Arch. Math. 69 (1997) 507–514
0003-889X/97/060507-08 $ 3.10/0

Archiv der Mathematik

A result related to Ricceri's conjecture on generalized quasi-variational inequalities

By

PAOLO CUBIOTTI and NGUYEN DONG YEN

Abstract. We obtain an existence theorem for generalized quasi-variational inequalities in infinite-dimensional normed spaces which improves some aspects of a recent result by P. Cubiotti [5], and gives a partial affirmative answer to a conjecture formulated by B. Ricceri [8].

1. Introduction. The aim of this paper is to establish an existence theorem for a class of generalized quasi-variational inequalities in normed spaces.

Let E be a real Hausdorff topological vector space with the topological dual E'. Let there be given a closed convex set $X \subseteqq E$, a multifunction $\Gamma : X \to 2^X$, and a multifunction $\Phi : X \to 2^{E'}$. The generalized quasi-variational inequality defined by X, Γ and Φ, is the problem of finding a pair $(\hat{x}, \hat{\varphi}) \in X \times E'$ such that

$$(1.1) \qquad \hat{x} \in \Gamma(\hat{x}), \quad \hat{\varphi} \in \Phi(\hat{x}), \quad \sup_{y \in \Gamma(\hat{x})} \langle \hat{\varphi}, \hat{x} - y \rangle \leqq 0.$$

It is convenient to denote this problem by GQVI(X, Γ, Φ). In a finite-dimensional setting, such problem was considered firstly by D. Chan and J.S. Pang [2]. We refer to [5] and [8] for detailed discussions on sufficient conditions for generalized quasi-variational inequalities to have nonempty solution sets (the existence theorems). In [5], various applications of such problems were discussed. A new application to control theory has been recently given in [4].

When $\Gamma(x) \equiv X$ for all $x \in X$, problem (1.1) is said to be a generalized variational inequality, and is denoted by GVI(X, Φ). In [7] B. Ricceri proved a general existence theorem for generalized variational inequalities under the following assumptions on the multifunction Φ:

(A1) For every $x \in X$, the set $\Phi(x)$ is nonempty, convex, and weakly-$*$ compact;
(A2) For each $y \in X - X$, the set $\{x \in X : \inf_{\varphi \in \Phi(x)} \langle \varphi, y \rangle \leqq 0\}$ is compactly closed.

(A set $\Omega \subseteqq E$ is called compactly closed if its intersection with any compact subset of E is closed.) In [3] the finite-dimensional version of the just mentioned result was extended to generalized quasi-variational inequalities. Namely, the following theorem was obtained.

Mathematics Subject Classification (1991): 49J40.

Theorem 1.1 ([3, Theorem 1]). *Assume that E is a finite-dimensional Euclidean space, $X \subseteqq E$ is a closed convex set, and $K \subseteqq X$ is a nonempty compact set. The problem* GQVI(X, Γ, Φ) *has at least one solution in $K \times E'$ if the following conditions are satisfied:*

(i) *The set $\Phi(x)$ is nonempty and compact for each $x \in X$, and convex for each $x \in K$, with $x \in \Gamma(x)$;*
(ii) *for each $y \in X - X$ the set $\{x \in X : \inf_{\varphi \in \Phi(x)} \langle \varphi, y \rangle \leqq 0\}$ is closed;*
(iii) *Γ is a lower semicontinuous multifunction with closed graph, and, for each $x \in X$, $\Gamma(x)$ is a convex set which meets K;*
(iv) *for each $x \in X \setminus K$, where $x \in \Gamma(x)$, one has*

$$\sup_{y \in \Gamma(x) \cap K} \inf_{\varphi \in \Phi(x)} \langle \varphi, x - y \rangle > 0.$$

The last assumption of the above theorem is called the coercivity condition. It is essential in the case where the set X is unbounded. Note also that multifunctions having properties (i) and (ii) were characterized in [11], where a different proof for Theorem 1.1 was given.

It is interesting to look for an infinite-dimensional version of Theorem 1.1. (Of course, it is not easy to obtain such an extension.) To accelerate further research in this direction, the following conjecture was formulated.

Conjecture (B. Ricceri [8]; see also [5]). *Assume that: Φ satisfies assumptions* (A1) *and* (A2), *Γ is lower semicontinuous, with closed graph and, for each $x \in X$, $\Gamma(x)$ is a convex set which has nonempty interior in the affine hull of X. Moreover, let K, K_1 be two nonempty compact subsets of X, with $K_1 \subseteqq K$ and K_1 finite-dimensional, such that $\Gamma(x)$ has a nonempty intersection with K_1 for every $x \in X$, and, for every $x \in X \setminus K$, with $x \in \Gamma(x)$, one has*

$$\sup_{y \in \Gamma(x) \cap K_1} \inf_{\varphi \in \Phi(x)} \langle \varphi, x - y \rangle > 0. \tag{1.2}$$

Then, GQVI(X, Γ, Φ) *has some solution in $K \times E'$.*

One of us [5, Theorem 3.1] has proved that the conjecture is true under the following additional assumptions:

(α) E is a normed space;
(β) $\Gamma : X \to 2^X$ is a Lipschitzian multifunction;
(γ) there is $r > 0$ such that the coercivity condition (1.2) holds for any $x \in X \setminus K$ satisfying $d(x, \Gamma(x)) := \inf_{y \in \Gamma(x)} \|x - y\| \leqq r$.

In the sequel, by introducing a new technical construction we are able to remove condition (γ) and weaken condition (β). But we have to assume E to be a Banach space (or to assume that the closed convex hull of the set K is compact). In particular, we shall prove that the conjecture is true under the following additional assumptions:

$(\alpha)'$ E is a Banach space;
$(\beta)'$ $\Gamma : X \to 2^X$ is a Hausdorff lower semicontinuous multifunction.

Observe that $(\beta)'$ is weaker than (β), but stronger than the property that Γ is a lower semicontinuous multifunction. The question whether the above conjecture remains true without assuming $(\alpha)'$ and $(\beta)'$, is still open. The exact formulation of our result is given in Section 2. The proof is given in Section 3.

2. The result. In what follows we shall keep all the notations of the preceding introductory part. The multifunction $\Gamma : X \to 2^X$ is said to have closed graph if the set $\{(x, y) \in X \times X : y \in \Gamma(x)\}$ is closed in $X \times X$. A multifunction $F : X \to 2^Z$ from X into a normed space Z is said to be Hausdorff lower semicontinuous on X if for every $\overline{x} \in X$ and every $\delta > 0$ there exists a neighborhood $U(\overline{x})$ of $\overline{x}$ in X such that

$$F(\overline{x}) \subseteqq F(x) + \delta B_Z \quad \text{for all} \quad x \in U(\overline{x}),$$

where B_Z denotes the closed unit ball in Z. F is said to be lower semicontinuous on X if, for every $\overline{x} \in X$ and for every open set W in Z such that $F(\overline{x}) \cap W \neq \emptyset$, there is a neighborhood $U(\overline{x})$ of $\overline{x}$ with the property that $F(x) \cap W \neq \emptyset$ for all $x \in U(\overline{x})$. It is easily seen that Hausdorff lower semicontinuity implies lower semicontinuity.

Our result can be stated as follows.

Theorem 2.1. *Suppose that E is a real Banach space, $X \subseteqq E$ is a closed convex subset, $\Gamma : X \to 2^X$ and $\Phi : X \to 2^{E'}$ are certain multifunctions, K and K_1 are nonempty compact subsets of X such that K_1 is finite-dimensional and $K_1 \subseteqq K$. The problem* GQVI (X, Γ, Φ) *has some solution in $K \times E'$ if the following assumptions are valid:*

(i) *$\Phi(x)$ is a nonempty weakly-$*$ compact set for every $x \in X$, and $\Phi(x)$ is convex for each $x \in K$ such that $x \in \Gamma(x)$;*
(ii) *for every $y \in X - X$, the set $\{x \in X : \inf_{\varphi \in \Phi(x)} \langle \varphi, y \rangle \leqq 0\}$ is compactly closed;*
(iii) *Γ is Hausdorff lower semicontinuous, with closed graph;*
(iv) *for every $x \in X$, $\Gamma(x)$ is a convex set whose interior in the affine hull of X is nonempty, and $\Gamma(x)$ has a nonempty intersection with K_1;*
(v) *for every $x \in X \setminus K$, $x \in \Gamma(x)$, one has*

$$\sup_{y \in \Gamma(x) \cap K_1} \inf_{\varphi \in \Phi(x)} \langle \varphi, x - y \rangle > 0.$$

The proof of Theorem 2.1 will be given in the next section. Here we want only to point out that in our argument the completeness of the space E is used only to guarantee that the closed convex hull of the set K (denoted by $\overline{\text{co}}\, K$) is compact. Therefore, Theorem 2.1 also holds if we assume E to be any (even non-complete) normed space and that $\overline{\text{co}}\, K$ is compact in E.

3. Proof of Theorem 2.1. This proof is based on Theorem 1.1 and some well-known facts from functional analysis. Assuming the fulfilment of the assumptions, we divide the proof into a sequence of lemmas.

Let V denote the affine hull of X. (Note that V may not be closed). Let V_0 be the linear subspace corresponding to V. By $\overset{\circ}{\Gamma}(x) := \text{int}_V \Gamma(x)$ we denote the interior of $\Gamma(x)$ in the induced topology of V. By our assumption (iv), $\overset{\circ}{\Gamma}(x) \neq \emptyset$ for all $x \in X$. Besides, $\overset{\circ}{\Gamma}(x)$ is convex because $\Gamma(x)$ is convex.

Lemma 3.1. *The map* $\overset{\circ}{\Gamma} : X \to 2^V$ *has the following property: For every* $\bar{x} \in X$ *and every* $\bar{y} \in \overset{\circ}{\Gamma}(\bar{x})$*, there exist* $\eta > 0$ *and a neighborhood* $U(\bar{x})$ *of* $\bar{x}$ *in* E *such that*

$$\bar{y} + \eta B_{V_0} \subseteqq \overset{\circ}{\Gamma}(x) \quad \text{for all} \quad x \in U(\bar{x}) \cap X. \tag{3.1}$$

For proving this useful fact one can argue as with Proposition 2.4 in [5], where a reference to [10] was made. Here we give a simple direct proof.

Proof. Let $\bar{x} \in X$ and $\bar{y} \in \overset{\circ}{\Gamma}(\bar{x})$ be given. Since $\overset{\circ}{\Gamma}(\bar{x})$ is open in V, there is $\delta > 0$ such that

$$\bar{y} + 2\delta B_{V_0} \subseteqq \overset{\circ}{\Gamma}(\bar{x}). \tag{3.2}$$

By assumption (iii), there exists a neighborhood $U(\bar{x})$ of $\bar{x}$ in E such that

$$\Gamma(\bar{x}) \subseteqq \Gamma(x) + \delta B_{V_0} \quad \text{for all} \quad x \in U(\bar{x}) \cap X. \tag{3.3}$$

Combining (3.3) with (3.2) we obtain

$$\bar{y} + 2\delta B_{V_0} \subseteqq \Gamma(x) + \delta B_{V_0} \quad \text{for all} \quad x \in U(\bar{x}) \cap X. \tag{3.4}$$

We claim that

$$\bar{y} + \delta B_{V_0} \subseteqq \Gamma(x) \quad \text{for all} \quad x \in U(\bar{x}) \cap X. \tag{3.5}$$

On the contrary, suppose that there exists $\hat{x} \in U(\bar{x}) \cap X$ and $\hat{v} \in B_{V_0}$ such that $\bar{y} + \delta\hat{v} \notin \Gamma(\hat{x})$. Then $\delta\hat{v} \notin \Gamma(\hat{x}) - \bar{y}$. Since $\Gamma(\hat{x}) - \bar{y}$ is a closed convex subset of V_0, the separation theorem [9, Theorem 9.2, p. 65] shows that there is $v^* \in V_0'$ satisfying

$$\langle v^*, \delta\hat{v} \rangle > \sup\left\{\langle v^*, z - \bar{y}\rangle : z \in \Gamma(\hat{x})\right\}. \tag{3.6}$$

By (3.4),

$$\sup\left\{\langle v^*, 2\delta v\rangle : v \in B_{V_0}\right\} \leqq \sup\left\{\langle v^*, z - \bar{y} + \delta v\rangle : z \in \Gamma(\hat{x}),\ v \in B_{V_0}\right\},$$

hence

$$2\delta\|v^*\| \leqq \sup\left\{\langle v^*, z - \bar{y}\rangle : z \in \Gamma(\hat{x})\right\} + \delta\|v^*\|.$$

Thus,

$$\langle v^*, \delta\hat{v}\rangle \leqq \delta\|v^*\| \leqq \sup\left\{\langle v^*, z - \bar{y}\rangle : z \in \Gamma(\hat{x})\right\},$$

contrary to (3.6). From (3.5) it follows that (3.1) holds for any $\eta \in (0, \delta)$. □

We now describe a construction which will enable us to apply Theorem 1.1 to deal with our infinite-dimensional problem GQVI(X, Γ, Φ).

As before, denote by $\overline{\mathrm{co}}\, K$ the closed convex hull of K. We have $\overline{\mathrm{co}}\, K \subseteqq X$. Given any $z \in \overline{\mathrm{co}}\, K$ we choose a point $y(z) \in \overset{\circ}{\Gamma}(z)$. By Lemma 3.1, there exists an open neighborhood $U(z)$ of z in E such that

$$y(z) \in \overset{\circ}{\Gamma}(x) \quad \text{for all} \quad x \in U(z) \cap X. \tag{3.7}$$

As $\overline{\mathrm{co}}\, K$ is compact (see [6, Theorem 6, p. 416]), we find some points $z_1, z_2, \ldots, z_m \in \overline{\mathrm{co}}\, K$ such that

$$\overline{\mathrm{co}}\, K \subseteqq \bigcup_{i=1}^{m} (U(z_i) \cap V) =: \Omega_0\,.$$

Since

$$\rho := \inf\left\{d(a, V \setminus \Omega_0) : a \in \overline{\mathrm{co}}\, K\right\} > 0\,,$$

then the set $\Omega := \overline{\mathrm{co}}\, K + (\rho/2)B_{V_0}$ is convex and closed in V, and $\Omega \subset \Omega_0$. Let $\mathscr{F}$ be the family of all the finite-dimensional subspaces in E which contain K_1 and also the points $y(z_1), \ldots, y(z_m)$. Consider $\mathscr{F}$ as a directed set, with the set-theoretic inclusion. Fix any $S \in \mathscr{F}$. Note that $X_\Omega := X \cap \Omega$ is convex, and closed in V. For every $x \in X_\Omega$ there is some $i \in \{1, \ldots, m\}$ such that $x \in U(z_i)$. Hence $y(z_i) \in \mathring{\Gamma}(x)$. Therefore, $\mathring{\Gamma}(x) \cap S \neq \emptyset$. Let us set

$$\Gamma^S(x) = \Gamma(x) \cap \overline{X_\Omega \cap S} \quad \text{for every} \quad x \in \overline{X_\Omega \cap S}, \tag{3.8}$$

where, for any subset $A \subseteqq E$, $\overline{A}$ denotes the closure of A. Since the subspace S is finite-dimensional, it is closed in E, so $\Sigma := \overline{X_\Omega \cap S}$ is a closed convex subset of S. Since X_Ω is bounded, then Σ is also bounded, and hence it is a compact set in S. Every $\varphi \in E'$ generates a unique element $j(\varphi) \in S'$ by the formula

$$\langle j(\varphi), v\rangle := \langle \varphi, v\rangle \quad \text{for all} \quad v \in S.$$

For every $x \in X$, we define $\tilde{\Phi}(x) = \{j(\varphi) : \varphi \in \Phi(x)\}$.

The task is now to prove that Theorem 1.1 can be applied to the problem GQVI(Σ, Γ^S, $\tilde{\Phi}$). Take $K \cap S$ as the nonempty compact set in Theorem 1.1. The fact that $\tilde{\Phi}(x)$ is nonempty and compact for every $x \in \Sigma$, and convex for each $x \in K \cap S$ satisfying $x \in \Gamma^S(x)$, follows directly from our assumption (i) and the definition of $\tilde{\Phi}$. Also, it is easy to check that the coercivity condition of Theorem 1.1 holds by our assumption (v). Now, take any $y \in \Sigma - \Sigma \subseteqq X - X$. Since

$$\left\{x \in \Sigma : \inf_{\tilde{\varphi} \in \tilde{\Phi}(x)} \langle \tilde{\varphi}, y\rangle \leqq 0\right\} = \left\{x \in \Sigma : \inf_{\varphi \in \Phi(x)} \langle \varphi, y\rangle \leqq 0\right\}$$
$$= \left\{x \in X : \inf_{\varphi \in \Phi(x)} \langle \varphi, y\rangle \leqq 0\right\} \cap \Sigma,$$

then condition (ii) in Theorem 1.1 follows from our assumption (ii). What is left is to show that Γ^S is a lower semicontinuous multifunction with closed graph, $\Gamma^S(x)$ is nonempty and convex for each $x \in \Sigma$.

Since $\Gamma : X \to 2^X$ has closed graph, (3.8) implies that Γ^S is a multifunction with closed graph. As $\Gamma(x) \cap K_1 \neq \emptyset$ for every $x \in X$, and $K_1 \subset X_\Omega \cap S$, then (3.8) shows that $\Gamma^S(x)$ is nonempty for every $x \in \Sigma$. Given any $x \in \Sigma$, we deduce from formula (3.8) and the convexity property of $\Gamma(x)$ and X_Ω that $\Gamma^S(x)$ is a convex set.

Lemma 3.2. *For every $x \in \Sigma$, $\mathring{\Gamma}(x) \cap X_\Omega \cap S$ is nonempty.*

Proof. Let $x \in \Sigma = \overline{X_\Omega \cap S}$. Obviously, there is some $x' \in X_\Omega \cap S$ such that $\|x - x'\| \leqq \rho/4$. Hence $x - x' \in (\rho/4)B_{V_0}$. As $x' \in X_\Omega \subseteqq \overline{\mathrm{co}}\, K + (\rho/2)B_{V_0}$, then $x \in \overline{\mathrm{co}}\, K + (3\rho/4)B_{V_0} \subseteqq \mathring{\Omega}_0$. So there exists $i \in \{1, \ldots, m\}$ such that $x \in U(z_i)$. By (3.7), $y(z_i) \in \mathring{\Gamma}(x)$. In consequence, $\mathring{\Gamma}(x) \cap S \neq \emptyset$. Fix any $v_0 \in \mathring{\Gamma}(x) \cap S$ and any $v_1 \in \Gamma(x) \cap K_1$. The convexity of $\Gamma(x)$ implies that $\xi_t := v_1 + t(v_0 - v_1)$ is contained in $\mathring{\Gamma}(x) \cap S$ for all $t \in (0, 1]$. Since $v_1 + (\delta/2)B_{V_0} \subseteqq \Omega$, then $\xi_t \in \mathring{\Gamma}(x) \cap \mathring{X}_\Omega \cap S$ for all sufficiently small $t \in (0, 1]$. In particular, $\mathring{\Gamma}(x) \cap X_\Omega \cap S$ is nonempty. $\square$

Lemma 3.3. *$\Gamma^S : \Sigma \to 2^\Sigma$ is a lower semicontinuous multifunction.*

Proof. Suppose that $\bar{x} \in \Sigma$ and W is an open set in V such that $\Gamma^S(\bar{x}) \cap W \neq \emptyset$. By Lemma 3.2, there exists some $v_0 \in \overset{\circ}{\Gamma}(\bar{x}) \cap X_\Omega \cap S \subseteqq \Gamma^S(\bar{x})$. Let $\bar{v}$ be a point from $\Gamma^S(\bar{x}) \cap W = \Gamma(\bar{x}) \cap \overline{X_\Omega \cap S} \cap W$. Repeating the argument which has been used in the proof of Lemma 3.2, we find $\tau \in (0, 1]$ such that

$$\xi_\tau := \bar{v} + \tau(v_0 - \bar{v}) \in \overset{\circ}{\Gamma}(\bar{x}) \cap \overline{X_\Omega \cap S} \cap W. \tag{3.9}$$

By Lemma 3.1, there is a neighborhood $U(\bar{x})$ of $\bar{x}$ such that

$$\xi_\tau \in \overset{\circ}{\Gamma}(x) \quad \text{for all} \quad x \in U(\bar{x}). \tag{3.10}$$

From (3.9) and (3.10) it follows that

$$\overset{\circ}{\Gamma}(x) \cap \overline{X_\Omega \cap S} \cap W \neq \emptyset \quad \text{for all} \quad x \in U(\bar{x}).$$

Consequently, $\Gamma^S(x) \cap W \neq \emptyset$ for all $x \in U(\bar{x})$, as desired. □

For the problem GQVI $(\Sigma, \Gamma^S, \tilde{\Phi})$, all the assumptions of Theorem 1.1 have been verified. By that theorem, there exists a pair $(x_S, \tilde{\varphi}_S) \in (K \cap S) \times S'$ with the property that

$$x_S \in \Gamma^S(x_S), \quad \tilde{\varphi}_S \in \tilde{\Phi}(x_S), \quad \sup_{y \in \Gamma^S(x_S)} \langle \tilde{\varphi}_S, x_S - y \rangle \leqq 0. \tag{3.11}$$

Of course, $\tilde{\varphi}_S = j(\varphi_S)$ for some $\varphi_S \in \Phi(x_S)$. Clearly, the inequality in (3.11) implies

$$\sup_{y \in \Gamma(x_S) \cap X_\Omega \cap S} \langle \varphi_S, x_S - y \rangle \leqq 0. \tag{3.12}$$

Then we must have

$$\sup_{y \in \Gamma(x_S) \cap S} \langle \varphi_S, x_S - y \rangle \leqq 0. \tag{3.13}$$

Indeed, for any $y \in \Gamma(x_S) \cap S$, since $x_S \in K \subseteqq \overline{\text{co}}\, K \subseteqq X \subseteqq V$, $y \in \Gamma(x_S) \subseteqq X \subseteqq V$, $V - V \subseteqq V_0$, and X is a convex set, then $y_t := x_S + t(y - x_S)$ belongs to $X_\Omega = X \cap (\overline{\text{co}}\, K + (\rho/2) B_{V_0})$ for a sufficiently small $t \in (0, 1)$. By (3.12),

$$\langle \varphi_S, x_S - y_t \rangle = t \langle \varphi_S, x_S - y \rangle \leqq 0,$$

hence $\langle \varphi_S, x_S - y \rangle \leqq 0$, and (3.13) is proved.

It follows from (3.11) and the above arguments that: For every $S \in \mathscr{F}$ there exists $(x_S, \varphi_S) \in (K \cap S) \times E'$ such that

$$x_S \in \Gamma(x_S), \quad \varphi_S \in \Phi(x_S), \quad \sup_{y \in \Gamma(x_S) \cap S} \langle \varphi_S, x_S - y \rangle \leqq 0. \tag{3.14}$$

Since K is compact, then the net $\{x_S\}_{S \in \mathscr{F}}$ has a cluster point $\hat{x} \in K$. As Γ has closed graph, it follows from the first inclusion in (3.14) that $\hat{x} \in \Gamma(\hat{x})$.

The following lemma will complete the proof of Theorem 2.1.

Lemma 3.4. *There exists $\hat{\varphi} \in \Phi(\hat{x})$ such that $(\hat{x}, \hat{\varphi})$ is a solution of the problem* GQVI (X, Γ, Φ).

Proof. (We shall use the same method of reasoning as in [5] and in [7].) Let us first show that

$$\inf_{\varphi\in\Phi(\hat{x})} \langle \varphi, \hat{x} - y\rangle \leqq 0 \quad \text{for every} \quad y \in \overset{\circ}{\Gamma}(\hat{x}). \tag{3.15}$$

By contrary, assume that there exists $\tilde{y} \in \overset{\circ}{\Gamma}(\hat{x})$ such that

$$\inf_{\varphi\in\Phi(\hat{x})} \langle \varphi, \hat{x} - \tilde{y}\rangle > 0.$$

By Lemma 3.1, there is $\delta > 0$ such that

$$\tilde{y} + \delta B_{V_0} \subseteqq \Gamma(x) \quad \text{for all} \quad x \in (\hat{x} + \delta B_E) \cap X. \tag{3.16}$$

By our assumption (ii), there exists $\mu \in (0, \delta)$ such that

$$\inf_{\varphi\in\Phi(x)} \langle \varphi, \hat{x} - \tilde{y}\rangle > 0 \quad \text{for all} \quad x \in (\hat{x} + \mu B_E) \cap K.$$

Choose a subspace $S_1 \in \mathscr{F}$ such that $\tilde{y} \in S_1$ and $\hat{x} \in S_1$. Let $S_2 \in \mathscr{F}$ be such that $S_1 \subseteqq S_2$ and $\|x_{S_2} - \hat{x}\| < \mu$. Putting $w = x_{S_2} - \hat{x} + \tilde{y}$, we have $w \in V$ and

$$\|w - \tilde{y}\| = \|x_{S_2} - \hat{x}\| < \mu,$$

hence (3.16) yields $w \in \Gamma(x_{S_2})$. Since

$$\inf_{\varphi\in\Phi(x_{S_2})} \langle \varphi, x_{S_2} - w\rangle = \inf_{\varphi\in\Phi(x_{S_2})} \langle \varphi, \hat{x} - \tilde{y}\rangle > 0,$$

then $\langle \varphi_{S_2}, x_{S_2} - w\rangle > 0$. As $w \in \Gamma(x_{S_2}) \cap S_2$, then the last inequality is a contradiction to (3.14). We have then obtained (3.15), which can be rewritten as

$$\sup_{y\in\overset{\circ}{\Gamma}(\hat{x})} \inf_{\varphi\in\Phi(\hat{x})} \langle \varphi, \hat{x} - y\rangle \leqq 0.$$

According to Theorem 5 on p. 216 in [1], there exists $\hat{\varphi} \in \Phi(\hat{x})$ such that

$$\sup_{y\in\overset{\circ}{\Gamma}(\hat{x})} \langle \hat{\varphi}, \hat{x} - y\rangle = \sup_{y\in\overset{\circ}{\Gamma}(\hat{x})} \inf_{\varphi\in\Phi(\hat{x})} \langle \varphi, \hat{x} - y\rangle \leqq 0.$$

This yields $\sup_{y\in\Gamma(\hat{x})} \langle \hat{\varphi}, \hat{x} - y\rangle \leqq 0$, which completes the proof. □

Acknowledgements. The research of the second-named author was supported by the Italian Consiglio Nazionale delle Ricerche. He would like to thank Professor B. Ricceri and Dr. P. Cubiotti for warm hospitality at the Department of Mathematics, University of Messina, during his one-month stay in 1996.

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Eingegangen am 5. 10. 1996*)

Anschriften der Autoren:

Paolo Cubiotti
Department of Mathematics
University of Messina
98166 Sant'Agata-Messina
Italy

Nguyen Dong Yen
Hanoi Institute of Mathematics
P.O. Box 631
Bo Ho
10000 Hanoi
Vietnam

*) Eine Neufassung ging am 24. 3. 1997 ein.