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An expansion formula for the norm of a Jordan algebra

By

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Abstract. Matrices of a given size with entries from a field form an associative algebra, and can therefore be considered as a Jordan algebra. From that point of view, the determinant of a matrix is nothing but the reduced generic norm of the Jordan algebra. The basic expansion formulas of matrix determinants along a row or column do not make sense in the setting of arbitrary Jordan algebras. However, there also exists an expansion formula for matrix determinants along the diagonal of a matrix. In the paper, we show that, suitably interpreted, such an expansion formula holds for the reduced generic norm of a large class of Jordan algebras, including separable finite-dimensional Jordan algebras over fields of characteristic not 2.

There has recently been a great interest in the generic norm of finite-dimensional Jordan algebras, see for examples the papers by Jacobson, [4]–[8], and Petersson [15]. In this note, we give an algebraic proof for a formula, (3) below, for which an analytic proof was given in [12] which is written for a readership of statisticians and probabilists. At the same time, the formula is proven in a more general setting.

This formula can be motivated as follows. Let J be the Jordan algebra of $n \times n$ matrices over a field k. As is well-known, its generic norm as a Jordan algebra is the determinant. There are many formulas for determinants, but even the most basic ones, like expansion according to a row or column or *Leibniz's formula* for the determinant of the matrix $A = (a_{ij})$,

$$\det A = \sum_{\pi \in S_n} (-1)^{\pi} a_{1\pi(1)} \cdots a_{n\pi(n)}, \tag{1}$$

cannot be generalized to the setting of arbitrary Jordan algebras, at least not in an obvious way. This is due to the difficulty describing products of off-diagonal elements in terms of the Jordan algebra J. However, diagonal elements are well-adapted to the Jordan setting: they can be interpreted as coordinates with respect to an orthogonal system of primitive idempotents. Hence, any expansion of the determinant with respect to diagonal elements is likely to have a generalization to Jordan algebras. In fact, such a formula exists and follows easily from (1), namely

$$\det\,(A + \sum_{i=1}^{r} \lambda_i E_{ii}) = \sum_{T \subset \{1,\ldots,r\}} \det\,(A|T) \prod_{i \in T} \lambda_i \tag{2}$$

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where $A|T$ denotes the matrix obtained from A by deleting the rows and columns whose index is in T.

We will show that, with obvious interpretations, (2) holds for the reduced generic norm of any finite-dimensional Jordan algebra (Theorem 2), and even more generally, for certain "norm-like" systems of polynomials on an arbitrary not necessarily finite-dimensional Jordan algebra (Theorem 1). This general setting may be of interest if one wants to prove the formula for generically algebraic Jordan algebras ([9], [13]). For an application of our formula in the theory of Toeplitz C*-algebras the reader is referred to the recent preprint [2].

Setting. Throughout, J will denote a unital Jordan algebra over an infinite field k of characteristic $\neq 2$. We will denote by e the unit element of J, by $U(.)$ the quadratic representation of J, by $\{\ldots\}$ the corresponding Jordan triple product, by $B(.,.)$ the Bergman operator, i.e., $B(x,y)z = z - \{xyz\} + U(x)U(y)z$. For an idempotent c the Peirce spaces will be written as $J_i(c)$ and indexed by $i = 0, 1$ or 2.

1. The formula in a general setting. In this section, we will prove a generalization of (2) for certain types of polynomial functions on Jordan algebras, see Theorem 1 below. We will need:

Lemma 1. *Let $c \in J$ be an idempotent with Peirce spaces $J_i = J_i(c), = 0, 1$ or 2.*

a) *For $z_1 \in J_1$ and $w_i \in J_i, i = 0$ or 2, the Bergman operator $B(z_1, w_i)$ belongs to the structure group* Str (J) of J *and is a commutator in* Str (J).
b) *If for $x = x_0 + x_1 + x_2 \in J$ the Peirce-0-component x_0 is invertible in J_0, then $x = B(z,w)y$ for suitable $z \in J_1, w \in J_0$ and $y = y_2 \oplus y_0 \in J_2 \oplus J_0$.*

Proof. a) One knows ([10] 5.7) that (z_1, w_i) is quasi-invertible: (w_i, z_1) is nilpotent with $w_i^{(n,z_1)} = 0$ for $n \geqq 2$, so in particular $w_i^{z_1} = w_i$ and hence (z_1, w_i) is quasi-invertible by the symmetry principle. The Bergman operator $B(z_1, w_i)$ therefore belongs to the structure group Str (J) of J. By [10] (JP34), the map $J_1 \to$ Str $(J) : x_1 \mapsto B(x_1, w_i)$ is a group homomorphism. One also knows that $B(c, (1-\lambda)c) \in$ Str (J) for $\lambda \in k^*$ ([10] 5.3). But then, by [10] (3.9.5), (JP35) and 5.4, we obtain for an arbitrary $x_1 \in J_1$

$$\begin{aligned}&B(c,(1-\lambda)c)B(x_1,w_i)B(c,(1-\lambda)c)^{-1}B(x_1,w_i)^{-1}\\&= B(B(c,(1-\lambda)c)x_1, B(c,(1-\lambda)c)^{-1}w_i)B(-x_1,w_i)\\&= B(\lambda x_1, \lambda^{-i}w_i)B(-x_1,w_i) = B((\lambda-1)\lambda^{-i}x_1, w_i).\end{aligned}$$

Since $|k| > 2$ there exists $\lambda \in k^*$ such that $J_1 \to J_1 : x_1 \to (\lambda - 1)\lambda^{-i}x_1$ is surjective.

b) Let $w \in J_0$ be the inverse of x_0 in J_0. Then $\{x_0 w v\} = v$ for every $v \in J_1$ since $(e-c, e-c)$ and (x_0, w) are associated idempotents of the Jordan pair (J,J). Hence $x_1 = -\{z w x_0\}$ for $z = -x_1$.We put $y = x_0 \oplus x_2 - U(z)U(w)x_0$. A straightforward computation then shows $x = B(z,w)y$. $\square$

We recall that an idempotent $c \in J$ is *primitive* if 0 and c are the only idempotents in $J_2(c)$. We call an idempotent $c \in J$ *absolutely primitive* if every element of $J_2(c)$ has the form $\lambda c + n$ where $\lambda \in k$ and n is nilpotent. It is easily seen that an absolutely primitive idempotent is primitive ([3] p. 197).

We also need to recall the notions of Zariski topology and of polynomial functions on J([13]). Let $\{J_\alpha\}$ be the collection of finite-dimensional subspaces of J equipped with the usual Zariski topology. With respect to inclusion, $\{J_\alpha\}$ forms a directed set of topological spaces and the Zariski topology on J is the direct limit of these topological spaces. Hence, a subset V of J is open if and only if $V \cap J_\alpha$ is open for every α. As in the finite-dimensional case, non-empty open sets are dense.

Let Y be a vector space over k. We denote by $\mathrm{Pol}\,(J_\alpha, Y)$ the vector space of *polynomial functions from* J_α *to* Y, i.e., the span of the maps $x_\alpha \mapsto f_\alpha(x_\alpha)y$ where $y \in Y$ and $f_\alpha : J_\alpha \to k$ is an ordinary polynomial. For $J_\alpha \subset J_\beta$ one has the natural restriction map $\mathrm{Pol}\,(J_\beta, Y) \to \mathrm{Pol}\,(J_\alpha, Y)$ and one easily sees that $\{\mathrm{Pol}\,(J_\alpha, Y)\}$ forms an inverse system. We let $\mathrm{Pol}\,(J, Y)$ be its inverse limit and call its elements *polynomials on* J. Hence, a map $F : J \to Y$ is a polynomial if and only if $F|J_\alpha$ is a polynomial for all J_α. Note that polynomials are continuous with respect to the Zariski topology.

We consider a class $\mathscr{C}$ of Jordan algebras over k which is closed under taking Peirce subalgebras, i.e., for every J in $\mathscr{C}$ and for every idempotent $c \in J$ the subalgebras $J_i(c), i = 0$ or 2, again lie in $\mathscr{C}$. In Section 2 we will take as $\mathscr{C}$ the class of all finite-dimensional Jordan algebras. We fix an extension field L of k and assume that for every Jordan algebra J in $\mathscr{C}$ we are given a polynomial $\mathscr{N}_J : J \to L$. We call the collection $(\mathscr{N}_J | J \in \mathscr{C})$ a *norm-like system for* $\mathscr{C}$ *over* L if it has the following properties:

(N1) for every J in $\mathscr{C}$ there exists a character χ: $\mathrm{Str}\,(J) \to L$ such that $\mathscr{N}_J(gx) = \chi(g)\mathscr{N}_J(x)$ for every $g \in \mathrm{Str}\,(J)$ and $x \in J$;

(N2) for every absolutely primitive idempotent $c \in J$ and $y_i \in J_i(c)$ we have

$$\mathscr{N}_J(y_2 \oplus y_0) = \mathscr{N}_{J_2(c)}(y_2)\mathscr{N}_{J_0(c)}(y_0);$$

(N3) if $J = J_2(c)$ for an absolutely primitive idempotent c then $\mathscr{N}_J(\lambda c + n) = \lambda$ for $\lambda \in k$ and n nilpotent;

(N4) if $J = (0)$ then $\mathscr{N}_J(0) = 1$.

An important special case is the class $\mathscr{C}$ of all subalgebras $J_2(c)$ of a fixed separable finite-dimensional Jordan algebra J where c runs over all idempotents of J. For this class the generic norm of every $J_2(c)$ gives a norm-like system. This follows from standard properties of the generic norm, keeping in mind that every $J_2(c)$ inherits separability from J, and hence, in particular for an absolutely primitive idempotent c, we have $J_2(c) = kc$. Observe that, by separability, the generic norm of $J_2(c)$ coincides with the reduced generic norm. It is this way of looking at things that we will generalize in Theorem 2 below.

Theorem 1. *Let* $(\mathscr{N}_J)$ *be a norm-like system for* $\mathscr{C}$ *and let* $(c_1, \ldots, c_r)$ *be an orthogonal system of absolutely primitive idempotents of* J *where* J *is some algebra in* $\mathscr{C}$. *Then for all* $x \in J$ *and all* $\lambda_i \in k$ *we have*

$$\mathscr{N}_J\Big(x + \sum_{i=1}^{r} \lambda_i c_i\Big) = \sum_{T \subset \{1,\ldots,r\}} \mathscr{N}_{J_0(c_T)}(U(e - c_T)x)\lambda_T \tag{3}$$

where $c_T = \sum_{i \in T} c_i$ *and* $\lambda_T = \prod_{i \in T} \lambda_i$.

In particular, for an absolutely primitive idempotent $c \in J, \lambda \in k$ and $x \in J$

(4) $$\mathcal{N}_J(x+\lambda c) = \mathcal{N}_J(x) + \lambda \mathcal{N}_0(x)$$

where $\mathcal{N}_0(x) = \mathcal{N}_{J_0(c)}(U(e-c)x)$.

In (3) note that the case $T = \emptyset$ gives $c_T = 0$ and $\lambda_T = 1$, hence the corresponding term on the right side of (3) is $\mathcal{N}_J(x)$.

Proof by induction on r. For $r = 1$, (3) becomes (4), so we begin by proving (4).

Let $J_i = J_i(c)$ for $i = 0, 1, 2$. If $c = e$ then (4) follows from (N3) and (N4). In the following we can therefore assume that $c \neq e$. We first prove (4) in the special case where x has the form $x = B(z,w)y$ for $z \in J_1, w \in J_0$ and $y = y_2 \oplus y_0 \in J_2 \oplus J_0$. For such an x we have

$$\begin{aligned}\mathcal{N}_J(x+\lambda c) &= \mathcal{N}_J(B(z,w)(y+\lambda c)) \ \{\text{since } B(z,w)c = c\}\\ &= \mathcal{N}_J(y+\lambda c) \ \{\text{by (N1) and Lemma 1.a}\}\\ &= \mathcal{N}_J(y_0 \oplus (\kappa+\lambda)c + n_2) \ \{\text{where } y_2 = \kappa c + n_2 \text{ for } n_2 \in \operatorname{Rad}(J_2)\}\\ &= \mathcal{N}_0(y_0)\mathcal{N}_{J_2(c)}((\kappa+\lambda)c + n_2) \ \{\text{by (N2)}\}\\ &= \mathcal{N}_0(y_0)(\kappa+\lambda) \ \{\text{by (N3)}\}.\end{aligned}$$

On the other hand, $\mathcal{N}_J(x) = \mathcal{N}_J(B(z,w)y) = \mathcal{N}_J(y)$ {by (N1) and Lemma 1.a}$= \mathcal{N}_0(y_0)\kappa$ and $x_0 = (B(z,w)y)_0 = y_0$ since $\{zwy\} \in J_1$ and $U(z)U(w)y \in J_2$. This proves (4) assuming $x = B(z,w)y$.

Now fix $v = x_1 \oplus x_2 \in J_1 \oplus J_2, \lambda \in k$ and define

$$f: J_0 \to k : x_0 \mapsto \mathcal{N}_J(x_0 + v + \lambda c), \quad g: J_0 \to k : x_0 \mapsto \mathcal{N}_J(x_0 + v) + \lambda \mathcal{N}_0(x_0).$$

By what we have already shown and by Lemma 1.b, f and g agree on $\{x_0 \in J_0;\ x_0 \text{ is invertible in } J_0\}$ which is non-empty (containing $e - c$) and Zariski-open, hence Zariski-dense in J_0. Therefore $f = g$ on J_0, and (4) is proven.

We now assume $r \geqq 2$ and prove (3). By induction applied to $(c_1, \ldots, c_{r-1}) \subset J$ and by (4)

$$\begin{aligned}\mathcal{N}_J\Big(x + \sum_{i=1}^{r} \lambda_i c_i\Big) &= \mathcal{N}_J\Big(x + \lambda_r c_r + \sum_{i=1}^{r-1} \lambda_i c_i\Big)\\ &= \sum_{T \subset \{1,\ldots,r-1\}} \mathcal{N}_{J_0(c_T)}(U(e-c_T)(x+\lambda_r c_r))\lambda_T\\ &= \sum_{T \subset \{1,\ldots,r-1\}} \Big(\mathcal{N}_{J_0(c_T)}(U(e-c_T)x) + \lambda_r \mathcal{N}_{J_0(c_{T\cup\{r\}})}(U(e-c_{T\cup\{r\}})x)\Big)\lambda_T\\ &= \sum_{T \subset \{1,\ldots,r-1\}} \mathcal{N}_{J_0(c_T)}(U(e-c_T)x)\lambda_T\\ &\quad + \sum_{r \in T \subset \{1,\ldots,r\}} \mathcal{N}_{J_0(c_T)}(U(e-c_T)x)\lambda_T\end{aligned}$$

which equals the right side of (3). □

2. Finite-dimensional Jordan algebras. In this section we will show that the reduced generic norm is a norm-like system over $\bar{k}$, the algebraic closure of k, for the class of finite-dimensional Jordan algebras. *Throughout this section J will be a finite-dimensional Jordan algebra.*

We begin by recalling some definitions and notations. We denote by $m_x^0(\lambda)$ the *absolute reduced generic minimum polynomial of J*, i.e., the monic polynomial of smallest degree with coefficients in the polynomial functions on $\bar{J} = \bar{k} \otimes J$ such that $m_x^0(x)$ is nilpotent for every $x \in \bar{J}$. The *(absolute) reduced generic norm* N^0 is $(-1)^{m_0}$ times the constant term of $m_x^0(\lambda)$, where m_0 is the degree of $m_x^0(\lambda)$. Observe that in [1] the reduced generic norm N^0 is denoted by RN and $m_x^0(\lambda)$ by $RN(\lambda e - x)$. We denote by $\mu_x^0(\lambda)$ the reduced minimum polynomial of x, i.e., the monic polynomial of smallest degree among all polynomials $f \in \bar{k}[\lambda]$ with $f(x)$ nilpotent. Clearly, $\mu_x^0(\lambda) | m_x^0(\lambda)$. The following results are proven in [1] II.4.5 resp. [1] III Satz 4.3:

(5) the roots of $\mu_x^0(\lambda)$ and $m_x^0(\lambda)$ coincide, and
(6) $m_x^0(\lambda)$ is separable, i.e., the discriminant $\delta^0(x)$of $m_x^0(\lambda)$ is a non-zero polynomial function on J.

The main technique used in the following is the theory of Cartan subalgebras which are defined as follows. The Lie triple system associated with J is the triple system on J with triple product $[xyz] = (xy)z - x(yz) = R_{x,y}(z)$. One calls J *associator nilpotent* if the Lie triple system $[\ldots]$ is nilpotent, i.e., there exists an n such that any product of n of the $R_{x,y}$'s vanishes. A subalgebra $\mathfrak{C}$ of J is called *Cartan subalgebra* if it is associator nilpotent and equal to its own normalizer in the sense that $[\mathfrak{C}\mathfrak{C}_z] \subset \mathfrak{C}$ implies $z \in \mathfrak{C}$. Results on Cartan subalgebras of Jordan algebras can be found in [3] VIII.11 and [11].

Lemma 2 (cf. [10]16.15). *Let $\mathfrak{C}$ be a Cartan subalgebra of J. Denote by $m_x^0(\lambda)$ resp. $m_x^{0,\mathfrak{C}}(\lambda)$ the reduced generic minimum polynomial of J resp. $\mathscr{C}$ and denote by N^0 resp. $N^0_{\mathfrak{C}}$ the reduced generic norm of J resp. of $\mathfrak{C}$. Then*

$$m_x^0(\lambda) = m_x^{0,\mathfrak{C}}(\lambda) \text{ for } x \in \mathfrak{C} \text{ and } N^0_{\mathfrak{C}} = N^0|\mathfrak{C}.$$

Proof. We proceed as in the proof of [10] 16.15. We can assume that k is algebraically closed. Then $\mathfrak{C} = \oplus J_{ii}$ with respect to a complete orthogonal system of primitive idempotents ([3] VIII). Let H be the automorphism group of J. By [11] Lemma 3 we know that $H.\mathfrak{C} = H.(\oplus J_{ii})$ contains a Zariski-open and dense subset of J. On the other hand, it follows from (5) and (6) that

$$A = \{x \in J | m_x^0(\lambda) = \mu_x^0(\lambda)\}$$

is also a Zariski-open and dense subset of J. Hence $A \cap H.\mathfrak{C} \neq \emptyset$. But A is invariant under H, and therefore $A \cap \mathfrak{C} \neq \emptyset$. In obvious notation, we have $\mu_x^{0,\mathfrak{C}}(\lambda) = \mu_x^0(\lambda)$ for every $x \in \mathfrak{C}$, since $e \in \mathfrak{C}$. Hence, for $x \in A \cap \mathfrak{C}$ we obtain $m_x^0(\lambda) = \mu_x^0(\lambda) = \mu_x^{0,\mathfrak{C}}(\lambda)$. But $\mu_x^{0,\mathfrak{C}}(\lambda)$ divides $m_x^{0,\mathfrak{C}}(\lambda)$ which in turn divides $m_x^0(\lambda)$, so

$$m_x^0(\lambda) = \mu_x^0(\lambda) = \mu_x^{0,\mathfrak{C}}(\lambda) = m_x^{0,\mathfrak{C}}(\lambda)$$

holds for x in the Zariski-open subset $A \cap \mathfrak{C}$ of $\mathfrak{C}$, whence $m_x^0(\lambda) = m_x^{0,\mathfrak{C}}(\lambda)$ for every $x \in \mathfrak{C}$. This immediately implies the second assertion. □

Remark. Since J is finite-dimensional there exists a monic polynomial $f_x(\lambda)$ with coefficients in the ring of polynomial functions on J which is *satisfied by J* in the sense that $f_x(x) = 0$ holds for all $x \in J$. The unique monic polynomial of smallest degree which is satisfied by J is called the *generic minimum polynomial of* J and denoted $m_x(\lambda)$. The *generic norm* N is $(-1)^m$ times the constant term of $m_x(\lambda)$, where m is the degree of $m_x(\lambda)$. (In [1], the generic norm is denoted by HN and $m_x(\lambda)$ by $HN(\lambda e - x)$.) The proof above can be easily adapted for these polynomials, i.e., Lemma 2 is also true for the generic minimum polynomial and the generic norm.

Lemma 3. *Let c be an idempotent of J and put $B = J_2(c) \oplus J_0(c)$. Then:*

a) *Any Cartan subalgebra $\mathfrak{C}$ of B has the form $\mathfrak{C} = (\mathfrak{C} \cap J_2(c)) \oplus (\mathfrak{C} \cap J_0(c))$ and is also a Cartan subalgebra of J.*

b) *With the obvious notations, $m_x^0(\lambda) = m_x^{0,B}(\lambda) = m_{x_2}^{0,J_2}(\lambda).m_{x_0}^{0,J_0}(\lambda)$ for $x = x_2 + x_0 \in B$ and $N^0(x) = N_B^0(x) = N_2^0(x_2)N_0^0(x_0)$.*

Proof. a) Let $\mathfrak{C}$ be a Cartan subalgebra of B. Because $\mathfrak{C}$ is self-normalizing with respect to $[\ldots]$ one finds $c \in \mathfrak{C}$, which implies the first assertion. Moreover, since $[ccx] = \frac{1}{4}x_1$ for any $x = x_2 + x_1 + x_0 \in J$, every $x \in J$ normalizing $\mathfrak{C}$ must already lie in B, so $\mathfrak{C}$ is a Cartan subalgebra of J.

b) By [11] Thm. 3, B has a Cartan subalgebra $\mathfrak{C}$. By a) it is also a Cartan subalgebra of J, so by Lemma 2, $m_x^0(\lambda) = m_x^{0,\mathfrak{C}}(\lambda)$ for $x \in \mathfrak{C}$. On the other hand, one knows that $m_x^{0,\mathfrak{C}}(\lambda) | m_x^{0,B}(\lambda)$ for $x \in \mathfrak{C}$ and that $m_x^{0,B}(\lambda) | m_x^0(\lambda)$ for $x \in B$ (cf. [3] IV.3 Cor. 5 for the analogous statement for the generic minimum polynomials). But this implies $m_x^0(\lambda) = m_x^{0,B}(\lambda)$. The second equation,

$$m_x^{0,B}(\lambda) = m_{x_2}^{0,J_2}(\lambda).m_{x_0}^{0,J_0}(\lambda)$$

is true in general (cf. [3] VI.3 Thm. 2 for the analogous statement for the generic minimum polynomials). The claims on the reduced generic norms are then obvious consequences. □

Theorem 2. *Let $\mathscr{C}$ be the class of finite-dimensional Jordan algebras and for every J in $\mathscr{C}$ let $\mathscr{N}_J = N_J^0$ be the reduced generic norm of J. Then (N_J^0) is a norm-like system for $\mathscr{C}$ over k. In particular, the expansion formula* (3) *holds for N_J^0.*

Proof. Property (N1) follows from [1] II Satz 5.2 and (N2) was shown in Lemma 3.b. With respect to (N3) we have $N^0(\lambda c + n) = N^0(\lambda c)$ by [1] III Lemma 2.4. Moreover, we observe that an absolutely primitive idempotent c stays absolutely primitive in every base field extension: by [3] V. 4 Thm. 5 one knows that $J_2(c) = kc \oplus \mathfrak{N}$ where $\mathfrak{N} = \{n \in J_2(c) | n \text{ nilpotent}\} = \mathrm{Rad}(J_2(c))$ is a nil ideal, and $L \otimes \mathfrak{N}$ is nil for every extension field L of k (e.g. [1] I Satz 4.4). Hence c is absolutely primitive in the sense of [1], and $N^0(\lambda c) = \lambda$ then follows from [1] III Satz 7.5. Finally (N4) holds by definition. □

Concluding remarks. 1) In [13] the notion of a reduced generic norm is defined for so-called generically algebraic algebras, (cf. also [9] and [14]). One can show that the generic norm forms a norm-like system for the class of *unramified* generically algebraic Jordan algebras. It is therefore natural to conjecture that Theorem 2 holds in general for the reduced generic norm of generically algebraic Jordan algebras. A proof in this setting could probably be given along the lines of the arguments above, assuming that one can generalize the theory of Cartan subalgebras, which at present only exists for finite-dimensional algebras to the setting of generically algebraic algebras.

2) If J is the Jordan algebra of $n \times n$ matrices over k we have $N^0(x) = N(x) = \det(x)$ ([3]). After conjugation we can assume that the idempotents c_i in the expansion formula are the matrix units E_{ii}. Then (3) is an easy consequence of the usual (Leibniz) formula for the determinant. It is equally simple to derive (3) for the other standard examples of simple Jordan algebras (the exceptional case included), except for the case of skew symmetric matrices where the norm is related to the Pfaffian – this case seems to be rather messy.

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