)$; then $\frac{\partial}{\partial t}U(x,t) = A_2U(x,t)$ and so $\frac{\partial}{\partial t}U(x,t) = 0$ for $x = 0, 1$. Then $U(0,t) = u(0)$ and $U(1,t) = u(1)$ whence $Pu(0) = u(0)$, $Pu(1) = u(1)$ and the thesis follows. The proof for $(T_1(t))_{t \geqq 0}$ is similar. □

We strengthen now Theorem 3.1 by showing that the semigroups generated by A_1 and A_2 are bounded analytic of angle $\pi/2$. To this aim we need the following result:

Lemma 3.5. *The spectrum of A_2 (hence of A_1) is contained in $]-\infty, 0]$.*

Proof. If $b(0) \leqq -1$ and $b(1) \geqq 1$ this fact is proved in [14, Theorem 5.5].
Suppose, for example, $-1 < b(0) < 1$ and $-1 < b(1) < 1$ and let $\lambda \neq 0$ be an eigenvalue of A_2. If $u \in D(A_2)$ satisfies $A_2u = \lambda u$, then $u(0) = u(1) = 0$ because of the boundary conditions.

We write, using the function γ defined in (3.1),

$$\frac{d}{dx}\left[\gamma(x)u'(x)\right] = \lambda\frac{\gamma(x)}{m(x)}u(x).$$

Multiplying both sides by $\bar{u}$ and integrating by parts between ε and $1 - \varepsilon$ we get

$$-\int_\varepsilon^{1-\varepsilon}\gamma(x)|u'(x)|^2\,dx + \gamma(x)\bar{u}(x)u'(x)\Big|_\varepsilon^{1-\varepsilon} = \lambda\int_\varepsilon^{1-\varepsilon}\frac{\gamma(x)}{m(x)}|u(x)|^2\,dx.$$

By (3.2) the function $\gamma u'$ is bounded and $\gamma|u'|^2$ is summable on $[0,1]$. Letting $\varepsilon \to 0$ the boundary terms vanish and we obtain

$$-\int_0^1\gamma(x)|u'(x)|^2\,dx = \lambda\int_0^1\frac{\gamma(x)}{m(x)}|u(x)|^2\,dx$$

from which we deduce $\lambda \in \mathbb{R}$ and $\lambda < 0$.

The other cases are similar. $\square$

Using Theorem 3.1, Theorem 3.3, Lemma 3.5 together with [14, Proposition 5.6] we obtain the following result:

Theorem 3.6. *The semigroups generated by* $(A_1, D(A_1))$ *and* $(A_2, D(A_2))$ *are bounded analytic of angle* $\pi/2$.

4. Applications to some degenerate evolution problems. In this section we apply the results of Section 2 and Section 3 to some degenerate evolution problems on $[0, +\infty[$ and on $[0, 1]$. We shall consider more general degree of degeneracy and we shall see how, in many cases, a suitable change of variable allows to use our preceding results. We start with the following parabolic problem on $(0, +\infty)$:

$$\text{(4.1)} \qquad \begin{cases} u_t = a(x)\left[x^\alpha u_{xx} + b(x)x^\sigma u_x + c(x)u\right], \\ u(x,0) = u_0(x) \end{cases}$$

with suitable boundary conditions at $0, +\infty$.

Problem (4.1) has been considered by Vespri [16] in the case $\alpha \geqq 2, \sigma = \alpha/2$; we consider instead the case $0 < \alpha < 2$ which often occurs in the applications.

Considering the new variable $s = x^{1-\alpha/2}$, we observe that the operator

$$B = a(x)\left[x^\alpha D^2 + b(x)x^\sigma D + c(x)\right]$$

transforms into

$$C = a_1(s)\left[\left(1 - \frac{\alpha}{2}\right)^2 D^2 + \left(\left(1 - \frac{\alpha}{2}\right)b_1(s)s^{\frac{2\sigma-\alpha}{2-\alpha}} - \frac{\alpha}{2s}\left(1 - \frac{\alpha}{2}\right)\right)D + c_1(s)\right]$$

where a_1, b_1, c_1 are the functions a, b, c expressed in the variable s, that is, $a_1(s) = a(s^{2/(2-\alpha)})$ and similarly for b_1 and c_1.

Moreover, we denote by B_0 and respectively by C_0 the operators B and C with $c \equiv 0$.

In order to apply the results of Section 2, we assume the following conditions on the coefficients a_1, b_1, c_1:

a_1 uniformly continuous and bounded on $[0, +\infty)$ with $\inf_{s \geqq 0} a_1(s) > 0$;
b_1 and c_1 continuous and bounded on $[0, +\infty)$;
b_1 Hölder-continuous at $s = 0$, if $\alpha - 1 = \sigma$.

Moreover, for the exponents α, σ we have to require

$$\alpha - 1 \leqq \sigma \leqq \frac{\alpha}{2}$$

in order to get a degeneration like $1/s$ as $s \to 0$ and bounded coefficients as $s \to +\infty$.

Turning back to the variable x we make the following assumptions:

(i) a, b, c continuous and bounded on $[0, +\infty)$ with $\inf_{x \geqq 0} a(x) > 0$;
(ii) $a(x^{2/(2-\alpha)})$ uniformly continuous on $[0, +\infty)$;
(iii) $\alpha - 1 \leqq \sigma \leqq \dfrac{\alpha}{2}$;
(iv) b Hölder-continuous at $x = 0$, if $\alpha - 1 = \sigma$.

We observe that the condition $\sigma \leqq \alpha/2$ can be replaced by the boundedness of $b(x)x^{\sigma-\alpha/2}$ as $x \to +\infty$.

As regards the boundary condition at $s = 0$ for the operator C, we can impose Ventcel's conditions $\lim_{s \to 0} Cu(s) = 0$ if and only if

$$\alpha - 1 < \sigma \quad \text{or} \quad \alpha - 1 = \sigma,\ b(0) < 1.$$

In the sequel we shall always assume the above conditions on the exponents α and σ and consider for simplicity only the condition of boundedness at $+\infty$. Accordingly we define

$$D(C) = \big\{u \in C_b([0,+\infty[) \cap C^2(]0,+\infty[) : u', u'' \text{ bounded at } +\infty \\ \text{and } \lim_{s \to 0} C_0 u(s) = 0\big\}.$$

By Theorem 2.6 and noticing that the term $c(x)u$ is a bounded perturbation of the generator, we obtain that $(C, D(C))$ generates an analytic semigroup on $C_b([0,+\infty[)$.

With the above change of variable, it is easy to obtain

$$D(B) = \big\{u \in C_b([0,+\infty[) \cap C^2(]0,+\infty[) : x^{\alpha/2}u', x^{\alpha}u'' \text{ bounded at } +\infty \\ \text{and } \lim_{x \to 0} B_0 u(x) = 0\big\}.$$

By the above discussion, we obtain the following result:

Theorem 4.1. *The operator $(B, D(B))$ generates an analytic semigroup of angle $\pi/2$ on $C_b([0,+\infty[)$. The semigroup is positive and contractive.*

In the cases

$$\alpha < 1,\ \sigma > 0 \quad \text{or} \quad \alpha - 1 = \sigma,\ b(0) > \alpha - 1,$$

we can also impose Neumann's condition $\dfrac{du}{ds}(0) = 0$. Using the same method and the results of [2] and [14], one can prove the analyticity of the corresponding semigroup.

As a second application we consider differential operators like $L = a(x)D^2 + b(x)D$ on $C([0,1])$. Supposing $a(x) > 0$ on $]0,1[$, $a \in C^1(]0,1[)$ and $a^{-1/2} \in L^1(0,1)$ we can use the change of variable

$$s = \int_0^x a^{-1/2}(t)\,dt$$

and look for conditions on the coefficients a, b which allow to apply our preceding methods. For simplicity, we discuss only the case where the operator L has the form

$$L = m(x)\big[x^{\alpha}(1-x)^{\beta}D^2 + b(x)x^{\sigma}(1-x)^{\delta}D\big],$$

which is the most frequently met in the applications. We assume m continuous and strictly positive, b continuous and Hölder continuous at $x = 0, 1$. We require also $0 < \alpha, \beta < 2$ and, in order to apply Ventcel's conditions at $x = 0, 1$,

$$\alpha - 1 < \sigma \quad \text{or} \quad \alpha - 1 = \sigma,\ b(0) < 1;$$

$$\beta - 1 < \delta \quad \text{or} \quad \beta - 1 = \delta,\ b(1) > -1.$$

Observe that no condition on σ and δ is needed if $b \equiv 0$. Accordingly we define

$$D(L) = \{u \in C([0,1]) \cap C^2(]0,1[) : \lim_{x \to 0,1} Lu(x) = 0\}.$$

Using the change of variable $s = \int_0^x t^{-\alpha/2}(1-t)^{-\beta/2}\, dt$ we can repeat with minor changes the above discussion about the operator B obtaining the following result:

Theorem 4.2. *The operator $(L, (D(L))$ generates an analytic semigroup of angle $\pi/2$. The semigroup is positive and contractive.*

In particular we can state the following corollary:

Corollary 4.3. *The operator $m(x)\left[x^\alpha(1-x)^\beta D^2\right]$ generates an analytic semigroup of angle $\pi/2$ for every $\alpha, \beta > 0$.*

Proof. For $\alpha, \beta \geqq 2$ see [10]. For $0 < \alpha, \beta < 2$ the result is a consequence of the above theorem. The general case is deduced from these by using Proposition 2.4, arguing as in Theorem 3.1. □

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