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Moduli of uniform convexity for convex sets

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Abstract. Let C be a proper, closed subset with nonempty interior in a normed space X. We define four variants of modulus of convexity for C and prove that they all coincide. This result, which is classical and well-known for $C = B_X$ (the unit ball of X), requires a less easy proof than the particular case of B_X. We also show that if the modulus of convexity of C is not identically null, then C is bounded. This extends a result by M.V. Balashov and D. Repovš.

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1. Introduction. The notion of a uniformly convex norm was introduced by Clarkson [3] in 1936, and soon it became one of the basic concepts in geometry of Banach spaces. Among what is known, let us just recall that: uniform convexity of a complete norm implies reflexivity and is equivalent to uniform Fréchet differentiability of the dual norm (see, e.g., [7]); and a Banach space admits an equivalent uniformly convex norm if and only if it is superreflexive (Enflo [5]).

A mean for measuring uniform convexity of the norm $\|\cdot\|$ of a normed space X is its *modulus of convexity*

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\right\}, \quad \varepsilon \in [0,2]. \quad (1)$$

Then $\|\cdot\|$ is uniformly convex if and only if $\delta_{\|\cdot\|}(\varepsilon) > 0$ for each $\varepsilon \in (0,2]$ (equivalently, for each sufficiently small $\varepsilon > 0$). A useful, relatively easy fact is that:

(∗) the value of $\delta_{\|\cdot\|}(\varepsilon)$ does not change if we write $\|x\| \leq 1, \|y\| \leq 1$, and/or $\|x-y\| = \varepsilon$ in (1) (see, e.g., [7, p. 60]).

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It is clear that uniform convexity of a norm $\|\cdot\|$ is a property of the corresponding closed unit ball $B_{\|\cdot\|}$. In some situations, it turns out to be useful to extend the notion of uniform convexity to general convex bodies. (By a convex body in a normed space X we mean a proper closed convex subset of X with nonempty interior.)

It is natural to define the modulus of convexity of a convex body C in a normed space X as

$$\delta_C(\varepsilon) = \inf\left\{\operatorname{dist}\left(\frac{x+y}{2}, \partial C\right) : x, y \in \partial C, \|x-y\| \geq \varepsilon\right\},\ 0 \leq \varepsilon < \operatorname{diam}(C). \tag{2}$$

Then it is clear that the classical modulus of convexity $\delta_{\|\cdot\|}$ of the norm of X coincides with $\delta_{B_{\|\cdot\|}}$ as defined in (2). The modulus from (2), while natural, turns out to be not so comfortable to work with. More convenient is its variant

$$\widetilde{\delta}_C(\varepsilon) = \inf\left\{\operatorname{dist}\left(\frac{x+y}{2}, \partial C\right) : x, y \in C, \|x-y\| \geq \varepsilon\right\},\ 0 \leq \varepsilon < \operatorname{diam}(C), \tag{3}$$

defined by Balashov and Repovš [1]. (Actually, they used $\|x-y\| = \varepsilon$ in their definition, but it is easily seen to give the same value of $\widetilde{\delta}_C(\varepsilon)$; see Observation 3.3(a).) They proved some interesting properties of $\widetilde{\delta}_C$, among which the following boundedness result. *If X is a Banach space and C is uniformly convex in the sense that $\widetilde{\delta}_C(\varepsilon) > 0$ whenever $0 < \varepsilon < \operatorname{diam}(C)$, then C is bounded and there exists a constant $k_C > 0$ such that $\widetilde{\delta}_C(\varepsilon) \leq k_C\varepsilon^2$ for each ε.*

It is natural to ask about relations between the two moduli from (2) and (3). The definitions give immediately the inequality $\widetilde{\delta}_C \leq \delta_C$. Thanks to $(*)$, in the particular case of $C = B_{\|\cdot\|}$, we have $\delta_{B_{\|\cdot\|}} \equiv \delta_{\|\cdot\|} \equiv \widetilde{\delta}_{B_{\|\cdot\|}}$. However, the proof of $(*)$ cannot be easily modified to get the equality in the general case. Roughly speaking, the difficulty consists in the fact that the distances are measured by the norm $\|\cdot\|$, and the shape of its unit ball $B_{\|\cdot\|}$ can be quite different from the shape of C.

A first partial answer to the question about the relation between δ_C and $\widetilde{\delta}_C$ was given in our paper [4] dedicated to extendability of uniformly continuous quasiconvex functions from uniformly convex bodies. Therein, we have given a simple proof that the two moduli generate the same notion of a uniformly convex body.

The main aim of the present paper is to provide a proof that, indeed, the two moduli δ_C and $\widetilde{\delta}_C$ always coincide. More precisely, we define four variants of modulus of convexity of C and then prove that they all coincide. This is done in Theorem 3.6.

Our secondary aim is to show that the reasoning in [1] (by Balashov and Repovš) can be modified to generalize their aforementioned boundedness result from uniformly convex bodies in Banach spaces to convex bodies whose modulus of convexity is not identically null, in normed spaces (see Theorem 4.5).

As a simple corollary, we obtain a general upper estimate for δ_C, depending only on the diameter of the convex body C.

Finally, let us remark that we formulate our results for (nontrivial) convex sets C that are not necessarily closed. This approach, which is only formally different, is due to our previous paper [4] where we dealt mostly with open convex sets.

2. Notation and preliminaries. By a *normed space* we mean a real normed linear space of dimension at least two. If not specified otherwise, X denotes such a space, B_X is its closed unit ball, $S_X := \partial B_X$ is its unit sphere, and X^* is its dual Banach space.

The closed segment with endpoints $x, y \in X$ is denoted by $[x, y]$, and we also define $[x, y) := [x, y] \setminus \{y\}$, $(x, y) := [x, y] \setminus \{x, y\}$.

The distance of two sets $B, C \subset X$ is defined as

$$d(B, C) := \inf\{\|b - c\| : b \in B,\ c \in C\}$$

with the usual convention that $\inf \emptyset := \infty$. We also put $d(x, B) := d(\{x\}, B)$, $x \in X$. By $\operatorname{diam}(B)$ we denote the diameter of B; by $\operatorname{span}(B)$, $\operatorname{aff}(B)$, and $\operatorname{conv}(B)$ we mean the linear, affine, and convex hull of B, respectively.

Given a set E, a function $f \colon E \to \mathbb{R}$, and $\alpha \in \mathbb{R}$, we shall use a simplified notation like $[f > \alpha] := \{x \in E : f(x) > \alpha\}$, $[f = \alpha] := \{x \in E : f(x) = \alpha\}$, and similar.

For a convex set $C \subset X$, we shall use the following terminology. We shall say that:

- C is *nontrivial* if C contains at least two points and $C \neq X$;
- C is *strictly convex* if $\frac{x+y}{2} \in \operatorname{int} C$ whenever $x, y \in \partial C$ and $x \neq y$.

Observation 2.1. *The following properties are easy observations.*

(a) *C is nontrivial if and only if $\operatorname{card}(C) > 1$ and $\partial C \neq \emptyset$. Moreover, in this case $\operatorname{diam}(C) = \operatorname{diam}(\partial C)$.*
(b) *If C is nontrivial and ∂C contains no line, then $\overline{C}$ contains no line either.*
(c) *If C is nontrivial and strictly convex, then it has nonempty interior, and moreover, $\frac{x+y}{2} \in \operatorname{int} C$ whenever $x, y \in \overline{C}$, $x \neq y$.*
(d) *If C is nontrivial, then the distance-function $d(\cdot, \partial C)$ is concave on $\overline{C}$.*
(e) *If C is nontrivial, then either ∂C is connected or ∂C consists of two parallel closed hyperplanes.*

3. Four moduli of convexity, and their equality.

Definition 3.1 (The four moduli of convexity). Let C be a nontrivial convex set in a normed space X. For any $0 \leq \varepsilon < \operatorname{diam}(C)$, let us define:

$$\delta_C(\varepsilon) := \inf\left\{d\left(\frac{x+y}{2}, \partial C\right) :\ x, y \in \partial C,\ \|x-y\| \geq \varepsilon\right\},$$

$$\delta'_C(\varepsilon) := \inf\left\{d\left(\frac{x+y}{2}, \partial C\right) :\ x, y \in \partial C,\ \|x-y\| = \varepsilon\right\},$$

$$\widetilde{\delta}_C(\varepsilon) := \inf\left\{d\left(\frac{x+y}{2}, \partial C\right) :\ x, y \in \overline{C},\ \|x-y\| \geq \varepsilon\right\},$$

$$\widetilde{\delta}'_C(\varepsilon) := \inf\left\{d\left(\frac{x+y}{2}, \partial C\right) :\ x, y \in \overline{C},\ \|x-y\| = \varepsilon\right\}.$$

Each of these four functions is a kind of *modulus of convexity* of C.

Remark 3.2. As already remarked in the introduction, it is well known (see [7]) that the four moduli coincide for $C := B_X$. Moreover, it is clear that we can restrict ourselves to the case of nontrivial convex sets which are *closed*.

The aim of the present section, and our main result, is to show that the four moduli coincide for each nontrivial convex set. Before proving the theorem, we need some preparation.

Observation 3.3 (First properties). *Let $C \subset X$ be a nontrivial convex set.*

(a) $\widetilde{\delta}'_C(\varepsilon) = \widetilde{\delta}_C(\varepsilon) \leq \delta_C(\varepsilon) \leq \delta'_C(\varepsilon) \leq \varepsilon/2$ *for each* $\varepsilon \in [0, \operatorname{diam}(C))$.
(b) *At* $\varepsilon = 0$, *all four moduli are null and right-continuous.*
(c) δ_C *and* $\widetilde{\delta}_C \equiv \widetilde{\delta}'_C$ *are nondecreasing on* $[0, \operatorname{diam}(C))$.
(d) *If* ∂C *contains a segment of length* $0 < \ell < \operatorname{diam}(C)$, *then* $\delta'_C(\varepsilon) = 0$ *for each* $\varepsilon \in [0, \ell]$.

Proof. The equality in (a) follows easily: indeed, if $x, y \in \overline{C}$ and $\|x-y\| \geq \varepsilon$, then the segment $[x, y]$ contains two points x', y' such that $\|x'-y'\| = \varepsilon$ and $\frac{x'+y'}{2} = \frac{x+y}{2}$. To see the last inequality in (a), consider $x, y \in \partial C$ with $\|x-y\| = \varepsilon$, and observe that $d(\frac{x+y}{2}, \partial C) \leq \|\frac{x+y}{2} - y\| = \varepsilon/2$. The rest of (a) is obvious. The remaining parts of the statement are quite easy. □

The following lemma is a relative of [1, Lemma 2.2].

Lemma 3.4. *Assume that X is a normed space, $C \subset X$ is a closed, nontrivial convex set, and $0 < \varepsilon < \operatorname{diam}(C)$. Let $x, y \in \partial C$ and $x^*, y^* \in S_{X^*}$ be such that $\|x-y\| = \varepsilon$, $x^*(x) = \max x^*(C)$, and $y^*(y) = \max y^*(C)$. Then*

$$\|x^* - y^*\| \geq \frac{4\delta'_C(\varepsilon)}{\varepsilon}.$$

Proof. For simplicity, denote $\delta := \delta'_C(\varepsilon)$. Since $\frac{x+y}{2} + \delta B_X \subset C$, we have that $x^*(\frac{x+y}{2}) + \delta \leq x^*(x)$ and $y^*(\frac{x+y}{2}) + \delta \leq y^*(y)$. These two inequalities can be rewritten as $x^*(x-y) \geq 2\delta$ and $y^*(y-x) \geq 2\delta$. Summing them up, we obtain $(x^* - y^*)(x-y) \geq 4\delta$. Consequently, $4\delta \leq \|x^* - y^*\|\varepsilon$, and we are done. □

Let us first prove the following two-dimensional boundedness result about the modulus δ'_C.

Lemma 3.5. *Let Y be a two-dimensional normed space, let $D \subset Y$ be a closed, nontrivial convex set, and let $0 < \varepsilon < \operatorname{diam}(D)$ be such that $\delta'_D(\varepsilon) > 0$. Then D is bounded.*

Proof. Proceeding by contradiction, let us assume that D is unbounded. Since D is finite-dimensional, it is well known that then D contains a closed half-line, say $d+\mathbb{R}_+ v$ for some $d \in D$ and $v \neq 0$ (where $\mathbb{R}_+ := [0, \infty)$). We know that ∂D cannot contain line segments of length ε or more, hence it does not contain any half-line. Thus ∂D is an "unbounded simple curve" which is homeomorphic to the real line. Fix an arbitrary $a \in \partial D$; it divides ∂D into two unbounded branches. Let Γ be one of these two branches and let $a \in \Gamma$. Fix an onto homeomorphism $\varphi \colon \mathbb{R}_+ \to \Gamma$. By an elementary continuity argument, there exists an increasing sequence $\{t_n\}_n \subset \mathbb{R}_+$ such that $\|\varphi(t_{n+1}) - \varphi(t_n)\| = \varepsilon$ for each n. For each n, choose some $x_n^* \in S_{X^*}$ such that

$$\langle x_n^*, \varphi(t_n)\rangle = \max_{y \in D} \langle x_n^*, y\rangle.$$

Since x_n^* is upper bounded on D, we must have $x_n^*(v) \leq 0$. Now, the closed half-sphere $T := \{y^* \in S_{Y^*} : y^*(v) \leq 0\}$ is homeomorphic to $[0, 1]$, and this homeomorphism generates a natural ordering on T. By convexity of D, it is clear that the sequence $\{x_n^*\}_n \subset T$ is monotone in this ordering, and hence converging. But by Lemma 3.4, this sequence also satisfies $\|x_{n+1}^* - x_n^*\| \geq \frac{4\delta}{\varepsilon}$, which is clearly impossible (since it would imply that T has infinite length). This contradiction completes the proof. □

Now we are ready for our main result.

Theorem 3.6. *Let C be a nontrivial convex set in a normed space X. Then*

$$\delta_C(\varepsilon) = \delta'_C(\varepsilon) = \widetilde{\delta}_C(\varepsilon) = \widetilde{\delta}'_C(\varepsilon) \quad \textit{for each } 0 \leq \varepsilon < \operatorname{diam}(C).$$

Proof. Assume that C is closed (Remark 3.2). By Observation 3.3(a),(b), it suffices to show that $\delta'_C(\varepsilon) \leq \widetilde{\delta}'_C(\varepsilon)$ whenever $0 < \varepsilon < \operatorname{diam} C$. This is obvious when either $\delta'_C(\varepsilon) = 0$ or $\widetilde{\delta}'_C(\varepsilon) = \varepsilon/2$. So let us assume that

$$0 < \varepsilon < \operatorname{diam} C, \quad \delta'_C(\varepsilon) > 0, \quad \widetilde{\delta}'_C(\varepsilon) < \varepsilon/2.$$

Notice that the second condition implies that ∂C cannot contain line segments of length ε or greater.

Fix an arbitrary $0 < \eta < (\varepsilon/2) - \widetilde{\delta}'_C(\varepsilon)$. There exist points $x, y \in C$ such that $\|x - y\| = \varepsilon$ and $d(\frac{x+y}{2}, \partial C) < \widetilde{\delta}'_C(\varepsilon) + \eta$. By Observation 2.1(d), we can shift the segment $[x, y]$ along the line $\operatorname{aff}\{x, y\}$ in one of the two directions till the boundary in such a way that the new segment $[\hat{x}, \hat{y}]$ satisfies $d(\frac{\hat{x}+\hat{y}}{2}, \partial C) \leq d(\frac{x+y}{2}, \partial C)$. Hence we can (and do) assume that $x \in \partial C$. If also $y \in \partial C$, then we immediately get that $\delta'_C(\varepsilon) \leq \widetilde{\delta}'_C(\varepsilon) + \eta$; so let us assume that $y \in \operatorname{int} C$. By translation, we can also assume that $\frac{x+y}{2} = 0$. There exists $a \in \partial C$ such that $\|a\| = \|a - \frac{x+y}{2}\| < \widetilde{\delta}'_C(\varepsilon) + \eta < \varepsilon/2$. It is clear that a does not belong to $\operatorname{aff}\{x, y\} = \operatorname{span}\{x\}$, and hence $Y := \operatorname{span}\{a, x\}$ is a two-dimensional subspace of X.

Now, let us consider the closed, nontrivial convex set $D := C \cap Y$ in the normed space Y. We claim that D is bounded. Indeed, otherwise we would have $\varepsilon < \infty = \operatorname{diam}(D)$ and $\delta'_D(\varepsilon) \geq \delta'_C(\varepsilon) > 0$ and this leads to a contradiction by Lemma 3.5. So, our claim is proved.

Notice that $r := d_Y(0, \partial D) \leq \|a\| < \widetilde{\delta}'_C(\varepsilon) + \eta < \varepsilon/2$. By compactness, there exists $b \in \partial D$ such that $\|b\| = r$. Like a, the point b cannot belong to $\operatorname{span}\{x\}$. Fix some $f \in Y^*\backslash\{0\}$ such that $f(x) = 0$ and $f(b) > 0$, and denote $s := \max f(D)$. The set $D \cap [f = s]$ is a (possibly degenerate) line segment contained in ∂D, and hence of length less than ε. On the other hand, the segment $D \cap [f = 0] = D \cap \operatorname{span}\{x\}$ is of length greater than ε. By continuity, there exists $\alpha \in (0, s)$ such that the segment $D \cap [f = \alpha]$ has length exactly ε. Let us denote its endpoints by x', y' so that $x' - y' = x - y$, and let $z' := \frac{x'+y'}{2}$. Since $x' \notin \operatorname{span}\{x\}$, there exists $g \in Y^*\backslash\{0\}$ such that $g(z') = 0$ and $g(x) = \beta > 0$. Notice that

$$x' = x + z',\ y' = y + z',\ g(x') = g(x) = \beta,\ g(y') = g(y) = -\beta.$$

The parallelogram

$$\operatorname{conv}\{x, y, x', y'\} = [0 \leq f \leq \alpha] \cap [|g| \leq \beta]$$

is contained in D and its vertices x', y' belong to ∂D. This implies that the set $[f > \alpha] \cap [|g| > \beta]$ is disjoint from D, and hence the set $E := [f \geq \alpha] \cap [|g| \geq \beta]$ is disjoint from $\operatorname{int} D$. We claim that the point $b' := b + z'$ cannot belong to $\operatorname{int} D$.

Proceeding by contradiction, let $b' \in \operatorname{int} D$. If $|g(b')| < \beta$, then since $g(b) = g(b')$, there exists a unique $v \in (x, y)$ such that $b \in (v, b')$. But this is impossible since otherwise $b \in \operatorname{int} D$. So we must have $|g(b)| = |g(b')| \geq \beta$. Then $f(b') = f(b) + \alpha > \alpha$ which implies that $b' \in E$ which is impossible.

So we have proved our claim that $b' \notin \operatorname{int} D$. Since $z' \in \operatorname{int} D$, the segment $[z', b']$ contains a point of ∂D. Therefore

$$\delta'_C(\varepsilon) \leq \delta'_D(\varepsilon) \leq \operatorname{dist}(z', \partial D) \leq \|z' - b'\| = \|b\| < \widetilde{\delta}'_C(\varepsilon) + \eta.$$

Since we have $\delta'_C(\varepsilon) < \widetilde{\delta}'_C(\varepsilon) + \eta$ for each sufficiently small $\eta > 0$, we are done. □

Remark 3.7. Let us remark that the proof of Theorem 3.6 works also for the directional moduli of convexity (in which, roughly speaking, the points x, y are taken so that $x - y$ is a multiple of a certain fixed $v \neq 0$).

4. A general boundedness result.

Notation 4.1. As we have seen in the previous section, all four variants of moduli of convexity from Definition 3.1 coincide, and thus there is actually a unique modulus of convexity of a nontrivial convex set $C \subset X$. In what follows, we shall denote it by δ_C.

Let us recall that a nontrivial convex set $C \subset X$ is called *uniformly convex* if $\delta_C(\varepsilon) > 0$ for each $\varepsilon \in (0, \operatorname{diam}(C))$. In [1, Theorem 2.1], Balashov and Repovš proved that every uniformly convex set C in a Banach space is bounded. In the present section, we show that the ideas from [1] can be modified to prove a more general result: *if C is a nontrivial convex set in a normed space and $\delta_C(\varepsilon) > 0$ for some $\varepsilon \in (0, \operatorname{diam}(C))$, then C is bounded.* In this setting, an element of X^* which is bounded above on C does not necessarily attain its maximum over $\overline{C}$, and this represents certain difficulty.

As already remarked, there is no loss of generality in assuming that C is closed. The following lemma is just a simple modification of [1, Lemma 2.1], and we give its proof for sake of completeness. It is well known for $C = B_X$ (see [7, Lemma 1.e.8]).

Lemma 4.2. *Let C be a nontrivial closed convex set in a normed space X. Then the function $\varepsilon \mapsto \frac{\delta_C(\varepsilon)}{\varepsilon}$ is nondecreasing on $(0, \operatorname{diam}(C))$.*

Proof. Let $0 < \varepsilon_1 < \varepsilon_2 < \operatorname{diam}(C)$, and fix an arbitrary $\eta > 0$. Let $x, y \in C$ be such that $\|x - y\| = \varepsilon_2$ and $d(\frac{x+y}{2}, \partial C) < \delta_C(\varepsilon_2) + \eta$. There exists $a \in \partial C$ with $\|\frac{x+y}{2} - a\| < \delta_C(\varepsilon_2) + \eta$. Then the points $x' := (1 - \frac{\varepsilon_1}{\varepsilon_2})a + \frac{\varepsilon_1}{\varepsilon_2}x$ and $y' := (1 - \frac{\varepsilon_1}{\varepsilon_2})a + \frac{\varepsilon_1}{\varepsilon_2}y$ belong to C and satisfy $\|x' - y'\| = \varepsilon_1$ and $\|\frac{x'+y'}{2} - a\| = \frac{\varepsilon_1}{\varepsilon_2}\|\frac{x+y}{2} - a\|$. Thus $\delta_C(\varepsilon_1) \le \frac{\varepsilon_1}{\varepsilon_2}\|\frac{x+y}{2} - a\| < \frac{\varepsilon_1}{\varepsilon_2}[\delta_C(\varepsilon_2) + \eta]$. By letting $\eta \to 0^+$, we conclude that $\frac{\delta_C(\varepsilon_1)}{\varepsilon_1} \le \frac{\delta_C(\varepsilon_2)}{\varepsilon_2}$. □

For simplicity, let us introduce the following notation. Given a closed, nontrivial convex set $C \subset X$, we define

$$\Omega(C) := \{f \in X^* : \sup f(C) < \infty\},$$
$$S_\eta(f, C) := \{x \in \partial C : f(x) \ge \sup f(C) - \eta\}, \quad f \in \Omega(C), \eta \ge 0.$$

It is easy to see that $\Omega(C)$ is a convex cone (with vertex at 0) which does not reduce to a single point. Moreover, the cone $\Omega(C)$ is one-dimensional if and only if C is one of the following three types of sets: a half-space, a hyperplane, a strip between two parallel hyperplanes.

Also notice that if $f \in \Omega(C)$ and $\eta > 0$, then $S_\eta(f, S)$ is a closed slice of C with nonempty interior relative to C. Furthermore, for $f \in \Omega(C)\backslash\{0\}$, $S_0(f, C)$ is nonempty if and only if f attains its supremum over C.

The following lemma is a version of [1, Lemma 2.2] and Lemma 3.4 above, with approximate attaining instead of exact attaining.

Lemma 4.3. *Let $C \subset X$ be a nontrivial closed convex set. Let $\eta \ge 0$, $f, g \in \Omega(C) \cap S_{X^*}$, $x \in S_\eta(f, C)$, $y \in S_\eta(g, C)$, and $\eta < \delta_C(\|x - y\|)$. Then*

$$[f - g](x - y) \ge 4[\delta_C(\|x - y\|) - \eta] \tag{4}$$

and hence $\|f - g\| \ge \frac{4}{\|x-y\|}[\delta_C(\|x - y\|) - \eta]$.

Proof. Since $\delta := \delta_C(\|x - y\|) > 0$, C has nonempty interior and $x \ne y$. Since $\frac{x+y}{2} + \delta B_X \subset C$ and f has norm one, we have $f(\frac{x+y}{2}) + \delta \le \sup f(C) \le f(x) + \eta$, from which it follows that $f(x - y) \ge 2(\delta - \eta)$. Symmetrically, $g(y - x) \ge 2(\delta - \eta)$. Now, (4) follows by summing up the last two inequalities. The rest is obvious. □

Corollary 4.4. *Let $C \subset X$ be a nontrivial closed convex set, and let $0 < \varepsilon < \operatorname{diam}(C)$ be such that $\delta_C(\varepsilon) > 0$. Let $f, g \in \Omega(C) \cap S_{X^*}$, $\lambda \in (0, 1)$, $\eta := \lambda\delta_C(\varepsilon)$, $x \in S_\eta(f, C)$, and $y \in S_\eta(g, C)$. If $\|f - g\| < 4(1 - \lambda)\delta_C(\varepsilon)/\varepsilon$, then $\|x - y\| < \varepsilon$.*

Proof. Under the assumptions of the statement, assume that $\|x - y\| \ge \varepsilon$. Then $\delta_C(\|x - y\|) \ge \delta_C(\varepsilon) > \eta$. Using Lemmas 4.2 and 4.3, we obtain $\|f - g\| \ge \frac{4}{\|x-y\|}[\delta_C(\|x - y\|) - \lambda\delta_C(\varepsilon)] \ge \frac{4(1-\lambda)\delta_C(\|x-y\|)}{\|x-y\|} \ge \frac{4(1-\lambda)\delta_C(\varepsilon)}{\varepsilon}$, and we are done. □

As already mentioned in the introduction, the following Theorem 4.5 is obtained by an easy modification of the proof of [1, Theorem 2.1] by Balashov and Repovš.

Theorem 4.5. *Let C be a nontrivial convex set in a normed space X, and let $\varepsilon \in (0, \mathrm{diam}(C))$ be such that $\delta_C(\varepsilon) > 0$. Then*

$$\mathrm{diam}(C) \leq \varepsilon \left(\left[\frac{\varepsilon}{\delta_C(\varepsilon)} \right] + 1 \right)$$

where $[\cdot]$ denotes the lower integer part. In particular, C is bounded.

Proof. By Remark 3.2, we can (and do) assume that C is closed. Consider arbitrary $x, y \in \partial C$ such that $\varepsilon \leq \|x - y\| < \mathrm{diam}(C)$. Our assumptions imply that C cannot contain any line and has nonempty interior. By the Hahn–Banach theorem, there exist $f, g \in S_{X^*}$ such that $x \in S_0(f, C)$ and $y \in S_0(g, C)$. Then $f \neq g$ by Lemma 4.3 applied with $\eta = 0$.

Notice that the cone $\Omega(C)$ has dimension at least two and contains f, g. Thus there exists a two-dimensional subspace $Z \subset X^*$ containing f, g and such that the convex cone $K := Z \cap \Omega(C)$ is two-dimensional. The unit sphere S_Z of Z is a (symmetric) Jordan curve in Z containing f, g, and the length of S_Z is known to be at most 8 ([9, Theorem 4.2]; see also [6, Theorem 11.9]). Then it is clear that $S_Z \cap K$ contains an arc γ with endpoints f, g and of length at most 4. Let γ be oriented from f to g.

Consider an integer $N > \frac{\varepsilon}{\delta_C(\varepsilon)}$, and notice that $N > \frac{\varepsilon}{\varepsilon/2} = 2$ by Observation 3.3(a). Let $\lambda \in (0, 1)$ be such that $N > \frac{\varepsilon}{(1-\lambda)\delta_C(\varepsilon)}$. Let $\{f_0, f_1, \ldots, f_N\}$ be the partition of γ into N sub-arcs γ_i $(i = 1, \ldots, N)$ of equal length such that $f_0 := f$, $f_N = g$, and the indexing is increasing when moving along γ from f to g. Then, for each integer $1 \leq i \leq N$, we have

$$\|f_i - f_{i-1}\| \leq \ell(\gamma_i) = \ell(\gamma)/N \leq 4/N < 4(1 - \lambda)\delta_C(\varepsilon)/\varepsilon.$$

Put $\eta := \lambda\delta_C(\varepsilon)$, define $x_0 := x$ and $x_N := y$, and notice that $x_i \in S_0(f_i, C) \subset \mathrm{int}_C S_\eta(f_i, C)$ for $i \in \{0, N\}$. For each integer $1 \leq i < N$, choose some $x_i \in \mathrm{int}_C S_\eta(f_i, C)$, being careful to assure that $\|x_i - x_{i-1}\| < \mathrm{diam}(C)$ for all $i = 0, \ldots, N$. (This is clearly possible.) By Corollary 4.4, we obtain

$$\|x - y\| \leq \sum_{i=1}^{N} \|x_i - x_{i-1}\| < N\varepsilon.$$

This shows that $\mathrm{diam}(C) \leq N\varepsilon$ for each integer $N > \varepsilon/\delta_C(\varepsilon)$. Since the smallest such integer is $N = \left[\varepsilon/\delta_C(\varepsilon)\right] + 1$, we are done. □

Let us conclude with an easy corollary.

Corollary 4.6. *Let C be a nontrivial convex set in a normed space. Then*

$$\delta_C(\varepsilon) \leq \frac{3\varepsilon^2}{2\mathrm{diam}(C)} \quad \textit{for each } 0 \leq \varepsilon < \mathrm{diam}(C), \tag{5}$$

which for an unbounded C is intended as $\delta_C(\varepsilon) \equiv 0$.

Proof. If C is unbounded, then $\delta_C \equiv 0$ by Theorem 4.5. For $\varepsilon = 0$ or $\delta_C(\varepsilon) = 0$, (5) is obvious. Let $\varepsilon > 0$ and $\delta_C(\varepsilon) > 0$. By Theorem 4.5, $\operatorname{diam}(C) \leq \varepsilon\left(\frac{\varepsilon}{\delta_C(\varepsilon)} + 1\right)$, which can be rewritten as $\delta_C(\varepsilon) \leq \frac{\varepsilon^2}{\operatorname{diam}(C)-\varepsilon}$. Thus, taking into account that also $\delta_C(\varepsilon) \leq \varepsilon/2$ by Observation 3.3(a), we obtain

$$\frac{\delta_C(\varepsilon)}{\varepsilon^2} \leq \min\left\{\frac{1}{\operatorname{diam}(C)-\varepsilon}, \frac{1}{2\varepsilon}\right\} \leq \frac{3}{2\operatorname{diam}(C)}.$$

□

Remark 4.7. (a) Let us remark that one can obtain a finer estimate if the quantity $\Delta_C := \lim_{\varepsilon \to \operatorname{diam}(C)^-} \delta_C(\varepsilon) = \sup_{\varepsilon \in (0, \operatorname{diam}(C))} \delta_C(\varepsilon)$ is known. By Lemma 4.2, $\delta_C(\varepsilon) \leq \frac{\varepsilon \Delta_C}{\operatorname{diam}(C)}$. Using this instead of $\delta_C(\varepsilon) \leq \varepsilon/2$ in the above proof, one obtains that $\delta_C(\varepsilon) \leq \frac{\Delta_C + \operatorname{diam}(C)}{[\operatorname{diam}(C)]^2}\varepsilon^2$. However, usefulness of this estimate is somewhat doubtful.

(b) Let us also remark that Nordlander [8] proved that one always has $\delta_{B_X}(\varepsilon) \leq \delta_{B_H}(\varepsilon) = 1 - \sqrt{1 - (\varepsilon^2/4)}$ where B_H is the unit ball of any Hilbert space of dimension at least 2 (see also [2]). This easily implies that $\delta_{B_X}(\varepsilon) \leq \frac{1}{4}\varepsilon^2 = \frac{\varepsilon^2}{2\operatorname{diam}(B_X)}$ for each $\varepsilon \in [0, 2)$, and the constant $\frac{1}{4}$ is the best possible. Thus our estimate from Corollary 4.6 is not the best possible for $C = B_X$.

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