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Sections of convex bodies through their centroid

By

M. Fradelizi

Abstract. We give the best estimate in the comparison of the volume of the section of a convex body in $\mathbb{R}^n$ through its centroid by a k-dimensional affine subspace E^k with the volume of the section by any affine subspace parallel to E^k.

1. Introduction. In this paper, we answer a conjecture of Makai and Martini [2]. Namely, we prove the following: let K be a convex body in $\mathbb{R}^n$, let E^k be a k-dimensional affine subspace of $\mathbb{R}^n$ passing through the centroid of K and denote by $F(K, E^k)$ the ratio between the k-dimensional volume of the section of K by E^k and the k-dimensional volume of the maximal section of K parallel to E^k. Then

$$F(K, E^k) \geqq \left(\frac{k+1}{n+1}\right)^k$$

and the equality case is solved. For example, there is equality if K is a simplex with a k-face parallel to E^k.

For $k = 1$, the answer goes back to Bonnesen and Fenchel [1] or even further; it consists in comparison of length of chords. For $k = n - 1$, the result is due to Makai and Martini [2].

In the general case, we express the problem in terms of concave functions. Afterwards we give two proofs. Each of them has its interest. In the first one, which is geometric, we reduce the problem to the case when the concave function is affine. In the second one, more analytic, we reduce to a one-dimensional problem.

In the following, vol_k denotes the k-dimensional Lebesgue measure; $\langle\, ,\, \rangle$ denotes the usual scalar product with respect to the canonical Euclidean structure of $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$, the orthogonal of A is $A^\perp = \{x \in \mathbb{R}^n; \forall y \in A,\ \langle x, y\rangle = 0\}$, the convex hull of A is denoted by $\mathrm{conv}(A)$, its affine hull by $\mathrm{aff}(A)$ and the positive convex cone generated by A, with $x_0 \in \mathbb{R}^n$ as vertex is denoted by $\mathrm{pos}(x_0, A)$.

If $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^n$, then $\int f(x)dx \in \mathbb{R}^n$ denotes the vector obtained by integration of the coordinates of f. If $A \subset \mathbb{R}^n$ and $\varphi : A \to \mathbb{R}$, then the positive part of φ is denoted by $\varphi_+ = \max(\varphi, 0)$.

For general references on convex bodies and the Brunn-Minkowski theory, we refer to Schneider [3].

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Theorem 1. *Let E^k be a k-dimensional subspace of $\mathbb{R}^n$, with $1 \leqq k \leqq n-1$ and $K \subset \mathbb{R}^n$ be a convex body with g_K as centroid. Then*

$$\max_{x\in\mathbb{R}^n} \operatorname{vol}_k(K\cap(E^k+x)) \leqq \left(\frac{n+1}{k+1}\right)^k \operatorname{vol}_k(K\cap(E^k+g_K)) ,$$

with equality if and only if there exist $x_o \in \mathbb{R}^n$, F^{n-k} a subspace of $\mathbb{R}^n$ such that $F^{n-k} \oplus E^k = \mathbb{R}^n$ and two convex bodies K_1 and K_2 satisfying $K_1 \subset E^k$, $K_2 \subset F^{n-k}$ and $\dim(\operatorname{aff}(K_2)) = n-k-1$ *such that*

$$K = x_0 + \operatorname{conv}(K_1, K_2).$$

We will first prove that this theorem is a consequence of the following:

Theorem 2. *Let $q \geqq 1$ be an integer and $p > 0$. Let $C \subset \mathbb{R}^q$ be a convex body and $f : C \to \mathbb{R}$, $f \geqq 0$, $f \not\equiv 0$, such that $f^{\frac{1}{p}}$ is concave. Then*

$$\max_{x\in\mathbb{R}^q} f(x) \leqq \left(\frac{p+q+1}{p+1}\right)^p f(x_f) \quad \textit{where} \quad x_f = \frac{\int\limits_{\mathbb{R}^q} xf(x)\,dx}{\int\limits_{\mathbb{R}^q} f(x)\,dx} ,$$

with equality if and only if there exist $x_0 \in \mathbb{R}^q$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and $u \in \mathbb{R}^q$, $u \neq 0$ such that

$$f(x) = f(x_0)\left(1 - \frac{\langle x-x_0, u\rangle}{\alpha}\right)^p \quad \textit{for all } x \in C, \textit{ with } C = \operatorname{conv}(x_0, C_0),$$

where C_0 is a convex body in the affine hyperplane $\{x \in \mathbb{R}^q; \langle x-x_0, u\rangle = \alpha\}$.

Proof of Theorem 1. Under the hypotheses of Theorem 1, identify $\mathbb{R}^n$ with $(E^k)^\perp \times E^k$, denote by P the orthogonal projection onto $(E^k)^\perp$ and define f by:

$$f(x) = \operatorname{vol}_k(K\cap(E^k+x)) = \operatorname{vol}_k(\{y\in\mathbb{R}^k; (x,y)\in K\})$$

for all $x \in \mathbb{R}^{n-k}$. By the Brunn-Minkowski theorem, $f^{\frac{1}{k}}$ is concave on $P(K)$. Therefore we may apply Theorem 2 to f with $p = k$, $q = n-k$ and $C = P(K)$. Noticing that $\operatorname{vol}_k(K\cap(E^k+g_K)) = f(P(g_K))$ and using Fubini's theorem, we get

$$P(g_K) = \frac{\int\limits_K x\,dxdy}{\int\limits_K dxdy} = \frac{\int\limits_{\mathbb{R}^{n-k}} x \operatorname{vol}_k(\{y\in\mathbb{R}^k; (x,y)\in K\})dx}{\int\limits_{\mathbb{R}^{n-k}} \operatorname{vol}_k(\{y\in\mathbb{R}^k; (x,y)\in K\})\,dx}$$
$$= \frac{\int\limits_{\mathbb{R}^{n-k}} xf(x)\,dx}{\int\limits_{\mathbb{R}^{n-k}} f(x)\,dx} = x_f.$$

Therefore

$$\max_{x\in\mathbb{R}^n} \operatorname{vol}_k(K\cap(E^k+x)) = \max_{x\in\mathbb{R}^n} f(x) \leqq \left(\frac{n+1}{k+1}\right)^k f(x_f)$$
$$\leqq \left(\frac{n+1}{k+1}\right)^k \operatorname{vol}_k(K\cap(E^k+g_K)) ,$$

which is the inequality of Theorem 1.

Now we deduce the equality case of Theorem 1 from Theorem 2.

First, if K satisfies

$$\max_{x\in\mathbb{R}^n} \mathrm{vol}_k(K\cap(E^k+x)) = \binom{n+1}{k+1}^k \mathrm{vol}_k(K\cap(E^k+g_K)),$$

we get equality in Theorem 2 with $f(x)=\mathrm{vol}_k(K\cap(E^k+x))$. Therefore there exist $x_0\in(E^k)^\perp$, $\alpha\in\mathbb{R}$, $\alpha\neq 0$ and $u\in(E^k)^\perp$, $u\neq 0$ such that for all x in $P(K)$

$$\mathrm{vol}_k(K\cap(E^k+x)) = \mathrm{vol}_k(K\cap(E^k+x_0))\left(1-\frac{\langle x-x_0,u\rangle}{\alpha}\right)^k,$$

and $P(K)=\mathrm{conv}(x_0,C_0)$, where C_0 is a convex body satisfying

$$C_0\subset\{x\in(E^k)^\perp;\langle x-x_0,u\rangle=\alpha\}.$$

We first translate K such that $x_0=0$. Then we denote by $\widetilde{K}=S(K)$ the Schwarz symmetral of K with respect to $(E^k)^\perp$, that is the convex body such that for all $x\in(E^k)^\perp$, $\widetilde{K}\cap(E^k+x)$ is a k-euclidean ball, centered at x, with same volume as $K\cap(E^k+x)$. For all $x\in\mathrm{conv}(0,C_0)$, define $\widetilde{K}_x=\widetilde{K}\cap(E^k+x)$, then we have

$$\mathrm{vol}(\widetilde{K}_x)=\mathrm{vol}(\widetilde{K}_0)\left(1-\frac{\langle x,u\rangle}{\alpha}\right)^k, \tag{1}$$

which implies that there is equality in the Brunn-Minkowski theorem, hence $\widetilde{K}_x$ and $\widetilde{K}_0$ are homothetic. Therefore, for all $x\in\mathrm{conv}(0,C_0)$, there exist $\lambda_x\in\mathbb{R}$ and $y_x\in\mathbb{R}^n$ such that $\widetilde{K}_x=\lambda_x\widetilde{K}_0+y_x$. Because of (1), we have $\lambda_x=1-\dfrac{\langle x,u\rangle}{\alpha}$. Since $P(\widetilde{K}_x)=x$, we have $P(y_x)=x$. From its definition, $\widetilde{K}$ is symmetric with respect to $(E^k)^\perp$, hence $y_x=x$.

Finally, for all $x\in\mathrm{conv}(0,C_0)$, $\widetilde{K}_x=\left(1-\dfrac{\langle x,u\rangle}{\alpha}\right)\widetilde{K}_0+x$, hence $\widetilde{K}=\mathrm{conv}(\widetilde{K}_0,C_0)$. And we can conclude that $K=x_0+\mathrm{conv}(K_1,K_2)$, with $K_1\subset E^k$, such that $S(K_1)=\widetilde{K}_0$ and $K_2\subset F^{n-k}$, a subspace of $\mathbb{R}^n$ which satisfies $F^{n-k}\oplus E^k=\mathbb{R}^n$, with $P(K_2)=C_0$.

For the reverse implication, we see that if $K=\mathrm{conv}(K_1,K_2)$, then $S(K)=\widetilde{K}$ satisfies $\widetilde{K}=\mathrm{conv}(S(K_1),P(K_2))$. Since $P(K_2)$ is in an $(n-k-1)$-dimensional affine subspace of $(E^k)^\perp$, there exist $u\in(E^k)^\perp$, $u\neq 0$ and $\alpha\in\mathbb{R}$ such that $P(K_2)\subset\{x\in(E^k)^\perp;\langle x,u\rangle=\alpha\}$. And for all $x\in\mathrm{conv}(0,P(K_2))$ we get

$$\widetilde{K}_x=\left(1-\frac{\langle x,u\rangle}{\alpha}\right)S(K_1)+x\ ,$$

hence $\mathrm{vol}(K\cap(E^k+x))=\mathrm{vol}(\widetilde{K}_x)=\mathrm{vol}(K_1)\left(1-\dfrac{\langle x,u\rangle}{\alpha}\right)^k$, which means that the function $f(x)=\mathrm{vol}(K\cap(E^k+x))$ satisfies the equality case of Theorem 2. This implies that $\max\limits_{x\in\mathbb{R}^n}\mathrm{vol}_k(K\cap(E^k+x))=\dbinom{n+1}{k+1}^k\mathrm{vol}_k(K\cap(E^k+g_K))$.

2. Geometric proof of Theorem 2. Because of the translation invariance, we may assume that $\max\limits_{x\in\mathbb{R}^q} f(x)=f(0)$. Moreover, if $f(0)=f(x_f)$, the result is obvious, therefore we now assume $f(0)>f(x_f)$.

We first prove that there exists a section $H \cap C$ of C passing through x_f satisfying $\max_{H\cap C} f = f(x_f)$. In fact, more generally, we have:

Lemma 1. *Let $q \geqq 1$ be an integer, and $p > 0$. Let C be a convex body of $\mathbb{R}^q$ and $f : C \to \mathbb{R}$, $f \geqq 0$, $f \not\equiv 0$, such that $f^{\frac{1}{p}}$ is concave. Let x_0 be in the interior of C. Then there exists an affine hyperplane H in $\mathbb{R}^q$, passing through x_0, such that:*

$$f(x_0) = \max_{x\in H\cap C} f(x)\ .$$

Proof. We define $\widetilde{K} = \{(x,t) \in \mathbb{R}^q \times \mathbb{R};\ x \in C \text{ and } 0 \leqq t \leqq f^{\frac{1}{p}}(x)\}$. Since $f^{\frac{1}{p}}$ is concave, it is clear that $\widetilde{K}$ is a convex body in $\mathbb{R}^{q+1}$. Since the point $\left(x_0, f^{\frac{1}{p}}(x_0)\right)$ belongs to the boundary of $\widetilde{K}$, $\widetilde{K}$ admits a support hyperplane $\tilde{H}$ at this point. Therefore there exists $(y,s) \in \mathbb{R}^q \times \mathbb{R}$ such that for all $(x,t) \in \widetilde{K}$

$$\langle x - x_0, y\rangle + (t - f^{\frac{1}{p}}(x_0))s \leqq 0\ .$$

Let us prove that $s > 0$.

Since $(x_0, 0) \in \widetilde{K}$, we have $f^{\frac{1}{p}}(x_0)s \geqq 0$. Moreover, by definition, x_0 is in the interior of C so that $f^{\frac{1}{p}}(x_0) > 0$; it follows that $s \geqq 0$. But assume that $s = 0$; then $\langle x, y\rangle \leqq \langle x_0, y\rangle$ for all $x \in C$, which implies that the affine hyperplane $H = \{x \in \mathbb{R}^q; \langle x, y\rangle = \langle x_0, y\rangle\}$, separates x_0 from C. This is a contradiction because x_0 is in the interior of C and C is convex. Thus $s > 0$.

Now, for all $x \in H \cap C$, we have $(x, f^{\frac{1}{p}}(x)) \in \widetilde{K}$ so that $(f^{\frac{1}{p}}(x) - f^{\frac{1}{p}}(x_0))s \leqq 0$, therefore $f^{\frac{1}{p}}(x) \leqq f^{\frac{1}{p}}(x_0)$ and we get $f(x_0) = \max_{x\in H\cap C} f(x)$.

Proof of Theorem 2. Since C is a convex body, x_f belongs to the interior of C, so that we may apply Lemma 1 to x_f. Therefore, there exists an affine hyperplane H such that $f(x_f) = \max_{x\in H\cap C} f(x)$. Since we assumed that $\max f = f(0) > f(x_f)$, it implies that $0 \notin H$. Thus for some $u \in \mathbb{R}^q$, $H = \{x \in \mathbb{R}^q;\ \langle x, u\rangle = 1\}$. Define $H^- = \{x \in \mathbb{R}^q; \langle x, u\rangle \leqq 1\}$, $H^+ = \{x \in \mathbb{R}^q;\ \langle x, u\rangle \geqq 1\}$, $D_0 = C \cap H$ and $D = \operatorname{pos}(0, D_0)$.

Define $\varphi : \mathbb{R}^q \to \mathbb{R}^+$ by $\varphi(rx) = ((1-r)f^{\frac{1}{p}}(0) + rf^{\frac{1}{p}}(x))_+^p$ if $x \in D_0$ and $r \geqq 0$, and φ vanishes outside of D. Since $f^{\frac{1}{p}}$ is concave, we have for $x \in D_0$:

$\varphi^{\frac{1}{p}}(rx) \leqq f^{\frac{1}{p}}(rx)$ for $0 \leqq r \leqq 1$ hence $\varphi \leqq f$ on H^-.

$\varphi^{\frac{1}{p}}(rx) \geqq f^{\frac{1}{p}}(rx)$ for $r \geqq 1$ hence $\varphi \geqq f$ on H^+.

Define now $x_\varphi = \dfrac{\int_{\mathbb{R}^q} x\varphi(x)\ dx}{\int_{\mathbb{R}^q} \varphi(x)\ dx}$.

a) We first prove that $x_\varphi \in H^+$, with $x_\varphi \in H$ if and only if $f = \varphi$, that is:

$$\langle x_\varphi, u\rangle \geqq \langle x_f, u\rangle = 1 \text{ with equality if and only if f=}\varphi. \tag{2}$$

We have

$$x_\varphi - x_f = \frac{\int_{\mathbb{R}^q} x\varphi(x)\ dx}{\int_{\mathbb{R}^q} \varphi(x)\ dx} - x_f = \frac{\int_{\mathbb{R}^q} (x - x_f)\varphi(x)\ dx}{\int_{\mathbb{R}^q} \varphi(x)\ dx},$$

hence

$$\langle x_\varphi - x_f, u\rangle = \frac{\int_{\mathbb{R}^q} \langle x - x_f, u\rangle\ \varphi(x)\ dx}{\int_{\mathbb{R}^q} \varphi(x)\ dx} = \frac{\int_{\mathbb{R}^q} (\langle x, u\rangle - 1)\varphi(x)\ dx}{\int_{\mathbb{R}^q} \varphi(x)\ dx}.$$

Since $\varphi \leqq f$ on H^- and $\varphi \geqq f$ on H^+, we get that for all $x \in \mathbb{R}^q (\langle x,u\rangle - 1)\varphi(x) \geqq (\langle x,u\rangle - 1)f(x)$. Hence from the definition of x_f:

$$\int_{\mathbb{R}^q} (\langle x,u\rangle - 1)\varphi(x)\ dx \geqq \int_{\mathbb{R}^q} (\langle x,u\rangle - 1)f(x)\ dx = \left\langle \int_{\mathbb{R}^q} (x - x_f)f(x)\ dx, u \right\rangle = 0\ .$$

Thus $\langle x_\varphi, u\rangle \geqq \langle x_f, u\rangle = 1$ with equality if and only if:

$$\int_{\mathbb{R}^q} (\langle x,u\rangle - 1)(\varphi(x) - f(x))\ dx = 0.$$

Since $(\langle x,u\rangle - 1)(\varphi(x) - f(x)) \geqq 0$, this implies that $f = \varphi$.

b) We now prove that:

$$\langle x_\varphi, u\rangle \leqq \left(\frac{q}{p+q+1}\right) \frac{f^{\frac{1}{p}}(0)}{f^{\frac{1}{p}}(0) - f^{\frac{1}{p}}(x_f)}\ , \tag{3}$$

with equality if and only if φ is constant on D_0.

For all $x \in D$, $x \neq 0$, we have

$$\begin{aligned}\varphi(x) &= \left(\Big(1 - \langle x,u\rangle\Big) f^{\frac{1}{p}}(0) + \langle x,u\rangle f^{\frac{1}{p}}\left(\frac{x}{\langle x,u\rangle}\right)\right)_+^p \\ &= f(0)\left(1 - \langle x,u\rangle\left(1 - \frac{f^{\frac{1}{p}}\left(\frac{x}{\langle x,u\rangle}\right)}{f^{\frac{1}{p}}(0)}\right)\right)_+^p \\ &= f(0)\left(1 - \frac{\langle x,u\rangle}{\psi(x)}\right)_+^p \quad \text{where} \quad \psi(x) = \frac{f^{\frac{1}{p}}(0)}{f^{\frac{1}{p}}(0) - f^{\frac{1}{p}}\left(\frac{x}{\langle x,u\rangle}\right)}\ .\end{aligned}$$

Moreover for all $x \in D$, $x \neq 0$ and $t > 0$, we have $\psi(tx) = \psi(x)$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^q} \langle x,u\rangle \varphi(x)\ dx &= f(0) \int_D \langle x,u\rangle \left(1 - \frac{\langle x,u\rangle}{\psi(x)}\right)_+^p dx \\ &= f(0) \int_0^{+\infty} \int_{\langle x,u\rangle = t} t\left(1 - \frac{t}{\psi(x)}\right)_+^p dxdt\ .\end{aligned}$$

The change of variable $y = tx$ gives:

$$\begin{aligned}\int_{\mathbb{R}^q} \langle x,u\rangle \varphi(x)\ dx &= f(0) \int_0^{+\infty} \int_{D_0} t^q \left(1 - \frac{t}{\psi(y)}\right)_+^p dydt \\ &= f(0) \int_{D_0} \left(\int_0^{\psi(y)} t^q \left(1 - \frac{t}{\psi(y)}\right)^p dt\right) dy \\ &= f(0) \int_{D_0} \psi(y)^{q+1} \left(\int_0^1 s^q (1-s)^p\ ds\right) dy \\ &= f(0)\ \frac{\Gamma(q+1)\Gamma(p+1)}{\Gamma(p+q+2)} \int_{D_0} \psi(y)^{q+1}\ dy\ .\end{aligned}$$

The same calculation gives: $\int_{\mathbb{R}^q} \varphi(x)\, dx = f(0)\dfrac{\Gamma(q)\Gamma(p+1)}{\Gamma(p+q+1)} \int_{D_0} \psi(y)^q\, dy.$ Finally,

$$\langle x_\varphi, u\rangle = \frac{q}{p+q+1}\,\frac{\int_{D_0} \psi(y)^{q+1}\, dy}{\int_{D_0} \psi(y)^q\, dy} \leqq \frac{q}{p+q+1}\,\max_{D_0} \psi\ .$$

From Lemma 1, for all $x \in H \cap D$ we have:

$$\psi(x) = \frac{f^{\frac{1}{p}}(0)}{f^{\frac{1}{p}}(0) - f^{\frac{1}{p}}(x)} \leqq \frac{f^{\frac{1}{p}}(0)}{f^{\frac{1}{p}}(0) - f^{\frac{1}{p}}(x_f)}\ .$$

Thus: $\langle x_\varphi, u\rangle \leqq \left(\dfrac{q}{p+q+1}\right)\dfrac{f^{\frac{1}{p}}(0)}{f^{\frac{1}{p}}(0) - f^{\frac{1}{p}}(x_f)}$.

This proves inequality (3).

The case of equality gives $\dfrac{\int_{D_0} \psi(y)^{q+1}\, dy}{\int_{D_0} \psi(y)^q\, dy} = \max_{D_0} \psi$, which means that ψ is constant on D_0. It follows that there exist a real $\alpha \neq 0$, such that for all $x \in D$

$$\varphi(x) = f(0)\left(1 - \frac{\langle x, u\rangle}{\alpha}\right)_+^p\ .$$

c) Finally from (2) and (3), we get:

$$1 \leqq \langle x_\varphi, u\rangle \leqq \frac{q}{p+q+1} \times \frac{f^{\frac{1}{p}}(0)}{f^{\frac{1}{p}}(0) - f^{\frac{1}{p}}(x_f)}.$$

Hence:

$$f(0) \leqq \left(\frac{p+q+1}{p+1}\right)^p f(x_f)\ . \tag{4}$$

Moreover there is equality in (4) if and only if there is equality in (2) and (3) which means that $f(x) = \varphi(x) = f(0)\left(1 - \dfrac{\langle x, u\rangle}{\alpha}\right)_+^p$ for all x in C and $C = \operatorname{conv}(0, C_0)$ where $C_0 = D \cap \{x\ ;\ \langle x, u\rangle = \alpha\}$.

3. Analytic proof of Theorem 2. We prove the following version of Theorem 2:

Theorem 3. *Let $q \geqq 1$ be an integer, and $p > 0$. Let $C \subset \mathbb{R}^q$ be a convex body and let $g : C \to \mathbb{R}$, $g \geqq 0$, $g \neq 0$ be a concave function. Then*

$$g\left(\frac{\int_{\mathbb{R}^q} x g^p(x)\, dx}{\int_{\mathbb{R}^q} g^p(x)\, dx}\right) \geqq \left(\frac{p+1}{p+q+1}\right) \max_{x \in \mathbb{R}^q} g(x)\ , \tag{5}$$

with equality if and only if there exist $x_0 \in \mathbb{R}^q$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and $u \in \mathbb{R}^q$, $u \neq 0$ such that

$$g(x) = g(x_0)\left(1 - \frac{\langle x - x_0, u\rangle}{\alpha}\right) \quad \textit{for all} \quad x \in C \quad \textit{and} \quad C = \operatorname{conv}(x_0, C_0)$$

where C_0 is a convex body in the affine hyperplane $\{x \in \mathbb{R}^q; \langle x - x_0, u\rangle = \alpha\}$.

Proof. Since g is concave, it follows from Jensen's inequality that

$$g\left(\frac{\int\limits_{\mathbb{R}^q} x g^p(x)\ dx}{\int\limits_{\mathbb{R}^q} g^p(x)\ dx}\right) \geqq \frac{\int\limits_{\mathbb{R}^q} g^{p+1}(x)\ dx}{\int\limits_{\mathbb{R}^q} g^p(x)\ dx}$$

with equality if and only g is affine.

Define $\varphi(t) = \mathrm{vol}(\{x \in \mathbb{R}^q; g(x) \geqq t\})$ and $M = \max g$; we have

$$\begin{aligned}\int\limits_{\mathbb{R}^q} g^p(x)\ dx &= \int\limits_{\mathbb{R}^q}\left(\int\limits_0^{g(x)} p t^{p-1} dt\right) dx \\ &= \int\limits_0^M p t^{p-1} \mathrm{vol}_q(\{x \in \mathbb{R}^q; g(x) \geqq t\}) dt = p \int\limits_0^M t^{p-1}\varphi(t) dt.\end{aligned}$$

Thus to prove (5), it is enough to see that

$$\int\limits_0^M t^p \varphi(t) dt \geqq \frac{p}{p+q+1} M \int\limits_0^M t^{p-1}\varphi(t) dt. \tag{6}$$

We prove that $\varphi^{\frac{1}{q}}$ is concave on $[0, M]$:

Let x, y be in $[0, M]$ satisfying $g(x) \geqq t$ and $g(y) \geqq s$. Since g is concave,

$$g(\lambda x + (1-\lambda) y) \geqq \lambda t + (1-\lambda) s\ .$$

Thus $\lambda\{g \geqq t\} + (1-\lambda)\{g \geqq s\} \subset \{g \geqq \lambda t + (1-\lambda) s\}$. By the Brunn-Minkowski theorem and the definition of φ, this implies that

$$\varphi^{\frac{1}{q}}(\lambda t + (1-\lambda) s) \geqq \lambda \varphi^{\frac{1}{q}}(t) + (1-\lambda)\varphi^{\frac{1}{q}}(s),$$

which means that $\varphi^{\frac{1}{q}}$ is concave on $[0, M]$.

Moreover it is clear that φ is non-increasing on $[0, M]$. Therefore, to prove inequality (6), it is enough to show the following lemma and the case of equality in Theorem 3 follows from the case of equality in the lemma.

Lemma 2. *Let $p > 0$ and $q > 0$ be two real numbers. Let $\varphi : [0, M] \to \mathbb{R}$, $\varphi \geqq 0$, $\varphi \neq 0$ be a non-increasing function such that $\varphi^{\frac{1}{q}}$ is concave. Then*

$$\int\limits_0^M t^p \varphi(t) dt \geqq \frac{p}{p+q+1} M \int\limits_0^M t^{p-1}\varphi(t) dt \tag{7}$$

with equality if and only if $\varphi(t) = \varphi(0)\left(1 - \frac{t}{M}\right)^q$.

Proof. Define $c = \frac{p}{p+q+1}$. Let h be the non-increasing function on $[0, M]$ such that $h^{\frac{1}{q}}$ is affine, $h(M) = 0$ and $h(cM) = \varphi(cM)$. For some $a > 0$, $h(t) = a\left(1 - \frac{t}{M}\right)^q$ for all t in $[0, M]$.

Since $\varphi^{\frac{1}{q}}$ is concave on $[0, M]$, $\varphi(M) \geqq h(M) = 0$ and $h(cM) = \varphi(cM)$, we get $\varphi \leqq h$ on $[0, cM]$ and $\varphi \geqq h$ on $[cM, M]$. Hence:

$$\int\limits_0^M (t - cM)\ t^{p-1}\varphi(t) dt \geqq \int\limits_0^M (t - cM)\ t^{p-1} h(t) dt$$

with equality if and only if $\varphi = h$.

Therefore to prove (7), it is enough to prove that $\int_0^M (t - cM)\ t^{p-1} h(t) dt = 0$, which is clear by a simple calculation.

Remark. The same result is true if we assume that f is log-concave instead of $\frac{1}{q}$-concave. Both proofs, almost without change, can be adapted to prove it.

Theorem 4. *Let $q \geqq 1$ be an integer. Let $C \subset \mathbb{R}^q$ be a convex and $f : C \to \mathbb{R}$, $f \geqq 0$, $f \neq 0$ such that* $\log f$ *is concave. Then*

$$\max_{x \in \mathbb{R}^q} f(x) \leqq e^q f(x_f) \quad \textit{where} \quad x_f = \frac{\int_{\mathbb{R}^q} x f(x)\ dx}{\int_{\mathbb{R}^q} f(x)\ dx}$$

with equality if and only if there exist $x_0 \in \mathbb{R}^q$, $\alpha \in \mathbb{R}$, $\alpha > 0$ and $u \in \mathbb{R}^q$, $u \neq 0$ such that for all $x \in C$

$$f(x) = f(x_0) \exp\left(- \frac{\langle x - x_0, u \rangle}{\alpha} \right) \quad \textit{and} \quad C = \mathrm{pos}(x_0, C_0)$$

where C_0 is a convex body in the affine hyperplane $\{x \in \mathbb{R}^q; \langle x - x_0, u \rangle = \alpha\}$.

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Anschrift des Autors:

Matthieu Fradelizi
Université de Marne-la-Vallée
Equipe d'Analyse et de Mathématiques Appliquées
2, rue de la Butte Verte
93166 Noisy-le-grand Cedex
France

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