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On finite groups in which the twisted conjugacy classes of the unit element are subgroups

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Abstract. We consider groups G such that the set $[G, \varphi] = \{g^{-1}g^{\varphi} | g \in G\}$ is a subgroup for every automorphism φ of G, and we prove that there exists such a group G that is finite and nilpotent of class n for every $n \in \mathbb{N}$. Then there exists an infinite not nilpotent group with the above property and the Conjecture 18.14 of Khukhro and Mazurov (The Kourovka Notebook No. 20, 2022) is false.

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1. Introduction. Let G be a group and φ be an endomorphism of G; we say that elements $x, y \in G$ are φ-conjugate if there exists an element $z \in G$ such that $y = z^{-1}xz^{\varphi}$.

It is easy to check that the relation of φ-conjugation is an equivalence relation in G. In particular, it is the usual conjugation if $\varphi = \mathrm{id}_G$ and it is the total equivalence relation if $\varphi = 0_G$ is the zero endomorphism.

Equivalence classes are called twisted conjugacy classes or φ-conjugacy classes and their number $R(\varphi)$ is called the Reidemeister number of the endomorphism φ.

The φ-conjugacy class $[1]_{\varphi} = \{x^{-1}x^{\varphi} | x \in G\}$ of the unit element of the group G is a subset whose cardinality is equal to the index $|G : C_G(\varphi)|$ of the centralizer $C_G(\varphi) = \{x \in G | x^{\varphi} = x\}$ of φ in G; in what follows, we will put $[1]_{\varphi} =: [G, \varphi]$ and we will write $[x, \varphi] := x^{-1}x^{\varphi}$.

If $\varphi = \mathrm{id}_G$, then $[G, \varphi] = \{1\}$ and if $\varphi = 0_G$, then $[G, \varphi] = G$ so, in these cases, $[G, \varphi]$ is a subgroup of G. However, in the general case, $[G, \varphi]$ is not a subgroup, it is not if we consider an automorphism $\varphi \in \mathrm{Aut}(G)$ and not even if this automorphism $\varphi \in \mathrm{Inn}(G)$ is inner. For instance, if $G = S_3$ is the symmetric group of degree 3 and $\varphi = \bar{g}$ is the inner automorphism induced by $g = (123)$, then $[1]_{\bar{g}} = \{1, (132)\} \not\leq G$.

Notice that if $\varphi \in \mathrm{Aut}_C(G)$ is a central automorphism of a group G, that is, if $g^{-1}g^{\varphi} \in Z(G)$ for every $g \in G$, then $[G, \varphi] \leq G$. So, in particular, if the group G is nilpotent of class ≤ 2, then $[G, \varphi]$ is a subgroup for every $\varphi \in \mathrm{Inn}(G)$.

As it is easy to verify, if a group G is abelian, then not only $H = [G, \varphi] \leq G$ for every $\varphi \in \mathrm{End}(G)$, but for every element $x \in G$, its φ-conjugacy class is equal to the coset $[x]_{\varphi} = xH$, that is, φ-conjugation is a congruence in G.

It is possible to prove that this property characterizes abelian groups. In fact, if φ-conjugation is a congruence for every $\varphi \in \mathrm{End}(G)$, then in particular, conjugation is a congruence. This implies that every conjugacy class has order 1, that is, $g^x = g$ for every $g, x \in G$, hence G is abelian. Actually for a group to be abelian, it is enough that there exists an inner automorphism $\bar{g} \in \mathrm{Inn}(G)$ such that $\bar{g}$-conjugation is a congruence in G.

Proposition 1.1. *A group G is abelian if and only if there exists $g \in G$ such that $\bar{g}$-conjugation is a congruence in G.*

Proof. We only have to show that if $\bar{g}$-conjugation is a congruence for an element $g \in G$, then the group is abelian.

If $\bar{g}$ is a congruence, then $|[x]_{\bar{g}}| = |[1]_{\bar{g}}| = |\{h^{-1}h^g | h \in G\}|$ for every $x \in G$. In particular, $|\{h^{-1}h^g = [h, g] | h \in G\}| = |[g]_{\bar{g}}|$ so $|\{[h, g] | h \in G\}| = 1$, that is, $\{[h, g] | h \in G\} = \{1\}$ since $[g]_{\bar{g}} = \{g\}$. This means that $g \in Z(G)$ hence $\bar{g} = \mathrm{id}_G$, conjugation is a congruence and so G is abelian. □

In the paper [1], the authors prove that every finite group G in which the φ-conjugacy class of the unit element is a subgroup for every inner automorphism $\varphi \in \mathrm{Inn}(G)$ is nilpotent. Anyway, it is not possible to bound the nilpotency class of such groups; in fact, in [3], the authors construct, for any integer $n > 2$ and for any prime $p > 2$, a finite p-group G nilpotent of class $\geq n$ with this property. However, they notice that there exist automorphisms $\phi \in \mathrm{Aut}(G)\backslash\mathrm{Inn}(G)$ such that $[G, \phi]$ is not a subgroup of G.

It is therefore natural to ask if it is possible to bound the nilpotency class of a finite group G such that the φ-conjugacy class of the unit element is a subgroup for every $\varphi \in \mathrm{Aut}(G)$. In this regard, in [1], the conjecture is made that such groups could be abelian (cfr. also [5, 18.14]). This conjecture is certainly false, in fact, there exist finite non-abelian p-groups in which every automorphism is central (see for instance [2,7]). Of course, such groups are nilpotent of class 2 but we will prove the existence of nilpotent groups of every class n with this property. So also the answer to the previous question is negative.

Our main result is the following

Theorem 1.2 (Main Theorem). *For every integer $n \in \mathbb{N}$ and for every odd prime p, there exists a finite p-group G, of class n, in which the φ-conjugacy class of the unit element is a subgroup for every $\varphi \in \mathrm{Aut}(G)$.*

As we will see, these groups are abelian-by-cyclic, and this result will give a negative answer also to Problem 3 of [3].

Moreover, from Theorem 1.2, it follows the existence of an infinite non-nilpotent group G such that $[G,\varphi] \leq G$ for every $\varphi \in \mathrm{Aut}(G)$ and so Conjecture 18.14 of [5] is false.

2. The proof of the main theorem. Let n be an integer $n \geq 2$, p be an odd prime, and $G = A \rtimes \langle x\rangle$, where $A = \langle a\rangle \times \langle b\rangle \simeq \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$, $\langle x\rangle \simeq \mathbb{Z}_{p^{n-1}}$, and $c^x = c^{1+p}$ for every $c \in A$.

It is easy to verify that $G' = \langle a^p\rangle \times \langle b^p\rangle$ and that G is nilpotent of class n. Moreover, for every $c \in A$, we have that $\langle c\rangle$ is normal in G; in particular, for every $g_1, g_2 \in G$, we have that $\langle [g_1, g_2]\rangle$ is normal in G, then $\langle g_1, g_2\rangle' = \langle [g_1, g_2]\rangle$ and this implies that $\langle g_1, g_2\rangle$ is a regular p-group since its derived subgroup is cyclic ([4, III 10.2 (c)]). Therefore G is a regular p-group.

Let $\varphi \in \mathrm{Aut}(G)$ be an automorphism of G, we will prove that $[1]_\varphi = \{g^{-1}g^\varphi | g \in G\} = [G,\varphi]$ is a subgroup.

First of all we have that

$$[c,\varphi] \in A \quad \forall c \in A. \qquad (*)$$

In fact, if there exists $c \in A$ such that $[c,\varphi] \notin A$, then either $[a,\varphi] \notin A$ or $[b,\varphi] \notin A$. Without loss of generality. we may assume that $[a,\varphi] \notin A$, that is, $[a,\varphi] = yx^\alpha$ with $y \in A$ and $\alpha \in \mathbb{Z}$ such that p^{n-1} does not divide α.

Observe that $a^\varphi = ayx^\alpha$, $(a^\varphi)^{p^{n-1}} = (ay)^{p^{n-1}}(x^\alpha)^{p^{n-1}}z^{p^{n-1}}$ with $z \in G'$ since G is regular, and then $(a^\varphi)^{p^{n-1}} = (ay)^{p^{n-1}}$ because G' has exponent p^{n-1} and, in particular, ay has order p^n.

Let $[b,\varphi] = tx^\beta$ with $t \in A$ and $\beta \in \mathbb{Z}$, then $b^\varphi = btx^\beta$ and we have that also bt has order p^n since $(b^\varphi)^{p^{n-1}} = (bt)^{p^{n-1}}$.

Now $1 = [a,b] = [a^\varphi, b^\varphi] = [ayx^\alpha, btx^\beta] = [ay, x^\beta]^{x^\alpha}[x^\alpha, bt]^{x^\beta}$ and this implies that $[ay, x^\beta]^{x^\alpha} = [bt, x^\alpha]^{x^\beta}$. Therefore there exist integers $\gamma, \delta \in \mathbb{Z}$ such that $(ay)^\gamma = (bt)^\delta$ because $[ay, x^\beta]^{x^\alpha} \in \langle ay\rangle$ and $[bt, x^\alpha]^{x^\beta} \in \langle bt\rangle$.

From $\langle a\rangle \cap \langle b\rangle = \{1\}$, it follows that $\langle a^\varphi\rangle \cap \langle b^\varphi\rangle = \{1\}$, then

$$\langle (a^\varphi)^{p^{n-1}}\rangle \cap \langle (b^\varphi)^{p^{n-1}}\rangle = \langle (ay)^{p^{n-1}}\rangle \cap \langle (bt)^{p^{n-1}}\rangle = \{1\}.$$

This implies that $\langle ay\rangle \cap \langle bt\rangle = \{1\}$ because the intersection between the unique subgroups of order p of the cyclic p-groups $\langle ay\rangle$ and $\langle bt\rangle$ is the trivial subgroup. Hence p^n divides both γ and δ, so $x^\beta \in C_G(ay)$ and $x^\alpha \in C_G(bt)$. In particular, $(bt)^{x^\alpha} = (bt)^{(1+p)^\alpha} = bt$, then $(1+p)^\alpha \equiv 1 \pmod{p^n}$ and, since the multiplicative order of $(p+1)$ in $\mathbb{Z}_{p^n}$ is p^{n-1}, we have that p^{n-1} divides α, that is a contradiction. From $(*)$, it follows that $c^\varphi \in A$ for every $c \in A$, and in particular, $a^\varphi \in A$.

Now we prove that $x^\varphi = sx$ with $s \in A$ and so $[x,\varphi] = x^{-1}x^\varphi = s^x = s^{1+p} \in A$. Let $x^\varphi = sx^\lambda$ with $\lambda \in \mathbb{Z}$ and $s \in A$; from $a^x = a^{1+p}$, it follows that $(a^\varphi)^{x^\varphi} = (a^\varphi)^{(1+p)}$, that is, $(a^\varphi)^{x^\lambda} = (a^\varphi)^{1+p}$ and $(a^\varphi)^{(1+p)^\lambda} = (a^\varphi)^{1+p}$. Therefore $(1+p)^\lambda \equiv (1+p) \pmod{p^n}$, $\lambda \equiv 1 \pmod{p^n}$ and so $x^\varphi = sx$.

Since $[x,\varphi] \in A$, we have that $[x^\alpha, \varphi] = [x,\varphi]^\beta$ with $\beta \in \mathbb{Z}$, that is, $[x^\alpha,\varphi] \in \langle [x,\varphi]\rangle$ for every $\alpha \in \mathbb{Z}$ and so $V = \{[x^\alpha,\varphi] | \alpha \in \mathbb{Z}\} \subseteq \langle [x,\varphi]\rangle$.

Put $[x,\varphi] = c$, that is, $x^\varphi = xc$. If $|V| = |\langle x\rangle : C_{\langle x\rangle}(\varphi)| = p^k$, then $(x^\varphi)^{p^k} = (x^{p^k})^\varphi = x^{p^k}$, that is, $(xc)^{p^k} = x^{p^k}$. Hence $x^{p^k} = x^{p^k}c^{p^k}z^{p^k}$ with

$z \in (\langle x, c\rangle)' = \langle c^p\rangle$, that is, $x^{p^k} = x^{p^k} c^{p^k} (c^{lp})^{p^k} = x^{p^k} c^{(1+lp)p^k}$ which implies $c^{(1+lp)p^k} = 1$ and so the order $o(c)$ divides p^k. Then $p^k = |V| \leq |\langle c\rangle| \leq p^k$ and $|V| = |\langle c\rangle|$. So we have that $(**)$ $V = \langle [x, \varphi]\rangle$.

Let $B = \{[c, \varphi] | c \in A\}$. From $(*)$, we have that $B \subseteq A$ and we prove that $B \leq A$. In fact, for every $c, d \in A$, we get $[cd, \varphi] = [c, \varphi]^d [d, \varphi] = [c, \varphi][d, \varphi]$ and $[c^{-1}, \varphi] = [c, \varphi]^{-1}$. Moreover, $\langle [x, \varphi]\rangle \leq A$ since $[x, \varphi] \in A$. Then $B\langle [x, \varphi]\rangle \leq A$, and we show that $B\langle [x, \varphi]\rangle = [G, \varphi]$ so in particular $[G, \varphi] \leq G$.

For every $g \in G$, there exist $f \in A$ and $\alpha \in \mathbb{Z}$ such that $g = fx^\alpha$; then $[g, \varphi] = [fx^\alpha, \varphi] = [f, \varphi]^{x^\alpha} [x^\alpha, \varphi] = [f, \varphi]^\xi [x, \varphi]^\eta$ with $\xi, \eta \in \mathbb{Z}$ and so $[g, \varphi] \in B\langle [x, \varphi]\rangle$.

Conversely, if we consider $[d, \varphi][x, \varphi]^\eta$ with $d \in A$, then we have that there exists $\xi \in \mathbb{Z}$ such that $[d, \varphi][x, \varphi]^\eta = [d, \varphi][x^\xi, \varphi]$ since $(**)$ implies that there exists $\xi \in \mathbb{Z}$ such that $[x, \varphi]^\eta = [x^\xi, \varphi]$. Let $\beta \in \mathbb{Z}$ be such that $(1 + p)^\xi \beta \equiv 1 \pmod{p^n}$, then

$$[d^\beta x^\xi, \varphi] = [d^\beta, \varphi]^{x^\xi} [x^\xi, \varphi] = [d, \varphi]^{\beta x^\xi} [x, \varphi]^\eta = [d, \varphi]^{(1+p)^\xi \beta} [x, \varphi]^\eta = [d, \varphi][x, \varphi]^\eta,$$

that is, $[d, \varphi][x, \varphi]^\eta \in [G, \varphi]$ and this completes the proof.

3. Further remarks and open questions. From Theorem 1.2, it follows that for every $n \in \mathbb{N}$ and for every odd prime p, there exists a finite p-group P of class n such that $[P, \varphi] \leq P$ for every $\varphi \in \mathrm{Aut}(P)$, let us denote such a finite p-group by $G(n, p)$. It is possible to construct an infinite non-nilpotent group with the same property.

Corollary 3.1. There exists an infinite non-nilpotent group G such that $[G, \varphi] \leq G$ for every $\varphi \in \mathrm{Aut}(G)$.

Proof. For every $n \in \mathbb{N}$, fix a prime p_n such that $p_n \neq p_m$ for every $m < n$ and put $P_n := G(n, p_n)$. The restricted direct product ([8, p. 20]) $G := \mathrm{Dir}_{n \in \mathbb{N}} P_n$ is non-nilpotent and $[G, \varphi] \leq G$ for every $\varphi \in \mathrm{Aut}(G)$.

Note that, for every $\varphi \in G$ and for every $n \in \mathbb{N}$, we have that the restriction $\varphi_{|P_n} =: \varphi_n \in \mathrm{Aut}(P_n)$, so $[P_n, \varphi_n]$ is a subgroup of P_n. We will show that $[G, \varphi] = \langle [P_n, \varphi_n] | n \in \mathbb{N}\rangle$ so, in particular, it is a subgroup.

Let $g \in G$, then $g = g_{n_1} \cdots g_{n_t}$ with $t, n_1, \ldots, n_t \in \mathbb{N}$, and $g_{n_i} \in P_{n_i}$ for every $i \in \{1, \ldots, t\}$. So $[g, \varphi] = [g_{n_1} \cdots g_{n_t}, \varphi] = g_{n_t}^{-1} \cdots g_{n_1}^{-1} (g_{n_1} \cdots g_{n_t})^\varphi = g_{n_1}^{-1} g_{n_1}^{\varphi_{n_1}} \cdots g_{n_t}^{-1} g_{n_t}^{\varphi_{n_t}} = [g_{n_1}, \varphi_{n_1}] \cdots [g_{n_t}, \varphi_{n_t}]$ and $[G, \varphi] \subseteq \langle [P_n, \varphi_n] | n \in \mathbb{N}\rangle$.

Conversely, if $y \in \langle [P_n, \varphi_n] | n \in \mathbb{N}\rangle$, then $y = y_{n_1} \cdots y_{n_k}$ for some $k, n_1, \ldots, n_k \in \mathbb{N}$, $y_{n_i} = [z_{n_i}, \varphi_{n_i}] \in [P_{n_i}, \varphi_{n_i}]$ and, as before, it is easy to see that $y = [z_{n_1} \cdots z_{n_t}, \varphi] \in [G, \varphi]$. □

If $n = 2$, Theorem 1.2 gives an example of an abelian-by-(cyclic of order p) p-group that is a conterexample to Problem 3 of [3] for every odd prime p.

Regarding the case where $p = 2$, in [2], there is an example of a 2-group of class 2 such that $[G, \varphi] \leq G$ for every $\varphi \in \mathrm{Aut}(G)$, is it possible to have such a 2-group of class > 2?

Remark 3.2. Notice that if we consider the 2-group $G = A \rtimes \langle x\rangle$, with $A = \langle a\rangle \times \langle b\rangle \simeq \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, $\langle x\rangle \simeq \mathbb{Z}_{2^{n-1}}$, $n > 1$, and $c^x = c^3$ for every $c \in A$, then there exists $\varphi \in \mathrm{Aut}(G)$ such that $[G, \varphi] \not\leq G$.

In order to define such an automorphism, first of all, we prove, by induction, that for every $c \in A$ and for every $t \geq 1$, we have that

$$(xc)^{2^t} = x^{2^t} c^{2^{t+1}h} \text{ for some } h \in \mathbb{N}.$$

For if $t = 1$, then

$$(xc)^2 = xcxc = x^2 c^x c = x^2 c^4.$$

Suppose that, for some $t \in \mathbb{N}$, we have

$$(xc)^{2^t} = x^{2^t} c^{2^{t+1}h},$$

then $(xc)^{2^{t+1}} = ((xc)^{2^t})^2 = x^{2^{t+1}} (c^{2^{t+1}h})^{x^{2^t}} (c^{2^{t+1}h}) = x^{2^{t+1}} (c^{2^{t+1}h})^{3^{2^t}} c^{2^{t+1}h} = x^{2^{t+1}} c^{2^{t+1}h(3^{2^t}+1)} = x^{2^{t+1}} c^{2^{t+2}k}$ since $3^{2^t} + 1$ is even.

Therefore, if $c \in A$ has order 2^n, then xc has order 2^{n-1} and we may consider $\varphi \in \mathrm{Aut}(G)$ such that $y^\varphi = y$ for every $y \in A$ and $x^\varphi = xc$. We have that $[x, \varphi] \in A$ and, by induction, $[x^\alpha, \varphi] = [x, \varphi]^{x^{\alpha-1}} [x^{\alpha-1}, \varphi] \in A$ for every $\alpha \in \mathbb{N}$. Therefore $[G, \varphi] = [\langle x \rangle, \varphi]$ because for every $g = x^\alpha y \in G$ with $\alpha \in \mathbb{N}$ and $y \in A$, we have that $[g, \varphi] = [x^\alpha y, \varphi] = [x^\alpha, \varphi] \in [\langle x \rangle, \varphi]$. This implies that $|[G, \varphi]| = |[\langle x \rangle, \varphi]| = |\langle x \rangle : C_{\langle x \rangle}(\varphi)| \leq 2^{n-1}$. Now $x^{-1}x^\varphi = c \in [G, \varphi]$, so $[G, \varphi] \not\leq G$ since c has order 2^n.

Remark 3.3. Also every dihedral 2-group $G = D_{2^n} = A \rtimes \langle x \rangle$ with $A = \langle a \rangle \simeq \mathbb{Z}_{2^{n-1}}$ and $\langle x \rangle \simeq \mathbb{Z}_2$ $(n > 2)$ has an automorphism $\varphi \in \mathrm{Aut}(G)$ such that $[G, \varphi] \not\leq G$. In fact, if we consider the automorphism φ defined by $a^\varphi = a$ and $x^\varphi = xa^{-1}$, then we have that $[G, \varphi] = [\langle x \rangle, \varphi]$ so $|[G, \varphi]| \leq 2$. But $a^{-1} = x^{-1}x^\varphi \in [G, \varphi]$, then it is not a subgroup because a^{-1} has order $2^{n-1} > 2$.

Notice that, in particular, if $n = 3$, that is, $G = D_8$, then $\varphi_{|Z(G)} = \mathrm{id}_{Z(G)}$ because $[a, x]^\varphi = [a, xa^{-1}] = [a, x]$.

Also the quaternion group $Q_8 = \{1, -1, i, j, k, -i, -j, -k\}$ has an automorphism φ such that $[Q_8, \varphi] \not\leq Q_8$ and $\varphi_{|Z(Q_8)} = \mathrm{id}_{Z(Q_8)}$. Actually, if we consider the automorphism φ, defined by $i^\varphi = i$ and $j^\varphi = k$, then we have $[Q_8, \varphi] = \{1, -i\}$.

Indeed, it is possible to prove the following result:

Proposition 3.4. *For every prime p, if G is an extraspecial p-group, then there exists $\varphi \in \mathrm{Aut}(G)$ such that $[G, \varphi] \not\leq G$.*

In order to show this proposition, first of all, we recall that a group G is the central product of two subgroups H and K if $G = HK$, $[H, K] = 1$, and $H \cap K = Z(G)$. Then we prove the following two lemmas.

Lemma 3.5. *Suppose that a group G is the central product of two subgroups H and K; if there exists an automorphism $\phi \in \mathrm{Aut}(H)$ such that $z^\phi = z$ for every $z \in Z(G)$ and $[H, \phi] \not\leq H$, then there also exists $\varphi \in \mathrm{Aut}(G)$ such that $[G, \varphi] \not\leq G$.*

Proof. The group G is the central product of H and K, hence for every $g \in G$, we have that $g = hk$ with $h \in H$ and $k \in K$.

Let $\phi \in \mathrm{Aut}(H)$; if the restriction $\phi_{|Z(G)} = \mathrm{id}_{Z(G)}$, then $g^{\varphi} := (hk)^{\varphi} = h^{\phi}k$ defines a map $\varphi : G \to G$. In fact, $hk = h_1k_1$, with $h, h_1 \in H$ and $k, k_1 \in K$, if and only if $h_1^{-1}h = k_1k^{-1} \in Z(G)$; so $(h_1^{-1})^{\phi}h^{\phi} = (h_1^{-1}h)^{\phi} = (k_1k^{-1})^{\phi} = k_1k^{-1}$, that is, $h^{\phi}k = h_1^{\phi}k_1$.

It is easy to check that $\varphi \in \mathrm{Aut}(G)$, moreover $[G, \varphi] = \{g^{-1}g^{\varphi} | g \in G\} = \{k^{-1}h^{-1}h^{\phi}k | h \in H, k \in K\} = \{h^{-1}h^{\phi} | h \in H\} = [H, \phi]$. □

Lemma 3.6. *Let p be a prime, and G be a group of order p^3. If G is non-abelian, then there exists $\varphi \in Aut(G)$ such that $z^{\varphi} = z$ for every $z \in Z(G)$ and $[G, \varphi] \not\leq G$.*

Proof. If $p = 2$, then either $G \simeq D_8$ or $G \simeq Q_8$ and the claim is true, as we have seen before. So we may suppose that p is odd and in this case we have that either $G = \langle x, y, z | x^p = y^p = z^p = 1, [x, z] = [y, z] = 1, [x, y] = z\rangle$, that is, $G = H \rtimes \langle y\rangle$ with $H = \langle x, z\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $\langle y\rangle \simeq \mathbb{Z}_p$, and $x^y = xz$, $z^y = z$, or $G = \langle x, y | x^{p^2} = y^p = 1, x^y = x^{1+p}\rangle$, that is, $G = \langle x\rangle \rtimes \langle y\rangle$ with $x^{p^2} = 1$, $y^p = 1$, and $x^y = x^{1+p}$. Observe that in both cases, G is nilpotent of class 2, then for every $n \in \mathbb{N}$ such that $n \geq 2$, we have $(yx)^n = y^nx^n[x, y]^{\frac{n(n-1)}{2}}$, that is, $(yx)^n = y^nx^nz^{\frac{n(n-1)}{2}}$ in the first case, $(yx)^n = y^nx^nx^{p\frac{n(n-1)}{2}} = y^nx^{n+p\frac{n(n-1)}{2}}$ in the second case (see [8, 5.3.5 p. 137]).

In the first case, we may consider $\varphi \in \mathrm{Aut}(G)$ defined by $x^{\varphi} = x$ and $y^{\varphi} = yx$ and we have that $z^{\varphi} = [x, y]^{\varphi} = [x, yx] = [x, y] = z$, that is, $\varphi_{|Z(G)} = \mathrm{id}_{Z(G)}$. Moreover, if we consider $g = y^tx^nz^m \in G$ with $0 \leq t, n, m \leq p - 1$, then we have that $g^{\varphi} = (yx)^tx^nz^m = y^tx^tz^{\frac{t(t-1)}{2}}x^nz^m = y^tx^{t+n}z^{\frac{t(t-1)}{2}+m}$ and so $g^{-1}g^{\varphi} = (y^{-t}x^{-n}z^{nt-m})(y^tx^{t+n}z^{\frac{t(t-1)}{2}+m}) = (x^{-n})^{y^t}x^{t+n}z^{nt+\frac{t(t-1)}{2}} = (xz^t)^{-n}x^{t+n}z^{nt+\frac{t(t-1)}{2}} = x^tz^{\frac{t(t-1)}{2}}$ with $0 \leq t \leq p-1$. Then $[G, \varphi] = \{x^tz^{\frac{t(t-1)}{2}} | 0 \leq t \leq p - 1\}$ and this set is not a subgroup of G.

In the second case, we may consider the automorphism $\phi \in \mathrm{Aut}(G)$ defined by $y^{\phi} = y$ and $x^{\phi} = yx$; since $[x, y]^{\phi} = [yx, y] = [x, y]$, we have that $\phi_{|Z(G)} = \mathrm{id}_{Z(G)}$. Moreover, if we consider $g = y^nx^m \in G$ with $0 \leq n \leq p-1$ and $0 \leq m \leq p^2-1$, then we have that $g^{\phi} = y^n(yx)^m = y^{n+m}x^{m(1+p\frac{(m-1)}{2})}$. Therefore $g^{-1}g^{\phi} = y^{p-n}x^{p^2-m(1+p)^{p-n}}y^{n+m}x^{m(1+p\frac{(m-1)}{2})} = y^mx^{m(1-(1+p)^m+p\frac{(m-1)}{2})}$ and the set of these elements, with $0 \leq m \leq p^2 - 1$, is not a subgroup of G. □

For every prime p, an extra-special p-group is the iterated central product of non-abelian groups of order p^3 (see for instance Lemma 2.2.9 of [6]), then, from the two previous lemmas, Proposition 3.4 follows.

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